COMMUN. MATH. SCI.
Vol. 15, No. 3, pp. 587–591

WEAK-STRONG UNIQUENESS FOR COMPRESSIBLE NAVIER–STOKES SYSTEM WITH DEGENERATE VISCOSITY COEFFICIENT AND VACUUM IN ONE DIMENSION*

BORIS HASPOT†

Abstract. We prove weak-strong uniqueness results for the compressible Navier–Stokes system with degenerate viscosity coefficients and with vacuum in one dimension. In other words, we give conditions on the weak solution constructed in [Q.S. Jiu and Z.P. Xin, Kinet. Relat. Models, 1(2):313–330, 2008] so that it is unique. The novelty consists of dealing with initial density ρ_0 which contains vacuum. To do this we use the notion of *relative entropy* developed recently by Germain, Feireisl et al., and Mellet and Vasseur (see [P. Germain, J. Math. Fluid Mech., 13(1):137–146, 2011], [E. Feireisl, A. Novotný, and S. Yongzhong, Indiana University Mathematical Journal, 60(2):611–632, 2011], [A. Mellet and A. Vasseur, SIAM J. Math. Anal., 39(4):1344–1365, 2007/08]) combined with a new formulation of the compressible system ([B. Haspot, Journal of Mathematical Fluid Mechanics, HAL Id: hal-00770248, arXiv:1304.4502, 1, 2013], [B. Haspot, Eprint Arxiv, hal-01081580, 2014]); more precisely we introduce a new effective velocity v which makes the system parabolic on the density and hyperbolic on the velocity v.

Keywords. fluids mechanics; weak-strong uniqueness; relative entropy

AMS subject classifications. 35Q30; 35A02; 35A05; 35D30; 35D35

1. Introduction

We are interested in proving weak-strong uniqueness results for the compressible Navier–Stokes equation with degenerate viscosity in the euclidean space $\mathbb{R}$. The system reads:

$$\begin{cases} \partial_t\rho+\partial_x(\rho u)=0, \\ \partial_t(\rho u)+\partial_x(\rho u^2)-\partial_x(\mu(\rho)\partial_x u)+\partial_x P(\rho)=0. \end{cases} \tag{1.1}$$

The unknowns ρ and u stand for the density and the velocity of the fluid. They are functions of the space variable x and the time variable t, and are $\mathbb{R}^+$- and $\mathbb{R}$- valued respectively. Throughout the paper, we will assume that the pressure $P(\rho)$ obeys a γ-type law $P(\rho)=\rho^\gamma$ with $\gamma>1$. In addition, $\mu(\rho)$ corresponds to the viscosity coefficient, which is a regular function depending on ρ with $\mu(\rho)\geq 0$. Following the idea of [10–13], setting $v=u+\partial_x\varphi(\rho)$ with $\varphi'(\rho)=\frac{\mu(\rho)}{\rho^2}$, we have (see [12]):

$$\begin{cases} \partial_t\rho-\partial_x(\dfrac{\mu(\rho)}{\rho}\partial_x\rho)+\partial_x(\rho v)=0, \\ \rho\partial_t v+\rho u\partial_x v+\partial_x P(\rho)=0. \end{cases} \tag{1.2}$$

Roughly speaking the $1D$ compressible Navier Stokes equations can be considered as a compressible Euler system with a viscous regularization term on the density of the type $-\partial_x(\frac{\mu(\rho)}{\rho}\partial_x\rho)$. In the literature the constant viscosity case is often considered for mathematical purposes. Physically, however, the viscosity of a gas depends on the temperature and on the density (in the isentropic case). We can mention the case of the Chapman–Enskog viscosity law (see [5]) or the case of monoatomic gas ($\gamma=\frac{5}{3}$)

*Received: November 12, 2015; accepted (in revised form): June 24, 2016. Communicated by Song Jiang.

†Université Paris Dauphine, PSL Research University, Ceremade, Umr Cnrs 7534, Place du Maréchal De Lattre De Tassigny 75775 Paris cedex 16, France (haspot@ceremade.dauphine.fr).

when $\mu(\rho)=\rho^{\frac{1}{3}}$. More generally, we expect that the viscosity coefficient vanishes as a power of ρ on the vacuum. In this paper we are going to deal with degenerate viscosity coefficients of the form $\mu(\rho)=\mu\rho^{\alpha}$ for some $\alpha\geq 0$.

Since we want to establish weak-strong uniqueness results, let us recall briefly some results on the existence of global weak and strong solutions for the one dimensional case. There is a large amount of literature concerning the case when the viscosity coefficients are constant. The existence of global weak solutions for smooth enough data close to the equilibrium was first obtained by Kazhikhov and Shelukin [18]. In [15], Hoff extends the previous result by proving the existence of global weak solutions with large discontinuous initial data having different limits at $x=\pm\infty$. Let us mention, in passing, that the existence in any dimension of global weak solutions with constant viscosity coefficients was first proven by P-L Lions in [19], and the result was later refined by Feireisl et al [7]. The first results regarding the existence in dimensions two and higher of global weak solutions with degenerate viscosity in the case of cold pressure are due to Bresch and Desjardins. To do this they introduced a new entropy, the so-called BD entropy, under the condition that the viscosity coefficients satisfy an algebraic relation (see [2,3]). The stability of the global weak solution has been extended to the framework of a γ-law by Mellet and Vasseur in [21]. Recently the proof of existence of global weak solutions to the so called shallow water system has been delivered by Vasseur and Yu [24, 25]. Some ideas on the construction of global weak approximate solutions can be found in [2–4, 22, 27]. Jiu and Xin in [16] have proven the existence of global weak solutions for degenerate viscosity coefficient $\mu(\rho)=\rho^{\alpha}$ with $\alpha>\frac{1}{2}$ in one dimension.

Let us now focus on the problem of global strong solutions. Kanel in [17] gave the first proof of the existence of global classical solutions with large initial data for initial density far away from the vacuum when the viscosity coefficient is constant (see also [14]). Concerning global strong solutions with degenerate viscosity coefficients, Mellet and Vasseur have shown in [20] the existence of global strong solutions with large initial data for initial density far away from the vacuum and for viscosity coefficients of the form $\mu(\rho)=\rho^{\alpha}$ with $0\leq\alpha<\frac{1}{2}$. We extended this result in [12] to the case $\frac{1}{2}\leq\alpha\leq 1$.

A natural question is now to understand when the weak solutions are unique. The idea of weak-strong uniqueness is the following: assume that a weak solution has regular initial data such that there exists a strong solution in finite time associated to these initial data, then we can prove its uniqueness in the class of finite weak energy solutions. In recent years many results of weak-strong solutions have been obtained in fluid mechanics and kinetic equations. However, we can observe that in all these different situations a common natural tool is used, the so called relative entropy. It seems that this notion was first used to prove weak-strong uniqueness by Dafermos in [6] who was considering conservation laws. Mellet and Vasseur in [20] proved, also via the notion of relative entropy, weak-strong uniqueness for compressible Navier–Stokes equations in one dimension when the viscosity is non-degenerate (we also refer to [1] in the framework of kinetic equation). The existence of weak-strong solutions in the class of the weak solutions was first proved by Germain in [9] for constant viscosity coefficients in the multi-dimensional case. This result has been extended in an important work by Feireisl et al in [8] by constructing suitable weak solutions satisfying the relative entropy inequality proposed in [9].

In this paper we are interested in extending the results of [20] to the case of degenerate viscosity coefficients with initial data admitting vacuum. To do this we are going to combine the notion of relative entropy and a suitable formulation of the compressible Navier–Stokes equation proposed in the system (1.2). Indeed, the momentum equation in the system (1.2) is only an Euler equation in one dimension; roughly speaking it implies that all the difficulties (in particular the control of the gradient of the velocity) related to the degeneracy of the viscosity coefficients disappear. It enables us to apply a Gronwall argument. To our knowledge this is the first result of weak-strong uniqueness with degenerate viscosity coefficients and with vacuum for the compressible Navier–Stokes equation.

2. Main result

We present below our main theorem of weak-strong uniqueness for the system (1.2). It will be a convenient short-hand to denote $L^p_T L^q = L^p([0,T], L^q(\mathbb{R}))$. We denote, in the sequel, the energy $\mathcal{E}(\rho,u)$ and $\mathcal{E}(\rho,v)$ which is used in [16]:

$$\mathcal{E}(\rho,u)(t) = \int_{\mathbb{R}} [\frac{1}{2}\rho(t,x)|u(t,x)|^2 + \frac{1}{\gamma-1}\rho^\gamma(t,x)]dx + \int_0^t \int_{\mathbb{R}} \mu(\rho)|\partial_x u|^2(s,x)dsdx,$$
$$\mathcal{E}(\rho,v)(t) = \int_{\mathbb{R}} [\frac{1}{2}\rho(t,x)|v(t,x)|^2 + \frac{1}{\gamma-1}\rho^\gamma(t,x)]dx + \int_0^t \int_{\mathbb{R}} \partial_x P(\rho)\partial_x(\frac{\mu(\rho)}{\rho^2})dsdx.$$

THEOREM 2.1. *Let $P(\rho) = a\rho^\gamma$ with $\gamma > 1$ and $\mu(\rho) = \mu\rho^\gamma$ with $\mu > 0$. Assume that the initial data (ρ_0, u_0) satisfy:*

$$\rho_0 \in L^1 \cap L^\gamma, \quad \sqrt{\rho_0}u_0 \in L^2, \quad \sqrt{\rho_0}v_0 \in L^2, \quad \text{and} \quad \rho_0^{\frac{1}{2+\delta}} u_0 \in L^{2+\delta},$$

with $\delta > 0$ arbitrarily small. A solution $(\bar{\rho},\bar{v})$ of the system (1.2) is unique on $[0,T]$ in the set of solutions (ρ,v) such that:

$$\sup_{t\in(0,T)} \|\rho(t,\cdot)\|_{L^1} < +\infty, \quad \sup_{t\in(0,T)} \mathcal{E}(\rho,u)(t) < +\infty, \quad \text{and} \quad \sup_{t\in(0,T)} \mathcal{E}(\rho,v)(t) < +\infty, \tag{2.1}$$

provided $\bar{u},\bar{v},\partial_x\bar{u},\partial_x\bar{v} \in L^1_T(L^\infty)$ and $\bar{u} \in L^\infty((0,T)\times\mathbb{R})$.

REMARK 2.1. Let us mention that the assumption on $\bar{\rho}$ and $\bar{u}$ are satisfied in [26] in finite time in the case of a free boundary problem with vacuum.

Our main result, Theorem 2.1, is proved in Section 3 using the notion of relative entropy.

3. Proof of Theorem 2.1

Proof. We now consider two solution (ρ,v) and $(\bar{\rho},\bar{v})$ of the system (1.2), we set:

$$\begin{cases} U = v - \bar{v} \\ R = \rho - \bar{\rho} \\ U_1 = u - \bar{u}, \\ F(\bar{\rho},R) = \frac{1}{\gamma}(R+\bar{\rho})^\gamma - \bar{\rho}^{\gamma-1}R - \frac{1}{\gamma}\bar{\rho}^\gamma. \end{cases}$$

First, subtracting the mass conservation equation of the system (1.1) for (ρ,u) and $(\bar{\rho},\bar{u})$ gives:

$$\partial_t R + \partial_x(\rho U_1) + \partial_x(R\bar{u}) = 0. \tag{3.1}$$

Similarly, for the momentum equation of the system (1.2) we obtain:

$$(\rho\partial_t + \rho u\partial_x)U + a\rho\frac{\gamma}{\gamma-1}(\partial_x\rho^{\gamma-1} - \partial_x\bar{\rho}^{\gamma-1}) = -\rho U_1\partial_x\bar{v}. \tag{3.2}$$

Next, multiply Equation (3.2) by $U=U_1+\frac{\mu}{\gamma-1}\partial_x(\rho^{\gamma-1}-\bar\rho^{\gamma-1})$ and integrate over $(0,t)\times\mathbb{R}$. Following [9] we integrate by parts:

$$\begin{aligned}
&\int_{\mathbb{R}}\rho U_1\cdot\partial_x(\rho^{\gamma-1}-\bar\rho^{\gamma-1})dxds=-\int_0^t\int_{\mathbb{R}}\partial_x(\rho U_1)(\rho^{\gamma-1}-\bar\rho^{\gamma-1})dxds\\
&=\int_0^t\int_{\mathbb{R}}(\partial_tR+\partial_x(R\bar u))(\rho^{\gamma-1}-\bar\rho^{\gamma-1})dxds\qquad\text{using Equation (3.1)}\\
&=\int_0^t\int_{\mathbb{R}}\partial_tR\frac{\partial}{\partial R}Fdxds+\int_0^t\int_{\mathbb{R}}\bar u\partial_xR\frac{\partial}{\partial R}Fdxds+\int_0^t\int_{\mathbb{R}}\partial_x\bar uR\frac{\partial}{\partial R}Fdxds,\\
&=\int_0^t\int_{\mathbb{R}}\partial_tFdxds-\int_{\mathbb{R}}\partial_t\bar\rho\frac{\partial}{\partial\bar\rho}Fdxds+\int_0^t\int_{\mathbb{R}}\bar u\partial_xFdx-\int_0^t\int_{\mathbb{R}}\bar u\partial_x\bar\rho\frac{\partial}{\partial\bar\rho}Fdx\\
&\qquad\qquad+\int_0^t\int_{\mathbb{R}}\partial_x\bar uR\frac{\partial}{\partial R}Fdxds,\\
&=\int_0^t\int_{\mathbb{R}}\partial_tFdxds+\int_0^t\int_{\mathbb{R}}\partial_x\bar u(-F+\bar\rho\frac{\partial}{\partial\bar\rho}F+R\frac{\partial}{\partial R}F)dxds,\\
&=\int_0^t\int_{\mathbb{R}}\partial_tFdxds+(\gamma-1)\int_0^t\int_{\mathbb{R}}\partial_x\bar uFdxds.
\end{aligned}$$

REMARK 3.1. Let us observe that all the expressions written above converge properly, indeed let us deal with:

$$a\partial_x(\rho^\gamma)U_1=a\gamma\rho^{\gamma-\frac{3}{2}}\partial_x\rho\sqrt{\rho}U_1.$$

We know that $\gamma\rho^{\gamma-\frac{3}{2}}\partial_x\rho$ is in $L^\infty_T(L^2)$ via the energy $\mathcal{E}(\rho,v)$, and we have that $\sqrt{\rho}u$ belongs to $L^\infty_T(L^2)$. Thus it is enough to conclude convergence of the integral. Similarly since $\sqrt{\rho}$ is in $L^\infty_T(L^2)$ one can show that $\gamma\rho^{\gamma-\frac{3}{2}}\partial_x\rho\sqrt{\rho}\bar u$ is in $L^1_T(L^1)$ since $\bar u$ is in $L^1_T(L^\infty)$.
In the same way we have:

$$\frac{\rho}{\bar\rho}\partial_x\bar\rho^\gamma U_1=\gamma\bar\rho^{\gamma-2}\partial_x\bar\rho\sqrt{\rho}\sqrt{\rho}U.$$

Since $\sqrt{\rho}u$ is in $L^\infty_T(L^2)$, $\bar\rho^{\gamma-2}\partial_x\bar\rho$ is in $L^1_T(L^\infty)$, and $\sqrt{\rho}$ is in $L^\infty_T(L^2)$ we can conclude convergence by Hölder's inequality. We proceed similarly for the other term assuming that $\bar u$ belongs to $L^\infty_T(L^\infty)$.

Finally, after multiplying the momentum equation (3.2) by U we obtain:

$$\begin{aligned}
&\partial_t\|\sqrt{\rho}U\|^2_{L^2}+\frac{a\gamma}{\gamma-1}\partial_t\|F(\bar\rho,R)\|_{L^1}+\frac{a\mu\gamma}{(\gamma-1)^2}\|\sqrt{\rho}(\partial_x\rho^{\gamma-1}-\partial_x\bar\rho^{\gamma-1})\|^2_{L^2}\\
&\le\left|-a\gamma\int_{\mathbb{R}}\partial_x\bar uF(\bar\rho,R)dx-\int_{\mathbb{R}}\rho U_1\cdot\partial_x\bar v.Udx\right|
\end{aligned}$$

Next we have:

$$\int_{\mathbb{R}}(\rho U_1\partial_x\bar v)\cdot Udx=\int_{\mathbb{R}}(\rho U\partial_x\bar v)\cdot Udx-\mu\int_{\mathbb{R}}\sqrt{\rho}\big((\rho^{\gamma-\frac{3}{2}}\partial_x\rho-\sqrt{\rho}\bar\rho^{\gamma-2}\partial_x\bar\rho)\partial_x\bar v\big)\cdot Udx.$$

By a basic Gronwall inequality and a bootstrap argument, we conclude that if $\partial_x\bar u$ and $\partial_x\bar v$ belong to $L^1_T(L^\infty)$ then we have $\sqrt{\rho}v=\sqrt{\rho}\bar v$ and $\rho=\bar\rho$. ∎

REFERENCES

[1] F. Berthelin and A. Vasseur, *From kinetic equations to multidimensional isentropic gas dynamic before shocks*, SIAM J. Math. Anal., 36(6):1807–1835, 2005.

[2] D. Bresch and B. Desjardins, *Existence of global weak solutions to the Navier–Stokes equations for viscous compressible and heat conducting fluids*, J. Math. Pure Appl., 87(1):57–90, 2007.

[3] D. Bresch and B. Desjardins, *Existence of global weak solution for 2D viscous shallow water equations and convergence to the quasi-geostrophic model*, Comm. Math. Phys., 238(1-2):211–223, 2003.

[4] D. Bresch, B. Desjardins, and E. Zatorska, *Two-velocity hydrodynamics in fluid mechanics: Part II existence of global κ-entropy solutions to compressible Navier–Stokes systems with degenerate viscosities,* J. Math. Pures Appl., 104(4):801–836, 2015.

[5] S. Chapman and T.G. Cowling, *The Mathematical Theory of Non-Uniform Gases. An Account of the Kinetic Theory of Viscosity, Thermal Conduction and Diffusion in Gases*, Third Edition, Cambridge University Press, London, 1970.

[6] C.M. Dafermos, *The second law of thermodynamics and stability*, Arch. Rational Mech. Anal., 70(2):167–179, 1979.

[7] E. Feireisl, A. Novotný, and H. Petzeltòva, *On the existence of globally defined weak solutions to the Navier–Stokes equations*, J. Math. Fluid Mech., 3(4):358–392, 2001.

[8] E. Feireisl, A. Novotný, and S. Yongzhong, *Suitable weak solutions to the Navier–Stokes equations of compressible viscous fluids*, Indiana U. Math. J., 60(2):611–632, 2011.

[9] P. Germain, *Weak-strong uniqueness for the isentropic compressible Navier–Stokes system*, J. Math. Fluid Mech., 13(1):137–146, 2011.

[10] B. Haspot, *From the highly compressible Navier–Stokes equations to fast diffusion and porous media equations, existence of global weak solution for the quasi-solutions*, J. Math. Fluid Mech., 18(2):243–291, 2016

[11] B. Haspot, *New formulation of the compressible Navier–Stokes equations and parabolicity of the density*, preprint, arXiv:1411.5501, 2014.

[12] B. Haspot, *Existence of global strong solution for the compressible Navier–Stokes equations with degenerate viscosity coefficients in 1D*, preprint, arXiv:1411.5503, 2014.

[13] B. Haspot and E. Zatorska, *From the highly compressible Navier–Stokes equations to the Porous Media equation, rate of convergence*, Discrete Cont. Dyn.-A, 36:3107–3123, 2016.

[14] D. Hoff, *Global existence for 1D compressible, isentropic Navier–Stokes equations with large initial data*, Trans. Amer. Math. Doc., 303(1):169–181, 1987.

[15] D. Hoff, *Global solutions of the equations of one-dimensional, compressible flow with large data and forces, and with differing end states*, Z. Angew. Math. Phys., 49(5):774–785, 1998.

[16] Q.S. Jiu and Z.P. Xin, *The Cauchy problem for 1D compressible flows with density-dependent viscosity coefficients*, Kinet. Relat. Models, 1(2):313–330, 2008.

[17] Ya. I. Kanel, *On a model system of equations of one-dimensional gas motion*, Differentsial'nye Uravneniya, 4:721–734, 1968.

[18] A.V. Kazhikhov and V.V. Shelukhin, *Unique global solution with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas*, Prikl. Mat. Mech., 41(2):282–291, 1977.

[19] P.-L. Lions, *Mathematical Topics in Fluid Mechanics. Vol. 2. Compressible Models*, Oxford Lecture Series in Mathematics and its Applications, Oxford Science Publications, 1998.

[20] A. Mellet and A. Vasseur, *Existence and uniqueness of global strong solutions for one-dimensional compressible Navier–Stokes equations*, SIAM J. Math. Anal., 39(4):1344–1365, 2007.

[21] A. Mellet and A. Vasseur, *On the barotropic compressible Navier–Stokes equations*, Comm. Part. Diff. Eqs., 32(1-3):431–452, 2007.

[22] P.B. Mucha, M. Pokorný, and E. Zatorska, *Approximate solutions to model of two-component reactive flow*, Discrete Contin. Dyn. Syst. Ser. S, 7(5):1079–1099, 2014.

[23] V.V. Shelukin, *Boundary value problems for equations of a barotropic viscous gas with nonnegative initial density*, Dinamika Sploshn. Sredy, 74:108–125, 1986.

[24] A.F. Vasseur and C. Yu, *Existence of global weak solutions for 3D degenerate compressible Navier–Stokes equations*, Invent. Math., 206(3):935–974, 2016.

[25] A.F. Vasseur and C. Yu, *Global weak solutions to compressible quantum Navier–Stokes equations with damping*, SIAM J. Math. Anal., 48(2):1489–1511, 2016.

[26] T. Yang, *A vacuum problem for the one-dimensional compressible Navier–Stokes equations with density-dependent viscosity*, J. Diff. Eqs., 184:163–184, 2002.

[27] E. Zatorska, *On the flow of chemically reacting gaseous mixture*, J. Diff. Eqs., 253(12):3471–3500, 2012.