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STABILITY ANALYSIS OF A CLASS OF GLOBALLY HYPERBOLIC MOMENT SYSTEM*

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Abstract. This work studies the stability of a class of the globally hyperbolic moment system (GHMS) with the single relaxation-time collision model in the sense of hyperbolic relaxation systems. We prove the equilibrium stability of the GHMS in both one- and multi-dimensional space. For a five-moment system in one dimension, we prove its linear instability for some quiescent nonequilibrium states and demonstrate numerically the nonlinear instability of the nonequilibrium states.

Keywords. globally hyperbolic moment system; hyperbolic relaxation systems; stability criteria

AMS subject classifications. 82C40; 35L60; 35B35

1. Introduction

We are interested in the moment systems derived from the Boltzmann equation and its model equations. The Boltzmann equation in phase space $\mathbb{\Gamma} := (\boldsymbol{x}, \boldsymbol{\xi})$ and time t can be written as

$$\partial_t f + \boldsymbol{\xi} \cdot \boldsymbol{\nabla} f = \frac{1}{\varepsilon} Q(f, f), \tag{1.1}$$

where $f := f(\boldsymbol{x}, \boldsymbol{\xi}, t)$ is the mass density distribution function, $\boldsymbol{x} \in \mathbb{R}^d$ and $\boldsymbol{\xi} \in \mathbb{R}^d$ are spatial position and particle velocity, respectively, d is the dimensions of space, $Q(f, f)$ denotes the bi-linear collision term, and ε is a parameter, *i.e.*, the Knudsen number Kn. There are several approaches to approximate the Boltzmann equation (1.1). One is the Chapman–Enskog procedure to obtain the *normal* solutions of the Boltzmann equation. For flows at equilibrium or near equilibrium, the proper closures are the Euler equations or the Navier–Stokes–Fourier (NSF) equations, respectively. With the Chapman–Enskog analysis, one can also derive Burnett or super-Burnett equations [6, 7, 12], both of which suffer from instabilities and are ill-posed [2]. There has been a continuous endeavor to construct well-posed generalized hydrodynamic models by various regularizations [3, 4, 15, 5]. The Chapman–Enskog approach to construct generalized hydrodynamic equations beyond the NSF system proves to be difficult.

Another approach is the moment method for *non-normal* solutions of the Boltzmann equation. The Boltzmann equation (1.1) can be recast as a hierarchy of partial differential equations for the velocity moments of the distribution function in space $\boldsymbol{x}$

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and time t:

$$\partial_t \mathbf{M}^{(n)} + \nabla \cdot \mathbf{M}^{(n+1)} = \frac{1}{\varepsilon}\langle \boldsymbol{\xi}^n Q\rangle, \quad n \in \mathbb{N}_0 := \{0,1,2,\ldots\}, \tag{1.2a}$$

$$\mathbf{M}^{(n)}(\boldsymbol{x},t) := \langle \boldsymbol{\xi}^n f\rangle := \int_{\mathbb{R}^d} \boldsymbol{\xi}^n f(\boldsymbol{x},\boldsymbol{\xi},t)\,\mathrm{d}\boldsymbol{\xi}, \quad \boldsymbol{\xi}^n := \underbrace{\boldsymbol{\xi}\boldsymbol{\xi}\cdots\boldsymbol{\xi}}_{n}, \tag{1.2b}$$

$$\langle \boldsymbol{\xi}^n Q\rangle(\boldsymbol{x},t) := \int_{\mathbb{R}^d} \boldsymbol{\xi}^n Q\,\mathrm{d}\boldsymbol{\xi}, \tag{1.2c}$$

where $\mathbf{M}^{(n)}$ denotes the nth order velocity moment of the distribution function f, and $\langle \boldsymbol{\xi}^n Q\rangle$ denotes the production term, or the change of moment, due to the collision Q. In addition to the term $\mathbf{M}^{(n+1)}$, the production term $\langle \boldsymbol{\xi}^n Q\rangle$ may have to be expressed by an infinite number of moments in general, depending on the nature of molecular interaction. Thus the moment equation (1.2a) forms a hierarchy of an infinite number of equations in general, and the moment closure problem remains as one of the most interesting and challenging problem to be solved [35, 36, 37, 25, 32]. There are two essential requirements for a proper moment closure. First, the system must be hyperbolic so it is well-posed [24]. And second, as an approximation to the Boltzmann equation, the system must preserve the dissipation characterized by the H-theorem. The latter requirement is far more difficult to achieve than the former [25]; and an immediate question is how to characterize the dissipation.

In Maxwell's moment method [28, 29, 20, 34, 30, 43], which leads to the r^{-5} force law for the celebrated model of Maxwell molecule, the production term $\langle \boldsymbol{\xi}^n Q\rangle$ does not lead to any moments of the order higher than $\mathbf{M}^{(n)}$. Nevertheless, the moment $\mathbf{M}^{(n+1)}$ generated by the advection term $\nabla\cdot(\boldsymbol{\xi} f)$ remains as a higher order moment to be closed. In Grad's moment method [16, 17, 20, 32], the distribution function is expanded in terms of multi-dimensional Hermite polynomials $\mathbf{H}^{(n)}(\bar{\boldsymbol{c}})$ of the normalized peculiar velocity $\bar{\boldsymbol{c}} := (\boldsymbol{\xi} - \boldsymbol{u})/\sqrt{\theta}$ [18], and the system of moments $\{\langle \mathbf{H}^{(n)}(\bar{\boldsymbol{c}}) f\rangle | n = 0,1,2,\ldots\}$ is indeed not closed. In Grad's approach to close the system, $\langle \mathbf{H}^{(m)} f\rangle$ is set to be zero for $m > n$ with $n \geq 2$, and the equations of $\partial_t \langle \mathbf{H}^{(m)} f\rangle$ with $m > n$ are neglected. The system so obtained is the so-called Grad's nth order moment system.

Within the hierarchy of Grad's closures beyond the NSF equations, the most celebrated one is the thirteen moment system [16, 17]. Unfortunately, Grad's moment closure system is not endowed with global hyperbolicity, as explicitly shown by Müller and Ruggeri [30] for one dimensional cases and by Cai, Fan, and Li [10] for three dimensional cases. It is particularly worth mentioning that Grad's thirteen moment system is *not* hyperbolic even in the vicinity of the equilibrium $f = f^{(0)}$, that is, the system is ill-posed even when arbitrary close to the equilibrium. There has been a persistent effort to impose hyperbolicity on Grad's moment closures by various regularizations [25, 31], but with rather limited success [22]. It is only very recently that Grad's moment closures were systematically formulated as a globally hyperbolic moment system (GHMS) [8, 9, 11, 13].

The aim of this work is to analyze the stability of the globally hyperbolic moment system proposed by Cai, Fan, and Li [8, 11]. Specifically, we show that at the equilibrium the GHMS satisfy the second structural stability condition proposed by Yong [41] for hyperbolic relaxation systems. Further, we prove the linear instability and show the nonlinear instability numerically for a five-moment system in one dimension (1D) at certain nonequilibrium states. Our results show that, while the global hyperbolicity is a necessary condition to ensure well-posedness of the moment closure system, it alone cannot guarantee the system to be physical; in fact, it is far from it.

The remainder of the paper is organized as follows. Section 2 briefly discusses the Boltzmann equation with the single-relaxation-time collision model [1, 39] and the hydrodynamic equations derived from it. Section 3 discusses Grad's moment system in one dimension (1D). Section 4 describes a class of the globally hyperbolic moment system in 1D, and the stability of the GHMS in 1D is the subject of interest in this work. Sections 5, 6 and 7 present the main results of this work. Section 5 proves the equilibrium stability of the GHMS in general. Section 6 proves the existence of some nonequilibrium states which are linearly unstable for a five-moment system in 1D, and Section 7 shows numerically the nonlinear instability of the nonequilibrium states. Finally, Section 8 concludes the paper. The paper also includes three appendices. Appendix A succinctly describes the second structural stability condition and the relaxation criterion proposed by Yong [41, 42] for hyperbolic relaxation systems, which are the theoretical basis of our stability analysis. Appendix B extends the proof of equilibrium stability of the GHMS in 1D in Sec. 6 to the GHMS in multi-dimensional space. And Appendix C provides a validation for the first-order upwind scheme which is used to carry out the nonlinear stability analysis in Section 7.

2. The Boltzmann–BGKW equation

We will study the Boltzmann equation with the single-relaxation-time approximation for the collision term due to Bhatnagar, Gross, and Krook (BGK) [1] and Welander [39]:

$$\partial_t f + \boldsymbol{\xi}\cdot\boldsymbol{\nabla} f = -\frac{1}{\tau}\left[f - f^{(0)}\right], \tag{2.1}$$

where τ is the relaxation time, and $f^{(0)}$ is the local Maxwellian distribution function defined as

$$f^{(0)}(\rho,\boldsymbol{u},T;\boldsymbol{x},t) = \frac{\rho}{(2\pi\theta)^{d/2}}\exp\left(-\frac{c^2}{2\theta}\right) := \rho\omega(\bar{\boldsymbol{c}}), \tag{2.2}$$

where $\rho(\boldsymbol{x},t)$, $\boldsymbol{u}(\boldsymbol{x},t)$, and $T(\boldsymbol{x},t)$ are local flow mass density, the macroscopic flow velocity, and the temperature, respectively, $\theta = RT$, R is the gas constant, $\boldsymbol{c} := \boldsymbol{\xi} - \boldsymbol{u}$ is the peculiar velocity, $\bar{\boldsymbol{c}} := \boldsymbol{c}/\sqrt{\theta}$, and $c^2 := \boldsymbol{c}\cdot\boldsymbol{c}$.

The macroscopic quantities ρ, $\boldsymbol{u}$, and T are obtained through the lowest order moments of f and $f^{(0)}$:

$$\rho = \int_{\mathbb{R}^d} f\,\mathrm{d}\boldsymbol{\xi} = \int_{\mathbb{R}^d} f^{(0)}\mathrm{d}\boldsymbol{\xi}, \tag{2.3a}$$

$$\rho\boldsymbol{u} = \int_{\mathbb{R}^d} \boldsymbol{\xi} f\,\mathrm{d}\boldsymbol{\xi} = \int_{\mathbb{R}^d} \boldsymbol{\xi} f^{(0)}\mathrm{d}\boldsymbol{\xi}, \tag{2.3b}$$

$$\frac{1}{2}\rho\left(u^2 + d\theta\right) = \frac{1}{2}\int_{\mathbb{R}^d} \xi^2 f\,\mathrm{d}\boldsymbol{\xi} = \frac{1}{2}\int_{\mathbb{R}^d} \xi^2 f^{(0)}\mathrm{d}\boldsymbol{\xi}, \tag{2.3c}$$

where $u^2 := \boldsymbol{u}\cdot\boldsymbol{u}$ and $\xi^2 := \boldsymbol{\xi}\cdot\boldsymbol{\xi}$.

The first three moments of the Boltzmann equation yield the following equations for ρ, $\boldsymbol{u}$, and T:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}+\rho\boldsymbol{\nabla}\cdot\boldsymbol{u}=0, \tag{2.4a}$$

$$\rho\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}+\boldsymbol{\nabla}p+\boldsymbol{\nabla}\cdot\mathbf{S}=0, \tag{2.4b}$$

$$\frac{d}{2}\rho\frac{\mathrm{d}\theta}{\mathrm{d}t}+\boldsymbol{\nabla}\cdot\boldsymbol{q}+\mathbf{P}:\boldsymbol{\nabla}\boldsymbol{u}=0, \tag{2.4c}$$

where $\mathrm{d}/\mathrm{d}t:=\partial_t+\boldsymbol{u}\cdot\boldsymbol{\nabla}$, and the pressure tensor $\mathbf{P}$, the deviatoric pressure p, the stress tensor $\mathbf{S}$ and the heat flux $\boldsymbol{q}$ are given by:

$$\mathbf{P}:=\int_{\mathbb{R}^d}\boldsymbol{c}\boldsymbol{c}f\mathrm{d}\boldsymbol{\xi}, \tag{2.5a}$$

$$p:=\rho\theta=\rho RT, \tag{2.5b}$$

$$\mathbf{S}:=\mathbf{P}-p\mathbf{I}, \tag{2.5c}$$

$$\boldsymbol{q}:=\frac{1}{2}\int_{\mathbb{R}^d}c^2\boldsymbol{c}f\mathrm{d}\boldsymbol{\xi}, \tag{2.5d}$$

where $\mathbf{I}$ is the $d\times d$ identity matrix.

A challenging closure problem in kinetic theory is how to approximate the pressure tensor $\mathbf{P}$ and the heat flux $\boldsymbol{q}$ beyond the Navier–Stokes–Fourier closure. When $f\approx f^{(0)}$, Equations (2.4) become the Euler equations, *i.e.*, $\mathbf{P}=p\mathbf{I}$ and $\boldsymbol{q}=\mathbf{0}$. When $f\approx f^{(0)}+f^{(1)}$, where $f^{(1)}$ is the first order deviation from the equilibrium in terms of the small Knudsen number in Chapman–Enskog expansion, Equations (2.4) become the Navier–Stokes–Fourier equations, *i.e.*,

$$\mathbf{P}=p\mathbf{I}-\mu\left[(\boldsymbol{\nabla}\boldsymbol{u})+(\boldsymbol{\nabla}\boldsymbol{u})^{\dagger}-\frac{2}{d}(\boldsymbol{\nabla}\cdot\boldsymbol{u})\mathbf{I}\right], \tag{2.6a}$$

$$\boldsymbol{q}=-\kappa\boldsymbol{\nabla}T, \tag{2.6b}$$

where μ and κ are the coefficients of viscosity and heat diffusivity, respectively.

As mentioned in the introduction, higher order closures are far more challenging and less successful. One of higher-order approximation of the moment closures is Grad's moment method [16, 17], which will be discussed briefly next.

3. Grad's moment method

In Grad's moment method [16, 17], the distribution function $f(\boldsymbol{x},\boldsymbol{\xi},t)$ is expanded in terms of multi-dimensional Hermite tensor polynomials [18]. In this work, we will restrict ourselves to the one dimensional case, in which the distribution function is expanded in terms of the Hermite polynomials as the following:

$$f(x,\xi,t)=\frac{1}{\sqrt{2\pi\theta}}\exp\left(-\frac{\bar{c}^2}{2}\right)\sum_{n=0}^{\infty}f_n\theta^{-n/2}H^{(n)}(\bar{c}):=\omega(\bar{c})\sum_{n=0}^{\infty}f_n\theta^{-n/2}H^{(n)}(\bar{c}), \tag{3.1}$$

where $\bar{c}:=(\xi-u)/\sqrt{\theta}$ is the dimensionless peculiar velocity, $H^{(n)}$ is the nth-order Hermite polynomial which can be obtained by its generating function:

$$H^{(n)}(x)=(-1)^n\exp\left(\frac{x^2}{2}\right)\frac{\mathrm{d}^n}{\mathrm{d}x^n}\exp\left(-\frac{x^2}{2}\right), \tag{3.2}$$

and f_n, $n=0, 1, \cdots, \infty$, are the expansion coefficients which are state variables. In particular, the first few f_n's are:

$$f_0=\rho, \quad f_1=f_2=0, \quad f_3=\frac{1}{3}q, \tag{3.3}$$

where q is the heat flux.

For the Boltzmann–BGKW equation in 1D, the equations of f_n are:

$$n=0: \quad \frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}=0, \tag{3.4a}$$

$$n=1: \quad \frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho\theta+\rho u u)}{\partial x}=0, \tag{3.4b}$$

$$n=2: \quad \frac{1}{2}\frac{\partial(\rho\theta)}{\partial t}+\frac{1}{2}\frac{\partial(\rho u\theta)}{\partial x}+\frac{\partial(3f_3)}{\partial x}+\rho\theta\frac{\partial u}{\partial x}=0, \tag{3.4c}$$

$$\begin{aligned} n\geq 3: \quad & \frac{\partial f_n}{\partial t}+(n+1)\frac{\partial f_{n+1}}{\partial x}+u\frac{\partial f_n}{\partial x}+(n+1)f_n\frac{\partial u}{\partial x}-f_{n-1}\frac{\theta}{\rho}\frac{\partial \rho}{\partial x}+\theta\frac{\partial f_{n-1}}{\partial x} \\ & +\frac{1}{2}[(n-1)f_{n-1}+\theta f_{n-3}]\frac{\partial\theta}{\partial x}-\frac{3}{\rho}f_{n-2}\frac{\partial f_3}{\partial x}=-\frac{1}{\tau}f_n. \end{aligned} \tag{3.4d}$$

It is obvious that a truncated system of the first n moments is not closed because of the terms of f_{n+1} in the above equation of f_n. In Grad's approach to close the system, f_n is set to be zero for $n>m$ with $m\geq 3$, and the equations of $\partial_t f_n$ with $n>m$ are truncated.

4. A class of the globally hyperbolic moment system in 1D

Grad's moment system in 1D can be written in a concise vector form by using the following vector notation to denote the state of moments up to nth order:

$$\mathbf{W}:=(\rho, u, \theta, f_3, \ldots, f_n)^\dagger \in \mathbb{R}^{n+1}, \tag{4.1}$$

where $\dagger$ denotes the transpose, and $n\in\mathbb{N}_0$ and $n\geq 3$. For the *collisionless* Boltzmann equation with the closure that $f_m=0$, $\forall m\geq n+1$, Grad's moment system in 1D is

$$\frac{\partial \mathbf{W}}{\partial t}+\mathbf{A}\frac{\partial \mathbf{W}}{\partial x}=0, \tag{4.2}$$

where $\mathbf{A}$ is the following $(n+1)\times(n+1)$ matrix:

$$\mathbf{A}=\begin{pmatrix} u & \rho & 0 & 0 & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & 0 \\ \theta/\rho & u & 1 & 0 & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & 0 \\ 0 & 2\theta & u & 6/\rho & 0 & \cdot & \cdot & \cdots & \cdot & \cdot & 0 \\ 0 & 4f_3 & \rho\theta/2 & u & 4 & 0 & \cdot & \cdots & \cdot & \cdot & 0 \\ -\theta f_3/\rho & 5f_4 & 3f_3/2 & \theta & u & 5 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -\theta f_{n-2}/\rho & nf_{n-1} & [(n-2)f_{n-2}+\theta f_{n-4}]/2 & -3f_{n-3}/\rho & 0 & 0 & 0 & \cdots & \theta & u & n \\ -\theta f_{n-1}/\rho & (n+1)f_n & [(n-1)f_{n-1}+\theta f_{n-3}]/2 & -3f_{n-2}/\rho & 0 & 0 & 0 & \cdots & 0 & \theta & u \end{pmatrix}. \tag{4.3}$$

The characteristic polynomial associated with $\mathbf{A}$ is [8]:

$$|\lambda\mathbf{I}-\mathbf{A}|=\theta^{\frac{n+1}{2}}H^{(n+1)}(\bar{\lambda})-\frac{(n+1)!}{2\rho}\left[\theta(\bar{\lambda}^2-1)f_{n-1}+2\sqrt{\theta}\bar{\lambda}f_n\right], \tag{4.4}$$

where $\bar{\lambda}:=(\lambda-u)/\sqrt{\theta}$. Clearly, it is not straightforward to solve the above eigenvalue problem for large n, because of the terms including both f_{n-1} and f_n. Therefore it is not clear at all if the above system is hyperbolic.

To circumvent the difficulty of solving the eigenvalue problem (4.4) and to ensure the hyperbolicity of the moment system, Cai, Fan, and Li [8] propose a modified system in which the term including both f_{n-1} and f_n is eliminated in the characteristic equation (4.4), so the eigenvalue problem can be easily solved. This means that the matrix $\mathbf{A}$ in system (4.2) is replaced by the following one:

$$\hat{\mathbf{A}}=\begin{pmatrix} u & \rho & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \theta/\rho & u & 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 2\theta & u & 6/\rho & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 4f_3 & \rho\theta/2 & u & 4 & 0 & \cdot & \cdot & \cdot & 0 \\ -\theta f_3/\rho & 5f_4 & 3f_3/2 & \theta & u & 5 & 0 & \cdot & \cdot & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\theta f_{n-2}/\rho & nf_{n-1} & [(n-2)f_{n-2}+\theta f_{n-4}]/2 & -3f_{n-3}/\rho & 0 & \cdot & 0 & \theta & u & n \\ -\theta f_{n-1}/\rho & 0 & [-2f_{n-1}+\theta f_{n-3}]/2 & -3f_{n-2}/\rho & 0 & \cdot & \cdot & 0 & \theta & u \end{pmatrix}, \tag{4.5}$$

that is, $\hat{\mathbf{A}}$ differs from $\mathbf{A}$ only in the second and third entries of the last row. Then the characteristic polynomial associated with $\hat{\mathbf{A}}$ is [8]:

$$|\lambda\mathbf{I}-\hat{\mathbf{A}}|=\theta^{(n+1)/2}H^{(n+1)}(\bar{\lambda}), \tag{4.6}$$

of which the eigenvalues are

$$\lambda_i=\sqrt{\theta}\bar{\lambda}_i+u, \tag{4.7}$$

where $\bar{\lambda}_i$, $i=1$, 2, …, $(n+1)$, are the distinctive roots of the $(n+1)$st order Hermite polynomial $H^{(n+1)}$. This ensures the hyperbolicity of the modified system with $\hat{\mathbf{A}}$ (instead of $\mathbf{A}$), and the modified system is the so-called globally hyperbolic moment system [8].

The moment system with the modified coefficient matrix $\hat{\mathbf{A}}$ of (4.5) can be written in the following concise form [11, 13]:

$$\mathbf{D}\frac{\partial \mathbf{W}}{\partial t}+\mathbf{MD}\frac{\partial \mathbf{W}}{\partial x}=\frac{1}{\tau}\mathbf{BW}, \tag{4.8}$$

where $\mathbf{B}$ is the following diagonal matrix for the BGKW model:

$$\mathbf{B}=\mathrm{diag}(0,0,0,-1,\ldots,-1), \tag{4.9}$$

$\mathbf{M}$ is a tridiagonal matrix with all diagonal entries equal to u, all subdiagonal entries equal to θ, and the superdiagonal entries equal to 1, 2, 3, …, n:

$$\mathbf{M}=\begin{pmatrix} u & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \theta & u & 2 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \theta & u & 3 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \theta & u & 4 & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & \ddots & \ddots & \ddots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & \theta & u & n \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & \theta & u \end{pmatrix}, \tag{4.10}$$

and $\mathbf{D}$ is block-structured matrix in the following form:

$$\mathbf{D} := \mathbf{D}(\mathbf{W}) = \begin{pmatrix} \mathbf{D}_{11} & \mathbf{0} \\ \mathbf{D}_{21} & \mathbf{I}_{n-3} \end{pmatrix}, \tag{4.11}$$

with $\mathbf{D}_{11} = \text{diag}(1, \rho, \rho/2, 1)$, $\mathbf{I}_{n-3}$ the $(n-3)\times(n-3)$ identity matrix, and

$$\mathbf{D}_{21} = \begin{pmatrix} 0 & f_3 & 0 & 0 \\ 0 & f_4 & \frac{1}{2}f_3 & 0 \\ 0 & f_5 & \frac{1}{2}f_4 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & f_{n-1} & \frac{1}{2}f_{n-2} & 0 \end{pmatrix}. \tag{4.12}$$

Therefore, matrix $\hat{\mathbf{A}}$ can be decomposed as $\hat{\mathbf{A}} = \mathbf{D}^{-1}\mathbf{M}\mathbf{D}$, which can is used in the stability analysis of next Section.

5. Equilibrium stability of GHMS

The moment closure systems in kinetic theories are of the form of hyperbolic relaxation systems [41, 42], which are systems of first-order partial differential equations with a small parameter ε in the denominator of the source term. Although hyperbolicity of the system is required to ensure the existence of its solution, the stability of solution in the limit of vanishingly small parameter is even more important [42]. To this end, the second structural stability condition proposed by Yong [41] for hyperbolic relaxation systems can be directly applied to analyze the moment system. The structural stability condition has been shown to be satisfied by numerous well-known models in physics [45] as well as various numerical schemes for conservation laws [42], indicating the universality of the stability condition [41, 42]. For example, the moment closure proposed by Levermore [25] has been shown to satisfy the condition [42]. The precise statement of the second structural stability condition is provided in Appendix A. In what follows, we will prove that the globally hyperbolic moment system (4.8) satisfies the structural stability condition.

We note that the matrix $\mathbf{M}$ can be symmetrized by the following diagonal matrix:

$$\mathbf{G} = \text{diag}\left(1, \frac{1}{\theta}, \frac{2}{\theta^2}, \ldots, \frac{(i-1)!}{\theta^{i-1}}, \ldots, \frac{n!}{\theta^n}\right), \tag{5.1}$$

that is,

$$\mathbf{GM} = \begin{pmatrix} u & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & u/\theta & 2/\theta & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2/\theta & 2u/\theta^2 & 6/\theta^2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 6/\theta^2 & 6u/\theta^3 & 24/\theta^3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & (n-1)!/\theta^{n-2} & (n-1)!u/\theta^{n-1} & n!/\theta^{n-1} \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & n!/\theta^{n-1} & n!u/\theta^n \end{pmatrix} \tag{5.2}$$

is a symmetric matrix. Define matrix $P := \sqrt{2\mathbf{G}}$ and $\mathbf{A}_0 := \mathbf{D}^{\dagger}\mathbf{G}\mathbf{D}$, then we can proceed to verify the second structural stability condition due to Yong [41, 42], which is also given in Appendix A.

For the BGKW model, the linearized collision term in (4.8), $\tau^{-1}\mathbf{D}^{-1}\mathbf{B} = \tau^{-1}\mathbf{B}$, is diagonal and is obviously stable. Thus the first requirement of the stability condition is easily met with $\mathbf{S}=-\mathbf{I}_3$ in (A.4) (cf. (A.4) in Appendix A). The second requirement of the stability condition of for the flux term is satisfied because

$$\mathbf{A}_0\hat{\mathbf{A}}=\mathbf{D}^\dagger\mathbf{G}\mathbf{D}\mathbf{D}^{-1}\mathbf{M}\mathbf{D}=\mathbf{D}^\dagger\mathbf{G}\mathbf{M}\mathbf{D} \tag{5.3}$$

is indeed symmetric (cf. (A.5) in Appendix A).

To verify the third requirement in the stability condition, we note that, for any equilibrium state $\mathbf{W}$,

$$f_i=0, \quad \forall i\geq 3. \tag{5.4}$$

Consequently $\mathbf{D}$ becomes a diagonal matrix:

$$\mathbf{D}=\mathrm{diag}(1,\rho,\rho/2,1,1,\dots,1), \tag{5.5}$$

thus $\mathbf{D}$, $\mathbf{G}$, and $\mathbf{B}$ are diagonal, too, and for matrix $P:=\sqrt{2\mathbf{G}}$, we have

$$\begin{aligned}\mathbf{A}_0\mathbf{D}^{-1}\mathbf{B}=\mathbf{D}^\dagger\mathbf{G}\mathbf{B}&=-\mathrm{diag}\left(0,0,0,\frac{6}{\theta^3},\dots,\frac{(i-1)!}{\theta^{i-1}},\dots,\frac{n!}{\theta^n}\right)\\&=\frac{1}{2}P^\dagger\,\mathrm{diag}(0,0,0,-1,-1,\dots,-1)\,P,\end{aligned} \tag{5.6}$$

which means the third requirement in the stability condition is satisfied (cf. (A.6) in Appendix A).

Thus we prove that the globally hyperbolic moment system (4.8) satisfies the second structural stability condition for hyperbolic relaxation systems. With the stability condition satisfied, the solution of the moment system will go to the solution of the corresponding equilibrium system, the Euler equations, as ϵ goes to 0. The above proof for the system in 1D can be extended to its multi-dimensional counterparts and the result is presented in Appendix B.

6. Linear instability of a 5-moment system at nonequilibrium states

In this section we will analyze the linear stability of the GHMS at an arbitrary state $\mathbf{W}$. Specifically, we will consider the following system with $n=4$ in 1D:

$$\frac{\partial\mathbf{W}}{\partial t}+\hat{\mathbf{A}}\frac{\partial\mathbf{W}}{\partial x}=\frac{1}{\tau}\mathbf{B}\mathbf{W}, \tag{6.1}$$

were $\mathbf{W}=(\rho,u,\theta,f_3,f_4)^\dagger\in\mathbb{R}^5$ and $\mathbf{B}=\mathrm{diag}(0,0,0,-1,-1)$. For this system, we have $\mathbf{B}=\mathbf{D}^{-1}\mathbf{B}$ and $\hat{\mathbf{A}}=\mathbf{D}^{-1}\mathbf{M}\mathbf{D}$ with $\mathbf{M}$ and $\mathbf{D}$ given by

$$\mathbf{M}=\begin{pmatrix}u&1&0&0&0\\\theta&u&2&0&0\\0&\theta&u&3&0\\0&0&\theta&u&4\\0&0&0&\theta&u\end{pmatrix},\quad \mathbf{D}=\begin{pmatrix}1&0&0&0&0\\0&\rho&0&0&0\\0&0&\frac{\rho}{2}&0&0\\0&0&0&1&0\\0&f_3&0&0&1\end{pmatrix}. \tag{6.2}$$

We will identify some states at which system (6.1) is linearly unstable.

The linear stability of (6.1) is determined by the eigenvalues of the following matrix derived from the linearized system (6.1) (cf. Appendix A and [42]):

$$\tilde{\mathbf{H}}_\mathrm{r}(\eta,\zeta)=\eta\mathbf{D}\mathbf{B}\mathbf{D}^{-1}+\imath\zeta\mathbf{M},\quad \tilde{\mathbf{H}}_\mathrm{r}:=\mathbf{D}\mathbf{H}_\mathrm{r}\mathbf{D}^{-1}, \tag{6.3}$$

where $\imath := \sqrt{-1}$, and $\eta = 1/\tau$ and ζ are the relaxation rate and the wave-vector, respectively. Note that the characteristic polynomials of $\tilde{\mathbf{H}}_{\rm r}$ and $\mathbf{H}_{\rm r}$ are identical, and $\tilde{\mathbf{H}}_{\rm r}$ is easier to be dealt with (cf. Appendix A and [42] for the definition of $\mathbf{H}_{\rm r}$), so we will use $\tilde{\mathbf{H}}_{\rm r}$ instead of $\mathbf{H}_{\rm r}$ in what follows. However, we will not solve the eigenvalue problem of $\tilde{\mathbf{H}}_{\rm r}$. Instead, we will find some special solution of the eigenvalue problem analytically to demonstrate the linear instability of the system.

First, we choose $\eta = 1$ and $u = 0$ so $\tilde{\mathbf{H}}_{\rm r}(\eta, \zeta)$ becomes

$$\tilde{\mathbf{H}}_{\rm r}(\eta = 1, \zeta) = \begin{pmatrix} 0 & \imath\zeta & 0 & 0 & 0 \\ \imath\theta\zeta & 0 & 2\imath\zeta & 0 & 0 \\ 0 & \imath\theta\zeta & 0 & 3\imath\zeta & 0 \\ 0 & 0 & \imath\theta\zeta & -1 & 4\imath\zeta \\ 0 & f_3/\rho & 0 & \imath\theta\zeta & -1 \end{pmatrix}, \tag{6.4}$$

and the corresponding characteristic polynomial is:

$$\left|\lambda\mathbf{I} - \tilde{\mathbf{H}}_{\rm r}(\eta = 1, \zeta)\right| = \lambda^5 + 2\lambda^4 + (1 + 10a)\lambda^3 + 9a\lambda^2 + (3a + 15a^2 + \imath b)\lambda + 3a^2, \tag{6.5}$$

where $a := \theta\zeta^2$ and $b := 24 f_3 \zeta^3/\rho$. The above equation for λ cannot be solved analytically in general. To simplify the problem, we solve the equation of a and b, equivalently ζ and f_3, with a given complex-valued λ, such that $\Re(\lambda) > 0$.

Substitute $\lambda = 1 + \imath$ into the characteristic equation (6.5), which becomes the following equation for a and b:

$$\left(18a^2 - 17a - b - 14\right) + \imath\left(15a^2 + 41a + b - 2\right) = 0, \tag{6.6}$$

the real and imaginary parts of the above equation are:

$$\begin{cases} 18a^2 - 17a - b - 14 = 0, \\ 15a^2 + 41a + b - 2 = 0, \end{cases}$$

which leads to the following equation for a alone:

$$33a^2 - 24a - 16 = 0.$$

Because $a := \theta\zeta^2 \geq 0$, therefore, the admissible solutions for (a, b) are:

$$a = \frac{4(\sqrt{42} - 3)}{33}, \quad b = 18a^2 - 17a - 14 = \frac{2(1029 - 662\sqrt{42})}{363}. \tag{6.7}$$

In summary, with the parameter $\eta = 1/\tau = 1$, the linearized system (6.1) of the GHMS in 1D with $n = 4$ is unstable with the initial quiescent state $u = 0$, $\rho > 0$, and $\theta > 0$ at mode ζ and the third-order moment f_3 given below:

$$\zeta = \sqrt{\frac{a}{\theta}}, \quad a = \frac{4(\sqrt{42} - 3)}{33}, \tag{6.8a}$$

$$f_3 = \frac{(1029 - 662\sqrt{42})\rho}{4356\zeta^3} = \frac{(749 - 29\sqrt{42})\sqrt{3 + \sqrt{42}}}{1056}\rho\theta^{3/2} \approx 1.635\rho\theta^{3/2}. \tag{6.8b}$$

In other words, we have found the state $\mathbf{W} = (\rho, u, \theta, f_3, f_4)^\dagger$ with which the linearized GHMS (6.1) is linearly unstable at the wave-length $2\pi\sqrt{\theta/a}$ (a is a constant given by Equations (6.7)) — the growth rate of the unstable mode is 1. This unstable state is nonequilibrium because the heat flux $q = 3f_3 \neq 0$.

7. Nonlinear Instability — Numerical Results

We now investigate the nonlinear stability of the moment system with the BGKW collision term given by Equation (6.1). Specifically, we numerically solve the following system with spatial periodic boundary condition:

$$\frac{\partial \mathbf{W}}{\partial t}+\hat{\mathbf{A}}\frac{\partial \mathbf{W}}{\partial x}=\frac{1}{\tau}\mathbf{B}\mathbf{W}, \quad x\in[0,1),\ t>0, \tag{7.1}$$

with the initial data $\mathbf{W}=(W_0,W_1,\ldots,W_4)^{\dagger}$ and

$$W_0(x,0)=\rho_0, \tag{7.2a}$$
$$W_1(x,0)=u_0, \tag{7.2b}$$
$$W_2(x,0)=\theta_0\,[2+\sin(2\pi x)], \tag{7.2c}$$
$$W_3(x,0)=\frac{(749-29\sqrt{42})(\sqrt{42}+3)^{1/2}}{1056}\rho_0\,[W_2(x,0)]^{3/2}, \tag{7.2d}$$
$$W_4(x,0)=1, \tag{7.2e}$$

where $\rho_0=1$, $u_0=0$, and $\theta_0>0$ is an arbitrary constant which has no effect on the stability analysis and can be scaled out. The above initial conditions are constructed based on the unstable states given by Equations (6.8).

There are two choices of τ in system (7.1). One is $\tau=1$, which is consistent with choice $\eta=1/\tau=1$ in our analysis. The other is $\tau=\mathrm{Kn}/\rho$ for which our linear stability analysis remains valid. With $\tau=\mathrm{Kn}/\rho$, system (7.1) can be written as

$$\frac{1}{\rho}\frac{\partial \mathbf{W}}{\partial t}+\frac{1}{\rho}\hat{\mathbf{A}}\frac{\partial \mathbf{W}}{\partial x}=\frac{1}{\mathrm{Kn}}\mathbf{B}\mathbf{W}. \tag{7.3}$$

If the above system is linearized about $\rho=1$, the resulting system is identical to what is analyzed previously. Also, with $\tau=\mathrm{Kn}/\rho$, we can set $\mathrm{Kn}=1$ hence $\tau=1/\rho\approx 1$, and the linear stability analysis remains valid, so system (7.1) remains linearly unstable with the initial conditions given by (7.2). Most importantly, τ can be scaled out by $t\to t/\tau$ and $x\to x/\tau$ in system (7.1), thus the choice of τ is inconsequential.

We use a validated first-order upwind scheme to solve the initial value problem of system (7.1). The validation of the numerics is given in Appendix C. The solution of system (7.1) always blows up with the initial conditions (7.2), regardless of the value of θ_0, the time step size Δt, or the choice of τ. The temperature $\theta(x,t)$ always becomes negative some where at some finite time. Figure 7.1 shows the density $\rho(x,t)$, the velocity $u(x,t)$, and the temperature $\theta(x,t)$, and nonequilibrium variables $f_3(x,t)$ and $f_4(x,t)$ right before they blow up. The parameters used in the simulation are: $\tau=1$, $\theta_0=20000$, $\Delta x=2.5\times 10^{-4}$, and $\Delta t/\Delta x=10^{-4}$. Clearly, the density $\rho(x,t)$ becomes highly oscillatory at the location where the temperature $\theta(x,t)$ approaches to zero.

To understand the cause of the blow-up of the solution, a qualitative analysis of

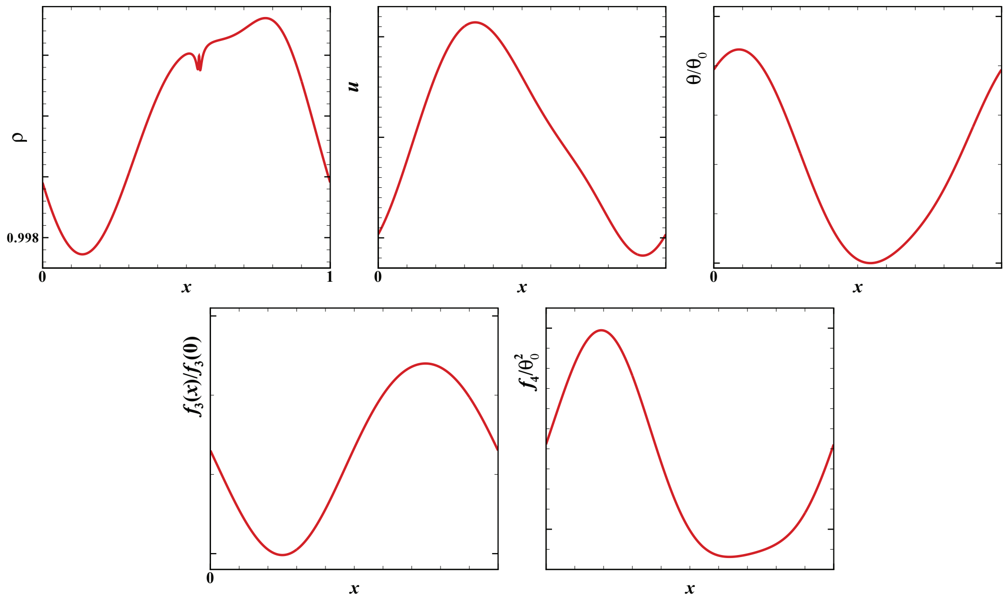

FIG. 7.1. *The numerical solution of* (7.1) *with the initial conditions* (7.2) *just before blow up. Top row, from left to right: the density* $\rho(x)$*, the velocity* $u(x)$*, and the temperature* $\theta(x)/\theta_0$*. Bottom row, from left to right: the third-order moment* $f_3(x)/f_3(t=0)$ *and the fourth-order moment* $f_4(x)/\theta^2$*.* $\tau=1$*,* $\theta_0=20000$*,* $\Delta x=2.5\times 10^{-4}$*,* $\Delta t/\Delta x=10^{-4}$*.*

system (7.1) would help. The system (7.1) in long-form is:

$$\partial_t\rho+u\partial_x\rho+\rho\partial_x u=0, \tag{7.4a}$$

$$\partial_t u+\frac{\theta}{\rho}\partial_x\rho+u\partial_x u+\partial_x\theta=0, \tag{7.4b}$$

$$\partial_t\theta+2\theta\partial_x u+u\partial_x\theta+\frac{6}{\rho}\partial_x f_3=0, \tag{7.4c}$$

$$\partial_t f_3+4f_3\partial_x u+\frac{\rho\theta}{2}\partial_x\theta+u\partial_x f_3+4\partial_x f_4=-\frac{1}{\tau}f_3, \tag{7.4d}$$

$$\partial_t f_4-\frac{\theta f_3}{\rho}\partial_x\theta-f_3\partial_x\rho+\theta\partial_x f_3+u\partial_x f_4=-\frac{1}{\tau}f_4. \tag{7.4e}$$

The first two equations above are the mass and momentum equations in the Euler equations in 1D. However, the third equation has the terms $\theta\partial_x u+(6/\rho)\partial_x f_3$ in addition to the energy equation in the Euler equations. With the quiescent initial state (7.2), the temperature θ has a sinusoidal component. Initially, the gradient of θ, $\partial_x\theta$, derives the velocity u, θ, f_3, and f_4, then the gradient of u, $\partial_x u$, derives the density ρ, and the gradient of f_3, $\partial_x f_3$, derives θ and f_4, so on and so forth. This is a nonlinearly coupled system. Without sufficient dissipation, the system cannot be stable, and that is what observed in the data shown in Fig. 7.1. In Fig. 7.1, θ is normalized by θ_0, f_3 by $f_3(0):=f_3(t=0):=(2\theta_0)^{3/2}(749-29\sqrt{42})(3+\sqrt{42})/1056$, and f_4 by θ_0^2. The data shown in Fig. 7.1 indicate that, just before the solution blows up,

$$|\rho| = O(1) < |u| = O(1) \ll |\theta| = O(10^4) \ll |f_3| = O(10^8) \approx |f_4| = O(10^8),$$
$$|\partial_x\rho| = O(10^{-3}) \ll |\partial_x u| = O(1) \ll |\partial_x\theta| = O(10^4) \ll |\partial_x f_3| = O(10^8) \approx |\partial_x f_4| = O(10^8).$$

Clearly, both f_3 and f_4, and especially f_3, play the dominant role in the system. The dissipation in the system, *i.e.*, the terms of $1/\tau$ in the equations of f_3 and f_4 is insufficient to keep the system in check — the magnitudes of the oscillatory component in both θ and f_4 grow rapidly. Note that τ can be scaled out by $t \to t/\tau$ and $x \to x/\tau$. Therefore, the solution will blow up independent of the value of τ.

To see how system (7.4) evolves, Fig. 7.2 shows the evolution of the system at $x = x_*$, where, just before the blow-up, θ attains its minimum, *i.e.*, $\theta \gtrsim 0$ and

$$\partial_x\theta\big|_{x=x_*} = 0. \tag{7.5}$$

Figure 7.2 shows that $f_3(x_*,t)/f_3(0)$, and in turn, $\partial_t\theta(x_*,t)/\theta_0^2$, are slow varying in time, and both are negative; and both $\theta(x_*,t)$ and $f_4(x_*,t)$ decrease monotonically in time. The temperature θ at $x = x_*$ appears to decay linearly in time, *i.e.*,

$$\frac{\theta(x_*,t)}{\theta_0} \approx 1 - \frac{t}{t_\infty}, \quad 0 < t < t_\infty, \tag{7.6}$$

where t_∞ is the critical time when the temperature θ reaches zero, before it becomes negative and the system blows up. We have also observed that the system with the initial conditions (7.2) so long as $W_3(x,0) \geq 0.1\rho_0\,[W_2(x,0)]^{3/2}$.

To ensure that the numerical solutions are not affected by the time step size Δt, we conduct a convergence study of Δt. The spatial grid size is fixed at $\Delta x = 2.5\times10^{-4}$. For a given value of θ_0, we vary the ratio $\Delta t/\Delta x$ and to compute the critical time t_∞ when the solution blows up. Table 7.1 summarizes the results of the critical time t_∞ depending on both θ_0 and $\Delta t/\Delta x$.

TABLE 7.1. *The dependence of the critical time t_∞ when the solution blows up on $\Delta t/\Delta x$ and θ_0. $\tau = 1$ (or $\tau = \mathrm{Kn}/\rho$) and $\Delta x = 2.5\times10^{-4}$. The symbols "—" and "×" indicate, respectively, that either the computation is not carried for it takes too long and it cannot be done because of numerical instability due to the fact that Δt is not small enough. The asymptotic values of the critical time, t_∞, in the last row are obtained by the least-square fitting of the cubic polynomial given by* (7.7).

$\Delta t/\Delta x$	$\theta_0 = 0.02$	$\theta_0 = 0.2$	$\theta_0 = 20$	$\theta_0 = 200$	$\theta_0 = 20000$
$1\cdot10^{-2}$	0.649475	0.20085000	0.01995000	0.006325000	×
$5\cdot10^{-3}$	0.649475	0.20083750	0.01992500	0.006312500	×
$1\cdot10^{-3}$	0.649470	0.20082750	0.01990500	0.006295000	0.000632500
$5\cdot10^{-4}$	0.649470	0.20082625	0.01990250	0.006291250	0.000630000
$1\cdot10^{-4}$	0.649469	0.20082475	0.01990100	0.006289250	0.000629000
$1\cdot10^{-5}$	—	—	0.01990005	0.006288875	0.000628775
$1\cdot10^{-6}$	—	—	—	0.006288835	0.000628710
t_∞	0.6494688	0.2008245	0.0199002	0.0062888	0.0006287

The results in Table 7.1 show that the critical time t_∞ depends very weakly on the value of time step size Δt, indicating that the values of Δt we use are sufficiently small. The results also show that the dependence of t_∞ on θ_0. By using the least-square method, we fit with the critical time in Table 7.1 with the following cubic polynomial:

$$t(\Delta t) = t_\infty + c_1\Delta t + c_2\Delta t^2 + c_3\Delta t^3, \tag{7.7}$$

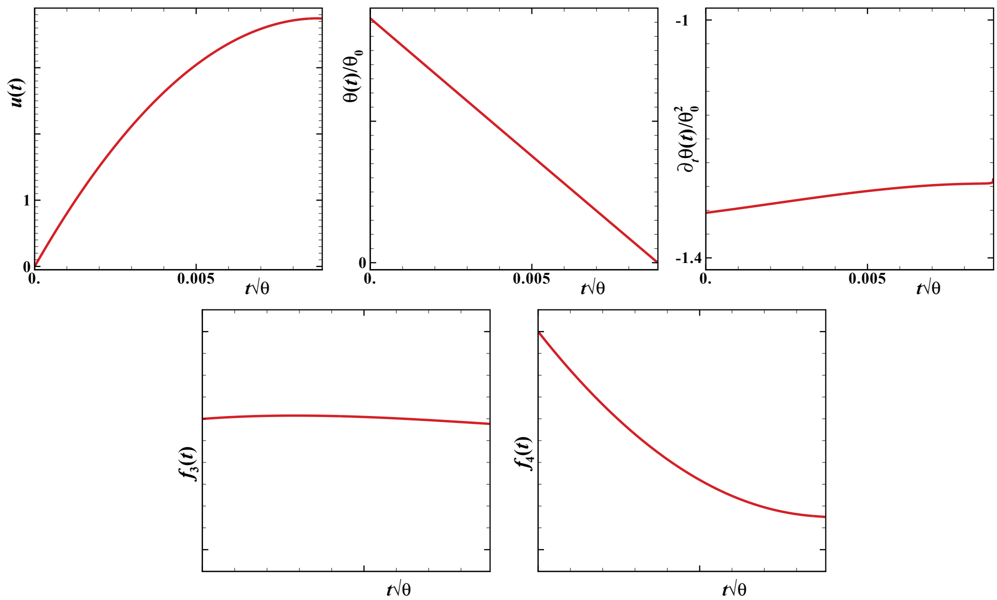

FIG. 7.2. *The numerical solution of* (7.1) *with the initial conditions* (7.2) *at* $x=x_*$ *where* $\partial_x\theta(x_*)=0$ *and* θ *is very small before the solution blow up. Top row, from left to right: the velocity* $u(x_*,t)$, *and the temperature* $\theta(x_*,t)/\theta_0$, *and* $\partial\theta(x_*,t)/\theta^2$. *Bottom row, from left to right: the third-order moment* $f_3(x_*,t)/f_3(0)$, $f_3(0)$ *and the fourth-order moment* $f_4(x_*,t)/\theta^2$. *The time* t *is rescaled to* $t\sqrt{\theta}$.

where t_∞ is the asymptotic value of the critical time in the limit $\Delta t\to 0$. It should be noted that, due to the low fluctuation in the density ρ, as seen in Fig. 7.1, the critical times obtained with $\tau=1$ and $\tau=\mathrm{Kn}/\rho$ are nearly identical to that obtained with $\mathrm{Kn}=1$. With $\theta_0=0.2$, 20, 200, and 20 000, we observe that the critical time t_∞ scales with θ_0 as the following:

$$t_\infty \propto \theta_0^{-1/2}, \tag{7.8}$$

which can be easily derived from the linear dispersion relationship $\eta t=\zeta\lambda$, where η, ζ, and λ are the frequency, wave-number, and wave-length, respectively, and the fact that $a:=\theta\zeta^2>0$ is an arbitrary *constant*. The scaling (7.8) is a consequence of the fact that system (3.4) can be normalized by $\theta_0^{(n+1)/2}$ so $t\to t\sqrt{\theta_0}$, so θ_0 is scaled out of the system. The scaling of t_∞ indicates that the system is unstable so long as $\theta_0>0$, *i.e.*, with any finite initial temperature $\theta_0>0$, the system always blows up in a finite time $t_\infty\propto\theta_0^{-1/2}<\infty$. Therefore our numerical results indicate that the *nonlinear* system (7.1) is unstable with the initial conditions (6.8).

8. Conclusions and Discussion

In this work, we study the stability of a class of the globally hyperbolic moment system [8, 9] in the sense of hyperbolic relaxation systems [41, 42]. First, we prove that the GHMS derived from the Boltzmann equation with the single-relaxation-time collision model is stable at equilibrium [41] in both 1D and multi-dimensions. The equilibrium stability ensures that, in the limit of relaxation time, the GHMS goes to its proper limit — the Euler equations.

Second, we prove the *linear* instability of a five-moment system at some *nonequilibrium* states, and show numerically the *nonlinear* instability of the system. While the global hyperbolicity is an important feature for a moment system to retain, it alone is insufficient for the system to be physical, for without stability, the system cannot be physical. We may speculate the cause of the instability in the five-moment system. The first two equations in system (7.4) are the mass and momentum equations in the Euler equations in 1D, which are attained only when the system at the equilibrium, *i.e.*, with no heat flux. The initial state includes a non-zero heat flux, which provides a deriving source in the system. We note the fact that the relaxation time τ can be scaled out. We observe numerically that the system is unstable so long as the magnitude of the oscillatory component in the initial heat flux is larger than some critical value. This observation indicates that the system does not have a proper conservation-dissipation mechanism to ensure its stability.

Third, the moment system is derived from the Boltzmann equation, of which the H-theorem is an essential characteristics — the H-theorem ensures the solution of the Boltzmann equation always approaches to the equilibrium. It is, therefore, appropriate to ask the question if and to what extend the H-theorem of the Boltzmann equation is retained in a truncated moment system. When the moment system is cast as a hyperbolic relaxation system, the stability theory developed for hyperbolic relaxation systems [41, 42] can be directly applied to analyze the moment system. It should also be emphasized that the stability theory for hyperbolic relaxation systems [41, 42] has been verified for numerous well-known systems of PDEs in physics [41, 42], and it can also be used to check numerical schemes for conservation laws. The present work further demonstrates the universality and the significance of the stability theory for hyperbolic relaxation systems [41, 42].

Finally, we would like to point out the connection between the second structural stability condition [41, 42] and the H-theorem: to the moment closure systems, the stability condition characterizes the same dissipative or irreversible effect as the H-theorem to the Boltzmann equation. The conservation-dissipation criteria for hyperbolic relaxation systems due to Yong [44, 45] and the H-theorem have the same mathematical formalism, *i.e.*, the entropy inequality. In fact, the conservation-dissipation criteria can be viewed as the counterpart of the H-theorem in hyperbolic relaxation systems. The criteria are stronger than the structural stability condition [44, 45], and they have led to a nonlinear extension of extended irreversible thermodynamics [47, 33]. We are convinced that a physical moment closure should pass the test of the conservation-dissipation criteria, as observed in this and other previous studies [45, 40, 46, 19, 26].

While we have demonstrated the non-equilibrium instabilities in a specific GHMS in this work, we have yet to show how to overcome the instabilities observed in this work, which shall be the subject of our future work. In addition, we are also interested in the stability analysis of other moment systems [8, 9, 23, 11, 13].

Appendix A. Stability criteria for hyperbolic relaxation systems.

The moment system (4.2) is a special case of the hyperbolic relaxation system [41, 42], which is a system of first-order partial differential equations (PDEs) with a small parameter $\epsilon > 0$ defined as the following:

$$\partial_t \mathbf{W} + \boldsymbol{\nabla} \cdot \mathsf{F}^\dagger = \frac{1}{\epsilon}\mathbf{Q}, \tag{A.1}$$

or more generally,

$$\partial_t \mathbf{W} + \sum_{\alpha=1}^{d} \mathbf{R}_\alpha \frac{\partial \mathbf{W}}{\partial x_\alpha} = \frac{1}{\epsilon}\mathbf{Q}, \tag{A.2}$$

where $\mathbf{W}$ is an unknown n-vector valued function of $(\boldsymbol{x}, t) := (x_1, x_2, \ldots, x_d, t) \in \mathbb{R}^d \times \mathbb{R}^+$; $\mathsf{F} := (\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_d) \in \mathbb{R}^{n\times d}$ and $\mathbf{F}_j := \mathbf{F}_j(\mathbf{W})$ is an n-vector; $\mathbf{R}_\alpha := \mathbf{R}_\alpha(\mathbf{W}) \in \mathbb{R}^{n\times n}$ is a second-order tensor whose components are smooth functions of $\mathbf{W} \in \mathbb{G}$, and $\mathbb{G}$ is the state space which is an open subset in $\mathbb{R}^n$; and $\mathbf{Q} := \mathbf{Q}(\mathbf{W})$ is an n-vector associated with the equilibrium manifold $\mathbb{E}$ defined as the following:

$$\mathbb{E} := \{\mathbf{W} \in \mathbb{G} | \, \mathbf{Q}(\mathbf{W}) = \mathbf{0}\} \neq \emptyset. \tag{A.3}$$

The sufficient stability condition for the nonlinear system (A.2) consists of the following three parts [41]:

1. There exist an invertible $n \times n$ matrix $P := P(\mathbf{W})$ and a stable $r \times r$, $0 < r \leq n$, matrix $\mathbf{S}(\mathbf{W})$ defined on the equilibrium manifold $\mathbb{E}$, such that

$$P[\boldsymbol{\nabla}_{\mathrm{w}} \mathbf{Q}] = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S} \end{pmatrix} P, \quad \forall \mathbf{W} \in \mathbb{E}. \tag{A.4}$$

   Note that $P := P(\mathbf{W})$ is *not* the pressure tensor defined in equation (2.5b).

2. As a system of first-order PDEs, (A.2) is symmetrizable, that is, there exists a positive definite Hermitian matrix $\mathbf{A}_0(\mathbf{W}) \in \mathbb{R}^{n\times n}$ such that

$$\mathbf{A}_0 \mathbf{R}_\alpha = \mathbf{R}_\alpha^\dagger \mathbf{A}_0, \quad \forall \mathbf{W} \in \mathbb{G} \text{ and } \mathbf{R}_\alpha \in \mathbb{R}^{n\times n}, \quad \alpha = 1, \ldots, d; \tag{A.5}$$

3. The hyperbolic part and the source term are coupled in the sense that

$$\mathbf{A}_0[\boldsymbol{\nabla}_{\mathrm{w}} \mathbf{Q}] + [\boldsymbol{\nabla}_{\mathrm{w}} \mathbf{Q}]^\dagger \mathbf{A}_0 \leq -P^\dagger \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{I}_{\mathrm{r}} \end{pmatrix} P, \quad \forall \mathbf{W} \in \mathbb{E}. \tag{A.6}$$

the system (A.2) is stable at the equilibrium if (A.4), (A.5), and (A.6) are satisfied simultaneously. Variants of the above stability condition can be found in [41, 42]. The condition consisting of (A.4), (A.5), and (A.6) is the so-called the second structural stability condition due to Yong [41, 42].

The first requirement (A.4) for the collision or source term is the essential condition of the Tikhonov theorem (cf., *e.g.*, [38]) for the system of ordinary differential equations (*i.e.*, (A.1) or (A.2) without the spatial derivatives) to have a well-behaved limit $\epsilon \to 0^+$. The second requirement (A.5) states that the system of first-order partial differential equations is symmetrizable hyperbolic in the sense of Friedrichs [27]. And the third requirement (A.6) specifies a proper coupling of the ODE and hyperbolic parts.

The linearized counterpart of the full nonlinear system (A.2) is:

$$\partial_t \mathbf{W} + \sum_{\alpha=1}^{d} \mathbf{R}_\alpha \frac{\partial \mathbf{W}}{\partial x_\alpha} = \frac{1}{\epsilon}\mathbf{B}\mathbf{W}, \tag{A.7}$$

where $\mathbf{R}_\alpha \in \mathbb{R}^{n\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times n}$ are constant second-order tensors. Fourier transform in space of Equation (A.7) with constant $\mathbf{R}_\alpha$ leads to:

$$\partial_t \tilde{\mathbf{W}} = \mathbf{H}_{\mathrm{r}}(\eta, \boldsymbol{\zeta})\tilde{\mathbf{W}}, \tag{A.8}$$

where $\tilde{\mathbf{W}}$ is the Fourier transform of $\mathbf{W}$,

$$\mathbf{H}_{\mathrm{r}}(\eta,\boldsymbol{\zeta}) = \eta\mathbf{B} + \imath\sum_{\alpha=1}^{d}\zeta_\alpha\mathbf{R}_\alpha, \tag{A.9}$$

$\eta = 1/\epsilon \in \mathbb{R}^+$ is the frequency, and $(\zeta_1,\ldots,\zeta_d) := \boldsymbol{\zeta} \in \mathbb{R}^d$ is the wave-vector. Clearly, a *necessary* stability condition [42] for the linearized system (A.7) is that $\mathbf{H}_{\mathrm{r}}(\eta,\boldsymbol{\zeta})$ has no eigenvalues with positive real part for any $\eta \geq 0$ and $\boldsymbol{\zeta} \in \mathbb{R}^d$, that is, for any eigenvalue λ_i of $\mathbf{H}_{\mathrm{r}}(\eta,\boldsymbol{\zeta})$, the stability requires that:

$$\Re(\lambda_i) \leq 0, \qquad 1 \leq i \leq n, \quad \forall \boldsymbol{\zeta} \in \mathbb{R}^d, \quad \eta \geq 0. \tag{A.10}$$

The above condition (A.10) is a direct corollary of the so-called relaxation criterion due to Yong [42].

Appendix B. The equilibrium stability of the multi-dimensional GHMS.

In this section we prove the equilibrium stability of the multi-dimensional GHMS derived from the Boltzmann–BGKW equation (**??**).

B.1. Derivation of the hierarchy of moment equations

Suppose $\boldsymbol{x} = (x_1,\ldots,x_d) \in \mathbb{R}^d$ and t are fixed, the distribution function f is expended with the basis functions of multi-dimensional Hermite tensor polynomials $H_{\boldsymbol{n}}(\bar{\boldsymbol{c}})$ [18]:

$$f(\boldsymbol{\xi}) = \sum_{\boldsymbol{n}\in\mathbb{N}_0^d} f_{\boldsymbol{n}} H_{\boldsymbol{n}}(\bar{\boldsymbol{c}}) := \langle \mathbf{H}, \mathbf{f}\rangle_\infty, \tag{B.1}$$

where $\langle\cdot,\cdot\rangle_\infty$ denotes the inner product of two vectors in infinite dimensional space, and $\bar{\boldsymbol{c}} := (\boldsymbol{\xi} - \boldsymbol{u})/\sqrt{\theta}$, $\boldsymbol{n} := (n_1,\cdots,n_d)$ is the d-dimensional index. To make the ordering of coefficients $f_{\boldsymbol{n}}$ unique, we follow the convention given by Fan *et al.* [13], *i.e.*, $f_{\boldsymbol{n}}$ is ordered in the ascending order n first, $n := |\boldsymbol{n}| := \sum_\alpha n_\alpha$, and second, in the ordering of $n_1 \succ n_2 \succ \cdots \succ n_d$:

$$\begin{aligned}\mathbf{f} = (&f_{\boldsymbol{0}},\\ &f_{\hat{\boldsymbol{n}}_1}, f_{\hat{\boldsymbol{n}}_2}, \ldots, f_{\hat{\boldsymbol{n}}_d},\\ &f_{2\hat{\boldsymbol{n}}_1}, f_{\hat{\boldsymbol{n}}_1+\hat{\boldsymbol{n}}_2}, f_{\hat{\boldsymbol{n}}_1+\hat{\boldsymbol{n}}_3}, \ldots, f_{\hat{\boldsymbol{n}}_1+\hat{\boldsymbol{n}}_d}, f_{2\hat{\boldsymbol{n}}_2}, f_{\hat{\boldsymbol{n}}_2+\hat{\boldsymbol{n}}_3}, \ldots, f_{\hat{\boldsymbol{n}}_2+\hat{\boldsymbol{n}}_d}, \ldots, f_{2\hat{\boldsymbol{n}}_d},\\ &f_{3\hat{\boldsymbol{n}}_1}, f_{2\hat{\boldsymbol{n}}_1+\hat{\boldsymbol{n}}_2}, f_{2\hat{\boldsymbol{n}}_1+\hat{\boldsymbol{n}}_3}, \ldots, f_{2\hat{\boldsymbol{n}}_1+\hat{\boldsymbol{n}}_d}, f_{\hat{\boldsymbol{n}}_1+2\hat{\boldsymbol{n}}_2},\ f_{\hat{\boldsymbol{n}}_1+\hat{\boldsymbol{n}}_2+\hat{\boldsymbol{n}}_3}, \ldots, f_{3\hat{\boldsymbol{n}}_d},\\ &f_{4\hat{\boldsymbol{n}}_1}, \ldots),\end{aligned} \tag{B.2}$$

where $\hat{\boldsymbol{n}}_\alpha$ is the unit vector in the αth direction. The vector $\mathbf{H} := (H_{\boldsymbol{n}}(\bar{\boldsymbol{c}}))_{\boldsymbol{n}\in\mathbb{N}_0^d}$ consists of functions $H_{\boldsymbol{n}}$, which are the multi-dimensional Hermite tensor polynomials [18]:

$$H_{\boldsymbol{n}}(\bar{\boldsymbol{c}}) = (2\pi\theta)^{-d/2}\prod_{\alpha=1}^{d}\theta^{-\frac{n_\alpha}{2}} H^{(n_\alpha)}(\bar{c}_\alpha)\exp\left(-\frac{\bar{c}_\alpha^2}{2}\right), \tag{B.3}$$

where $H^{(n_\alpha)}(\bar{c}_\alpha)$ is the Hermite tensor polynomial defined by Equation (3.1) and $H_{\boldsymbol{n}}$ is zero if any component of $\boldsymbol{n}$, n_α, is negative.

The following properties of the Hermite polynomials $H_{\boldsymbol{n}}$ will be used later:

$$\langle H_{\boldsymbol{n}}, H_{\boldsymbol{m}} \rangle = \frac{\boldsymbol{n}!}{\theta^{\boldsymbol{n}}} \delta_{\boldsymbol{n},\boldsymbol{m}}, \tag{B.4}$$

$$\frac{\partial H_{\boldsymbol{n}}(\bar{\boldsymbol{c}})}{\partial s} = \sum_{\alpha=1}^{d} \frac{\partial u_\alpha}{\partial s} H_{\boldsymbol{n}+\hat{\boldsymbol{n}}_\alpha}(\bar{\boldsymbol{c}}) + \frac{1}{2}\frac{\partial \theta}{\partial s} \sum_{\alpha=1}^{d} H_{\boldsymbol{n}+2\hat{\boldsymbol{n}}_\alpha}(\bar{\boldsymbol{c}}), \tag{B.5}$$

$$\xi_\beta H_{\boldsymbol{n}}(\bar{\boldsymbol{c}}) = \theta H_{\boldsymbol{n}+\hat{\boldsymbol{n}}_\beta}(\bar{\boldsymbol{c}}) + u_\beta H_{\boldsymbol{n}}(\bar{\boldsymbol{c}}) + n_\beta H_{\boldsymbol{n}-\hat{\boldsymbol{n}}_\beta}(\bar{\boldsymbol{c}}), \tag{B.6}$$

where $s = t$ or x_α, $\alpha = 1, 2, \dots, d$, and the inner product here, $\langle \cdot, \cdot \rangle$, is defined as the following:

$$\langle f, g \rangle := \int_{\mathbb{R}^d} \frac{1}{\omega(\bar{\boldsymbol{c}})} f(\bar{\boldsymbol{c}}) g(\bar{\boldsymbol{c}}) \mathrm{d}\bar{\boldsymbol{c}}, \quad \omega(\bar{\boldsymbol{c}}) := \frac{1}{(2\pi\theta)^{d/2}} \exp\left(-\frac{\bar{c}^2}{2}\right). \tag{B.7}$$

The properties of $H_{\boldsymbol{n}}$ and the definition of macroscopic quantities immediately yield the following:

$$f_{\boldsymbol{0}} = \rho, \quad f_{\hat{\boldsymbol{n}}_\alpha} = 0, \quad \textstyle\sum_{\beta=1}^{d} f_{2\hat{\boldsymbol{n}}_\beta} = 0, \qquad \alpha = 1, \cdots, d, \tag{B.8}$$

and the stress tensor $\boldsymbol{\sigma}$ and heat flux $\boldsymbol{q}$ are given in the following simple form:

$$\sigma_{\alpha\beta} = f_{\hat{\boldsymbol{n}}_\alpha + \hat{\boldsymbol{n}}_\beta}, \quad \sigma_{\alpha\alpha} = 2 f_{2\hat{\boldsymbol{n}}_\alpha}, \qquad \alpha, \beta = 1, \cdots, d, \quad \alpha \neq \beta, \tag{B.9a}$$

$$q_\alpha = 2 f_{3\hat{\boldsymbol{n}}_\alpha} + \sum_{\beta=1}^{d} f_{2\hat{\boldsymbol{n}}_\beta + \hat{\boldsymbol{n}}_\alpha}, \qquad \alpha = 1, \cdots, d. \tag{B.9b}$$

With the expansion (B.1) for the distribution f, the derivative of f can be written as

$$\frac{\partial f}{\partial s} = \sum_{\boldsymbol{n} \in \mathbb{N}_0^d} \left[\frac{\partial f_{\boldsymbol{n}}}{\partial s} + \sum_{\alpha=1}^{d} \frac{\partial u_\alpha}{\partial s} f_{\boldsymbol{n}-\hat{\boldsymbol{n}}_\alpha} + \frac{1}{2}\frac{\partial \theta}{\partial s} \sum_{\alpha=1}^{d} f_{\boldsymbol{n}-2\hat{\boldsymbol{n}}_\alpha} \right] H_{\boldsymbol{n}}, \tag{B.10}$$

where $s = t$ or x_α, $\alpha = 1, 2, \dots, d$, which can be concisely written as the following:

$$\frac{\partial f}{\partial s} = \left\langle \mathbf{H}, \mathsf{D} \frac{\partial \mathbf{W}}{\partial s} \right\rangle_\infty, \tag{B.11}$$

where $\mathbf{W}$ is the same as $\mathbf{f}$ except that $f_{\hat{\boldsymbol{n}}_\alpha}$ is replaced by u_α and $f_{2\hat{\boldsymbol{n}}_1}$ by $\theta/2$, *i.e.*,

$$\begin{aligned} \mathbf{W} = (&f_{\boldsymbol{0}}, \\ &u_1, u_2, \dots, u_d, \\ &\theta/2, f_{\hat{\boldsymbol{n}}_1+\hat{\boldsymbol{n}}_2}, f_{\hat{\boldsymbol{n}}_1+\hat{\boldsymbol{n}}_3}, \dots, f_{\hat{\boldsymbol{n}}_1+\hat{\boldsymbol{n}}_\alpha}, f_{2\hat{\boldsymbol{n}}_2}, f_{\hat{\boldsymbol{n}}_2+\hat{\boldsymbol{n}}_3}, \dots, f_{\hat{\boldsymbol{n}}_2+\hat{\boldsymbol{n}}_\alpha}, \dots, f_{2\hat{\boldsymbol{n}}_d}, \\ &f_{3\hat{\boldsymbol{n}}_1}, f_{2\hat{\boldsymbol{n}}_1+\hat{\boldsymbol{n}}_2}, f_{2\hat{\boldsymbol{n}}_1+\hat{\boldsymbol{n}}_3}, \dots, f_{2\hat{\boldsymbol{n}}_1+\hat{\boldsymbol{n}}_\alpha}, f_{\hat{\boldsymbol{n}}_1+2\hat{\boldsymbol{n}}_2}, f_{\hat{\boldsymbol{n}}_1+\hat{\boldsymbol{n}}_2+\hat{\boldsymbol{n}}_3}, \dots, f_{3\hat{\boldsymbol{n}}_d}, \\ &f_{4\hat{\boldsymbol{n}}_1}, \dots), \end{aligned} \tag{B.12}$$

where the matrix $\mathsf{D} \in \mathbb{R}^{\infty \times \infty}$ is determined by Equation (B.10) with its non-zero entries given below:

$$\mathsf{D}_{ii} = 1, \quad \mathsf{D}_{ij} = f_{\boldsymbol{n}-\hat{\boldsymbol{n}}_\alpha}, \qquad i = K(\boldsymbol{n}),\ j = K(\hat{\boldsymbol{n}}_\alpha),\ |\boldsymbol{n}| > 1,\ \boldsymbol{n} \neq 2\hat{\boldsymbol{n}}_1, \tag{B.13a}$$

$$\mathsf{D}_{ij} = \textstyle\sum_{\alpha=1}^{d} f_{\boldsymbol{n}-2\hat{\boldsymbol{n}}_\alpha}, \qquad i = K(\boldsymbol{n}),\ j = K(2\hat{\boldsymbol{n}}_1),\ |\boldsymbol{n}| > 1,\ \boldsymbol{n} \neq 2\hat{\boldsymbol{n}}_1, \tag{B.13b}$$

$$\mathsf{D}_{ii} = \rho, \qquad i = K(\hat{\boldsymbol{n}}_\alpha) \text{ or } K(2\hat{\boldsymbol{n}}_1), \tag{B.13c}$$

$$\mathsf{D}_{ij} = -1, \qquad i = K(2\hat{\boldsymbol{n}}_1) \neq j = K(2\hat{\boldsymbol{n}}_\beta), \tag{B.13d}$$

where $K(\boldsymbol{n}) \in \mathbb{N}_0$ is the index of $f_{\boldsymbol{n}}$ in $\mathbf{f}$ defined in Equation (B.2). The above results are obtained through the following calculations.

First, Equation (B.13b) is directly obtained from Equation (B.10). Then we set $\boldsymbol{n} = \hat{\boldsymbol{n}}_\alpha$ to calculate

$$\frac{\partial f_{\hat{\boldsymbol{n}}_\alpha}}{\partial s} + \sum_{\beta=1}^{d} \frac{\partial u_\beta}{\partial s} f_{\hat{\boldsymbol{n}}_\alpha - \hat{\boldsymbol{n}}_\beta} + \frac{1}{2}\frac{\partial \theta}{\partial s} \sum_{\beta=1}^{d} f_{\hat{\boldsymbol{n}}_\alpha - 2\hat{\boldsymbol{n}}_\beta} = f_{\mathbf{0}} \frac{\partial u_\alpha}{\partial s} = \rho \frac{\partial u_\alpha}{\partial s}, \tag{B.14}$$

which means the only nonzero entry in the row $i = K(\hat{\boldsymbol{n}}_\alpha)$ of $\mathbf{D}$ is $\mathbf{D}_{ii} = \rho$. We further set $\boldsymbol{n} = 2\hat{\boldsymbol{n}}_1$ and use Equation (B.8) to obtain

$$\frac{\partial f_{2\hat{\boldsymbol{n}}_1}}{\partial s} + \sum_{\alpha=1}^{d} \frac{\partial u_\alpha}{\partial s} f_{2\hat{\boldsymbol{n}}_1 - \hat{\boldsymbol{n}}_\alpha} + \frac{1}{2}\frac{\partial \theta}{\partial s} \sum_{\alpha=1}^{d} f_{2\hat{\boldsymbol{n}}_1 - 2\hat{\boldsymbol{n}}_\alpha} = \frac{1}{2}\frac{\partial \theta}{\partial s}\rho - \sum_{\alpha=2}^{d} \frac{\partial f_{2\hat{\boldsymbol{n}}_\alpha}}{\partial s}.$$

Next we compute the convection term as

$$\begin{aligned}
\xi_\beta \frac{\partial f}{\partial s} &= \sum_{\boldsymbol{n}\in\mathbb{N}_0^d} \left[\frac{\partial f_{\boldsymbol{n}}}{\partial s} + \sum_{\alpha=1}^{d} \frac{\partial u_\alpha}{\partial s} f_{\boldsymbol{n}-\hat{\boldsymbol{n}}_\alpha} + \frac{1}{2}\frac{\partial \theta}{\partial s}\sum_{\alpha=1}^{d} f_{\boldsymbol{n}-2\hat{\boldsymbol{n}}_\alpha} \right] \xi_\beta H_{\boldsymbol{n}} \\
&= \sum_{\boldsymbol{n}\in\mathbb{N}_0^d} \left[\frac{\partial f_{\boldsymbol{n}}}{\partial s} + \sum_{\alpha=1}^{d} \frac{\partial u_\alpha}{\partial s} f_{\boldsymbol{n}-\hat{\boldsymbol{n}}_\alpha} + \frac{1}{2}\frac{\partial \theta}{\partial s}\sum_{\alpha=1}^{d} f_{\boldsymbol{n}-2\hat{\boldsymbol{n}}_\alpha} \right] \\
&\qquad \cdot \left(\theta H_{\boldsymbol{n}+\hat{\boldsymbol{n}}_\beta} + u_\beta H_{\boldsymbol{n}} + n_\beta H_{\boldsymbol{n}-\hat{\boldsymbol{n}}_\beta}\right) \\
&= \underbrace{\sum_{\boldsymbol{n}\in\mathbb{N}_0^d} \left[\frac{\partial f_{\boldsymbol{n}}}{\partial s} + \sum_{\alpha=1}^{d} \frac{\partial u_\alpha}{\partial s} f_{\boldsymbol{n}-\hat{\boldsymbol{n}}_\alpha} + \frac{1}{2}\frac{\partial \theta}{\partial s}\sum_{\alpha=1}^{d} f_{\boldsymbol{n}-2\hat{\boldsymbol{n}}_\alpha} \right] \theta H_{\boldsymbol{n}+\hat{\boldsymbol{n}}_\beta}}_{\left\langle \mathbf{N}_\theta \mathbf{H}, \mathbf{D}\frac{\partial \mathbf{W}}{\partial s} \right\rangle_\infty} \\
&\quad + \underbrace{\sum_{\boldsymbol{n}\in\mathbb{N}_0^d} \left[\frac{\partial f_{\boldsymbol{n}}}{\partial s} + \sum_{\alpha=1}^{d} \frac{\partial u_\alpha}{\partial s} f_{\boldsymbol{n}-\hat{\boldsymbol{n}}_\alpha} + \frac{1}{2}\frac{\partial \theta}{\partial s}\sum_{\alpha=1}^{d} f_{\boldsymbol{n}-2\hat{\boldsymbol{n}}_\alpha} \right] u_\beta H_{\boldsymbol{n}}}_{\left\langle \mathbf{N}_{u_\beta} \mathbf{H}, \mathbf{D}\frac{\partial \mathbf{W}}{\partial s} \right\rangle_\infty} \\
&\quad + \underbrace{\sum_{\boldsymbol{n}\in\mathbb{N}_0^d} \left[\frac{\partial f_{\boldsymbol{n}}}{\partial s} + \sum_{\alpha=1}^{d} \frac{\partial u_\alpha}{\partial s} f_{\boldsymbol{n}-\hat{\boldsymbol{n}}_\alpha} + \frac{1}{2}\frac{\partial \theta}{\partial s}\sum_{\alpha=1}^{d} f_{\boldsymbol{n}-2\hat{\boldsymbol{n}}_\alpha} \right] n_\beta H_{\boldsymbol{n}-\hat{\boldsymbol{n}}_\beta}}_{\left\langle \mathbf{N}_{n_\beta} \mathbf{H}, \mathbf{D}\frac{\partial \mathbf{W}}{\partial s} \right\rangle_\infty},
\end{aligned} \tag{B.15}$$

where $\mathbf{N}_\theta$, $\mathbf{N}_{u_\beta}$, $\mathbf{N}_{n_\beta} \in \mathbb{R}^{\infty\times\infty}$, and their nonzero entries are

$$\mathbf{N}_{\theta ij} = \theta,\ \mathbf{N}_{u_\beta ii} = u_\beta,\ \mathbf{N}_{n_\beta ik} = n_\beta, \quad i = K(\boldsymbol{n}),\ j = K(\boldsymbol{n}+\hat{\boldsymbol{n}}_\beta),\ k = K(\boldsymbol{n}-\hat{\boldsymbol{n}}_\beta). \tag{B.16}$$

Thus $\xi_\beta \partial f/\partial s$ in Equation (B.15) can be expressed as

$$\xi_\beta \frac{\partial f}{\partial s} = \left\langle \mathbf{H}, \mathbf{M}_\beta \mathbf{D}\frac{\partial \mathbf{W}}{\partial s} \right\rangle_\infty, \quad \mathbf{M}_\beta := \mathbf{N}_\theta^\dagger + \mathbf{N}_{u_\beta}^\dagger + \mathbf{N}_{n_\beta}^\dagger, \tag{B.17}$$

where the nonzero entries of $\mathbf{M}_\beta$ are

$$\mathbf{M}_{\beta ij} = \begin{cases} \theta, & i = K(\boldsymbol{n} + \hat{\boldsymbol{n}}_\beta),\ j = K(\boldsymbol{n}), \\ u_\beta, & i = j = K(\boldsymbol{n}), \\ n_\beta, & i = K(\boldsymbol{n} - \hat{\boldsymbol{n}}_\beta),\ j = K(\boldsymbol{n}). \end{cases} \tag{B.18}$$

It can be seen that $\mathbf{M}_\beta$ is a tridiagonal-like matrix with positive off-diagonal entries.

Similarly, the collision term can be written as the following:

$$-\frac{1}{\tau}\left[f - f^{(0)}\right] = -\frac{1}{\tau}\sum_{|\boldsymbol{n}|\geq 2} f_{\boldsymbol{n}} H_{\boldsymbol{n}}(\bar{\boldsymbol{c}}) = \frac{1}{\tau}\langle \mathbf{H}, \mathbf{B}\mathbf{W}\rangle_\infty, \tag{B.19}$$

where $f^{(0)} = f_{\mathbf{0}} H_{\mathbf{0}}(\bar{\boldsymbol{c}})$, and the non-zero elements of the matrix $\mathbf{B} \in \mathbb{R}^{\infty\times\infty}$ are:

$$\mathbf{B}_{ij} = \begin{cases} 1, & i = K(2\hat{\boldsymbol{n}}_1) \neq j = K(2\hat{\boldsymbol{n}}_\beta), \\ -1, & i = K(\boldsymbol{n}) = j,\ |\boldsymbol{n}| > 1,\ \boldsymbol{n} \neq 2\hat{\boldsymbol{n}}_1. \end{cases} \tag{B.20}$$

Thus, with Equations (B.11), (B.17) and (B.19), the Boltzmann–BGKW equation can be written as

$$\left\langle \mathbf{H}, \mathbf{D}\frac{\partial \mathbf{W}}{\partial t}\right\rangle_\infty + \sum_{\beta=1}^{d}\left\langle \mathbf{H}, \mathbf{M}_\beta \mathbf{D}\frac{\partial \mathbf{W}}{\partial x_\beta}\right\rangle_\infty = \frac{1}{\tau}\langle \mathbf{H}, \mathbf{B}\mathbf{W}\rangle_\infty. \tag{B.21}$$

With the orthogonality of the Hermite tensor polynomials in $\mathbf{H}$, the above equation becomes the following infinite hierarchy of moment equations:

$$\mathbf{D}\frac{\partial \mathbf{W}}{\partial t} + \sum_{\beta=1}^{d} \mathbf{M}_\beta \mathbf{D}\frac{\partial \mathbf{W}}{\partial x_\beta} = \frac{1}{\tau}\mathbf{B}\mathbf{W}, \tag{B.22}$$

where $\mathbf{D}$, $\mathbf{M}_\beta$ and $\mathbf{B}$ are defined in Equation (B.13), (B.18) and (B.20), respectively.

B.2. The multi-dimensional GHMS

To derive the multi-dimensional GHMS from the Boltzmann–BGKW equation (**??**) [13], we first introduce a projection operator $\mathcal{P}$ that truncates the expansion of g to a finite order M, *i.e.*,

$$\mathcal{P}g = g_M, \quad g = \sum_{\boldsymbol{n}\in\mathbb{N}_0^d} g_{\boldsymbol{n}} H_{\boldsymbol{n}}(\bar{\boldsymbol{c}}), \quad g_M := \sum_{|\boldsymbol{n}|\leq M} g_{\boldsymbol{n}} H_{\boldsymbol{n}}(\bar{\boldsymbol{c}}). \tag{B.23}$$

The representation of $\mathcal{P}$, $\mathbf{T} \in \mathbb{R}^{N\times\infty}$, is given by $\mathbf{T} := (\mathbf{I}_N, \mathbf{0})$, $\mathbf{I}_N$ is the Nth order identity matrix, and N denotes the number of entries in $\mathbf{g}_M$, thus

$$\mathbf{T}\mathbf{g} = \mathbf{g}_M, \quad \mathbf{g} := (g_{\mathbf{0}}, \dots, g_{\boldsymbol{n}}, \dots)^\dagger_{\boldsymbol{n}\in\mathbb{N}_0^d}, \quad \mathbf{g}_M := (g_{\mathbf{0}}, \dots, g_{\boldsymbol{n}})^\dagger_{|\boldsymbol{n}|\leq M}. \tag{B.24}$$

The procedure to derive the GHMS consists of the following steps:

Step 1. Expand the distribution function f in terms of $\{H_{\boldsymbol{n}}(\bar{\boldsymbol{c}})\}$ with the ordering of $\{f_n\}$ specified by Equation (B.2);

Step 2. Construct the unknown $\mathbf{W}$ in the form of Equation (B.12);

Step 3. Truncate the expansion of f with a given order $M \geq 2$:

$$\mathcal{P}f = \sum_{|\boldsymbol{n}|\leq M} f_{\boldsymbol{n}} H_{\boldsymbol{n}}(\bar{\boldsymbol{c}}) = \langle \mathbf{T}\mathbf{H}, \mathbf{T}\mathbf{f}\rangle_N. \tag{B.25}$$

Step 4. Carry out spatial and temporal derivatives:

$$\frac{\partial(\mathcal{P}f)}{\partial s} = \left\langle \mathbf{H}, \mathbf{D}\mathbf{T}^\dagger \frac{\partial \mathbf{T}\mathbf{W}}{\partial s} \right\rangle_\infty, \quad s = t,\ x_\alpha. \tag{B.26}$$

Step 5. Truncate spatial and temporal derivatives of f:

$$\mathcal{P}\frac{\partial(\mathcal{P}f)}{\partial x_\beta} = \left\langle \mathbf{T}\mathbf{H}, \mathbf{T}\mathbf{D}\mathbf{T}^\dagger \frac{\partial \mathbf{T}\mathbf{W}}{\partial x_\beta} \right\rangle_N. \tag{B.27}$$

Step 6. Multiplication with velocity:

$$\xi_\beta \mathcal{P}\frac{\partial(\mathcal{P}f)}{\partial x_\beta} = \left\langle \mathbf{H}, \mathbf{M}_\beta \mathbf{T}^\dagger \mathbf{T}\mathbf{D}\mathbf{T}^\dagger \frac{\partial \mathbf{T}\mathbf{W}}{\partial x_\beta} \right\rangle_\infty. \tag{B.28}$$

Step 7. Expand the collision term:

$$Q(\mathcal{P}f) = \frac{1}{\tau}\left\langle \mathbf{H}, \mathbf{B}\mathbf{T}^\dagger \mathbf{T}\mathbf{W} \right\rangle_\infty, \quad Q(f) := -\frac{1}{\tau}\left[f - f^{(0)}\right]. \tag{B.29}$$

Step 8. Truncate Equation (B.26), (B.28) and (B.29), and match the coefficients of the basis functions to obtain the following regularized moment system:

$$\mathbf{T}\mathbf{D}\mathbf{T}^\dagger \frac{\partial \mathbf{T}\mathbf{W}}{\partial t} + \sum_{\beta=1}^{d} \mathbf{T}\mathbf{M}_\beta \mathbf{T}^\dagger \mathbf{T}\mathbf{D}\mathbf{T}^\dagger \frac{\partial \mathbf{T}\mathbf{W}}{\partial x_\beta} = \frac{1}{\tau}\mathbf{T}\mathbf{B}\mathbf{T}^\dagger \mathbf{T}\mathbf{W}. \tag{B.30}$$

Since $\mathbf{T} = (\mathbf{I}_N, \mathbf{0})$ is a cut-off, hence $\mathbf{T}\mathbf{D}\mathbf{T}^\dagger$ is the Nth order principal minor determinant of $\mathbf{D}$. For the sake of conciseness, in what follows we will replace $\mathbf{T}\mathbf{D}\mathbf{T}^\dagger$, $\mathbf{T}\mathbf{M}_\beta\mathbf{T}^\dagger$, $\mathbf{T}\mathbf{B}\mathbf{T}^\dagger$, and $\mathbf{T}\mathbf{W}$ by $\mathbf{D}$, $\mathbf{M}_\beta$, $\mathbf{B}$, and $\mathbf{W}$, respectively, so the regularized moment system (B.30) can be concisely written as the following:

$$\mathbf{D}\frac{\partial \mathbf{W}}{\partial t} + \sum_{\beta=1}^{d} \mathbf{M}_\beta \mathbf{D}\frac{\partial \mathbf{W}}{\partial x_\beta} = \frac{1}{\tau}\mathbf{B}\mathbf{W}, \tag{B.31}$$

where the elements of $\mathbf{D}$, $\mathbf{M}_\beta$ and $\mathbf{B}$ are defined in Equation (B.13), (B.18), and (B.20), respectively. The above equation will be used in the stability analyses in the next section.

B.3. Stability of the multi-dimensional GHMS

To prove the stability of the GHMS (B.31), we first show that $\mathbf{D}$ is invertible. Introduce an upper triangular matrix $\mathbf{V}$ with the following nonzero elements:

$$\mathbf{V}_{ij} = 1, \quad i = K(\boldsymbol{n}) = j, \text{ or } i = K(2\hat{\boldsymbol{n}}_1) \neq j = K(2\hat{\boldsymbol{n}}_\beta), \tag{B.32}$$

Since the diagonal entries are positive, $\mathbf{V}$ is invertible. It is easy to see that $\mathbf{V}\mathbf{D}$ differs from $\mathbf{D}$ with only

$$\mathbf{V}\mathbf{D}_{ii} = d\rho,\ \mathbf{V}\mathbf{D}_{ij} = 0, \quad i = K(2\hat{\boldsymbol{n}}_1) \neq j = K(2\hat{\boldsymbol{n}}_\beta), \tag{B.33}$$

and all the rest entries are equal to those of $\mathbf{D}$. This means $\mathbf{V}\mathbf{D}$ is a lower triangular matrix with all diagonal entries positive. Thus $\mathbf{D}$ is an invertible matrix.

B.3.1. Verification of the first condition satisfied by the collision term When $\mathbf{W}$ is in the equilibrium space, we have

$$f_{\boldsymbol{n}} = 0, \quad |\boldsymbol{n}| \geq 1. \tag{B.34}$$

Therefore we can show that $\mathsf{D}^{-1}\mathsf{B}$ is the following diagonal matrix:

$$\mathsf{D}^{-1}\mathsf{B} = \mathrm{diag}(\underbrace{0, 0, \ldots, 0}_{d+2}, -1, -1, \ldots, -1). \tag{B.35}$$

From Equation (B.34), we have the elements of D at the equilibria as the following:

$$\mathsf{D}_{ii} = 1, \qquad i = K(\boldsymbol{n}),\ |\boldsymbol{n}| > 1,\ \boldsymbol{n} \neq 2\hat{\boldsymbol{n}}_1, \tag{B.36a}$$

$$\mathsf{D}_{ij} = \rho, \qquad i = K(\hat{\boldsymbol{n}}_\alpha) = j;\ i = K(2\hat{\boldsymbol{n}}_\alpha),\ j = K(2\hat{\boldsymbol{n}}_1), \tag{B.36b}$$

$$\mathsf{D}_{ij} = -1, \qquad i = K(2\hat{\boldsymbol{n}}_1) \neq j = K(2\hat{\boldsymbol{n}}_\beta), \tag{B.36c}$$

from which we can obtain the nonzero elements of D^{-1}:

$$\mathsf{D}^{-1}_{ii} = 1, \qquad i = K(\boldsymbol{n}),\ |\boldsymbol{n}| \neq 1,\ \boldsymbol{n} \neq 2\hat{\boldsymbol{n}}_\alpha, \tag{B.37a}$$

$$\mathsf{D}^{-1}_{ii} = \frac{1}{\rho}, \qquad i = K(\hat{\boldsymbol{n}}_\alpha), \tag{B.37b}$$

$$\mathsf{D}^{-1}_{ij} = \frac{1}{d\rho}, \qquad i = K(2\hat{\boldsymbol{n}}_1),\ j = K(2\hat{\boldsymbol{n}}_\alpha), \tag{B.37c}$$

$$\mathsf{D}^{-1}_{ij} = -\frac{1}{d} + \delta_{\alpha\beta}, \qquad i = K(2\hat{\boldsymbol{n}}_\beta),\ j = K(2\hat{\boldsymbol{n}}_\alpha),\ \beta \neq 1. \tag{B.37d}$$

With D^{-1} known, it is straightforward to verify Equation (B.35). Thus the first condition is satisfied with any diagonal matrix P which is positive definite; and P will be given later in Section B.3.3.

B.3.2. Verification of the second condition satisfied by the convective term

We introduce an $M \times M$ diagonal matrix U with the following diagonal elements:

$$\mathsf{U}_{ii} = \sqrt{\frac{\theta^{\boldsymbol{n}}}{\boldsymbol{n}!}}, \quad i = K(\boldsymbol{n}),\ |\boldsymbol{n}| \leq M. \tag{B.38}$$

We can prove the following two propositions:

1. The following matrices are symmetric:

$$\tilde{\mathsf{M}}_\alpha := \mathsf{U}^{-1}\mathsf{M}_\alpha\mathsf{U}, \quad \alpha = 1, 2, \ldots, d. \tag{B.39}$$

2. The GHMS (B.31) can be recast as a symmetric hyperbolic system:

$$\tilde{\mathsf{D}}^\dagger\tilde{\mathsf{D}}\frac{\partial \mathbf{W}}{\partial t} + \sum_{\alpha=1}^{d}\tilde{\mathsf{D}}^\dagger\tilde{\mathsf{M}}_\alpha\tilde{\mathsf{D}}\frac{\partial \mathbf{W}}{\partial s} = \frac{1}{\tau}\tilde{\mathsf{D}}^\dagger\mathsf{U}^{-1}\mathsf{B}\mathbf{W}, \quad \tilde{\mathsf{D}} := \mathsf{U}^{-1}\mathsf{D}. \tag{B.40}$$

The proof of Equations (B.39) and (B.40) is given as the following. Define $\mathsf{G}_\beta := \mathsf{U}^{-1}\mathsf{M}_\beta$ and we have:

$$\mathsf{G}_{\beta ij} = \begin{cases} \theta\sqrt{\dfrac{\boldsymbol{n}!}{\theta^{\boldsymbol{n}}}\dfrac{n_\beta+1}{\theta}}, & i = K(\boldsymbol{n}+\hat{\boldsymbol{n}}_\beta),\ j = K(\boldsymbol{n}), \\ u_\beta\sqrt{\dfrac{\boldsymbol{n}!}{\theta^{\boldsymbol{n}}}}, & i = j = K(\boldsymbol{n}), \\ n_\beta\sqrt{\dfrac{\boldsymbol{n}!}{\theta^{\boldsymbol{n}}}\dfrac{\theta}{n_\beta}}, & i = K(\boldsymbol{n}-\hat{\boldsymbol{n}}_\beta),\ j = K(\boldsymbol{n}). \end{cases} \tag{B.41}$$

We can then compute $\tilde{\mathbf{M}}_\beta := \mathbf{G}_\beta \mathbf{U}$:

$$\tilde{\mathbf{M}}_{\beta ij} = \begin{cases} \sqrt{(n_\beta+1)\theta}, & i = K(\boldsymbol{n}+\hat{\boldsymbol{n}}_\beta),\ j = K(\boldsymbol{n}), \\ u_\beta, & i = j = K(\boldsymbol{n}), \\ \sqrt{n_\beta \theta}, & i = K(\boldsymbol{n}-\hat{\boldsymbol{n}}_\beta),\ j = K(\boldsymbol{n}), \end{cases} \tag{B.42}$$

which means

$$\tilde{\mathbf{M}}_{\beta ij} = \sqrt{(n_\beta+1)\theta} = \tilde{\mathbf{M}}_{\beta ji}, \quad i = K(\boldsymbol{n}+\hat{\boldsymbol{n}}_\beta),\ j = K(\boldsymbol{n}). \tag{B.43}$$

Thus $\tilde{\mathbf{M}}_\beta$, $\beta = 1, 2, \ldots, d$, is symmetric. Multiplying Equation (B.31) with $\tilde{\mathbf{D}}^\dagger \mathbf{U}^{-1}$ from the left, $\tilde{\mathbf{D}} := \mathbf{U}^{-1}\mathbf{D}$, we immediately obtain the symmetric hyperbolic equation (B.40). Thus the second condition is satisfied by choosing the symmetric matrix $\mathbf{A}_0 = \tilde{\mathbf{D}}^\dagger \tilde{\mathbf{D}}$.

B.3.3. Verification of the third condition

With $\mathbf{A}_0 = \tilde{\mathbf{D}}^\dagger \tilde{\mathbf{D}}$ and the collision term $\mathbf{D}^{-1}\mathbf{B}$, we directly compute $\mathbf{L} = \mathbf{A}_0 \mathbf{D}^{-1}\mathbf{B}$ at the equilibria. First we compute

$$\mathbf{L} = \mathbf{A}_0 \mathbf{D}^{-1}\mathbf{B} = \tilde{\mathbf{D}}^\dagger \tilde{\mathbf{D}} \mathbf{D}^{-1}\mathbf{B} = \mathbf{D}^\dagger \mathbf{U}^{-\dagger}\mathbf{U}^{-1}\mathbf{B}, \qquad \mathbf{U}^{-\dagger} := (\mathbf{U}^\dagger)^{-1}. \tag{B.44}$$

Note that $\mathbf{U}$ is a diagonal matrix, thus for $|\boldsymbol{n}| \le M$ we have

$$\mathbf{U}^{-\dagger}\mathbf{U}^{-1} = \mathbf{U}^{-2} = \operatorname{diag}\left(\frac{1}{U_1^2}, \frac{1}{U_2^2}, \ldots, \frac{1}{U_N^2}\right), \quad U_i = \mathbf{U}_{ii} = \sqrt{\frac{\theta^{\boldsymbol{n}}}{\boldsymbol{n}!}}, \quad i = K(\boldsymbol{n}). \tag{B.45}$$

Then the nonzero elements of $\mathbf{C} := \mathbf{U}^{-\dagger}\mathbf{U}^{-1}\mathbf{B}$ are

$$\mathbf{C}_{ij} = \begin{cases} \dfrac{2}{\theta^2}, & i = K(2\hat{\boldsymbol{n}}_1) \ne j = K(2\hat{\boldsymbol{n}}_\beta), \\ -\dfrac{\boldsymbol{n}!}{\theta^{\boldsymbol{n}}}, & |\boldsymbol{n}| > 1,\ \boldsymbol{n} \ne 2\hat{\boldsymbol{n}}_1. \end{cases} \tag{B.46}$$

With $\mathbf{C}$ given, the nonzero elements of $\mathbf{L} := \mathbf{A}_0\mathbf{D}^{-1}\mathbf{B} = \mathbf{D}^\dagger\mathbf{C}$ can be easily computed:

$$\mathbf{L}_{ii} = \begin{cases} -\dfrac{4}{\theta^2}, & i = K(2\hat{\boldsymbol{n}}_\alpha),\ \alpha \ne 1, \\ -\dfrac{\boldsymbol{n}!}{\theta^{\boldsymbol{n}}}, & i = K(\boldsymbol{n}),\ |\boldsymbol{n}| \ge 2,\ \boldsymbol{n} \ne 2\hat{\boldsymbol{n}}_\alpha,\ \forall \alpha. \end{cases} \tag{B.47}$$

With the following P:

$$P = \sqrt{-2\mathbf{L}} + \operatorname{diag}(\underbrace{1, 1, \ldots, 1}_{d+2}, 0, 0, \ldots, 0), \tag{B.48}$$

the third condition is satisfied. Thus we proved that the multi-dimensional globally hyperbolic moment system (B.31) admits the stability conditions in Appendix A.

Appendix C. An upwind scheme.

The following first-order upwind scheme is used to solve system (7.1):

$$\frac{\mathbf{W}_j^{n+1} - \mathbf{W}_j^n}{\Delta t} + \hat{\mathbf{A}}_j^{n+} \frac{\mathbf{W}_j^n - \mathbf{W}_{j-1}^n}{\Delta x} + \hat{\mathbf{A}}_j^{n-} \frac{\mathbf{W}_{j+1}^n - \mathbf{W}_j^n}{\Delta x} = \frac{1}{\tau}\mathbf{B}\mathbf{W}_j^{n+1}, \tag{C.1}$$

where $\mathbf{W}_j^n := \mathbf{W}(x_j, t_n)$, and $\hat{\mathbf{A}}_j^{n+}$ and $\hat{\mathbf{A}}_j^{n-}$ include only the non-negative and non-positive eigenvalues of $\hat{\mathbf{A}}_j^n$, respectively; specifically, they are obtained as follows:

$$\hat{\mathbf{A}} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^*, \quad \mathbf{\Lambda}^{\pm} := \frac{1}{2}(\mathbf{\Lambda} \pm |\mathbf{\Lambda}|), \quad \hat{\mathbf{A}}^{\pm} := \mathbf{U}\mathbf{\Lambda}^{\pm}\mathbf{U}^*, \tag{C.2}$$

where $\mathbf{U}$ is the unitary matrix which diagonalizes $\hat{\mathbf{A}}$ and $\mathbf{U}^*$ is the complex conjugate of $\mathbf{U}$, $\mathbf{\Lambda}$ is eigen-matrix of $\hat{\mathbf{A}}$, *i.e.*, the elements of $\mathbf{\Lambda}$ are the eigenvalues of $\hat{\mathbf{A}}$, $\{\lambda_\beta\}$, and $|\mathbf{\Lambda}|$ denotes the diagonal matrix with the elements $\{|\lambda_\beta|\}$. In the above equations, both the subscript "j" and the superscript "n" are omitted.

The source term in system (7.1), $\tau^{-1}\mathbf{B}\mathbf{W}$, can be stiff due to small τ, thus it should be treated implicitly in order to allow $\Delta t \gg \tau$. Because $\mathbf{B}$ is a constant diagonal matrix, thus the scheme (C.1) can indeed be implemented explicitly. In addition, the upwind scheme of (C.1) can be shown to be asymptotically preserving [21, 14].

The upwind scheme is validated with the following shock tube problem:

$$\frac{\partial \mathbf{W}}{\partial t} + \hat{\mathbf{A}}\frac{\partial \mathbf{W}}{\partial x} = \frac{1}{\tau}\mathbf{B}\mathbf{W}, \quad \mathbf{W} \in \mathbb{R}^5,\ x \in \mathbb{R},\ t \in \mathbb{R}^+, \tag{C.3}$$

where $\mathbf{W} := (\rho, u, \theta, f_3, f_4)^\dagger$, with the initial data given below:

$$\mathbf{W}(x,0) = \begin{cases} (7,0,1,0,0)^\dagger, & x < 0, \\ (1,0,1,0,0)^\dagger, & x > 0. \end{cases} \tag{C.4}$$

The relaxation time is chosen to be $\tau = \mathrm{Kn}/\rho$ in this case [8]. With a fixed spatial grid size $\Delta x = 10^{-3}$ and a time step size $\Delta t = 2 \cdot 10^{-4}$, the results for cases of $\mathrm{Kn} = 0.05$ and 0.5 are compared with the results obtained with the discontinuous Galerkin finite element method (DG-FEM) [8]. As shown in Fig. C.1, our results agree very well with the existing ones [8]. This test validates the upwind scheme (C.1).

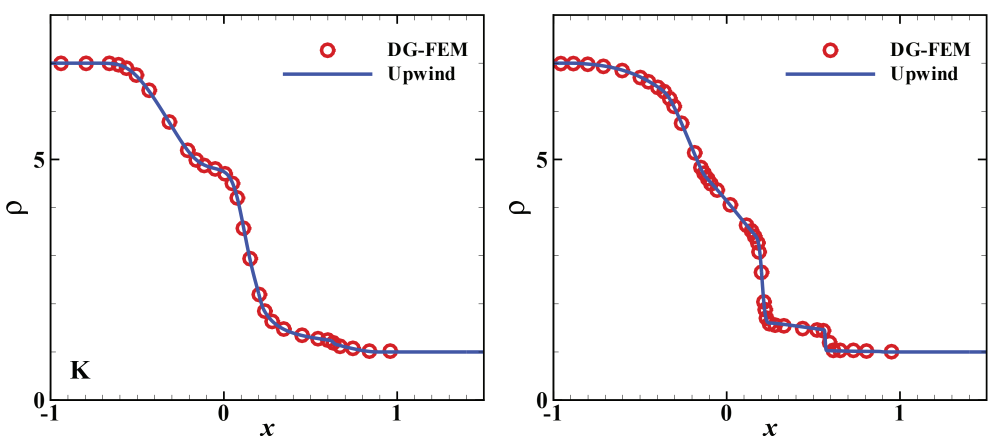

FIG. C.1. *The density profile $\rho(x,t)$ for the shock tube problem given by* (C.3) *and* (C.4) *at time* $t = 0.3$. *Left:* $\mathrm{Kn} = 0.05$, *right:* $\mathrm{Kn} = 0.5$. *The results obtained with the upwind scheme* (C.1) *and DG-FEM* [8] *are denoted by solid lines and symbols* (○), *respectively.*

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