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MIXED BOUNDARY CONDITIONS FOR A SIMPLIFIED QUANTUM ENERGY-TRANSPORT MODEL IN MULTI-DIMENSIONAL DOMAINS*

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Abstract. In this paper we obtain a weak solution to a quantum energy-transport model for semiconductors. The model is formally derived from the quantum hydrodynamic model in the large-time and small-velocity regime by Jüngel and Milišić [Nonlinear Anal.: Real World Appl., 12:1033–1046, 2011]. It consists of a fourth-order nonlinear parabolic equation for the electron density, an elliptic equation for the electron temperature, and the Poisson equation for the electric potential. Our solution is global in the time variable, while the N space variables lie in a bounded Lipschitz domain with a mixed boundary condition. The existence proof is based upon a carefully-constructed approximation scheme which generates a sequence of positive approximate solutions. These solutions are so regular that they can be used to form a variety of test functions to produce a priori estimates. Then these estimates are shown to be enough to justify passing to the limit in the approximate problems.

Keywords. existence; Lipschitz domains; mixed boundary conditions; temperature gradient; degenerate fourth-order parabolic equations; quantum energy-transport model for semiconductors

AMS subject classifications. 35Q99; 35B50; 35K65; 35K52; 35A01; 74K35; 82D25

1. Introduction

In this paper we are concerned with the existence of a weak solution to the initial boundary-value problem

$$\partial_t n = -\mathrm{div}\left[\frac{\varepsilon^2}{6} n\nabla\left(\frac{\Delta\sqrt{n}}{\sqrt{n}}\right) - \nabla(nT) + n\nabla V\right] \quad \text{in} \quad \Omega_T, \tag{1.1}$$

$$-\mathrm{div}(n\nabla T) = \frac{n}{d_l}(T_l(x) - T) \quad \text{in} \quad \Omega_T, \tag{1.2}$$

$$\lambda^2 \Delta V = n - C(x) \quad \text{in} \quad \Omega_T, \tag{1.3}$$

$$n = n_D, \quad \Delta\sqrt{n} = 0, \quad T = T_D, \quad V = V_D \quad \text{on} \quad \Sigma_D, \tag{1.4}$$

$$n\nabla\left(\frac{\Delta\sqrt{n}}{\sqrt{n}}\right)\cdot\nu = \nabla n\cdot\nu = n\nabla T\cdot\nu = \nabla V\cdot\nu = 0 \quad \text{on} \quad \Sigma_N, \tag{1.5}$$

$$n(x,0) = n_0(x) \quad \text{on} \quad \Omega. \tag{1.6}$$

Here $T>0$, Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$, $\Omega_T = \Omega\times(0,T)$, Γ_D an open subset of $\partial\Omega$, $\Gamma_N = \partial\Omega\setminus\overline{\Gamma_D}$, $\Sigma_D = \Gamma_D\times(0,T)$, $\Sigma_N = \Gamma_N\times(0,T)$, and ν the unit outward normal to the boundary $\partial\Omega$. The div (divergence), ∇ (gradient), and Δ (Laplacian) are all taken with respect to the space variables x. The system (1.1)–(1.3) can be formally derived from the quantum hydrodynamic equations [8]. In this case, $n = n(x,t)$ is the electron density, $V = V(x,t)$ the electrostatic potential, and T the electron temperature. The three physical parameters ε, λ, d_l are the scaled Planck constant, the scaled Debye length, and the energy relaxation time, respectively. The doping profile $C(x)$, the lattice temperature $T_l(x)$, the boundary data $n_D = n_D(x)$, $V_D = V_D(x)$, $T_D = T_D(x)$, and the initial function $n_0(x)$ are known functions of their arguments whose precise assumptions will be made at a later time.

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The system (1.1)–(1.3) is first studied in [8], where the domain Ω is assumed to be an N-dimensional torus with $N \le 3$, which implies that only the periodic boundary conditions are considered. Its mathematical approach relies on the identity

$$\operatorname{div}\left[2n\nabla\left(\frac{\Delta\sqrt{n}}{\sqrt{n}}\right)\right]=\sum_{i=1,j=1}^{N}\left[n(\ln n)_{x_ix_j}\right]_{x_ix_j}. \tag{1.7}$$

Obviously, our boundary conditions here are physically more realistic. Unfortunately, they also render identity (1.7) useless to us. Thus we must develop a new approach. A well-known difficulty in the study of fourth-order parabolic equations is that the maximum principle is no longer valid. In fact, the heat kernel for the biharmonic heat equation changes signs. Therefore, we have to rely on the nonlinear structure of our equations to obtain a non-negative n. It turns out that the term $\frac{1}{\sqrt{n}}$ in Equation (1.1) plays a significant role in this. In terms of a priori estimates, we set

$$F=\frac{\varepsilon^2}{6}\frac{\Delta\sqrt{n}}{\sqrt{n}}+V.$$

Write Equation (1.1) in the form

$$\partial_t n+\operatorname{div}(n\nabla F)=\operatorname{div}(n\nabla T+T\nabla n). \tag{1.8}$$

Then use $F-F_D$ as a test function in the above equation, where $F_D=F\,|_{\Gamma_D}$. Even though the boundary conditions (1.4) imply that $F_D=V_D$, this condition cannot be preserved in our approximate problems in Section 2. Consequently, the resulting inequality from the preceding test function is not closed in the sense that the upper bound still depends on n. To circumvent this problem, we couple this inequality with another one derived by using $\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n_D}}$ as a test function in Equation (1.8). The combination of the two yields our key estimates. By doing so, we have substantially generalized the results in both [8] and [14]. In particular, if we apply the method here to the problem considered in [14], the restriction $N\le 4$ in [14] becomes totally unnecessary.

The key to our development is to construct a sequence of approximate problems whose solutions are positive and so regular that $F-F_D$ and $\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n_D}}$ can be used as legitimate test functions. This effort is complicated by the fact that our boundary conditions are mixed and the fourth-order equation in the system involves the temperature gradient. In spite of this, we have managed to find a way to accomplish our goal. This is done in Sections 2 and 3, while in Section 4 we show the estimates obtained in the previous two sections are enough for us to pass to the limit in the approximate problems.

Recently, the optimal transport theory has been employed to treat a variety of fourth-order nonlinear parabolic equations. See, e.g., [4] and the references therein. The idea is to formulate these equations as gradient flows of various entropy functionals for various transportation metrics. The most famous example is the Fokker–Planck equation viewed by Jordan, Kinderlehrer and Otto [6] as the gradient flow of the Boltzmann entropy for the quadratic Monge–Kantorovich MK2 (frequently named Wasserstein metric). However, this theory does not seem to be capable of handling problems with mixed boundary conditions.

We are ready to introduce the notion of a weak solution.

Definition 1.1. *We say that a set of six measurable functions n,V,T,G,K,H is a weak solution of Problem (1.1)–(1.6) if:*

(1) $n\geq 0$, $\sqrt{n},V\in L^\infty(0,T;W^{1,2}(\Omega))$, $n^{\frac{1}{4}}\in L^2(0,T;W^{1,2}(\Omega))$, $n\in C([0,T];L^1(\Omega))$, $\Delta\sqrt{n}\in L^1(0,T;W^{1,1}(\Omega))$, $G\in L^2(\Omega_T)$, *and* $K,H\in (L^2(\Omega_T))^N$;

(2) $V=V_D$, $n=n_D$, $\Delta\sqrt{n}=0$ *on* Σ_D, *and* $\nabla\sqrt{n}\cdot\nu=0$ *on* Σ_N;

(3) $K=\operatorname{div}(\sqrt{n}T)-T\nabla\sqrt{n}$, $\quad T\sqrt{n}=T_D\sqrt{n_D}$ *on* Σ_D,

$$G=\frac{\Delta\sqrt{n}}{n^{\frac{1}{4}}} \quad \text{on the set } S_0\equiv\{(x,t)\in\Omega_T: n(x,t)>0\}, \text{ and} \tag{1.9}$$

$$H=\frac{\varepsilon^2}{6}\left(\nabla\Delta\sqrt{n}-2\frac{\Delta\sqrt{n}}{n^{\frac{1}{4}}}\nabla n^{\frac{1}{4}}\right)+\nabla V\sqrt{n} \quad \text{on} \quad S_0; \tag{1.10}$$

(4) *for each* $\xi\in C^\infty(R^N\times(-\infty,\infty))$ *such that* $\xi=0$ *on* Σ_D, *we have*

$$-\int_{\Omega_T}\xi_t n\,dxdt+\int_\Omega \xi(x,T)n(x,T)dx-\int_\Omega\xi(x,0)n_0(x)dx$$
$$+\int_{\Omega_T}\left(-\sqrt{n}H\nabla\xi+\sqrt{n}K\nabla\xi+T\nabla n\nabla\xi)\right)dxdt=0,$$
$$\int_{\Omega_T}\sqrt{n}K\nabla\xi dxdt=\int_{\Omega_T}\frac{n}{d_l}(T_l(x)-T)\xi dxdt,$$

and

$$-\lambda^2\int_{\Omega_T}\nabla V\nabla\xi dxdt=\int_{\Omega_T}(n-C(x))\xi dxdt.$$

Let us make some remarks about the definition. Since n may not be bounded below away from 0, no a priori bounds for ∇T can be expected. As a result, ∇T may not be a function, and the traces of T on Γ_D may not make sense. Thus the introduction of the function K is necessary, and the boundary condition in part (3) of the definition is a reasonable substitute for $T=T_D$ on Σ_D. Note that $K=\sqrt{n}\nabla T$ if ∇T exists as an L^p function with $p\geq 1$, and $T=T_D$ on Σ_D whenever it can be defined. Similarly, Equation (1.10) implies that the term $\sqrt{n}\nabla\left(\frac{\Delta\sqrt{n}}{\sqrt{n}}\right)$ has been replaced with

$$\nabla\Delta\sqrt{n}-2G\nabla n^{\frac{1}{4}}=\nabla\Delta\sqrt{n}-\frac{\Delta\sqrt{n}\nabla\sqrt{n}}{\sqrt{n}}.$$

We can only determine G and H on the set S_0, however their values on $\Omega_T\setminus S_0$ do not matter because there we always have $\sqrt{n}H=G\nabla n^{\frac{1}{4}}=0$. It does not seem possible to prove that $|\Omega_T\setminus S_0|=0$. We do have

$$|S_0|>0, \tag{1.11}$$

provided that the function n_D in condition (2) of the definition is positive a.e. on Σ_D. Recall that we only assume that $\partial\Omega$ is Lipschitz. As a result, the membership of $\Delta\sqrt{n}$ in $L^p(\Omega_T)$ for some $p\geq 1$ does not necessarily mean that $\nabla^2\sqrt{n}$, the Hessian of $\sqrt{n}$, lies in the same space. Thus the boundary condition

$$\nabla\sqrt{n}\cdot\nu=0 \text{ on } \Sigma_N$$

is understood in the sense that

$$-\int_{\Omega_T} \nabla\sqrt{n}\nabla\xi dxdt = \int_{\Omega_T} \Delta\sqrt{n}\xi dxdt$$

for all $\xi \in C^\infty(R^N \times R)$ with $\xi = 0$ on Σ_D.

The main result of this paper is that, under suitable assumptions on the given data, there is a weak solution to the initial boundary-value problem (1.1)-(1.6).

2. The approximation problems

The main result of this section is:

THEOREM 2.1. *Assume:*

(H1) Ω *is a bounded domain in* $\mathbb{R}^N$ *with Lipschitz boundary* $\partial\Omega$ *and* Γ_D *is a non-empty open subset of* $\partial\Omega$*;*

(H2) $N < 6$*;*

(H3) $\varepsilon, \lambda, d_l \in (0,\infty)$*;* $T_l(x), C(x) \in L^\infty(\Omega)$*;* $\rho_D, F_D, T_D, V_D \in W^{1,\infty}(\Omega)$ *with*

$$\min_{\overline{\Omega}} \rho_D > 0; \tag{2.1}$$

(H4) $\tau > 0$ *and* $p > 4$*;*

(H5) $f \in L^{\frac{4N}{N+2}}(\Omega)$.

Then there exists a quadruplet (ρ, F, T, V) *in the space* $\left(W^{1,2}(\Omega)\right)^4$ *with* $V, T, \rho \in L^\infty(\Omega)$*,* $\rho \geq 0$*, satisfying*

$$\frac{\rho^2 - f^2}{\tau} + \operatorname{div}\left[(\rho+\tau)^2\nabla F\right] = \operatorname{div}\left[(\rho+\tau)^2\nabla T + 2\rho T\nabla\rho\right] \quad in \quad \Omega, \tag{2.2}$$

$$-\operatorname{div}\left[(\rho+\tau)^2\nabla T\right] = \frac{\rho^2}{d_l}(T_l(x) - T) \quad in \quad \Omega, \tag{2.3}$$

$$-\frac{\varepsilon^2}{6}\Delta\rho + \tau\rho^{p-1} = (V - F)\rho + \tau \quad in \quad \Omega, \tag{2.4}$$

$$\lambda^2\Delta V = \rho^2 - C(x) \quad in \quad \Omega, \tag{2.5}$$

$$\rho = \rho_D, \quad T = T_D, \quad F = F_D, \quad V = V_D \quad on \quad \Gamma_D, \tag{2.6}$$

$$\nabla\rho\cdot\nu = \nabla F\cdot\nu = \nabla T\cdot\nu = \nabla V\cdot\nu = 0 \quad on \quad \Gamma_N \tag{2.7}$$

in the weak sense.

REMARK 2.1. Assumption (H1) implies that the Sobolev inequality

$$\|u\|_{q^*} \leq c\|\nabla u\|_q$$

holds for all $u \in W^{1,q}_{\Gamma_D}(\Omega) \equiv \{v \in W^{1,q}(\Omega), v = 0 \text{ on } \Gamma_D\}$, where $1 \leq q < N, q^* = \frac{Nq}{N-q}$, $\|\cdot\|_q$ is the norm in $L^q(\Omega)$ and c is a positive number determined by N, q, Γ_D and Ω. In Equations (2.2)–(2.7), we have simply transformed a fourth-order equation into a system of two second-order equations, discretized the time derivative, and then regularized the resulting system by adding a positive number to the possibly degenerate elliptic coefficients. However, this approximation has worked mainly because we have added the two terms $\tau, \tau\rho^{p-1}$ in Equation (2.4). As we shall see, the latter term has a regularizing effect, while the former term leads us to the conclusion that ρ is bounded below away from 0. This enables us to solve Equation (2.4) for F and to obtain regularity properties for $\frac{1}{\rho}$. The idea was first employed by the author in [13]. Note that this τ term is not needed in [14] because the solution there is classical, and as a result the strong maximum principle is applicable. Finally, Assumption (H2) is due to the presence of the temperature gradient in Equation (2.2).

Before we prove Theorem 2.1, we state a few preparatory lemmas. The first one collects some results that are useful to us.

LEMMA 2.1. *The following statements hold true:*

(i) Let $\{v_j\}$ be a sequence in a Banach space. If every subsequence of $\{v_j\}$ has a further subsequence converging to the same limit v, the whole sequence $\{v_j\}$ also converges to v.

(ii) Let $\{u_j\}$ be a sequence in a Hilbert space with the property $u_j \rightharpoonup u$ weakly. If $\lim_{j\to\infty}\|u_j\|^2 = \|u\|^2$, then $u_j \to u$ strongly.

(iii) If $\theta(s)$ is an increasing (decreasing) function on $\mathbb{R}$, then

$$(s-t)\theta(s) \geq (\leq) \int_t^s \theta(r)dr \quad \text{for all } s,t\in\mathbb{R}.$$

(iv) Let $\{u_j\}$ be a bounded sequence in $W^{1,2}(\Omega)$ satisfying

$$\begin{aligned} -\operatorname{div}[a(x)\nabla u_j] &= f_j \quad \text{in} \quad \Omega, \\ u_j &= u_D \quad \text{on} \quad \Gamma_D, \\ \nabla u_j\cdot\nu &= 0 \quad \text{on} \quad \Gamma_N, \end{aligned}$$

where $\{f_j\}$ is a sequence in $L^2(\Omega)$ with $\|f_j\|_1 \leq c$, Ω, Γ_D, Γ_N are given as in Theorem 2.1, $u_D \in W^{1,2}(\Omega)$, and $a(x)$ satisfies

$$0 < c_0 \leq a(x) \leq c_1 \quad \text{a.e. on } \Omega \text{ for some } c_0\leq c_1 \text{ in } (0,\infty). \tag{2.8}$$

Then $\{u_j\}$ is precompact in $W^{1,q}(\Omega)$ for each $q\in[1,2)$. If, in addition, $\|f_j\|_r \leq c$ for some $r > \frac{2N}{N+2}$, then $\{u_j\}$ is precompact in $W^{1,2}(\Omega)$.

Items (i)-(iii) are well-known. The first part of (iv) can also be found in many places. See, e.g., [15]. The second part of (iv) can be seen from the simple estimate

$$\begin{aligned} \int_\Omega |\nabla u_i - \nabla u_j|^2 dx &= \int_\Omega (f_i - f_j)(u_i - u_j)dx \\ &\leq \|f_i - f_j\|_r \|u_i - u_j\|_{\frac{r}{r-1}}. \end{aligned} \tag{2.9}$$

If $r > \frac{2N}{N+2}$, then $\frac{r}{r-1} < \frac{2N}{N-2} = 2^*$ and $\{u_j\}$ is precompact in $L^q(\Omega)$ for each $q < 2^*$.

LEMMA 2.2. *Let $a(x)$, Ω, Γ_D, Γ_N be given as in the preceding lemma and $u\in W^{1,2}(\Omega)$ a solution of the problem*

$$\begin{aligned} -\operatorname{div}[a(x)\nabla u] &= h(x) \quad \text{in} \quad \Omega, \\ u &= u_D \quad \text{on} \quad \Gamma_D, \\ \nabla u\cdot\nu &= 0 \quad \text{on} \quad \Gamma_N. \end{aligned}$$

Assume that $u_D \in W^{1,2}(\Omega)\cap L^\infty(\Omega)$ and $h\in L^q(\Omega)$ for some $q > \frac{N}{2}$. Then

$$\|u\|_\infty \leq c(\|u_D\|_\infty + \|h\|_q), \tag{2.10}$$

where the constant c depends only on Ω, q, and N.

This lemma is well-known.

We are ready to prove the main result.

Proof. **(Proof of Theorem 2.1.)** We only need to focus our attention on the case where $N>2$. The case where $N=2$ is easier to handle, and therefore it will be left untreated. A solution will be constructed via the Leray–Schauder Fixed Point Theorem ([5], Theorem 11.3). For this purpose, we set

$$\mathcal{A}=W^{1,2}(\Omega)\cap L^\infty(\Omega).$$

Define an operator B from $\mathcal{A}$ into $\mathcal{A}$ as follows: Given $\rho\in\mathcal{A}$, we solve the problem

$$\lambda^2\Delta V=\rho^2-C(x) \quad \text{in} \quad \Omega, \tag{2.11}$$

$$V=V_D \quad \text{on} \quad \Gamma_D, \tag{2.12}$$

$$\nabla V\cdot\nu=0 \quad \text{on} \quad \Gamma_N \tag{2.13}$$

for V. The classical theory for linear elliptic equations asserts that this problem has a unique solution V in the space $\mathcal{A}$. Then we proceed to consider the problem

$$-\mathrm{div}\left[(\rho^+ +\tau)^2\nabla T\right]=\frac{\rho^2}{d_l}(T_l(x)-T) \quad \text{in} \quad \Omega, \tag{2.14}$$

$$T=T_D \quad \text{on} \quad \Gamma_D, \tag{2.15}$$

$$\nabla T\cdot\nu=0 \quad \text{on} \quad \Gamma_N, \tag{2.16}$$

where $\rho^+=\rho$ if $\rho>0$ and 0 otherwise.

CLAIM 2.1. *There is a unique function T in $W^{1,2}(\Omega)$ satisfying Equations (2.14)-(2.16) in the weak sense. Furthermore, we have:*

(C1) $m\le T\le M$, where

$$M=\max\{ess\ sup_\Omega T_l(x),\max_\Omega T_D\},$$

$$m=\min\{ess\ inf_\Omega T_l(x),\min_\Omega T_D\};$$

(C2) $\int_\Omega(\rho^+ +\tau)^2|\nabla T|^2dx\le c\int_\Omega(|\rho|+\tau)^2dx$. Here and in what follows c denotes a positive number that depends on Ω and the given data except τ and f.

The existence and uniqueness of a weak solution to Problem (2.14)-(2.16) are well-known. To see that $T\le M$, we write Equation (2.14) in the form

$$-\mathrm{div}\left[(\rho^+ +\tau)^2\nabla T\right]+\frac{\rho^2}{d_l}(T-M)=\frac{\rho^2}{d_l}(T_l(x)-M) \tag{2.17}$$

and then use $(T-M)^+$ as a test function in the equation. The other inequality in (C1) can be established in a similar manner. To obtain (C2), we write Equation (2.14) in the form

$$-\mathrm{div}\left[(\rho^+ +\tau)^2\nabla T\right]+\frac{\rho^2}{d_l}(T-T_D)=\frac{\rho^2}{d_l}(T_l(x)-T_D). \tag{2.18}$$

Using $T-T_D$ as a test function in the above equation yields

$$\int_\Omega(\rho^+ +\tau)^2|\nabla T|^2dx\le c\int_\Omega(\rho^+ +\tau)^2|\nabla T_D|^2dx+c\int_\Omega\rho^2|T_l(x)-T_D|^2dx$$

$$\leq c \int_{\Omega} (|\rho|+\tau)^2 dx. \tag{2.19}$$

Having the functions V, T in hand, we form the problem

$$-\mathrm{div}\left[(\rho^+ + \tau)^2 \nabla F\right] = \frac{\rho^2 - f^2}{\tau} - \mathrm{div}\left((\rho+\tau)^2 \nabla T + 2\rho T \nabla \rho\right) \quad \text{in} \quad \Omega, \tag{2.20}$$

$$F = F_D, \quad \text{on} \quad \Gamma_D, \tag{2.21}$$

$$\nabla F \cdot \nu = 0 \quad \text{on} \quad \Gamma_N. \tag{2.22}$$

This problem also has a unique solution F in $W^{1,2}(\Omega)$. The image of ρ under B is defined to be the unique solution of the following problem

$$-\frac{\varepsilon^2}{6}\Delta\psi = -\tau|\rho|^{p-2}\rho + (V-F)\rho^+ + \tau \quad \text{in} \quad \Omega, \tag{2.23}$$

$$\psi = \rho_D \quad \text{on} \quad \Gamma_D, \tag{2.24}$$

$$\nabla\psi \cdot \nu = 0 \quad \text{on} \quad \Gamma_N. \tag{2.25}$$

To see that B is well-defined, first observe from the Sobolev inequality that $F \in L^{\frac{2N}{N-2}}(\Omega)$. By Assumption (H2), we have $\frac{2N}{N-2} > \frac{N}{2}$. Thus we can conclude, with the aid of Lemma 2.2, that Problem (2.23)–(2.25) has a unique solution ψ in the space $\mathcal{A}$.

We still need to prove the following assertions:

(F1) B is continuous;

(F2) B maps bounded sets into precompact ones;

(F3) there is a positive number c such that $\|\rho\|_{\mathcal{A}} \leq c$ for each $\rho \in \mathcal{A}$ and each $\sigma \in [0,1]$ satisfying

$$\sigma B(\rho) = \rho. \tag{2.26}$$

To claim Assertions (F1) and (F2), it is enough to show that

$$\rho_j \rightharpoonup \rho \quad \text{in } \mathcal{A} \text{ implies} \quad B(\rho_j) \to B(\rho) \quad \text{in } \mathcal{A}. \tag{2.27}$$

To this end, we infer from the weak convergence of $\{\rho_j\}$ in $W^{1,2}(\Omega)$ that there is a subsequence of $\{\rho_j\}$, still denoted by $\{\rho_j\}$, such that

$$\rho_j \to \rho \quad \text{a.e. in } \Omega. \tag{2.28}$$

Consequently,

$$\rho_j \to \rho \quad \text{strongly in } L^r(\Omega) \text{ for each } r > 1. \tag{2.29}$$

Denote by ψ_j, F_j, T_j, V_j the respective solutions of Problems (2.23)–(2.25), (2.20)–(2.22), (2.14)–(2.16), and (2.11)–(2.13) with ρ_j in place of ρ. We claim that F_j, T_j, V_j, ψ_j are all bounded in $W^{1.2}(\Omega)$. To see this, substitute ρ_j for ρ and V_j for V in Equation (2.11) and use $V_j - V_D$ as a test function in the resulting equation to derive

$$\begin{aligned}\|\nabla(V_j - V_D)\|_2 &\leq c\left(\|\nabla V_D\|_2 + \|\rho_j\|^2_{\frac{4N}{N+2}} + \|C(x)\|_{\frac{2N}{N+2}}\right)\\ &\leq c\|\rho_j\|^2_\infty + c.\end{aligned} \tag{2.30}$$

The boundedness of $\{T_j\}$ in $\mathcal{A}$ can easily be seen from Claim 2.1. This together with Equation (2.20) and a calculation similar to Equation (2.30) implies that $\{F_j\}$ is bounded in $W^{1,2}(\Omega)$. As for $\{\psi_j\}$, first note that $\{(V_j-F_j)\rho_j\}$ is bounded in $L^{\frac{2N}{N-2}}(\Omega)$ and $\frac{2N}{N-2}>\frac{N}{2}$ due to Assumption (H2). We can easily infer from Lemma 2.2 that $\{\psi_j\}$ is bounded in $L^\infty(\Omega)$. On account of (iv) of Lemma 2.1, $\{V_j\}$, $\{T_j\}$ and ψ_j are also precompact in $W^{1,2}(\Omega)$. Thus we may assume that

$$\begin{aligned} V_j &\to V \quad \text{strongly in } W^{1,2}(\Omega),\\ T_j &\to T \quad \text{strongly in } W^{1,2}(\Omega),\\ F_j &\rightharpoonup F \quad \text{weakly in } W^{1,2}(\Omega),\\ \psi_j &\to \psi \quad \text{strongly in } W^{1,2}(\Omega). \end{aligned}$$

It is fairly easy to see that the limits V,T,F,ψ are the respective solutions of problems (2.11)–(2.13), (2.14)–(2.16), (2.20)–(2.22), and (2.23)–(2.25). By virtue of the uniqueness of a solution to these problems, the above convergences hold for the entire sequences. This is a consequence of (i) of Lemma 2.1. Remember that the sequence $\{(V_j-F_j)\rho_j\}$ is bounded in $L^{\frac{2N}{N-2}}(\Omega)$. We pick a number r from the interval $(\frac{N}{2},\frac{2N}{N-2})$. It immediately follows that

$$(V_j-F_j)\rho_j \to (V-F)\rho \quad \text{strongly in } L^r(\Omega). \tag{2.31}$$

From the boundary value problems satisfied by $\{\psi_j\}$ and ψ, we calculate, with the aid of Lemma 2.2, that

$$\|\psi_j-\psi\|_\infty \le c\|\tau(|\rho_j|^{p-2}\rho_j-|\rho|^{p-2}\rho)+(V_j-F_j)\rho_j-(V-F)\rho\|_r. \tag{2.32}$$

Thus $\{\psi_j\}$ is precompact in $L^\infty(\Omega)$.

It remains to show Assertion (F3). Without loss of generality, assume $\sigma>0$. Then Equation (2.26) is equivalent to

$$\frac{\rho^2-f^2}{\tau}+\operatorname{div}\left[(\rho^+ +\tau)^2\nabla F\right]=\operatorname{div}\left[(\rho+\tau)^2\nabla T+2\rho T\nabla\rho\right] \quad \text{in} \quad \Omega, \tag{2.33}$$

$$-\operatorname{div}\left[(\rho^+ +\tau)^2\nabla T\right]=\frac{\rho^2}{d_l}(T_l(x)-T) \quad \text{in} \quad \Omega, \tag{2.34}$$

$$-\frac{\varepsilon^2}{6}\Delta\frac{\rho}{\sigma}+\tau|\rho|^{p-2}\rho=(V-F)\rho^+ +\tau \quad \text{in} \quad \Omega, \tag{2.35}$$

$$\lambda^2\Delta V=\rho^2-C(x) \quad \text{in} \quad \Omega, \tag{2.36}$$

$$\rho=\sigma\rho_D, \quad T=T_D, \quad F=F_D, \quad V=V_D \quad \text{on} \quad \Gamma_D, \tag{2.37}$$

$$\nabla\rho\cdot\nu=\nabla F\cdot\nu=\nabla T\cdot\nu=\nabla V\cdot\nu=0 \quad \text{on} \quad \Gamma_N. \tag{2.38}$$

Use ρ^- as a test function in Equation (2.35) to obtain

$$\frac{\varepsilon^2}{6\sigma}\int_\Omega |\nabla\rho^-|^2dx+\tau\int_\Omega \left(\rho^-\right)^p dx \le 0. \tag{2.39}$$

This implies

$$\rho\ge 0 \quad \text{a.e. on } \Omega. \tag{2.40}$$

Thus we can rewrite Equation (2.35) as

$$-\frac{\varepsilon^2}{6}\Delta\frac{\rho}{\sigma}+\tau\rho^{p-1}=(V-F)\rho+\tau \quad \text{in} \quad \Omega. \tag{2.41}$$

Using $F-F_D$ as a test function in Equation (2.33) and keeping in mind (C1) and (C2) in Claim 2.1, we arrive at

$$\begin{aligned}
&\int_\Omega (\rho+\tau)^2|\nabla F-\nabla F_D|^2 dx\\
&\leq -\int_\Omega (\rho+\tau)^2\nabla F_D(\nabla F-\nabla F_D)dx+\int_\Omega \frac{\rho^2-f^2}{\tau}(F-F_D)dx\\
&\quad +\int_\Omega (\rho+\tau)^2\nabla T(\nabla F-\nabla F_D)dx+\int_\Omega 2\rho T\nabla\rho(\nabla F-\nabla F_D)dx\\
&\leq \delta\int_\Omega (\rho+\tau)^2|\nabla F-\nabla F_D|^2dx+c\|\rho^2-f^2\|_{\frac{2N}{N+2}}\|F-F_D\|_{\frac{2N}{N-2}}\\
&\quad +c(\delta)\int_\Omega (\rho+\tau)^2|\nabla T|^2dx+c(\delta)\int_\Omega T^2|\nabla\rho|^2dx+c(\delta)\int_\Omega (\rho+\tau)^2dx\\
&\leq 2\delta\int_\Omega (\rho+\tau)^2|\nabla F-\nabla F_D|^2dx+c(\delta)(\|f\|^4_{\frac{4N}{N+4}}+\|\rho\|^4_{\frac{4N}{N+2}})\\
&\quad +c(\delta)\int_\Omega |\nabla\rho|^2dx+c(\delta), \quad \delta>0,
\end{aligned} \tag{2.42}$$

from whence follows

$$\int_\Omega |\nabla F|^2dx\leq c\|\rho\|^4_{\frac{4N}{N+2}}+c\int_\Omega |\nabla\rho|^2dx+c. \tag{2.43}$$

Here c also depends on τ and f. By a similar argument, we can deduce

$$\int_\Omega |\nabla V|^2dx\leq c\|\rho\|^4_{\frac{4N}{N+2}}+c. \tag{2.44}$$

Our next step is to select $\rho-\sigma\rho_D$ as a test function in Equation (2.41). This gives rise to the equation

$$\begin{aligned}
&\frac{\varepsilon^2}{6\sigma}\int_\Omega |\nabla\rho|^2dx+\tau\int_\Omega \rho^{p-1}(\rho-\sigma\rho_D)dx\\
&=\frac{\varepsilon^2}{6}\int_\Omega \nabla\rho\nabla\rho_D dx+\int_\Omega (V-F)\rho(\rho-\sigma\rho_D)dx+\tau\int_\Omega (\rho-\sigma\rho_D)dx.
\end{aligned} \tag{2.45}$$

Note from (iii) of Lemma 2.1 that

$$\rho^{p-1}(\rho-\sigma\rho_D)\geq \frac{1}{p}[\rho^p-(\sigma\rho_D)^p] \quad \text{a.e. on } \Omega. \tag{2.46}$$

The fourth integral in Equation (2.45) can be estimated as follows:

$$\begin{aligned}
\int_\Omega (V-F)\rho(\rho-\sigma\rho_D)dx&\leq \|V-F\|_{\frac{2N}{N-2}}\left(\|\rho\|^2_{\frac{4N}{N+2}}+c\|\rho\|_{\frac{2N}{N+2}}\right)\\
&\leq \delta\int_\Omega |\nabla\rho|^2dx+c\|\rho\|^4_{\frac{4N}{N+2}}+c,
\end{aligned} \tag{2.47}$$

where $\delta>0$. Collecting the preceding two estimates in Equation (2.45), we derive

$$\frac{1}{\sigma}\int_\Omega |\nabla\rho|^2dx+\int_\Omega \rho^p dx\leq c\|\rho\|^4_{\frac{4N}{N+2}}+c$$

$$\begin{aligned} &\leq c\int_\Omega \rho^4 dx + c \\ &\leq \delta\int_\Omega \rho^p dx + c(\delta), \quad \delta>0. \end{aligned} \tag{2.48}$$

The last step is due to the fact that $p>4$. Thus we obtain

$$\int_\Omega |\nabla\rho|^2 dx \leq c, \qquad \int_\Omega \rho^p dx \leq c. \tag{2.49}$$

This, in turns, implies that

$$\|\nabla V\|_2, \quad \|\nabla F\|_2 \leq c. \tag{2.50}$$

It remains to show

$$\|\rho\|_\infty \leq c. \tag{2.51}$$

Owing to conditions (2.37), (2.38), and Equation (2.41), ρ satisfies

$$-\frac{\varepsilon^2}{6}\Delta\rho + \tau\sigma\rho^{p-1} = \sigma(V-F)\rho + \sigma\tau \quad \text{in } \Omega, \tag{2.52}$$

$$\rho = \sigma\rho_D \quad \text{on } \Gamma_D, \tag{2.53}$$

$$\nabla\rho\cdot\nu = 0 \quad \text{on } \Gamma_N. \tag{2.54}$$

Observe that results in ([2, 9], and [11]) are local. Hence they are not directly applicable here. To prove estimate (2.51), we define, for $u\in L^q(\Omega)$ with $q\geq 1$, that

$$\eta(r;u\chi_\Omega) = \sup_{x\in\mathbb{R}^N}\int_{|x-y|<r}\frac{|u(y)|\chi_\Omega(y)}{|x-y|^{N-2}}dy,$$

where χ_Ω is the characteristic function of Ω. If $q>\frac{N}{2}$, a simple application of Hölder's inequality yields

$$\eta(r;u\chi_\Omega) \leq c\|u\|_q r^{\frac{2q-N}{q}}, \quad c = c(N,q). \tag{2.55}$$

That is, $L^q(\Omega)\subset K_N(\Omega)$ if $q>\frac{N}{2}$, where $K_N(\Omega)$ is the so-called Kato class of functions [2]. A result in [16] asserts

$$\int_\Omega |u|w^2 dx \leq c\eta(r;u\chi_\Omega)\left(\int_\Omega |\nabla w|^2 dx + \frac{1}{r^2}\int_\Omega w^2\right) \tag{2.56}$$

for each $r>0$ and each $w\in W^{1,2}(\Omega)$, where c depends only on N and the Lipschitz boundary of Ω. If Ω is a ball with radius r, then result (2.56) first appeared in [3].

Now we are ready to employ the classical Moser iteration technique. For this purpose, we let

$$M = \max_{\overline{\Omega}}\rho_D, \tag{2.57}$$

$$K = \|M|V-F| + \tau\|_q, \quad \frac{N}{2} < q \leq \frac{2N}{N-2}, \tag{2.58}$$

$$v = (\rho - M)^+ + K. \tag{2.59}$$

The existence of such a q is due to the fact that $\frac{2N}{N-2} > \frac{N}{2}$, which also indicates that we can apply result (2.56) to obtain

$$\int_\Omega |V-F|w^2 \le cr^{\frac{2q-N}{q}} \int_\Omega |\nabla w|^2 dx + \frac{c}{r^{\frac{N}{q}}} \int_\Omega w^2 dx \quad \text{for each } w \in W^{1,2}(\Omega). \tag{2.60}$$

For each $\beta > 0$, the function $v^\beta - K^\beta$ is non-negative and satisfies $v^\beta - K^\beta |_{\Gamma_D} = 0$. Upon using it as a test function in Equation (2.52), we obtain

$$\begin{aligned}
\frac{\varepsilon^2}{6} \frac{4\beta}{(\beta+1)^2} \int_\Omega \left|\nabla v^{\frac{\beta+1}{2}}\right|^2 dx &= \frac{\varepsilon^2}{6} \int_\Omega \beta v^{\beta-1} \nabla\rho \nabla v dx \\
&\le \int_\Omega [\sigma(V-F)\rho + \sigma\tau](v^\beta - K^\beta) dx \\
&\le \int_\Omega (|V-F|\rho + \tau)(v^\beta - K^\beta) dx \\
&\le \int_\Omega (|V-F|(\rho - M) + M|V-F| + \tau) v^\beta dx \\
&\le \int_\Omega |V-F|(\rho-M)^+ v^\beta dx + \int_\Omega (M|V-F|+\tau) v^\beta dx.
\end{aligned} \tag{2.61}$$

Notice that $(\rho - M)^+ \le v$. We can deduce from estimate (2.60) that

$$\begin{aligned}
\int_\Omega |V-F|(\rho-M)^+ v^\beta dx &\le \int_\Omega |V-F| v^{\beta+1} dx \\
&\le cr^{\frac{2q-N}{q}} \int_\Omega |\nabla v^{\frac{\beta+1}{2}}|^2 dx + \frac{c}{r^{\frac{N}{q}}} \int_\Omega v^{\beta+1} dx, \quad r > 0.
\end{aligned} \tag{2.62}$$

The second integral on the right-hand side of Equation (2.61) can be estimated as follows:

$$\begin{aligned}
\int_\Omega (M|V-F|+\tau) v^\beta dx &\le \|M|V-F|+\tau\|_q \left(\int_\Omega v^{\frac{\beta q}{q-1}} dx\right)^{\frac{q-1}{q}} \\
&= \left(\int_\Omega (K v^\beta)^{\frac{q}{q-1}} dx\right)^{\frac{q-1}{q}} \\
&\le \left(\int_\Omega v^{\frac{(\beta+1)q}{q-1}} dx\right)^{\frac{q-1}{q}}.
\end{aligned} \tag{2.63}$$

We choose r so that the coefficient of the third integral in Equation (2.62) satisfies

$$cr^{\frac{2q-N}{q}} = \frac{\varepsilon^2}{12} \frac{4\beta}{(\beta+1)^2}.$$

Then plugging Inequalities (2.62) and (2.63) into (2.61) yields

$$\begin{aligned}
\int_\Omega \left|\nabla v^{\frac{\beta+1}{2}}\right|^2 dx &\le c\left(\frac{(\beta+1)^2}{\beta}\right)^{\frac{2q}{2q-N}} \int_\Omega v^{\beta+1} dx + c\frac{(\beta+1)^2}{\beta} \left(\int_\Omega v^{\frac{(\beta+1)q}{q-1}} dx\right)^{\frac{q-1}{q}} \\
&\le c\left(\frac{(\beta+1)^2}{\beta}\right)^{\frac{2q}{2q-N}} \left(\int_\Omega v^{\frac{(\beta+1)q}{q-1}} dx\right)^{\frac{q-1}{q}}.
\end{aligned} \tag{2.64}$$

Here we have used the fact that $\frac{(\beta+1)^2}{\beta}>1$. Note that $K^{\beta+1}\leq\frac{1}{|\Omega|}\int_\Omega v^{\beta+1}dx$. We calculate, with the aid of the Sobolev inequality, that

$$\begin{aligned}\left(\int_\Omega v^{\frac{(\beta+1)N}{N-2}}dx\right)^{\frac{N-2}{N}} &= \|v^{\frac{\beta+1}{2}}\|^2_{\frac{2N}{N-2}}\\ &\leq 2\|v^{\frac{\beta+1}{2}}-K^{\frac{\beta+1}{2}}\|^2_{\frac{2N}{N-2}}+2\|K^{\frac{\beta+1}{2}}\|^2_{\frac{2N}{N-2}}\\ &\leq c\int_\Omega\left|\nabla v^{\frac{\beta+1}{2}}\right|^2dx+cK^{\beta+1}\\ &\leq c\left(\frac{(\beta+1)^2}{\beta}\right)^{\frac{2q}{2q-N}}\left(\int_\Omega v^{\frac{(\beta+1)q}{q-1}}dx\right)^{\frac{q-1}{q}}. \end{aligned}\tag{2.65}$$

Set

$$\kappa=\frac{N(q-1)}{(N-2)q}.$$

Our assumption on q implies that $\kappa>1$. For $\beta\geq\kappa-1$, we can write Equation (2.65) in the form

$$\|v\|_{\kappa\frac{(\beta+1)q}{q-1}}\leq\left(c(\beta+1)^{\frac{4q}{2q-N}}\right)^{\frac{1}{\beta+1}}\|v\|_{\frac{(\beta+1)q}{q-1}}.\tag{2.66}$$

Let us take $\beta+1=\kappa^m$, $m=1,2,\cdots$, so that by Equation (2.66)

$$\begin{aligned}\|v\|_{\kappa^j\frac{N}{N-2}} &\leq\prod_{m=1}^{j-1}\left(c\kappa^{\frac{4qm}{2q-N}}\right)^{\kappa^{-m}}\|v\|_{\frac{N}{N-2}}\\ &\leq c^{\sum_{m=1}^{j-1}\kappa^{-m}}\kappa^{\sum_{m=1}^{j-1}\frac{4qm}{2q-N}\kappa^{-m}}\|v\|_{\frac{N}{N-2}}\\ &\leq c\|v\|_{\frac{N}{N-2}}. \end{aligned}\tag{2.67}$$

Letting $j\to\infty$, we therefore obtain

$$\|v\|_\infty\leq c\|v\|_{\frac{N}{N-2}},\tag{2.68}$$

whence by the interpolation inequality ([5], p. 146) we have

$$\|v\|_\infty\leq c\|v\|_1.\tag{2.69}$$

This implies that

$$\|\rho\|_\infty\leq cM+cK+c\|\rho\|_1.\tag{2.70}$$

This completes the proof of Theorem 2.1. □

Remark 2.2. Let us remark that the above proof of estimate (2.51) can easily be modified to show Lemma 2.2. We can also infer that an easy way to obtain estimate (2.51) is by making a further assumption

$$p>\frac{2N}{6-N}.\tag{2.71}$$

Then this combined with estimates (2.49) and (2.50) implies that $|V-F|\rho\in L^r(\Omega)$ for some $r>\frac{N}{2}$, and thus Lemma 2.2 becomes applicable. Under Assumption (H4), condition (2.71) already holds true for $N\le 4$. We pursue the generality here mainly because later we will have to employ the same proof to show that $\frac{1}{\rho}$ is also bounded. The classical regularity for linear elliptic equations [5] asserts that ρ is locally Hölder continuous. In fact, we even have $\rho\in W^{2,\frac{2N}{N-2}}_{\text{loc}}(\Omega)$. The mixed boundary condition prevents us from obtaining good estimates over the whole domain. However, if we further assume that $\partial\Gamma_D$ is a Lipschitz hyper surface in $\partial\Omega$ then $\rho\in C^\alpha(\overline{\Omega})$ for some $\alpha\in(0,1)$.

3. Properties of solutions

In this section we turn our attention to the properties of the solution obtained in Theorem 2.1.

LEMMA 3.1. *Let Assumptions (H1)–(H5) hold and (V,F,T,ρ) be the solution obtained in Theorem 2.1. Then ρ is bounded below away from 0.*

Proof. We begin by showing

$$\frac{1}{\rho^{\frac{N+2}{2(N-2)}}}\in W^{1,2}(\Omega). \tag{3.1}$$

To this end, we use $(\rho+\delta)^{-s}-(\rho_D+\delta)^{-s}$, where $\delta>0,\ s\ge 1$, as a test function in Equation (2.4) to derive

$$\begin{aligned}
&\tau\int_\Omega\frac{1}{(\rho+\delta)^s}dx+\frac{\varepsilon^2 s}{6}\int_\Omega\frac{1}{(\rho+\delta)^{s+1}}|\nabla\rho|^2dx\\
&\le\frac{\varepsilon^2 s}{6}\int_\Omega\frac{1}{(\rho_D+\delta)^{s+1}}\nabla\rho\nabla\rho_D dx+\tau\int_\Omega(\rho+\delta)^{p-1-s}dx\\
&\quad+\tau\int_\Omega\frac{1}{(\rho_D+\delta)^s}dx+\int_\Omega|V-F|\frac{1}{(\rho+\delta)^{s-1}}dx+\int_\Omega|V-F|\frac{\rho}{(\rho_D+\delta)^s}dx.
\end{aligned}\tag{3.2}$$

Obviously, we only need to worry about the second and fourth integrals on the right-hand side of the preceding inequality. We take

$$s=\frac{2N}{N-2}.$$

Subsequently,

$$\begin{aligned}
\int_\Omega|V-F|\frac{1}{(\rho+\delta)^{s-1}}dx&\le\left(\int_\Omega|V-F|^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{2N}}\left(\int_\Omega\frac{1}{(\rho+\delta)^{\frac{2N}{N-2}}}dx\right)^{\frac{N+2}{2N}}\\
&\le\sigma\int_\Omega\frac{1}{(\rho+\delta)^{\frac{2N}{N-2}}}dx+c(\sigma)\int_\Omega|V-F|^{\frac{2N}{N-2}}dx,\quad\sigma>0.
\end{aligned}\tag{3.3}$$

Next we consider the second integral on the right-hand side of Equation (3.2). Owing to the fact that $\rho\in L^\infty(\Omega)$, we only need to deal with the case where $p-s-1<0$. Obviously, $1+s-p<s=\frac{2N}{N-2}$. Consequently, we have

$$(\rho+\delta)^{p-s-1}=\frac{1}{(\rho+\delta)^{1+s-p}}\le\sigma\frac{1}{(\rho+\delta)^{\frac{2N}{N-2}}}+c(\sigma). \tag{3.4}$$

Using this and estimate (3.3) in Equation (3.2), choosing σ so small in the resulting inequality that the term

$$\sigma\int_\Omega \frac{1}{(\rho+\delta)^{\frac{2N}{N-2}}}dx$$

can be absorbed into the first term on the left-hand side of Equation (3.2), and then taking $\delta\to 0$, we yield condition (3.1).

Set

$$u=\frac{1}{\rho+\delta},\quad \delta>0.$$

Recall the regularity properties of ρ in Remark 2.2. We compute $\Delta\rho$ to obtain

$$\Delta\rho=-\frac{1}{u^2}\Delta u+\frac{2}{u^3}|\nabla u|^2.$$

Substituting this into Equation (2.4), we arrive at

$$-\frac{\varepsilon^2}{6}\Delta u+\frac{\varepsilon^2}{6}\frac{2}{u}|\nabla u|^2+\tau u^2=\left[\tau\frac{\rho^{p-1}}{\rho+\delta}-(V-F)\frac{\rho}{\rho+\delta}\right]u \tag{3.5}$$

from whence follows

$$-\frac{\varepsilon^2}{6}\Delta u\le\left[\tau\rho^{p-2}+|V-F|\right]u. \tag{3.6}$$

Since $\rho\in L^\infty(\Omega)$, the coefficient function of u in the above inequality lies in $L^{\frac{2N}{N-2}}(\Omega)$. As a result, the proof of estimate (2.51) is totally applicable here. We thereby obtain

$$\|u\|_\infty\le c+\|u\|_1\le c. \tag{3.7}$$

The last step is due to condition (3.1). The proof is complete. ∎

We remark that the last term τ in Equation (2.4) has played a critical role in the establishment of condition (3.1). The introduction of this term is the second key ingredient in our method, with the first one being the $\tau\rho^{p-1}$ term in Equation (2.4). It is not difficult to see that condition (3.1) implies $|V-F|u\in L^r(\Omega)$ for some $r>\frac{N}{2}$, provided that $N\le 4$. This means that Lemma 2.2 is enough for our method and our proof of estimate (2.51) is not needed under the assumption $N\le 4$.

For other properties of the solution, we need to replace Assumption (H5) with the following stronger assumption.

(H6) $f\in\mathcal{A},\quad f\ge 0,\quad f|_{\Gamma_D}=\rho_D$.

LEMMA 3.2. *Let Assumptions (H1)–(H4) and (H6) hold and (V,F,T,ρ) be the solution obtained in Theorem 2.1. Then we have*

$$-\int_\Omega(\rho^2-f^2)\frac{\Delta\rho}{\rho}\ge\int_\Omega\left(|\nabla\rho|^2-|\nabla f|^2\right)dx. \tag{3.8}$$

A more general version of this inequality is established in [12] for more regular ρ.

Proof. By Lemma 3.1, we can divide Equation (2.4) through by ρ to arrive at

$$F=\frac{\varepsilon^2}{6}\frac{\Delta\rho}{\rho}-\tau\rho^{p-2}+\tau\rho^{-1}+V\quad\text{a.e. on }\Omega, \tag{3.9}$$

from whence follows

$$\frac{\Delta\rho}{\rho}\in L^{\frac{2N}{N-2}}(\Omega). \tag{3.10}$$

This together with Assumption (H6) implies that

$$(\rho^2-f^2)\frac{\Delta\rho}{\rho}\in L^{\frac{2N}{N-2}}(\Omega) \text{ and } \rho^2(x)-f^2(x)=0 \text{ on } \Gamma_D. \tag{3.11}$$

Since $f\in L^\infty(\Omega)$, we have

$$\frac{\rho^2-f^2}{\rho}\in W^{1,2}(\Omega).$$

We calculate

$$\begin{aligned}
-\int_\Omega(\rho^2-f^2)\frac{\Delta\rho}{\rho}dx &= \int_\Omega \nabla\rho\nabla\left(\rho-\frac{f^2}{\rho}\right)dx\\
&= \int_\Omega\left(|\nabla\rho|^2-\frac{2f}{\rho}\nabla\rho\nabla f+\frac{f^2}{\rho^2}|\nabla\rho|^2\right)dx\\
&= \int_\Omega\left(|\nabla\rho|^2-|\nabla f|^2\right)dx+\int_\Omega\left|\nabla f-\frac{f}{\rho}\nabla\rho\right|^2dx\\
&\geq \int_\Omega\left(|\nabla\rho|^2-|\nabla f|^2\right)dx.
\end{aligned} \tag{3.12}$$

This completes the proof. □

LEMMA 3.3. *Let U be the solution of the problem*

$$\lambda^2\Delta U=f^2-C(x) \quad in \quad \Omega, \tag{3.13}$$
$$U=V_D \quad on \quad \Gamma_D, \tag{3.14}$$
$$\nabla U\cdot\nu=0 \quad on \quad \Gamma_N. \tag{3.15}$$

Then we have

$$\begin{aligned}
&\frac{\varepsilon^2}{6\tau}\int_\Omega(|\nabla\rho|^2-|\nabla f|^2)dx+\frac{2}{p}\int_\Omega(\rho^p-f^p)dx\\
&\quad+\frac{\lambda^2}{2\tau}\int_\Omega\left(|\nabla(V-V_D)|^2-|\nabla(U-V_D)|^2\right)dx+\int_\Omega(\rho+\tau)^2|\nabla F|^2dx\\
&\leq c\int_\Omega(\rho+\tau)^2dx+c\int_\Omega|\nabla\rho|^2dx+2\int_\Omega(\rho-f)dx+\frac{1}{\tau}\int_\Omega(\rho^2-f^2)(V_D-F_D)dx+c,
\end{aligned}$$

where c depends on the given data except τ and f.

Proof. Subtracting Equation (3.13) from Equation (2.11) yields

$$\lambda^2\Delta(V-U)=\rho^2-f^2 \quad \text{in} \quad \Omega. \tag{3.16}$$

Using $V-V_D$ as a test function in this equation, we derive

$$-\int_\Omega(\rho^2-f^2)V\,dx$$

$$
\begin{aligned}
&= -\int_\Omega (\rho^2 - f^2) V_D dx + \lambda^2 \int_\Omega [|\nabla(V - V_D)|^2 - \nabla(U - V_D)\nabla(V - V_D)] dx \\
&\geq -\int_\Omega (\rho^2 - f^2) V_D dx + \frac{1}{2}\lambda^2 \int_\Omega \left(|\nabla(V - V_D)|^2 - |\nabla(U - V_D)|^2\right) dx.
\end{aligned}
$$

Observe from (iii) of Lemma 2.1 that

$$
\rho^{p-2}(\rho^2 - f^2) \geq \int_{f^2}^{\rho^2} s^{\frac{p-2}{2}} ds = \frac{2}{p}(\rho^p - f^p), \tag{3.17}
$$

$$
\rho^{-1}(\rho^2 - f^2) \leq \int_{f^2}^{\rho^2} s^{-\frac{1}{2}} ds = 2(\rho - f). \tag{3.18}
$$

We are ready to estimate

$$
\begin{aligned}
\int_\Omega (\rho^2 - f^2) F dx &= \int_\Omega (\rho^2 - f^2) \left(\frac{\varepsilon^2}{6} \frac{\Delta \rho}{\rho} - \tau \rho^{p-2} + \frac{\tau}{\rho} + V \right) dx \\
&\leq -\frac{\varepsilon^2}{6} \int_\Omega \left(|\nabla \rho|^2 - |\nabla f|^2\right) - \frac{2\tau}{p} \int_\Omega (\rho^p - f^p) dx \\
&\quad + 2\tau \int_\Omega (\rho - f) dx + \int_\Omega (\rho^2 - f^2) V_D dx \\
&\quad - \frac{1}{2}\lambda^2 \int_\Omega \left(|\nabla(V - V_D)|^2 - |\nabla(U - V_D)|^2\right) dx.
\end{aligned} \tag{3.19}
$$

Use $F - F_D$ as a test function in Equation (2.20) to obtain

$$
\begin{aligned}
\int_\Omega (\rho + \tau)^2 |\nabla F|^2 dx &\leq c \int_\Omega (\rho + \tau)^2 |\nabla F_D|^2 dx + c \int_\Omega (\rho + \tau)^2 |\nabla T|^2 dx \\
&\quad + c \int_\Omega T^2 |\nabla \rho|^2 dx + \frac{1}{\tau} \int_\Omega (\rho^2 - f^2)(F - F_D) dx \\
&\leq c \int_\Omega (\rho + \tau)^2 dx + c \int_\Omega |\nabla \rho|^2 dx + \frac{1}{\tau} \int_\Omega (\rho^2 - f^2)(F - F_D) dx.
\end{aligned} \tag{3.20}
$$

The last step is due to Claim 2.1. Combining the last two inequalities yields the desired result. The proof is complete. ∎

LEMMA 3.4. *Let Assumptions (H1)–(H4) and (H6) hold and assume*

$$
F_D = -\tau \rho_D^{p-2} + \frac{\tau}{\rho_D} + V_D. \tag{3.21}
$$

Then we have

$$
\begin{aligned}
&\frac{\varepsilon^2}{6} \int_\Omega \frac{(\Delta \rho)^2}{\rho} dx + (p-2)\tau \int_\Omega \rho^{p-3} |\nabla \rho|^2 dx + \tau \int_\Omega \rho^{-2} |\nabla \rho|^2 dx + \int_\Omega \frac{\rho^2 - f^2}{\tau} \frac{1}{\rho_D + \tau} dx \\
&\leq c \int_\Omega (\rho + \tau)^2 |\nabla F| dx + c \int_\Omega |\nabla V| dx + c \int_\Omega |\nabla \rho| dx \\
&\quad + c \int_\Omega \rho |\nabla \rho| dx + \frac{1}{\tau} \int_\Omega \int_{f^2}^{\rho^2} \frac{1}{\sqrt{s} + \tau} ds dx + c \int_\Omega (\rho + \tau)^2 dx + c.
\end{aligned} \tag{3.22}
$$

Proof. We begin by first showing

$$\left|\int_\Omega \nabla\rho\nabla T dx\right| \le c\int_\Omega (\rho+\tau)^2 dx + c. \tag{3.23}$$

To this end, we use $\frac{1}{\rho+\tau}-\frac{1}{\rho_D+\tau}$ as a test function in Equation (2.14) to obtain

$$\begin{aligned}&\int_\Omega (\rho+\tau)^2\nabla T\left(-\frac{1}{(\rho+\tau)^2}\nabla\rho+\frac{1}{(\rho_D+\tau)^2}\nabla\rho_D\right)dx\\ &=\int_\Omega \frac{\rho^2}{d_l}(T_l(x)-T)\left(\frac{1}{\rho+\tau}-\frac{1}{\rho_D+\tau}\right)dx.\end{aligned} \tag{3.24}$$

We deduce from Claim 2.1 that

$$\begin{aligned}\left|\int_\Omega \nabla\rho\nabla T dx\right| &\le \left|\int_\Omega (\rho+\tau)^2\nabla T\frac{1}{(\rho_D+\tau)^2}\nabla\rho_D dx\right| + c\int_\Omega \rho^2 dx + c\int_\Omega \rho dx\\ &\le c\int_\Omega (\rho+\tau)^2|\nabla T|^2 dx + c\int_\Omega (\rho+\tau)^2 dx + c\\ &\le c\int_\Omega (\rho+\tau)^2 dx + c.\end{aligned} \tag{3.25}$$

Note that $\frac{1}{4\rho}|\nabla\rho|^2=|\nabla\sqrt{\rho}|^2$. Thus $\sqrt{\rho}\in W^{1,2}(\Omega)$. Obviously, $\frac{1}{\rho+\tau}-\frac{1}{\rho_D+\tau}$ is a legitimate test function for Equation (2.20). Upon using it, we derive

$$\begin{aligned}&\int_\Omega (\rho+\tau)^2\nabla F\left(-\frac{1}{(\rho+\tau)^2}\nabla\rho+\frac{1}{(\rho_D+\tau)^2}\nabla\rho_D\right)dx\\ &=\int_\Omega \frac{\rho^2-f^2}{\tau}\left(\frac{1}{\rho+\tau}-\frac{1}{\rho_D+\tau}\right)dx+\int_\Omega (\rho+\tau)^2\nabla T\left(-\frac{1}{(\rho+\tau)^2}\nabla\rho+\frac{1}{(\rho_D+\tau)^2}\nabla\rho_D\right)dx\\ &\quad+\int_\Omega 2\rho T\nabla\rho\left(-\frac{1}{(\rho+\tau)^2}\nabla\rho+\frac{1}{(\rho_D+\tau)^2}\nabla\rho_D\right)dx\\ &\le \int_\Omega \frac{\rho^2-f^2}{\tau}\left(\frac{1}{\rho+\tau}-\frac{1}{\rho_D+\tau}\right)dx-\int_\Omega \nabla T\nabla\rho dx\\ &\quad+c\int_\Omega (\rho+\tau)^2|\nabla T|^2 dx+c\int_\Omega (\rho+\tau)^2 dx+c\|T\|_\infty\int_\Omega |\nabla\sqrt{\rho}|^2 dx+c\|T\|_\infty\int_\Omega \rho|\nabla\rho| dx\\ &\le \int_\Omega \frac{\rho^2-f^2}{\tau}\left(\frac{1}{\rho+\tau}-\frac{1}{\rho_D+\tau}\right)dx+c\int_\Omega (\rho+\tau)^2 dx+c\int_\Omega |\nabla\sqrt{\rho}|^2 dx+c\int_\Omega \rho|\nabla\rho| dx+c.\end{aligned} \tag{3.26}$$

The last step is due to Claim 2.1. In view of Equation (3.9), we have $\frac{\Delta\rho}{\rho}\in W^{1,2}(\Omega)$. By Assumption (3.21), there holds

$$\left.\frac{\Delta\rho}{\rho}\right|_{\Gamma_D}=0.$$

Keeping these in mind, we calculate

$$\begin{aligned}-\int_\Omega \nabla F\nabla\rho dx &= -\int_\Omega \nabla\left(\frac{\varepsilon^2}{6}\frac{\Delta\rho}{\rho}-\tau\rho^{p-2}+\tau\rho^{-1}+V\right)\nabla\rho dx\\ &=\frac{\varepsilon^2}{6}\int_\Omega \frac{(\Delta\rho)^2}{\rho}dx+(p-2)\tau\int_\Omega \rho^{p-3}|\nabla\rho|^2 dx\end{aligned}$$

$$+\tau\int_\Omega \rho^{-2}|\nabla\rho|^2dx-\int_\Omega \nabla\rho\nabla V dx. \tag{3.27}$$

Use $\rho-\rho_D$ as a test function in Equation (2.11) to obtain

$$\begin{aligned}\int_\Omega \nabla V\nabla\rho dx&=\int_\Omega \nabla V\nabla\rho_D dx-\frac{1}{\lambda^2}\int_\Omega \rho^2(\rho-\rho_D)dx+\frac{1}{\lambda^2}\int_\Omega C(x)(\rho-\rho_D)dx\\ &\le c\int_\Omega|\nabla V|dx+c\int_\Omega \rho^2dx+c.\end{aligned} \tag{3.28}$$

Set

$$\frac{\Delta\rho}{\sqrt{\rho}}=G. \tag{3.29}$$

Then we have $G\in L^{\frac{2N}{N-2}}(\Omega)$ and

$$\Delta\rho=\sqrt{\rho}G. \tag{3.30}$$

By Lemma 3.1, we have that $\ln\sqrt{\rho}\in W^{1,2}(\Omega)$. Thus we can use $\ln\sqrt{\rho}-\ln\sqrt{\rho_D}$ as a test function in the above equation to obtain

$$-\int_\Omega \nabla\rho\left(\frac{1}{\sqrt{\rho}}\nabla\sqrt{\rho}-\frac{1}{\sqrt{\rho_D}}\nabla\sqrt{\rho_D}\right)dx=\int_\Omega \sqrt{\rho}G(\ln\sqrt{\rho}-\ln\sqrt{\rho_D})dx, \tag{3.31}$$

from which we can derive

$$\begin{aligned}\int_\Omega|\nabla\sqrt{\rho}|^2dx&\le\delta\int_\Omega G^2dx+c\int_\Omega|\nabla\rho|dx+c\int_\Omega\rho^2dx+c\\ &\le\delta\int_\Omega\frac{(\Delta\rho)^2}{\rho}dx+c\int_\Omega|\nabla\rho|dx+c\int_\Omega\rho^2dx+c,\quad \delta>0.\end{aligned} \tag{3.32}$$

Here we have used the fact that $|\rho\ln^2\sqrt{\rho}|\le c\rho^2+c$ for some $c>0$. Plugging this result, Equation (3.28), and Equation (3.27) into Equation (3.26) and selecting δ suitably small give the desired result. □

We remark that in [8] it is assumed that the number m in Claim (C1) is positive. We have been able to remove this restriction due to the estimate (3.32).

4. Existence theorem

The main result of this section is:

THEOREM 4.1. *Assume that Assumptions (H1)–(H3) in the preceding section hold with* $\rho_D=\sqrt{n_D}$. *If* $\sqrt{n_0(x)}\in\mathcal{A}$ *with* $\sqrt{n_0(x)}=\rho_D$ *on* Γ_D, *then there is a weak solution to Problem (1.1)-(1.6).*

Assumption (H3) implies that the boundary data are time-independent. This is done largely for technical convenience. But there is also a physical explanation for this. See [7] for details.

We introduce the new variable $\rho=\sqrt{n}$. Let $T>0$ be given. We divide the time interval $[0,T]$ into j equal subintervals, $j\in\{1,2,\cdots\}$. Set $\tau=\frac{T}{j}$. Fix some $p>4$. We approximate the problem (1.1)-(1.6) as follows. For $k=1,\cdots,j$ solve recursively the systems

$$\tfrac{\rho_k^2-\rho_{k-1}^2}{\tau}+\mathrm{div}\left[(\rho_k+\tau)^2\nabla F_k\right]=\mathrm{div}\left[(\rho_k+\tau)^2\nabla T_k+2T_k\rho_k\nabla\rho_k\right]\quad\text{in }\ \Omega, \tag{4.1}$$

$$-\text{div}\left[(\rho_k+\tau)^2\nabla T_k\right]=\frac{{\rho_k}^2}{d_l}(T_l(x)-T_k) \ \text{ in } \ \Omega, \tag{4.2}$$

$$-\frac{\varepsilon^2}{6}\Delta\rho_k+\tau{\rho_k}^{p-1}=(V_k-F_k)\rho_k+\tau \ \text{ in } \ \Omega, \tag{4.3}$$

$$\lambda^2\Delta V_k=\rho_k^2-C(x) \ \text{ in } \ \Omega, \tag{4.4}$$

$$\rho_k=\rho_D, \ \ T_k=T_D, \ \ F_k=F_D, \ \ V_k=V_D \ \text{ on } \ \Gamma_D, \tag{4.5}$$

$$\nabla\rho_k\cdot\nu=\nabla F_k\cdot\nu=\nabla T_k\cdot\nu=\nabla V_k\cdot\nu=0 \quad \text{on} \quad \Gamma_N, \tag{4.6}$$

$$\rho_0(x)=\sqrt{n_0(x)}, \tag{4.7}$$

where F_D is given as in Assumption (3.21). Introduce the functions

$$\begin{aligned}
\tilde{n}_j(x,t)&=(t-t_{k-1})\frac{{\rho_k}^2(x)-\rho_{k-1}^2(x)}{\tau}+\rho_{k-1}^2(x), \quad x\in\Omega, \ \ t\in(t_{k-1},t_k],\\
\overline{n}_j(x,t)&=\rho_k^2(x), \quad x\in\Omega, \ \ t\in(t_{k-1},t_k],\\
\overline{T}_j(x,t)&=T_k(x), \quad x\in\Omega, \ \ t\in(t_{k-1},t_k],\\
\overline{F}_j(x,t)&=F_k(x), \quad x\in\Omega, \ \ t\in(t_{k-1},t_k],\\
\overline{V}_j(x,t)&=V_k(x) \quad x\in\Omega, \ \ t\in(t_{k-1},t_k],
\end{aligned}$$

where $t_k=k\tau$. We can rewrite the system (4.1)–(4.7) as

$$\frac{\partial\tilde{n}_j}{\partial t}+\text{div}\left[(\sqrt{\overline{n}_j}+\tau)^2\nabla\overline{F}_j\right]=\text{div}\left[(\sqrt{\overline{n}_j}+\tau)^2\nabla\overline{T}_j\right]+\text{div}\left(2\overline{T}_j\sqrt{\overline{n}_j}\nabla\sqrt{\overline{n}_j}\right) \ \text{ in } \ \Omega_T, \tag{4.8}$$

$$-\text{div}\left[(\sqrt{\overline{n}_j}+\tau)^2\nabla\overline{T}_j\right]=\frac{\overline{n}_j}{d_l}(T_l(x)-\overline{T}_j) \ \text{ in } \ \Omega_T, \tag{4.9}$$

$$-\frac{\varepsilon^2}{6}\Delta\sqrt{\overline{n}_j}+\tau\sqrt{\overline{n}_j}^{p-1}=(\overline{V}_j-\overline{F}_j)\sqrt{\overline{n}_j}+\tau \ \text{ in } \ \Omega_T, \tag{4.10}$$

$$\lambda^2\Delta\overline{V}_j=\overline{n}_j-C(x) \ \text{ in } \ \Omega_T, \tag{4.11}$$

$$\sqrt{\overline{n}_j}=\rho_D, \ \ \overline{T}_j=T_D, \ \ \overline{F}_j=F_D, \ \ \overline{V}_j=V_D \ \text{ on } \ \Sigma_D, \tag{4.12}$$

$$\nabla\sqrt{\overline{n}_j}\cdot\nu=\nabla\overline{F}_j\cdot\nu=\nabla\overline{T}_j\cdot\nu=\nabla\overline{V}_j\cdot\nu=0 \quad \text{on} \quad \Sigma_N, \tag{4.13}$$

$$\tilde{n}_j(x,0)=n_0(x) \quad \text{on } \Omega. \tag{4.14}$$

Lemma 4.1. *Let the assumptions of Theorem 4.1 hold. Then we have:*

$$\begin{aligned}
&\max_{0\le t\le T}\left(\int_\Omega|\nabla\sqrt{\overline{n}_j}|^2dx+\tau\int_\Omega\sqrt{\overline{n}_j}^{p}dx+\int_\Omega|\nabla\overline{V}_j|^2dx\right)\\
&\quad+\tau\int_{\Omega_T}\sqrt{\overline{n}_j}^{p-3}|\nabla\sqrt{\overline{n}_j}|^2dxdt+\tau\int_{\Omega_T}\frac{|\nabla\sqrt{\overline{n}_j}|^2}{\sqrt{\overline{n}_j}^2}dxdt\\
&\quad+\int_{\Omega_T}(\sqrt{\overline{n}_j}+\tau)^2|\nabla\overline{F}_j|^2dxdt+\int_{\Omega_T}\frac{(\Delta\sqrt{\overline{n}_j})^2}{\sqrt{\overline{n}_j}}dxdt\le c,
\end{aligned} \tag{4.15}$$

$$m\le\overline{T}_j\ \le M \quad \textit{a.e. on } \Omega_T, \tag{4.16}$$

$$\max_{0\le t\le T}\int_\Omega(\sqrt{\overline{n}_j}+\tau)^2|\nabla\overline{T}_j|^2dx\le c, \tag{4.17}$$

where c depends only on the given data in Problem (1.1)–(1.6).

Proof. The Inequalities (4.16) and (4.17) are an easy consequence of Claim 2.1 and Equation (4.15). We claim that we can derive Equation (4.15) from Lemmas 3.3 and 3.4. First we can conclude from Lemma 3.4 that

$$\begin{aligned}
&\frac{\varepsilon^2}{6}\int_\Omega \frac{(\Delta\rho_k)^2}{\rho_k}dx+(p-2)\tau\int_\Omega \rho_k^{p-3}|\nabla\rho_k|^2dx \\
&\quad +\tau\int_\Omega \rho_k^{-2}|\nabla\rho_k|^2dx+\int_\Omega \frac{{\rho_k}^2-\rho_{k-1}^2}{\tau}\frac{1}{\rho_D+\tau}dx \\
&\le c\int_\Omega (\rho_k+\tau)^2|\nabla F_k|dx+\frac{1}{\tau}\int_\Omega\int_{\rho_{k-1}^2}^{\rho_k^2}\frac{1}{\sqrt{s}+\tau}dsdx+c\int_\Omega|\nabla V_k|dx \\
&\quad +c\int_\Omega(\rho_k+\tau)^2dx+c\int_\Omega|\nabla\rho_k|dx+c\int_\Omega\rho_k|\nabla\rho_k|dx+c. \qquad (4.18)
\end{aligned}$$

Multiply through the above inequality by τ, then sum up over k, and thereby derive

$$\begin{aligned}
&\int_{\Omega_s}\frac{(\Delta\sqrt{\overline{n}_j})^2}{\sqrt{\overline{n}_j}}dxdt+(p-2)\tau\int_{\Omega_s}\sqrt{\overline{n}_j}^{\,p-3}|\nabla\sqrt{\overline{n}_j}|^2dxdt \\
&\quad +\tau\int_{\Omega_s}\frac{|\nabla\sqrt{\overline{n}_j}|^2}{\sqrt{\overline{n}_j}^{\,2}}dxdt+\int_\Omega(\overline{n}_j-n_0)\frac{1}{\rho_D+\tau}dx \\
&\le c\int_{\Omega_s}(\sqrt{\overline{n}_j}+\tau)^2|\nabla\overline{F}_j|dxdt+\int_\Omega\int_{n_0}^{\overline{n}_j}\frac{1}{\sqrt{s}+\tau}dsdx+c\int_{\Omega_s}(\sqrt{\overline{n}_j}+\tau)^2dxdt \\
&\quad +c\int_{\Omega_s}|\nabla\overline{V}_j|dxdt+c\int_{\Omega_s}|\nabla\sqrt{\overline{n}_j}|dxdt+c\int_{\Omega_s}\sqrt{\overline{n}_j}|\nabla\sqrt{\overline{n}_j}|dxdt+c, \qquad (4.19)
\end{aligned}$$

where $\Omega_s=\Omega\times(0,s), s\in(0,T]$. Observe that

$$\int_{n_0}^{\overline{n}_j}\frac{1}{\sqrt{s}+\tau}ds\le\int_{n_0}^{\overline{n}_j}\frac{1}{\sqrt{s}}ds=2(\sqrt{\overline{n}_j}-\sqrt{n_0}). \qquad (4.20)$$

Utilizing this in Equation (4.19), we obtain

$$\begin{aligned}
\int_\Omega\overline{n}_jdx&\le c\int_{\Omega_s}\overline{n}_jdxdt+c\int_{\Omega_s}(\sqrt{\overline{n}_j}+\tau)^2|\nabla\overline{F}_j|dxdt \\
&\quad +c\int_{\Omega_s}|\nabla\overline{V}_j|dxdt+c\int_{\Omega_s}|\nabla\sqrt{\overline{n}_j}|dxdt+c\int_{\Omega_s}\sqrt{\overline{n}_j}|\nabla\sqrt{\overline{n}_j}|dxdt+c. \qquad (4.21)
\end{aligned}$$

In view of Lemma 3.3, we have

$$\begin{aligned}
&\frac{\varepsilon^2}{6\tau}\int_\Omega(|\nabla\rho_k|^2-|\nabla\rho_{k-1}|^2)dx+\frac{2}{p}\int_\Omega(\rho_k^p-\rho_{k-1}^p)dx \\
&\quad +\frac{\lambda^2}{2\tau}\int_\Omega\left(|\nabla(V_k-V_D)|^2-|\nabla(V_{k-1}-V_D)|^2\right)dx+\int_\Omega(\rho_k+\tau)^2|\nabla F_k|^2dx \\
&\le c\int_\Omega|\nabla\rho_k|^2dx+c\int_\Omega(\rho_k+\tau)^2dx \\
&\quad +2\int_\Omega(\rho_k-\rho_{k-1})dx+\frac{1}{\tau}\int_\Omega({\rho_k}^2-\rho_{k-1}^2)(V_D-F_D)dx+c. \qquad (4.22)
\end{aligned}$$

Multiply through this inequality by τ and sum up over k to obtain

$$
\begin{aligned}
&\int_\Omega |\nabla\sqrt{\overline{n}_j}|^2dx+\tau\int_\Omega \sqrt{\overline{n}_j}^p dx+\int_\Omega |\nabla\overline{V}_j|^2dx+\int_{\Omega_s}(\sqrt{\overline{n}_j}+\tau)^2|\nabla\overline{F}_j|^2dxdt\\
\le& c\int_{\Omega_s}|\nabla\sqrt{\overline{n}_j}|^2dxdt+c\int_{\Omega_s}(\sqrt{\overline{n}_j}+\tau)^2dxdt\\
&+2\tau\int_\Omega(\sqrt{\overline{n}_j}-\sqrt{n_0})dx+\int_\Omega(\overline{n}_j-n_0)(V_D-F_D)dx+c\\
\le& c\int_{\Omega_s}|\nabla\sqrt{\overline{n}_j}|^2dxdt+c\int_{\Omega_s}\overline{n}_jdxdt+c\int_\Omega\overline{n}_jdx+c\\
\le& c\int_{\Omega_s}|\nabla\sqrt{\overline{n}_j}|^2dxdt+c\int_{\Omega_s}\overline{n}_jdxdt+c\int_{\Omega_s}(\sqrt{\overline{n}_j}+\tau)^2|\nabla\overline{F}_j|dxdt+c\int_{\Omega_s}|\nabla\overline{V}_j|dxdt\\
&+c\int_{\Omega_s}|\nabla\sqrt{\overline{n}_j}|dxdt+c\int_{\Omega_s}\sqrt{\overline{n}_j}|\nabla\sqrt{\overline{n}_j}|dxdt+c. \qquad (4.23)
\end{aligned}
$$

The last step is due to Equation (4.21). By the Sobolev inequality, we have

$$\int_{\Omega_s}\overline{n}_jdxdt\le c\int_{\Omega_s}|\nabla\sqrt{\overline{n}_j}|^2dxdt+c. \qquad (4.24)$$

This combined with Equation (4.23) implies

$$\int_\Omega|\nabla\sqrt{\overline{n}_j}|^2dx+\int_\Omega|\nabla\overline{V}_j|^2dx\le c\int_{\Omega_s}|\nabla\overline{V}_j|^2dxdt+c\int_{\Omega_s}|\nabla\sqrt{\overline{n}_j}|^2dxdt+c. \qquad (4.25)$$

This puts us in a position to apply Gronwall's inequality, from whence follows

$$\int_\Omega|\nabla\sqrt{\overline{n}_j}|^2dx+\int_\Omega|\nabla\overline{V}_j|^2dx\le c. \qquad (4.26)$$

Using this in Equation (4.23) and then Equation (4.19) yields the desired result. □

Lemma 4.2. *Both* $\{\sqrt{\overline{n}_j}\}$ *and* $\{\overline{n}_j^{\frac14}\}$ *are precompact in* $L^2(0,T;W^{1,2}(\Omega))$.

Proof. We first show that $\{\overline{n}_j\}$ is precompact in $L^2((0,T);L^p(\Omega))$ for each $1\le p<\frac{N}{N-2}$. By the Sobolev inequality and Lemma 4.1, we have

$$\|\sqrt{\overline{n}_j}\|_{\frac{2N}{N-2}}\le\|\sqrt{\overline{n}_j}-\rho_D\|_{\frac{2N}{N-2}}+\|\rho_D\|_{\frac{2N}{N-2}}\le\|\nabla(\sqrt{\overline{n}_j}-\rho_D)\|_2+c\le c. \qquad (4.27)$$

Subsequently, there hold the inequalities

$$
\begin{aligned}
&\int_\Omega|\nabla\overline{n}_j|^{\frac{N}{N-1}}dx\\
=&\int_\Omega|2\sqrt{\overline{n}_j}\nabla\sqrt{\overline{n}_j}|^{\frac{N}{N-1}}dx\\
\le&\left(\int_\Omega|\nabla\sqrt{\overline{n}_j}|^2dx\right)^{\frac{N}{2(N-1)}}\left(\int_\Omega|\sqrt{\overline{n}_j}|^{\frac{2N}{N-2}}dx\right)^{\frac{N-2}{2(N-1)}}\le c, \qquad (4.28)\\
&\int_\Omega\left|(\sqrt{\overline{n}_j}+\tau)^2\nabla\overline{T}_j\right|^{\frac{N}{N-1}}dx\\
=&\int_\Omega\left|(\sqrt{\overline{n}_j}+\tau)\nabla\overline{T}_j(\sqrt{\overline{n}_j}+\tau)\right|^{\frac{N}{N-1}}dx
\end{aligned}
$$

$$\leq \left(\int_\Omega (\sqrt{\overline{n}_j}+\tau)^2 |\nabla \overline{T}_j|^2 dx\right)^{\frac{N}{2(N-1)}} \cdot \left(\int_\Omega |\sqrt{\overline{n}_j}+\tau|^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{2(N-1)}} \leq c, \quad (4.29)$$

$$\begin{aligned}
&\int_\Omega \left|(\sqrt{\overline{n}_j}+\tau)^2 \nabla \overline{F}_j\right|^{\frac{N}{N-1}} dx \\
&= \int_\Omega \left|(\sqrt{\overline{n}_j}+\tau)\nabla \overline{F}_j (\sqrt{\overline{n}_j}+\tau)\right|^{\frac{N}{N-1}} dx \\
&\leq \left(\int_\Omega (\sqrt{\overline{n}_j}+\tau)^2 |\nabla \overline{F}_j|^2 dx\right)^{\frac{N}{2(N-1)}} \cdot \left(\int_\Omega |\sqrt{\overline{n}_j}+\tau|^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{2(N-1)}} \\
&\leq c \left(\int_\Omega (\sqrt{\overline{n}_j}+\tau)^2 |\nabla \overline{F}_j|^2 dx\right)^{\frac{N}{2(N-1)}}. \quad (4.30)
\end{aligned}$$

Thus we can conclude that $\{\nabla \overline{n}_j\}$ and $\{(\sqrt{\overline{n}_j}+\tau^2)\nabla \overline{T}_j\}$ are bounded in $L^\infty(0,T;L^{\frac{N}{N-1}}(\Omega))$, while $\{(\sqrt{\overline{n}_j}+\tau)^2 \nabla \overline{F}_j\}$ is bounded in $L^2(0,T;L^{\frac{N}{N-1}}(\Omega))$. This together with estimate (4.8) asserts that $\{\frac{\partial \tilde{n}_j}{\partial t}\}$ is bounded in the space $L^2\left(0,T;\left(W^{1,N}_{\Gamma_D}(\Omega)\right)^*\right)$. In view of the definitions of $\tilde{n}_j, \overline{n}_j$, we have that

$$\max_{0\leq t\leq T} \int_\Omega |\tilde{n}_j|^{\frac{N}{N-2}} dx \leq 2 \max_{0\leq t\leq T} \int_\Omega |\overline{n}_j|^{\frac{N}{N-2}} dx \quad (4.31)$$

$$\leq 2 \max_{0\leq t\leq T} \int_\Omega |\sqrt{\overline{n}_j}|^{\frac{2N}{N-2}} dx \leq c, \quad (4.32)$$

$$\max_{0\leq t\leq T} \int_\Omega |\nabla \tilde{n}_j|^{\frac{N}{N-1}} dx \leq 2 \max_{0\leq t\leq T} \int_\Omega |\nabla \overline{n}_j|^{\frac{N}{N-1}} dx \leq c. \quad (4.33)$$

Hence $\{\tilde{n}_j\}$ is bounded in $L^\infty(0,T;W^{1,\frac{N}{N-1}}(\Omega))$. We infer from the Sobolev inequality that for each $1 \leq p < \frac{N}{N-2}$ the embedding $W^{1,\frac{N}{N-1}}(\Omega) \hookrightarrow L^p(\Omega)$ is compact and $L^p(\Omega) \hookrightarrow \left(W^{1,N}_{\Gamma_D}(\Omega)\right)^*$ is continuous if p is also bigger than 1. Thus we are in a position to invoke the results in [10] to conclude that $\{\tilde{n}_j\}$ is precompact in both $C([0,T];L^p(\Omega))$ and $L^2\left(0,T;\left(W^{1,N}_{\Gamma_D}(\Omega)\right)^*\right)$. For $t_{k-1} < t \leq t_k$, we calculate from the definitions of $\tilde{n}_j, \overline{n}_j$ that

$$\begin{aligned}
\tilde{n}_j - \overline{n}_j &= (t-t_k)\frac{\rho_k^2-\rho_{k-1}^2}{\tau} \\
&= (t-t_k)\operatorname{div}\left[-(\rho_k+\tau)^2 \nabla F_k + (\rho_k+\tau)^2 \nabla T_k + 2T_k \rho_k \nabla \rho_k\right],
\end{aligned}$$

from which it follows

$$\|\tilde{n}_j - \overline{n}_j\|_{\left(W^{1,N}_{\Gamma_D}(\Omega)\right)^*} \leq c(t-t_k)\| -(\rho_k+\tau)^2 \nabla F_k + (\rho_k+\tau)^2 \nabla T_k + 2T_k\rho_k \nabla \rho_k\|_{\frac{N}{N-1}}.$$

We estimate from Inequalities (4.28)–(4.30) that

$$\begin{aligned}
&\int_0^T \|\tilde{n}_j - \overline{n}_j\|^2_{\left(W^{1,N}_{\Gamma_D}(\Omega)\right)^*} dt \\
&= \sum_{k=1}^{k=j} \int_{t_{k-1}}^{t_k} \|\tilde{n}_j - \overline{n}_j\|^2_{\left(W^{1,N}_{\Gamma_D}(\Omega)\right)^*} dt
\end{aligned}$$

$$\leq c\sum_{k=1}^{k=j}\tau^3\|-(\rho_k+\tau)\nabla F_k+(\rho_k+\tau)^2\nabla T_k+2T_k\rho_k\nabla\rho_k\|^2_{\frac{N}{N-1}}$$

$$=c\tau^2\int_{\Omega_T}\left[(\sqrt{\overline{n}_j}+\tau)^2\left(|\nabla\overline{F}_j|^2+|\nabla\overline{T}_j|^2\right)+|\nabla\sqrt{\overline{n}_j}|^2\right]dxdt\leq c\tau^2. \tag{4.34}$$

Thus $\{\overline{n}_j\}$ is also precompact in $L^2\left(0,T;\left(W^{1,N}_{\Gamma_D}(\Omega)\right)^*\right)$. This, again, puts us in position to conclude from the results in [10] that $\{\overline{n}_j\}$ is precompact in $L^2(0,T;L^p(\Omega))$ for each $p<\frac{N}{N-2}$. Hence there exists a subsequence of $\{\overline{n}_j\}$ converging a.e. on Ω_T, from whence follows

$$\{\sqrt{\overline{n}_j}\}\text{ is precompact in }L^p(\Omega_T)\text{ for each }p<\tfrac{2N}{N-2}. \tag{4.35}$$

Set

$$G_j=\frac{\Delta\sqrt{\overline{n}_j}}{\overline{n}_j^{\frac{1}{4}}}. \tag{4.36}$$

Then we have from Lemma 4.1 that $\{G_j\}$ is bounded in $L^2(\Omega_T)$. Write Equation (4.36) in the form

$$-\Delta\sqrt{\overline{n}_j}=-G_j\overline{n}_j^{\frac{1}{4}}. \tag{4.37}$$

We compute that

$$\int_{\Omega_T}\left(G_j\overline{n}_j^{\frac{1}{4}}\right)^{\frac{4N}{3N-2}}dxdt\leq\left(\int_{\Omega_T}G_j^2dxdt\right)^{\frac{2N}{3N-2}}\left(\int_{\Omega_T}n_j^{\frac{N}{N-2}}dxdt\right)^{\frac{N-2}{3N-2}}\leq c. \tag{4.38}$$

Observe from Assumption (H2) that $\frac{4N}{3N-2}>\frac{2N}{N+2}$. This combined with Equation (4.35) enables us to use a calculation almost identical to estimate (2.9) to conclude that $\{\sqrt{\overline{n}_j}\}$ is precompact in $L^2(0,T;W^{1,2}(\Omega))$.

To show that $\{\overline{n}_j^{\frac{1}{4}}\}$ is precompact in $L^2(0,T;W^{1,2}(\Omega))$, we first extract a subsequence of $\{j\}$, still denoted by $\{j\}$, such that

$$\overline{n}_j\to n\quad\text{a.e. on }\Omega_T\text{ and strongly in }L^2(0,T;L^p(\Omega))\text{ for }p<\tfrac{N}{N-2}, \tag{4.39}$$

$$\sqrt{\overline{n}_j}\to\sqrt{n}\quad\text{strongly in }L^2(0,T;W^{1,2}(\Omega)), \tag{4.40}$$

$$G_j\rightharpoonup G\quad\text{weakly in }L^2(\Omega_T). \tag{4.41}$$

We can take $j\to\infty$ in Equation (4.37) to obtain

$$-\Delta\sqrt{n}=-Gn^{\frac{1}{4}}. \tag{4.42}$$

By virtue of (ii) in Lemma 2.1, it is enough for us to show that

$$\int_{\Omega_T}|\nabla\overline{n}_j^{\frac{1}{4}}|^2dxdt\to\int_{\Omega_T}|\nabla n^{\frac{1}{4}}|^2dxdt. \tag{4.43}$$

To this end, we use $\ln\overline{n}_j^{\frac{1}{4}}-\ln\sqrt{\rho_D}$ as a test function in Equation (4.37) to derive

$$\int_{\Omega_T}\frac{1}{\overline{n}_j^{\frac{1}{4}}}\nabla\sqrt{\overline{n}_j}\nabla\overline{n}_j^{\frac{1}{4}}dxdt$$

$$
\begin{aligned}
&= \int_{\Omega_T} \frac{1}{\sqrt{\rho_D}} \nabla \sqrt{\overline{n}_j} \nabla \sqrt{\rho_D} dxdt - \int_{\Omega_T} G_j (\overline{n}_j{}^{\frac{1}{4}} \ln \overline{n}_j{}^{\frac{1}{4}} - \overline{n}_j{}^{\frac{1}{4}} \ln \sqrt{\rho_D}) dxdt \\
&\to \int_{\Omega_T} \frac{1}{\sqrt{\rho_D}} \nabla \sqrt{n} \nabla \sqrt{\rho_D} dxdt - \int_{\Omega_T} G (n^{\frac{1}{4}} \ln n^{\frac{1}{4}} - n^{\frac{1}{4}} \ln \sqrt{\rho_D}) dxdt. \qquad (4.44)
\end{aligned}
$$

Here we have used the fact that $s\ln s$ is a continuous function on $[0,\infty)$ and $|s\ln s| \le cs^2 + c$ on $[0,\infty)$ for some $c > 0$. The left-hand side of Equation (4.44) is equal to $2\int_{\Omega_T} |\nabla \overline{n}_j{}^{\frac{1}{4}}|^2 dxdt$. Thus we may assume

$$
\overline{n}_j{}^{\frac{1}{4}} \rightharpoonup n^{\frac{1}{4}} \quad \text{weakly in } L^2(0,T;W^{1,2}(\Omega)). \qquad (4.45)
$$

Now use $\ln(n^{\frac{1}{4}} + \delta) - \ln(\sqrt{\rho_D} + \delta)$ as a test function in Equation (4.42) to obtain

$$
\begin{aligned}
2\int_{\Omega_T} \frac{n^{\frac{1}{4}}}{n^{\frac{1}{4}} + \delta} |\nabla n^{\frac{1}{4}}|^2 dxdt &= \int_{\Omega_T} \frac{1}{n^{\frac{1}{4}} + \delta} \nabla \sqrt{n} \nabla n^{\frac{1}{4}} dxdt \\
&= \int_{\Omega_T} \frac{1}{\sqrt{\rho_D} + \delta} \nabla \sqrt{n} \nabla \sqrt{\rho_D} dxdt \\
&\quad - \int_{\Omega_T} G (n^{\frac{1}{4}} \ln(n^{\frac{1}{4}} + \delta) - n^{\frac{1}{4}} \ln(\sqrt{\rho_D} + \delta)) dxdt. \qquad (4.46)
\end{aligned}
$$

In view of the Lebesgue Dominated Convergence Theorem, we can take $\delta \to 0$ in the above equation to obtain

$$
\begin{aligned}
2\int_{\Omega_T} |\nabla n^{\frac{1}{4}}|^2 dxdt &= \int_{\Omega_T} \frac{1}{\sqrt{\rho_D}} \nabla \sqrt{n} \nabla \sqrt{\rho_D} dxdt \\
&\quad - \int_{\Omega_T} G (n^{\frac{1}{4}} \ln n^{\frac{1}{4}} - n^{\frac{1}{4}} \ln \sqrt{\rho_D}) dxdt. \qquad (4.47)
\end{aligned}
$$

This together with Equation (4.44) gives Equation (4.43). The proof is complete. □

Without loss of generality, we assume that

$$
\begin{aligned}
\overline{T}_j &\rightharpoonup T \quad \text{weak}^* \text{ in } L^\infty(\Omega_T), && (4.48) \\
(\sqrt{\overline{n}_j} + \tau) \nabla \overline{T}_j &\rightharpoonup K \quad \text{weakly in } \left(L^2(\Omega_T)\right)^N, && (4.49) \\
(\sqrt{\overline{n}_j} + \tau) \nabla \overline{F}_j &\rightharpoonup H \quad \text{weakly in } \left(L^2(\Omega_T)\right)^N, && (4.50) \\
\overline{n}_j{}^{\frac{1}{4}} &\to n^{\frac{1}{4}} \quad \text{strongly in } L^2(0,T;W^{1,2}(\Omega)), && (4.51) \\
\overline{V}_j &\rightharpoonup V \quad \text{weak}^* \text{ in } L^\infty(0,T;W^{1,2}(\Omega)). && (4.52)
\end{aligned}
$$

By virtue of estimate (4.34), we also have

$$
\tilde{n}_j \to n \quad \text{strongly in } C([0,T];L^q(\Omega)) \text{ for each } q < \tfrac{N}{N-2}. \qquad (4.53)
$$

On account of Equation (4.28) and Lemma 4.2, there holds

$$
\overline{n}_j \to n \quad \text{strongly in } L^2(0,T;W^{1,q}(\Omega)) \text{ for each } q < \tfrac{N}{N-1}. \qquad (4.54)
$$

Lemma 4.3. *We have:*

(C3) $K = \operatorname{div}(\sqrt{n} T) - T \nabla \sqrt{n}$*;*

(C4) $\Delta\sqrt{n}\in L^1(0,T;W^{1,1}(\Omega))$ *and*

$$H=\frac{\varepsilon^2}{6}\left(\nabla\Delta\sqrt{n}-\frac{\Delta\sqrt{n}\nabla\sqrt{n}}{\sqrt{n}}\right)+\sqrt{n}\nabla V \quad \text{on } S_0,$$

where $S_0=\{(x,t)\in\Omega_T:n(x,t)>0\}$ *as defined in (1.9).*

Proof. We claim that the sequence $\{\nabla\left[(\sqrt{\overline{n}_j}+\tau)\overline{T}_j\right]\}$ is bounded in $\left(L^2(\Omega_T)\right)^N$. To see this, we compute

$$\nabla\left[(\sqrt{\overline{n}_j}+\tau)\overline{T}_j\right]=(\sqrt{\overline{n}_j}+\tau)\nabla\overline{T}_j+\overline{T}_j\nabla\sqrt{\overline{n}_j}.$$

We can easily conclude from Inequalities (4.17), (4.16), and (4.15) that the last two terms in the above equation are both bounded in $\left(L^2(\Omega_T)\right)^N$. In view of the limits (4.40) and (4.48), we have that

$$\begin{aligned}(\sqrt{\overline{n}_j}+\tau)\nabla\overline{T}_j &= \nabla\left[(\sqrt{\overline{n}_j}+\tau)\overline{T}_j\right]-\overline{T}_j\nabla\sqrt{\overline{n}_j}\\ &\rightharpoonup \nabla\left(\sqrt{n}T\right)-T\nabla\sqrt{n}=K \quad \text{weakly in } \left(L^2(\Omega_T)\right)^N. \qquad (4.55)\end{aligned}$$

This gives Claim (C3).

To establish Claim (C4), we observe from Lemma 4.1 that

$$\begin{aligned}\int_{\Omega_T}|\tau\overline{F}_j|^2dxdt &\le 2\int_{\Omega_T}|\tau\overline{F}_j-\tau F_D|^2dxdt+c\\ &\le c\int_{\Omega_T}|\nabla(\tau\overline{F}_j-\tau F_D)|^2dxdt+c\\ &\le c\int_{\Omega_T}(\sqrt{\overline{n}_j}+\tau)^2|\nabla\overline{F}_j|^2dxdt+c\le c. \qquad (4.56)\end{aligned}$$

By (Equation 3.9), we have that

$$\frac{\varepsilon^2}{6}\frac{\Delta\sqrt{\overline{n}_j}}{\sqrt{\overline{n}_j}}+\frac{\tau}{\sqrt{\overline{n}_j}}=\overline{F}_j-\overline{V}_j+\tau\sqrt{\overline{n}_j}^{\,p-2}. \qquad (4.57)$$

Multiply through this equation by $\frac{\tau^3}{\sqrt{\overline{n}_j}}$ and then integrate the resulting equation over Ω_T to derive

$$\begin{aligned}&\int_{\Omega_T}\frac{\tau^4}{\overline{n}_j}dxdt+\frac{\varepsilon^2\tau^3}{6}\int_{\Omega_T}\frac{\Delta\sqrt{\overline{n}_j}}{\overline{n}_j}dxdt\\ &=\int_{\Omega_T}\left(\tau\overline{F}_j\frac{\tau^2}{\sqrt{\overline{n}_j}}+\tau^4\sqrt{\overline{n}_j}^{\,p-3}-\tau\overline{V}_j\frac{\tau^2}{\sqrt{\overline{n}_j}}\right)dxdt\\ &\le\delta\int_{\Omega_T}\frac{\tau^4}{\overline{n}_j}dxdt+c\int_{\Omega_T}\left(|\tau\overline{F}_j|^2+|\tau\overline{V}_j|^2\right)dxdt+c\\ &\le\delta\int_{\Omega_T}\frac{\tau^4}{\overline{n}_j}dxdt+c, \qquad \delta>0. \qquad (4.58)\end{aligned}$$

We calculate, with the aid of Equation (4.37) and Equation (4.38), that

$$\int_{\Omega_T}\frac{\Delta\sqrt{\overline{n}_j}}{\overline{n}_j}dxdt=\int_{\Omega_T}\Delta\sqrt{\overline{n}_j}\left(\frac{1}{\overline{n}_j}-\frac{1}{\rho_D^2}\right)dxdt+\int_{\Omega_T}\Delta\sqrt{\overline{n}_j}\frac{1}{\rho_D^2}dxdt$$

$$\geq -\int_{\Omega_T} \nabla\sqrt{\overline{n}_j}\left(\frac{-2}{\sqrt{\overline{n}_j}^3}\nabla\sqrt{\overline{n}_j}+\frac{2}{\rho_D^3}\nabla\rho_D\right)dxdt-c$$

$$\geq \int_{\Omega_T}\frac{2}{\sqrt{\overline{n}_j}^3}|\nabla\sqrt{\overline{n}_j}|^2 dxdt-c. \tag{4.59}$$

Plugging this into Equation (4.58), we arrive at

$$\int_{\Omega_T}\frac{\tau^4}{\overline{n}_j}dxdt+\tau^3\int_{\Omega_T}\frac{1}{\sqrt{\overline{n}_j}^3}|\nabla\sqrt{\overline{n}_j}|^2 dxdt\leq c. \tag{4.60}$$

We estimate

$$\int_{\Omega_T}\left|\frac{\tau\Delta\sqrt{\overline{n}_j}}{\sqrt{\overline{n}_j}}\right|^{\frac{4}{3}}dxdt=\int_{\Omega_T}\left|\frac{\Delta\sqrt{\overline{n}_j}}{\overline{n}_j^{\frac{1}{4}}}\right|^{\frac{4}{3}}\left|\frac{\tau}{\overline{n}_j^{\frac{1}{4}}}\right|^{\frac{4}{3}}dxdt$$

$$\leq\left(\int_{\Omega_T}\left(\frac{\Delta\sqrt{\overline{n}_j}}{\overline{n}_j^{\frac{1}{4}}}\right)^2 dxdt\right)^{\frac{2}{3}}\left(\int_{\Omega_T}\left(\frac{\tau}{\overline{n}_j^{\frac{1}{4}}}\right)^4 dxdt\right)^{\frac{1}{3}}\leq c. \tag{4.61}$$

Consequently, we may assume

$$\frac{\tau^2}{\sqrt{\overline{n}_j}}\rightharpoonup I \quad \text{weakly in } L^2(\Omega_T), \tag{4.62}$$

$$\frac{\tau\Delta\sqrt{\overline{n}_j}}{\sqrt{\overline{n}_j}}\rightharpoonup J \quad \text{weakly in } L^{\frac{4}{3}}(\Omega_T). \tag{4.63}$$

Recall that $S_0=\{(x,t)\in\Omega_T : n(x,t)>0\}$. It immediately follows from Equation (4.39) that

$$\frac{1}{\sqrt{\overline{n}_j}}\to\frac{1}{\sqrt{n}} \quad \text{a.e. on } S_0. \tag{4.64}$$

This together with Equation (4.60) asserts that

$$\frac{\tau^2}{\sqrt{\overline{n}_j}}\to 0 \quad \text{a.e. on } S_0 \text{ and strongly in } L^r(S_0) \text{ for each } 1\leq r<2. \tag{4.65}$$

Hence by Equation (4.41), we have

$$\frac{\tau\Delta\sqrt{\overline{n}_j}}{\sqrt{\overline{n}_j}}=\frac{\Delta\sqrt{\overline{n}_j}}{\overline{n}_j^{\frac{1}{4}}}\cdot\frac{\tau}{\overline{n}_j^{\frac{1}{4}}}=G_i\cdot\frac{\tau}{\overline{n}_j^{\frac{1}{4}}}\rightharpoonup 0 \quad \text{weakly in } L^1(S_0). \tag{4.66}$$

That is,

$$I=J=0 \quad \text{a.e. on } S_0. \tag{4.67}$$

Recall from Equation (4.38) that $\{\Delta\sqrt{\overline{n}_j}\}$ is bounded in $L^{\frac{4N}{3N-2}}(\Omega_T)$. Thus we may assume

$$\Delta\sqrt{\overline{n}_j}\rightharpoonup\Delta\sqrt{n} \quad \text{weakly in } L^{\frac{4N}{3N-2}}(\Omega_T). \tag{4.68}$$

Note from Lemma 4.1 that for each $\delta > 0$ there holds

$$\tau \int_\Omega \sqrt{\overline{n}_j}^{p-\delta} \le c\tau^{\frac{\delta}{p}} \left(\tau \int_\Omega \sqrt{\overline{n}_j}^{p} dx \right)^{\frac{p-\delta}{p}} \le c\tau^{\frac{\delta}{p}} \to 0. \tag{4.69}$$

We can conclude that

$$\begin{aligned}(\sqrt{\overline{n}_j}+\tau)\overline{F}_j &= \frac{\varepsilon^2}{6}\frac{\Delta\sqrt{\overline{n}_j}}{\sqrt{\overline{n}_j}}(\sqrt{\overline{n}_j}+\tau) - \tau\sqrt{\overline{n}_j}^{p-2}(\sqrt{\overline{n}_j}+\tau) \\ &\quad + \frac{\tau}{\sqrt{\overline{n}_j}}(\sqrt{\overline{n}_j}+\tau) + \overline{V}_j(\sqrt{\overline{n}_j}+\tau) \\ &\rightharpoonup \frac{\varepsilon^2}{6}\Delta\sqrt{n} + \frac{\varepsilon^2}{6}J + I + V\sqrt{n} \quad \text{at least weakly in } L^1(\Omega_T). \end{aligned} \tag{4.70}$$

On the other hand, we can compute

$$\overline{F}_j \nabla\sqrt{\overline{n}_j} = \frac{\varepsilon^2}{6}\frac{\Delta\sqrt{\overline{n}_j}}{\overline{n}_j^{\frac{1}{4}}} 2\nabla \overline{n}_j^{\frac{1}{4}} - \tau\sqrt{\overline{n}_j}^{p-2}\nabla\sqrt{\overline{n}_j} + \frac{\tau}{\sqrt{\overline{n}_j}}\nabla\sqrt{\overline{n}_j} + \overline{V}_j\nabla\sqrt{\overline{n}_j}. \tag{4.71}$$

In view of Lemma 4.2, Equations (4.41), and (4.52), we can pass to the limit in the first and last terms on the right-hand side of the above equation. As for the remaining two terms there, we estimate from Lemma 4.1 that

$$\begin{aligned}\left|\int_{\Omega_T} \tau\sqrt{\overline{n}_j}^{p-2}\nabla\sqrt{\overline{n}_j}\,dxdt\right| &\le \left(\tau\int_{\Omega_T}\sqrt{\overline{n}_j}^{p-3}\left|\nabla\sqrt{\overline{n}_j}\right|^2 dxdt\right)^{\frac{1}{2}} \left(\tau\int_{\Omega_T}\sqrt{\overline{n}_j}^{p-1}dxdt\right)^{\frac{1}{2}} \\ &\le c\left(\tau\int_{\Omega_T}\sqrt{\overline{n}_j}^{p-1}dxdt\right)^{\frac{1}{2}} \to 0, \\ \left|\int_{\Omega_T}\frac{\tau\nabla\sqrt{\overline{n}_j}}{\sqrt{\overline{n}_j}}dxdt\right| &\le c\tau^{\frac{1}{2}}\left(\tau\int_{\Omega_T}\frac{|\nabla\sqrt{\overline{n}_j}|^2}{\sqrt{\overline{n}_j}^2}dxdt\right)^{\frac{1}{2}} \le c\tau^{\frac{1}{2}}.\end{aligned}$$

Thus we have

$$\overline{F}_j\nabla\sqrt{\overline{n}_j} \rightharpoonup \frac{\varepsilon^2}{3}G\nabla n^{\frac{1}{4}} + V\nabla\sqrt{n} \quad \text{weakly in } L^1(\Omega_T). \tag{4.72}$$

In view of Equation (4.42), there holds

$$G = \frac{\Delta\sqrt{n}}{n^{\frac{1}{4}}} \quad \text{on } S_0. \tag{4.73}$$

Observe from Equations (4.70) and (4.72) that

$$\begin{aligned}(\sqrt{\overline{n}_j}+\tau)\nabla\overline{F}_j &= \nabla\left[(\sqrt{\overline{n}_j}+\tau)\overline{F}_j\right] - \overline{F}_j\nabla\sqrt{\overline{n}_j} \\ &\rightharpoonup \nabla\left(\frac{\varepsilon^2}{6}\Delta\sqrt{n} + \frac{\varepsilon^2}{6}J + I + V\sqrt{n}\right) - \frac{\varepsilon^2}{3}G\nabla n^{\frac{1}{4}} - V\nabla\sqrt{n} = H\end{aligned} \tag{4.74}$$

in the sense of distributions. To see that $\Delta\sqrt{n} \in L^1(0,T;W^{1,1}(\Omega))$, we estimate

$$\int_{\Omega_T}\left|\tau^2\sqrt{\overline{n}_j}^{p-3}\nabla\sqrt{\overline{n}_j}\right|^{\frac{2p}{2p-3}}dxdt$$

$$\leq \left(\tau \int_{\Omega_T} \sqrt{\overline{n}_j}^{p-3} \nabla \sqrt{\overline{n}_j}|^2 dxdt\right)^{\frac{p}{2p-3}} \cdot \left(\tau^{\frac{3p}{p-3}} \int_{\Omega_T} \sqrt{\overline{n}_j}^{p} dxdt\right)^{\frac{p-3}{2p-3}}$$
$$\leq c\tau^{\frac{2p+3}{2p-3}} \leq c. \tag{4.75}$$

Note that from Equation (4.57)

$$\nabla\left(\frac{\varepsilon^2}{6}\frac{\tau\Delta\sqrt{\overline{n}_j}}{\sqrt{\overline{n}_j}} + \frac{\tau^2}{\sqrt{\overline{n}_j}}\right) = \tau\nabla\overline{F}_j - \tau\nabla\overline{V}_j + \tau^2(p-2)\sqrt{\overline{n}_j}^{p-3}\nabla\sqrt{\overline{n}_j}. \tag{4.76}$$

The right-hand side of the above equation is bounded in $L^{\frac{2p}{2p-3}}(0,T;L^{\frac{2p}{2p-3}}(\Omega))$. This implies that

$$\frac{\varepsilon^2}{6}J + I \in L^{\frac{2p}{2p-3}}(0,T;W^{1,\frac{2p}{2p-3}}(\Omega)). \tag{4.77}$$

By Equation (4.74), we have

$$\frac{\varepsilon^2}{6}\nabla\Delta\sqrt{n} = H - \nabla\left(\frac{\varepsilon^2}{6}J + I\right) - \sqrt{n}\nabla V + \frac{\varepsilon^2}{3}G\nabla n^{\frac{1}{4}} \in L^1(\Omega_T). \tag{4.78}$$

On the set S_0, we have

$$H = \frac{\varepsilon^2}{6}\left(\nabla\Delta\sqrt{n} - \frac{\Delta\sqrt{n}\nabla\sqrt{n}}{\sqrt{n}}\right) + \sqrt{n}\nabla V.$$

This completes the proof. □

Now we can conclude the proof of Theorem 4.1. Observe that

$$(\sqrt{\overline{n}_j}+\tau)^2\nabla\overline{F}_j = (\sqrt{\overline{n}_j}+\tau)\cdot(\sqrt{\overline{n}_j}+\tau)\nabla\overline{F}_j \rightharpoonup \sqrt{n}H \quad \text{weakly in } L^1(\Omega_T),$$
$$(\sqrt{\overline{n}_j}+\tau)^2\nabla\overline{T}_j = (\sqrt{\overline{n}_j}+\tau)\cdot(\sqrt{\overline{n}_j}+\tau)\nabla\overline{T}_j \rightharpoonup \sqrt{n}K \quad \text{weakly in } L^1(\Omega_T).$$

Here we have used Equations (4.67) and (4.73). We can take $j\to\infty$ in system (4.8)–(4.14) to obtain the desired result.

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