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# COMPLETE BLOW UP FOR A PARABOLIC SYSTEM ARISING IN A THEORY OF THERMAL EXPLOSION IN POROUS MEDIA*

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**Abstract.** In this paper we consider a model of thermal explosion in porous media. The model consists of two reaction-diffusion equations in a bounded domain with Dirichlet boundary conditions and describes the initial stage of evolution of pressure and temperature fields. Under certain conditions, the classical solution of these equations exists only on finite time interval after which it forms a singularity and becomes unbounded (blows up). This behavior raises a natural question whether this solution can be extended, in a weak sense, after blow up time. We prove that the answer to this question is no, that is, the solution becomes unbounded in entire domain immediately after the singularity is formed. From a physical perspective our results imply that autoignition in porous materials occurs simultaneously in entire domain.

**Keywords.** Combustion in porous media, thermal explosion, blow up for parabolic systems.

**AMS subject classifications.** 35B44, 35D30, 35K51, 35K57, 80A25.

## 1. Introduction

Thermal explosion (autoignition) is a process of rapid development of reaction rates in combustible materials being initially in a non-reactive state [5,17]. The mathematical theory of thermal explosion in a context of thermo-diffusive combustion was developed in the classical works of Frank-Kamenetskii [5] and was studied rather intensively by mathematicians and engineers for several decades. The most notable mathematical works on the subject are [1, 2, 6, 12]. This theory, in physical literature, is commonly referred to as the Frank-Kamenetskii theory of thermal explosion, and, in mathematical literature, as the Gelfand problem. The latter name owes its origin to a chapter written by Barenblatt in a famous review of Gelfand [8]. The classical theory of thermal explosion is based on a short time approximation of the equation for conservation of energy under assumption of constant density and high activation energy and describes the initial stage of evolution of a temperature field in reactive media. The classical Frank-Kamnetskii theory of thermal explosion can be naturally extended to the case of autoignition in inert porous materials. In this situation pressure gradients may play an essential role in autoignition process. Consequently the constant density approximation used in the classical theory of thermal explosion becomes inapplicable. An appropriate modification of the Frank-Kamenetskii theory in this case leads to the following

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model [10]

$$\begin{cases} \gamma\Theta_t-(\gamma-1)\Pi_t=D\Delta\Theta+g(\Theta), & \\ \Pi_t-\Theta_t=\Delta\Pi & \text{in } \Omega\times(0,T), \\ \Pi=\Theta=0 & \text{on } \partial\Omega\times[0,T), \\ \Pi=\Theta=0 & \text{in } \Omega\times\{t=0\}. \end{cases} \tag{1.1}$$

Here, $\Theta$ and $\Pi$ are appropriately scaled temperature and pressure respectively, $\gamma>1$ is the specific heat ratio, $D<1$ is a ratio of pressure and molecular diffusivity, $\Omega$ is smooth bounded domain in $\mathbb{R}^n$ and $g$ is a reaction rate. In case of Arrihenius kinetics the reaction rate is given by

$$g(s)=\exp(s); \tag{1.2}$$

more generally and throughout this paper $g:[0,\infty)\to(0,\infty)$ is a $C^1$ non-decreasing function satisfying

$$\int_a^\infty \frac{ds}{g(s)}<\infty, \quad \text{for some} \quad a>0. \tag{1.3}$$

The first equation of system (1.1) represents a partially linearized equation of energy and the second equation of this system is a linearized equation of continuity taking into account the equation of state and momentum (Darcy law). The initial conditions listed in (1.1) state that at $t=0$ both temperature and pressure are distributed uniformly and are equal to the ambient. The boundary conditions in system (1.1) state that pressure and temperature are equal to the ambient at all times.

In a limiting case when $\gamma=1$ the system (1.1) decouples. Therefore, the system (1.1) effectively reduces to a single equation. This case corresponds to the classical Frnak-Kamenetskii model. In the case when $\gamma>1$ the first and the second equation of the system (1.1) are genuinely coupled. This coupling makes problem (1.1) more complex than the classical Gelfand problem and may lead to different time evolution.

It is known that in both cases $\gamma=1$ and $\gamma>1$ the long time behavior of the solution for problem (1.1) is fully determined by solutions of the stationary problem, that is solutions of

$$\begin{cases} -D\Delta\Theta=g(\Theta) & \text{in } \Omega, \\ \Theta=0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

Namely, the solution of problem (1.1) exists globally if system (1.4) admits a weak solution and blows up (becomes unbounded) if system (1.4) has no solutions (see [2] for $\gamma=1$ and [9] for $\gamma>1$). This condition is quite natural from the perspective of physics. Indeed, in a framework of a theory, existence of a global solution for evolutionary problem (1.1) can be viewed as ignition failure whereas blow up of solution indicates successful ignition (thermal explosion). Thus, thermal explosion occurs if and only if there is no stationary temperature profile. In physical literature the time of blow up of solution for problem (1.1) is commonly referred to as ignition delay.

In the case when the solution of problem (1.1) blows up it often happens at a single point. In this situation a natural question to ask is whether it is possible to extend a solution in a weak sense for times beyond the time of blow up. This question is not merely a mathematical exercise. Indeed, if a singularity is formed but the solution is extendible beyond blow up time and as a result is bounded in some portion of the domain $\Omega$, one may conclude that ignition occurs only in a part of the domain and certain parts of reactive material do not react. This type of behavior is quite common in applications [7]. In the case when a weak solution can be extended for a finite or infinite time, one can define a delay time (time of existence for classical solution) and complete reaction time (time when even a weak solution ceases to exist).

In [12] it was shown that if $\gamma=1$, this is not the case. That is the solution of problem (1.1) cannot be extended in any reasonable sense to times after blow up. In this paper we show that for $\gamma>1$ the situation is the same. This result is not automatically clear since pressure gradients can have a substantial influence on the dynamics of the system (1.1).

The paper is organized as follows: in Section 2 we state the main result and provide heuristic arguments. In Section 3 we give a proof of the main result.

## 2. Preliminaries and main result

To study the system (1.1) it is convenient to introduce new variables given by the following linear transformation [9,11]

$$\begin{aligned}\Theta(x,t) &= hu(x,t)+(1-h)v(x,t),\\ \Pi(x,t) &= \frac{1}{1-D}\left(u(x,t)-Dv(x,t)\right),\end{aligned} \tag{2.1}$$

where

$$d=D(1-\lambda)^2, \qquad h=\frac{\lambda}{1-D}, \tag{2.2}$$

and $\lambda$ is a unique positive solution of the quadratic equation

$$(1-\lambda)(\gamma+D\lambda)=1. \tag{2.3}$$

In these new variables after rescaling spatial coordinates $\Delta\to(1-\lambda)\Delta$ the system (1.1) reads [10]

$$\begin{cases} u_t=\Delta u+g(hu+(1-h)v), & \\ v_t=d\Delta v+g(hu+(1-h)v) & \text{in } \Omega\times(0,T),\\ u=0, \quad v=0 & \text{on } \partial\Omega\times[0,T),\\ u=0, \quad v=0 & \text{in } \Omega\times\{t=0\}.\end{cases} \tag{2.4}$$

Given restrictions on parameters $D$ and $\gamma$ stated in the introduction, we have $d\in(0,1)$ and $h\in[0,1]$. In what follows we will work with this alternative version of the system (1.1).

As was shown in [9] there exists a unique classical solution of system (2.4) on a time interval $(0,T_m)$. If the size of the domain $\Omega$ is sufficiently large, then the time of existence of a classical solution is finite (i.e. $T_m<\infty$ and $||u(\cdot,t)||_{L^\infty(\Omega)}+||v(\cdot,t)||_{L^\infty(\Omega)}\to\infty$ as $t\to T_m$ ). In this case we call $T_m$ the time of blow up.

Similarly to [12] we define complete blow up in the following manner. For any $n>0$ let $(g_n)=\min\{g,n\}$ and set $(u_n,v_n)$ to be the solution of a following problem

$$\begin{cases} (u_n)_t=\Delta u_n+g_n(hu_n+(1-h)v_n), & \\ (v_n)_t=d\Delta v_n+g_n(hu_n+(1-h)v_n) & \text{in } \Omega\times(0,\infty),\\ u_n=0, \quad v_n=0 & \text{on } \partial\Omega\times[0,\infty),\\ u_n=0, \quad v_n=0 & \text{in } \Omega\times\{t=0\}.\end{cases} \tag{2.5}$$

As follows from the standard theory [13, 16] for each fixed $n>0$ there exists a unique global classical solution for problem (2.5). Moreover, systems (2.4) and (2.5) are co-operative and their solutions are non-decreasing in time (see the next section for more details). Consequently $(u_n,v_n)\to(u,v)$ on $\Omega\times[0,T_m)$ as $n\to\infty$.

Let $\delta(x)=\mathrm{dist}(x,\partial\Omega)$. For $T>0$ we say that the solution of system (1.1) blows up completely if for every $\varepsilon>0$

$$\frac{u_n(x,t)+v_n(x,t)}{\delta(x)}\to\infty \quad \text{as} \quad n\to\infty \quad \text{uniformly on} \quad \Omega\times[T+\varepsilon,\infty). \tag{2.6}$$

Now we can formulate the main result of this paper.

THEOREM 2.1. *Let $(u,v)$ be the unique classical solution of system (2.4) defined on a maximal interval $[0,T_m)$. Then, $(u,v)$ blows up completely after $T_m$. Consequently $(\Theta,\Pi)$ (the solution of system (1.1)) blows up completely after $T_m$.*

Let us give several heuristic arguments that support the result of Theorem 2.1. According to this theorem the qualitative behavior of system (2.4) is similar to the one of a single equation. In general, the behavior of systems is strikingly different from those of single equations. However, the particular structure of system (2.4) leads to several restrictions on the dynamics of its solutions.

First using Duhamel's formula we observe that the integral solution for system (2.4) reads

$$\begin{aligned} u(x,t) &= \int_0^t S_1(t-s)g(hu(\cdot,s)+(1-h)v(\cdot,s))ds,\\ v(x,t) &= \int_0^t S_d(t-s)g(hu(\cdot,s)+(1-h)v(\cdot,s))ds, \end{aligned} \tag{2.7}$$

where $S_d(t)$ is a semi-group generated by $\partial_t-d\Delta$ on $\Omega\times(0,T)$ with Dirichlet boundary conditions. Setting $w(x,t):=hu(x,t)+(1-h)v(x,t)$ we have that the collective variable $w$ solves the single integral equation

$$w(x,t)=\int_0^t \mathcal{H}(t-s)g(w(\cdot,s))ds, \tag{2.8}$$

with $\mathcal{H}(t)=hS_1(t)+(1-h)S_d(t)$. Despite the fact that $\mathcal{H}(t)$ is not a semi-group, the behavior of this equation is expected to be similar to the integral solution of a single non-linear heat equation.

Another argument that favors the conclusion of Theorem 2.1 is that formally system (2.4) is a gradient flow. Indeed one can easily verify that

$$hu_t=-\frac{\delta E}{\delta u}, \quad (1-h)v_t=-\frac{\delta E}{\delta v}, \tag{2.9}$$

where the energy functional is given by

$$E(u,v)=\int_\Omega \frac{h}{2}|\nabla u|^2+(1-h)|\nabla v|^2+G(hu+(1-h)v), \tag{2.10}$$

and

$$G(s)=-\int_0^s g(s)ds. \tag{2.11}$$

Finally one can verify that the system (2.4) is quasi-monotone [13, 15] and thus the componentwise comparison principle applies.

## 3. Proof of Theorem 2.1

In this section we give a proof of Theorem 2.1 which is a generalization of a similar result of [12] for systems.

The proof of the main result is based on the comparison principle. To state this principle we need to introduce a standard notion of the super and sub-solution.

Let $(p,q)$ be the solution of the following problem,

$$\begin{cases} p_t = \Delta p + f(p,q), & \\ q_t = d\Delta q + f(p,q) & \text{in } \Omega\times(0,T), \\ p = q = 0 & \text{on } \partial\Omega\times[0,T), \\ p = p_0, \quad q = q_0 & \text{in } \Omega\times\{t=0\}. \end{cases} \tag{3.1}$$

We call a pair of functions $(\bar{p},\bar{q})$ a super-solution of problem (3.1) if

$$\begin{cases} \bar{p}_t \geq \Delta\bar{p} + f(\bar{p},\bar{q}), & \\ \bar{q}_t \geq d\Delta\bar{q} + f(\bar{p},\bar{q}) & \text{in } \Omega\times(0,T), \\ \bar{p} = \bar{q} = 0 & \text{on } \partial\Omega\times[0,T), \\ \bar{p}\geq p_0, \quad \bar{q}\geq q_0 & \text{in } \Omega\times\{t=0\}. \end{cases} \tag{3.2}$$

Similarly we call a pair of functions $(\underline{p},\underline{q})$ a sub-solution of problem (3.1) if

$$\begin{cases} \underline{p}_t \leq \Delta\underline{p} + f(\underline{p},\underline{q}), & \\ \underline{q}_t \leq d\Delta\underline{q} + f(\underline{p},\underline{q}) & \text{in } \Omega\times(0,T), \\ \underline{p} = \underline{q} = 0 & \text{on } \partial\Omega\times[0,T), \\ \underline{p}\leq p_0, \quad \underline{q}\leq q_0 & \text{in } \Omega\times\{t=0\}. \end{cases} \tag{3.3}$$

The following result is a reformulation of Lemma 2.2 and Corollary 2.3 of [4].

LEMMA 3.1. *Let $f$ be a continuous function satisfying $f(x,y)\geq f(x',y')$ for each $x\geq x'\geq 0$, $y\geq y'\geq 0$ and assume that $(\bar{p},\bar{q})$ is a super-solution of* (3.1). *Then, there exists a solution of* (3.1) *such that $(p,q)\leq(\bar{p},\bar{q})$. Moreover, if $(\underline{p},\underline{q})$ is a sub-solution of* (3.1), *then $(p,q)\geq(\underline{p},\underline{q})$.*

REMARK 3.1. It is clear that problem (2.4) satisfies all the conditions of Lemma 3.1 with $(p,q)=(u,v)$ $(p_0,q_0)=(0,0)$ and $f(p,q)=g(hp+(1-h)q)$. Thus problem (2.4) obeys the comparison principle. The same conclusion holds for a system (2.5).

Along with classical solutions of problem (3.1) we will also need to work with weak solutions of this problem. These solutions are defined as follows:

DEFINITION 3.1. *A weak solution of problem* (3.1) *is a pair of a non-negative functions $(p,q)$ such that for all $0<T'<T$ the functions $p,q,f(p,q)\delta$ belong to $L^1(\Omega,(0,T')$ and satisfy*

$$\begin{aligned} -\int_0^{T'}\int_\Omega p(\phi_t+\Delta\phi) &= \int_0^{T'}\int_\Omega f(p,q)\phi + \int_\Omega p_0\phi(\cdot,0), \\ -\int_0^{T'}\int_\Omega q(\psi_t+d\Delta\psi) &= \int_0^{T'}\int_\Omega f(p,q)\psi + \int_\Omega p_0\psi(\cdot,0), \end{aligned} \tag{3.4}$$

*for all $\phi,\psi\in C^2(\bar{\Omega},[0,T'])$ such that $\phi(\cdot,T')=\psi(\cdot,T')=0$ and $\phi=\psi=0$ on $\partial\Omega$.*

From the definition of a weak solution applied to system (2.4) we have that the maximal time of existence of a weak solution for this problem is given by

$$T^* = \sup\{T>0: \quad \rho(t)<\infty \quad \text{for all} \quad t<T\}, \tag{3.5}$$

where

$$\rho(t)=\lim_{n\to\infty}\left(||u_n(\cdot,t)\delta||_{L^1(\Omega)}+||v_n(\cdot,t)\delta||_{L^1(\Omega)}\right). \tag{3.6}$$

We call $T^*$ the time of complete blow up.

Classical solutions of problem (2.4) besides comparison principle enjoy the following properties:

LEMMA 3.2 (Lemma 3.3 [9]). *Let $(u,v)$ and $(u_n,v_n)$ be classical solutions of problems* (2.4) *and* (2.5) *respectively. Then, $(u_t,v_t)\geq 0$ and $((u_n)_t,(v_n)_t)\geq 0$.*

LEMMA 3.3 (Theorem 2.2 [9]). *If there exists a weak stationary solution of problem* (2.4)*, then the solution of problem* (2.4) *is global and classical.*

In addition the proof of Theorem 2.1 requires the following results:

LEMMA 3.4 (Lemma 6 [2]). *There exists a constant $K\geq 0$ such that for every $0<\varepsilon\leq\varepsilon_0$, there is a function $\Phi_\varepsilon\in C^2$, concave, increasing, with*

$$\begin{gathered}\Phi_\varepsilon(0)=0,\quad 0<\Phi_\varepsilon(x)\leq x\quad \textit{for}\quad x>0,\\ \frac{(g(\Phi_\varepsilon(x)))-\varepsilon K)^+}{g(x)}\leq\Phi_\varepsilon'(x)\leq 1\quad \textit{for}\quad x\geq 0,\end{gathered} \tag{3.7}$$

*where $(\cdot)^+$ denotes the positive part. Moreover, $\sup_{x\geq 0}\Phi_\varepsilon(x)<\infty$.*

REMARK 3.2. As can be clearly seen from the proof of Lemma 6 in [2] the function $\Phi_\varepsilon$ can be constructed in such a way that for any given $0<M<\infty$ we have:

$$\Phi_\varepsilon'(x)\to 1\quad \text{as}\quad \varepsilon\to 0\quad \text{uniformly on}\quad [0,M]. \tag{3.8}$$

LEMMA 3.5 (Lemma 7 [2]). *For every $0<T<\infty$ and $d>0$ there exists $\varepsilon_1(T)>0$ such that if $0<\varepsilon\leq\varepsilon_1\leq\varepsilon_0$ with $\varepsilon_0$ and $K$ as in Lemma 3.4, then the solution of problem*

$$\begin{cases} Z_t-d\Delta Z=-K\varepsilon & \text{in }\ \Omega\times(0,\infty),\\ Z=0 & \text{on }\partial\Omega\times[0,\infty),\\ Z=\delta & \text{in }\ \Omega\times\{t=0\}\end{cases} \tag{3.9}$$

*satisfies $Z\geq 0$ on $\Omega\times[0,T]$.*

LEMMA 3.6. *Let $\Phi\in C^2(\mathbb{R})$ be a concave increasing function, with bounded derivative satisfying $\Phi(0)=0$. Assume that for some $T>0$ and $R_0\in L^\infty(\Omega)$ and $F\delta\in L^1(\Omega\times(0,T))$, function $R\in L^1(\Omega,(0,T))$ solves*

$$\begin{cases} R_t-d\Delta R=F & \text{in }\ \Omega\times(0,T),\\ R=0 & \text{on }\partial\Omega\times[0,T),\\ R=R_0 & \text{in }\ \Omega\times\{t=0\},\end{cases} \tag{3.10}$$

*in a sense that*

$$-\int_0^T\int_\Omega R(\phi_t+d\Delta\phi)=\int_0^T\int_\Omega F\phi+\int_\Omega R_0\phi(\cdot,0), \tag{3.11}$$

*for all $\phi\in C^2(\bar\Omega,\times[0,T])$ such that $\phi(\cdot,T)=0$ and $\phi=0$ on $\partial\Omega$. Then,*

$$-\int_0^T\int_\Omega \Phi(R)(\phi_t+d\Delta\phi)\geq\int_0^T\int_\Omega \Phi'(R)F\phi+\int_\Omega \Phi(R_0)\phi(\cdot,0), \tag{3.12}$$

*with $\phi$ as above.*

This lemma is a straightforward adaption of Lemma 2 [2] given for an elliptic problem. We provide a proof of this result for completeness.

*Proof.* Consider an arbitrary sequence of smooth functions $(F_n)_{n\geq 0}$ such that $F_n\delta \to F\delta$ as $n\to\infty$ in $L^1(\Omega\times(0,T))$ and a sequence of smooth functions $(R_{0,n})_{n\geq 0}$ such that $R_{0,n}\to R_0$ as $n\to\infty$ in $L^1(\Omega)$. Let

$$\begin{cases} (R_n)_t - d\Delta R_n = F_n & \text{in } \Omega\times(0,T), \\ R_n = 0 & \text{on } \partial\Omega\times[0,T), \\ R_n = R_{0,n} & \text{in } \Omega\times\{t=0\}. \end{cases} \tag{3.13}$$

It follows from Lemma 3 [3] that $R_n\to R$ in $L^1(\Omega,(0,T))$ as $n\to\infty$. Also we have

$$(\Phi(R_n))_t - \Delta\Phi(R_n) = \Phi'(R_n)[(R_n)_t - d\Delta R_n] - \Phi''(R_n)|\nabla R_n|^2 \geq \Phi'(R_n)F_n. \tag{3.14}$$

The last inequality holds by (3.13) and concavity of $\Phi$.

Therefore,

$$-\int_0^T\int_\Omega \Phi(R_n)(\phi_t + d\Delta\phi) \geq \int_0^T \Phi'(R_n)F_n\phi + \int_\Omega \Phi(R_{0,n})\phi(\cdot,0), \tag{3.15}$$

for all $\phi\in C^2(\bar\Omega,\times[0,T])$ such that $\phi(\cdot,T)=0$ and $\phi=0$ on $\partial\Omega$. Letting $n\to\infty$ we have the result. ∎

LEMMA 3.7 (Lemma 2 [12]). *Let $S_d(t)$ be a semi-group generated by $\partial_t - d\Delta$ in $\Omega$ with Dirichlet boundary conditions. Then for every $t>0$, there exist $c(t),C(t)>0$ such that for all $F\geq 0$ such that $F\delta\in L^1(\Omega)$, one has*

$$c(t)||F\delta||_{L^1(\Omega)}\delta \leq S_d(t)F \leq C(t)||F\delta||_{L^1(\Omega)}\delta \quad on \quad \Omega. \tag{3.16}$$

We now turn to the proof of the theorem. The proof proceeds in several steps.

*Proof.* (**Proof of Theorem 2.1**.)

**Step 1.** In this step we show that if the solution of system (2.4) blows up at time $T_m$ while not blowing up completely after $T_m$ then the solution of this problem can be extended in a weak sense for a time interval $[T_m,T^*)$ for some $T_m < T^* \leq \infty$.

Assume that there exists $\tau>0$ such that $T_m+\tau<T^*$. For convenience we also assume that $\tau<T_m$. By definition of $T^*$ we have

$$||u_n(\cdot,s)\delta||_{L^1(\Omega)}, ||v_n(\cdot,s)\delta||_{L^1(\Omega)} \leq C_1 \quad \text{for} \quad s\in[\tau,T_m+\tau], \tag{3.17}$$

and all $n>0$ and some $C_1<\infty$ independent of $n$.

Let $\phi,\psi$ be arbitrary $C^2(\bar\Omega\times[\tau,T_m+\tau])$ functions such that $\phi=\psi=0$ on $\partial\Omega$. Multiplying the first and the second equation in system (2.5) by $\phi$ and $\psi$ respectively and integrating by parts we have

$$\begin{aligned} &\int_\tau^{T_m+\tau}\int_\Omega g_n(hu_n+(1-h)v_n)\phi \\ &= -\int_\tau^{T_m+\tau}\int_\Omega u_n[\phi_t+\Delta\phi] + \int_\Omega u_n(\cdot,T_m+\tau)\phi(\cdot,T_m+\tau) - \int_\Omega u_n(\cdot,\tau)\phi(\cdot,\tau), \end{aligned}$$

$$\int_\tau^{T_m+\tau}\int_\Omega g_n(hu_n+(1-h)v_n)\psi$$
$$=-\int_\tau^{T_m+\tau}\int_\Omega v_n[\psi_t+d\Delta\psi]+\int_\Omega v_n(\cdot,T_m+\tau)\psi(\cdot,T_m+\tau)-\int_\Omega v_n(\cdot,\tau)\psi(\cdot,\tau),\quad (3.18)$$

which is just a weak form of the solution of system (2.5) in a sense of Definition 3.1.

Now we show that $g_n(hu_n+(1-h)v_n)\delta\in L^1(\Omega\times(\tau,\tau+T_m))$.

Let $\xi$ be a solution of a following problem

$$\begin{cases} -\xi_t-\Delta\xi=0 & \text{in } \Omega\times(\tau,T_m+\tau),\\ \xi=0 & \text{on } \partial\Omega\times(\tau,T_m+\tau],\\ \xi=\delta & \text{in } \Omega\times\{t=T_m+\tau\}.\end{cases}\quad (3.19)$$

Setting $\phi=\xi$ in the first equation of (3.18) we have

$$\int_\tau^{T_m+\tau}\int_\Omega g_n(hu_n+(1-h)u_n)\xi$$
$$=-\int_\tau^{T_m+\tau}\int_\Omega u_n[\xi_t+\Delta\xi]+\int_\Omega u_n(\cdot,T_m+\tau)\xi(\cdot,T_m+\tau)-\int_\Omega u_n(\cdot,\tau)\xi(\cdot,\tau).\quad (3.20)$$

Since $\xi$ solves the heat equation (3.19) we can represent its solution in an integral form $\xi(x,s)=S(T_m+\tau-s)\delta$. By Lemma 3.7 we have $\xi(x,s)>C_2\delta$ for all $x\in\Omega$ and $s\in[\tau,T_m+\tau]$ and some constant $C_2>0$ independent of $n$. As a result equation (3.20) yields

$$C_2\int_\tau^{T_m+\tau}\int_\Omega g_n(hu_n+(1-h)u_n)\delta$$
$$\le\int_\tau^{T_m+\tau}\int_\Omega g_n(hu_n+(1-h)u_n)\xi\le\int_\Omega u_n(\cdot,T_m+\tau)\delta\le C_1,\quad (3.21)$$

and thus

$$\int_\tau^{T_m+\tau}\int_\Omega g_n(hu_n+(1-h)v_n)\delta\le C_1/C_2.\quad (3.22)$$

Consequently, $g_n(hu_n+(1-h)v_n)\delta\in L^1(\Omega\times(\tau,T_m+\tau))$ independently of $n$.

Next we claim that $u_n,v_n\in L^1(\Omega\times(\tau,T_m+\tau))$ independently of $n$. To show that, let $\eta,\chi$ be solutions of

$$\begin{cases} -\eta_t-\Delta\eta=1 & \text{in } \Omega\times(\tau,T_m+\tau),\\ \eta=0 & \text{on } \partial\Omega\times(\tau,T_m+\tau],\\ \eta=0 & \text{in } \Omega\times\{t=T_m+\tau\},\end{cases}\quad (3.23)$$

and

$$\begin{cases} -\chi_t-d\Delta\chi=1 & \text{in } \Omega\times(\tau,T_m+\tau),\\ \chi=0 & \text{on } \partial\Omega\times(\tau,T_m+\tau],\\ \chi=0 & \text{in } \Omega\times\{t=T_m+\tau\},\end{cases}\quad (3.24)$$

respectively. Multiplying the first and the second equation of system (2.5) by $\eta$ and $\chi$ respectively and taking into account system (3.23), (3.24) we have

$$\int_\tau^{T_m+\tau}\int_\Omega u_n=-\int_\tau^{T_m+\tau}\int_\Omega u_n[\eta_t+\Delta\eta]$$

$$=\int_\tau^{T_m+\tau}\int_\Omega g_n(hu_n+(1-h)v_n)\eta+\int_\Omega u_n(\cdot,\tau)\eta(\cdot,\tau),$$

$$\int_\tau^{T_m+\tau}\int_\Omega v_n=-\int_\tau^{T_m+\tau}\int_\Omega u_n[\chi_t+d\Delta\chi]$$

$$=\int_\tau^{T_m+\tau}\int_\Omega g_n(hu_n+(1-h)v_n)\chi+\int_\Omega u_n(\cdot,\tau)\chi(\cdot,\tau). \tag{3.25}$$

Using integral solution of system (3.23), (3.24) and Lemma 3.7 we have $\eta(\cdot,s),\chi(\cdot,s)<C_3\delta$ for $s\in(\tau,T_m+\tau)$. This observation and Equation (3.25) give

$$\int_\tau^{T_m+\tau}\int_\Omega u_n<C_4,\quad \int_\tau^{T_m+\tau}\int_\Omega v_n<C_5, \tag{3.26}$$

with some constants $C_4,C_5$ independent of $n$. That latter proves our claim.

Finally since $(u_n,v_n)$ is a non-decreasing sequence, monotone convergence theorem and inequality (3.26) imply that there exist $U,V\in L^1(\Omega\times(\tau,T_m+\tau))$ such that a sequence $(u_n,v_n)$ converges to $(U,V)$ in $L^1(\Omega\times(\tau,\tau+T_m))$ and almost everywhere on $\Omega\times(\tau,\tau+T_m)$ as $n\to\infty$. Moreover by inequality (3.22) we have that $g_n(hu_n+(1-h)v_n)\delta$ converges to $g(hU+(1-h)V)\delta$ in $L^1(\Omega\times(\tau,\tau+T_m))$ as $n\to\infty$. On the other hand letting $n\to\infty$ in (3.18) we have

$$\int_\tau^{T_m+\tau}\int_\Omega g(hU+(1-h)V)\phi$$

$$=-\int_\tau^{T_m+\tau}\int_\Omega U[\phi_t+\Delta\phi]+\int_\Omega U(\cdot,T_m+\tau)\phi(\cdot,T_m+\tau)-\int_\Omega U(\cdot,\tau)\phi(\cdot,\tau),$$

$$\int_\tau^{T_m+\tau}\int_\Omega g(hU+(1-h)V)\psi$$

$$=-\int_\tau^{T_m+\tau}\int_\Omega V[\psi_t+d\Delta\psi]+\int_\Omega V(\cdot,T_m+\tau)\psi(\cdot,T_m+\tau)-\int_\Omega V(\cdot,\tau)\psi(\cdot,\tau). \tag{3.27}$$

That is the solution of system (2.4) can be prolonged in a weak sense provided assumption (3.17) holds.

**Step 2.** In this step we show that time intervals of existence of both strong and weak solutions for (2.4) are the same. That is $T_m=T^*$

Let $\varepsilon_0$, $\Phi_\varepsilon$, $K$ be as in Lemma 3.4 and set $U_\varepsilon(x,t):=\Phi_\varepsilon(hU(x,t+\tau))$, $V_\varepsilon(x,t):=\Phi_\varepsilon((1-h)V(x,t+\tau))$, $W_\varepsilon(x,t):=\Phi_\varepsilon(hU(x,t+\tau)+(1-h)V(x,t+\tau))$.

We claim that

$$-\int_0^{T_m}\int_\Omega\frac{U_\varepsilon}{h}(\phi_t+\Delta\phi)$$

$$\geq\int_0^{T_m}\int_\Omega(g(W_\varepsilon)-K\varepsilon)^+\phi+\int_\Omega\frac{\Phi_\varepsilon(hu(\cdot,\tau))}{h}\phi(\cdot,0)-\int_0^{T_m}\int_\Omega\frac{V_\varepsilon}{1-h}(\psi_t+d\Delta\psi)$$

$$\geq\int_0^{T_m}\int_\Omega(g(W_\varepsilon)-K\varepsilon)^+\psi+\int_\Omega\frac{\Phi_\varepsilon((1-h)v(\cdot,\tau))}{1-h}\psi(\cdot,0). \tag{3.28}$$

Indeed, since $\Phi_\varepsilon$ is an increasing concave $C^2$ function we have $\Phi'_\varepsilon(x)\geq\Phi'_\varepsilon(y)>0$ for $y\geq x$. Therefore, in view of the non-negativity of $U$ and $V$ we have

$$\frac{\Phi'_\varepsilon(hU)}{\Phi'_\varepsilon(hU+(1-h)V)}\geq 1,\quad \frac{\Phi'_\varepsilon((1-h)V)}{\Phi'_\varepsilon(hU+(1-h)V)}\geq 1. \tag{3.29}$$

Using this simple observation and Lemma 3.4 we have

$$\begin{aligned}
&\int_0^{T_m}\int_\Omega g(hU+(1-h)V)\Phi_\varepsilon'(hU)\\
&=\int_0^{T_m}\int_\Omega g(hU+(1-h)V)\Phi_\varepsilon'(hU+(1-h)V)\frac{\Phi_\varepsilon'(hU)}{\Phi_\varepsilon'(hU+(1-h)V)}\\
&\geq\int_0^{T_m}\int_\Omega g(hU+(1-h)V)\Phi_\varepsilon'(hU+(1-h)V)\geq\int_0^{T_m}\int_\Omega(g(W_\varepsilon)-\varepsilon K)^+,
\end{aligned}\tag{3.30}$$

and

$$\begin{aligned}
&\int_0^{T_m}\int_\Omega g(hU+(1-h)V)\Phi_\varepsilon'((1-h)V)\\
&=\int_0^{T_m}\int_\Omega g(hU+(1-h)V)\Phi_\varepsilon'(hU+(1-h)V)\frac{\Phi_\varepsilon'((1-h)V)}{\Phi_\varepsilon'(hU+(1-h)V)}\\
&\geq\int_0^{T_m}\int_\Omega g(hU+(1-h)V)\Phi_\varepsilon'(hU+(1-h)V)\geq\int_0^{T_m}\int_\Omega(g(W_\varepsilon)-\varepsilon K)^+.
\end{aligned}\tag{3.31}$$

On the other hand since $U,V$ solve system (2.4) weakly in a sense of Definition 3.1 we have by Lemma 3.6

$$\begin{aligned}
&-\int_0^{T_m}\int_\Omega\frac{U_\varepsilon}{h}(\phi_t+\Delta\phi)\\
&\geq\int_0^{T_m}\int_\Omega\Phi_\varepsilon'(hU)g(hU+(1-h)V)\phi+\int_\Omega\frac{\Phi(hu(\cdot,\tau))}{h}\phi(\cdot,0),
\end{aligned}\tag{3.32}$$

and

$$\begin{aligned}
&-\int_0^{T_m}\int_\Omega\frac{V_\varepsilon}{1-h}(\psi_t+d\Delta\psi)\\
&\geq\int_0^{T_m}\int_\Omega\Phi_\varepsilon'((1-h)V)g(hU+(1-h)V)\phi+\int_\Omega\frac{\Phi((1-h)v(\cdot,\tau))}{1-h}\phi(\cdot,0).
\end{aligned}\tag{3.33}$$

In view of the fact that $V_\varepsilon$ and $U_\varepsilon$ are in $L^\infty(\Omega\times(0,\infty))$ and comparison principle (Lemma 3.1), we conclude that these two functions form a classical super-solution for the problem

$$\begin{cases}
\tilde u_t-\Delta\tilde u=(g(h\tilde u+(1-h)\tilde v)-K\varepsilon)^+, & \\
\tilde v_t-d\Delta\tilde v=(g(h\tilde u+(1-h)\tilde v)-K\varepsilon)^+ & \text{in }\ \Omega\times(0,T_m),\\
\tilde u=\tilde v=0 & \text{on }\partial\Omega\times[0,T_m),\\
\tilde u=\frac{\Phi_\varepsilon(hu(\cdot,\tau))}{h},\quad \tilde v=\frac{\Phi_\varepsilon((1-h)v(\cdot,\tau))}{1-h} & \text{on }\Omega\times\{t=0\},
\end{cases}\tag{3.34}$$

and thus $\tilde u,\tilde v\in L^\infty(\Omega\times(0,T_m))$ and $\tilde u\leq U_\varepsilon$, $\tilde v\leq V_\varepsilon$.

Next, since $g>0$ we have $u_t-\Delta u>0$, $v_t-d\Delta v>0$ in $\Omega\times(0,T_m)$. In view of this fact and zero boundary conditions it follows from maximum principle [14] that $u,v>0$ in $\Omega\times(0,T_m)$. This observation combined with Hopf lemma [14] imply that there exists $c_0>0$ such that $u(\cdot,\tau),v(\cdot,\tau)\geq c_0\delta$. It also follows directly from Remark 3.1 (see step 4 of Theorem 2 in [2] for more details) that there exists $0<\varepsilon_2<\varepsilon_1$ with $\varepsilon_1$ as in Lemma 3.5 such that for $0<\varepsilon\leq\varepsilon_2$ the following hold

$$\frac{\Phi_\varepsilon(hu(\cdot,\tau))}{h}\geq u(\cdot,\tau)-c_0\delta/2\geq c_0\delta/2,$$

$$\frac{\Phi_\varepsilon((1-h)v(\cdot,\tau))}{1-h} \geq v(\cdot,\tau) - c_0\delta/2 \geq c_0\delta/2. \tag{3.35}$$

Let now $0<\varepsilon\leq\varepsilon_2$ and let $(X,Y)(x,t)$ be a solution of the following problem

$$\begin{cases} X_t - \Delta X = -K\varepsilon, & \\ Y_t - d\Delta Y = -K\varepsilon & \text{in } \ \Omega\times(0,\infty), \\ X = Y = 0 & \text{on } \partial\Omega\times[0,\infty), \\ X = Y = c_0\delta/2 & \text{in } \ \Omega\times\{t=0\}, \end{cases} \tag{3.36}$$

that is non-negative on $\Omega\times[0,T_m]$ as follows from Lemma 3.5.

We now set $y(x,t)=\tilde{u}(x,t)-X(x,t)$ and $z(x,t)=\tilde{v}(x,t)-Y(x,t)$. From Equation (3.34), (3.35), (3.36) we have

$$\begin{cases} y_t - \Delta y = (g(h\tilde{u}+(1-h)\tilde{v}) - K\varepsilon)^+ + K\varepsilon, & \\ z_t - d\Delta z = (g(h\tilde{u}+(1-h)\tilde{v}) - K\varepsilon)^+ + K\varepsilon & \text{in } \ \Omega\times(0,T_m), \\ y = z = 0 & \text{on } \partial\Omega\times[0,T_m), \\ y = \frac{\Phi_\varepsilon(hu(\cdot,\tau))}{h} - c_0\delta/2, \quad z = \frac{\Phi_\varepsilon((1-h)v(\cdot,\tau))}{1-h} - c_0\delta/2 & \text{in } \ \Omega\times\{t=0\}. \end{cases} \tag{3.37}$$

Since

$$(g(h\tilde{u}+(1-h)\tilde{v}) - K\varepsilon)^+ + K\varepsilon \geq g(h\tilde{u}+(1-h)\tilde{v}) \geq g(hy+(1-h)z) \tag{3.38}$$

we have from system (3.37)

$$\begin{cases} y_t - \Delta y \geq g(hy+(1-h)z), & \\ z_t - d\Delta z \geq g(hy+(1-h)z) & \text{in } \ \Omega\times(0,T_m), \\ y = z = 0 & \text{on } \partial\Omega\times[0,T_m), \\ y \geq 0, \quad z \geq 0 & \text{in } \ \Omega\times\{t=0\}. \end{cases} \tag{3.39}$$

It thus follows from the comparison principle that $(u,v)\leq(y,z)$ on $\bar{\Omega}\times[0,T_m)$. This observation leads to a contradiction since $y,z\in L^\infty(\Omega\times(0,T_m))$. As a result we have $T^*=T_m$.

**Step 3.** In this step we describe the behavior of solution of system (2.4) right after blow up time.

Since $(u_n,v_n)$ is non-decreasing in $n$ and $T^*=T_m$ for any $\varepsilon>0$ we have

$$\lim_{n\to\infty} ||u_n(\cdot,T_m+\varepsilon)||_{L^1(\Omega)} = \infty, \quad \lim_{n\to\infty} ||v_n(\cdot,T_m+\varepsilon)||_{L^1(\Omega)} = \infty. \tag{3.40}$$

Since $(u_n,v_n)$ are super-caloric we have $u_n(x,T_m+2\varepsilon)\geq S_1(\varepsilon)u_n(\cdot,T_m+\varepsilon)$ and $v_n(x,T_m+2\varepsilon)\geq S_d(\varepsilon)v_n(\cdot,T_m+\varepsilon)$. By Lemma 3.7 it then follows

$$\frac{u_n(\cdot,T_m+2\varepsilon)}{\delta}, \frac{u_n(\cdot,T_m+2\varepsilon)}{\delta} \to \infty \quad as \quad n\to\infty \quad \text{uniformly on} \quad \Omega. \tag{3.41}$$

That proves complete blow up of solution of system (2.4). $\square$

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