COMMUN. MATH. SCI.
Vol. 15, No. 2, pp. 289–303

A DISCRETE MODEL FOR NONLOCAL TRANSPORT EQUATIONS WITH FRACTIONAL DISSIPATION*

TAM DO†

Abstract. In this note, we propose a discrete model to study one-dimensional transport equations with non-local drift and supercritical dissipation. The inspiration for our model is the equation

$$\theta_t + (H\theta)\theta_x + (-\Delta)^\alpha \theta = 0,$$

where H is the Hilbert transform. For our discrete model, we present blow-up results that are analogous to the known results for the above equation. In addition, we will prove regularity for our discrete model which suggests supercritical regularity in the range $1/4 < \alpha < 1/2$ in the continuous setting.

Keywords. Nonlocal transport, dyadic model, supercritical regularity.

AMS subject classifications. 35Q35.

1. Introduction

Consider the following the surface quasi-geostrophic Equation (1.1):

$$\theta_t(x,t) + u \cdot \nabla\theta(x,t) + (-\Delta)^\alpha \theta(x,t) = 0, \quad u = \nabla^\perp (-\Delta)^{-1/2}\theta, \quad (x,t) \in \mathbb{R}^2 \times [0,\infty), \tag{1.1}$$

where $(-\Delta)^\alpha$ with $\alpha > 0$ is the fractional Laplacian. For $0 \le \alpha < 1/2$, an important open question is whether solutions to the SQG equation can blow-up in finite time starting from smooth initial data. See [3] and [11] for recent progress and references.

One approach to studying (1.1) is to study simpler model equations. The most prominent of these is the following equation:

$$\theta_t(x,t) + (H\theta)(x,t)\theta_x(x,t) + (-\Delta)^\alpha \theta(x,t) = 0 \quad (x,t) \in \mathbb{R} \times [0,\infty), \tag{1.2}$$

where H is the Hilbert transform. The model possess the same scaling and type of nonlocality as (1.1). When $\alpha = 0$, by convention, we take the dissipation to be absent in the equation. In addition, (1.2) is a model for the Birkhoff–Rott equations for vortex sheets [14]. The regularity issues for the model (1.2) will serve as the inspiration for this note.

By virtue of (1.2) satisfying an L^∞ maximum principle, we can roughly divide the analysis of (1.2) into three categories according to the value of α. For $\alpha > 1/2$, the equation is *subcritical* in that one expects the dissipative term to dominate the nonlinearity and one has global regularity from functional analysis methods. For $\alpha = 1/2$, the equation is *critical* in the sense that dissipative term and nonlinear term are thought to be equally balanced. For $0 \le \alpha < 1/2$, the equation is said to be *supercritical* in that the nonlinear term is thought to dominate the dynamics due to scaling considerations. It was shown in [6] and [13] that when $0 \le \alpha < 1/4$, solutions can develop finite time blow-up. In [8], it was shown that for $\alpha \ge 1/2$ solutions exist for all time. In the supercritical range $1/4 \le \alpha < 1/2$, the question of global existence versus blow-up is open. It was shown in [7] that supercritical solutions derived in the vanishing viscosity limit, in a sense, become eventually regular. Also, it was shown that solutions supercritical by a logarithm are globally regular. We are not aware of any example where supercritical, with respect to the basic conservation laws, regularity has been proved for a fluid mechanics PDE.

*Received: November 23, 2015; accepted (in revised form): June 7, 2016. Communicated by Lenya Ryzhik.

†Rice Unversity 6100 Main St. Houston, TX 77005, (Tam.Do@rice.edu).

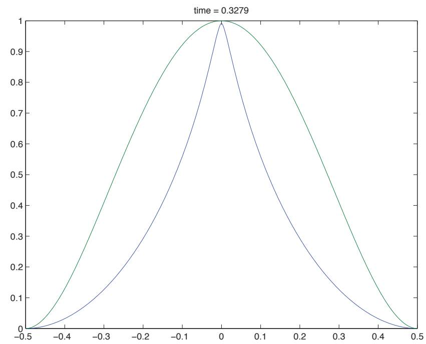

FIG. 1.1. *Plot of the solution for the inviscid equation with the following initial data:* $\theta_0(x) = (1-4x^2)^2\chi_{[-1/2,1/2]}(x)$.

1.1. Motivation for the model. The proof of blow-up for $0 \le \alpha < 1/4$ relies on the following novel inequality that for $f \in C_c^\infty(\mathbb{R}^+)$ and $0 < \delta < 1$, there exists a constant C_δ such that

$$-\int_0^\infty \frac{f_x(x)(Hf)(x)}{x^{1+\delta}}\,dx \ge C_\delta \int_0^\infty \frac{f^2(x)}{x^{2+\delta}}\,dx. \tag{1.3}$$

The proof given in [6] uses tools from complex analysis and is used to prove blow-up for even positive initial data with a maximum at 0. In [11], another more elementary proof of blow-up for $0 \le \alpha < 1/4$ was given. The proof also goes by way of (1.3) but without appealing to complex analysis. In particular, the complicated non-locality of the Hilbert transform is handled by the following key inequality from [11]:

PROPOSITION 1.1. *Suppose that the function $f(x)$ is C^1, even, $f'(x) \ge 0$ for $x > 0$ and f is bounded on $\mathbb{R}$ with $f(0) = 0$. Then for $1 < q < 2$,*

$$Hf(x) \le \log(q-1)(f(qx) - f(q^{-1}x)).$$

Consider initial data for (1.2) that is C^1, even, and monotone decaying away from the origin. These properties are preserved by (1.2). The proposition is then applied to $f(x) = \theta(0,t) - \theta(x,t)$. To derive our model, we will look at dyadic points. Consider the following system of ODEs:

$$a_k'(t) = -(a_k - a_{k-1})^2(t)2^k, \quad a_k(0) = a_k^0, \quad k \ge 1. \tag{1.4}$$

and we set $a_1'(t) = 0$, $a_0 = 0$ This system serves as a discrete model for

$$\theta_t(x,t) = -(H\theta)\theta_x(x,t). \tag{1.5}$$

One can think of $a_k(t)$ as approximating $\theta(2^{-k},t)$. In particular, one could think that $(a_{k-1} - a_k)(t)2^k \approx \theta_x(2^{-k},t)$ and that, using Proposition 1.1 as inspiration, we shall take $(a_k - a_{k-1})$ as our approximation of $H\theta(2^{-k},t)$. The reason for considering a dyadic model is the nature of conjectured blow up for (1.5). A cusp appears to form in finite time at the origin (see Figure 1.1) and it is our hope that studying (1.4) will help us understand such phenomenon. The fact that we are restricting ourselves to even, monotone decaying data is not too restricting. It has been shown that blow-up occurs at any local maximum [15] and it is believed that cusps are formed at local maximums. Furthermore, we will see many of the basic features from the continuous setting such as monotonicity are preserved in (1.4) (see Lemma 2.2).

Dyadic models have been used to study regularity properties of Navier–Stokes and Euler equations (see [10, 12, 2] as well as references therein). A key difference between with our model is that we use reductions in physical space to achieve our model rather than frequency space. One reason for this is that blow-up in (1.2) is believed to occur at a single point and as a cusp, which is not captured well by Fourier methods. In analyzing our model, we hope to gain insight into regularity of the continuous equation that is not apparent through standard norm estimates. Below, in Figure 1.2, we have plotted a solution to (1.4). We can observe that the discrete system seems to form a cusp at the origin much like (1.2), which is plotted in Figure 1.1.

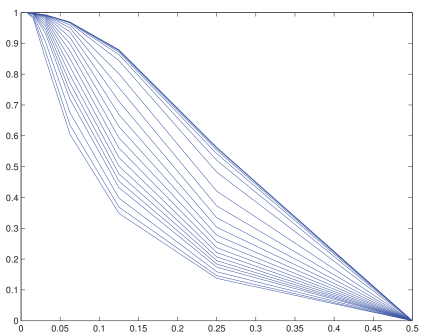

FIG. 1.2. *Plot of the solution on* $[0,1/2]$ *to* (1.4) *at various times with initial data corresponding to* $\theta_0(x) = (1-4x^2)^2\chi_{[-1/2,1/2]}(x)$.

1.2. Dyadic fractional Laplacian. Now, we would like to formulate a discrete version of the fractional Laplacian $(-\Delta)^\alpha$. Recall for θ smooth enough,

$$(-\Delta)^\alpha\theta(x) = P.V.\int_{-\infty}^{\infty}\frac{\theta(x)-\theta(y)}{|x-y|^{1+2\alpha}}\,dy,$$

see [5] for a derivation. If we take θ to be even then

$$(-\Delta)^\alpha\theta(x) = \int_0^\infty\left[\frac{1}{|x-y|^{1+2\alpha}}+\frac{1}{|x+y|^{1+2\alpha}}\right](\theta(x)-\theta(y))\,dy.$$

Let $x = 2^{-k}$ and $y = 2^{-n}$. If $k < n$ then $\frac{1}{|x-y|^{1+2\alpha}}+\frac{1}{|x+y|^{1+2\alpha}} \approx 2^{(1+2\alpha)k}$. Similarly, if $k > n$, then $\frac{1}{|x-y|^{1+2\alpha}}+\frac{1}{|x+y|^{1+2\alpha}} \approx 2^{(1+2\alpha)n}$. Also, $\theta(x)-\theta(y) \approx a_k - a_n$. Combining these observations, we can see that a reasonable model for fractional dissipation would be

$$(\mathcal{L}^\alpha a)_k = \sum_{n=0}^{k-1}(a_k-a_n)2^{2\alpha n} + \sum_{n=k+1}^{\infty}(a_k-a_n)2^{2\alpha k}2^{k-n}. \tag{1.6}$$

In our discrete formulation, we have ignored the tails of the integral defining $(-\Delta)^\alpha$. Adding the tails into (6) (i.e. letting the first sum go to $-\infty$) doesn't change the main results of this note and just adds more complications to the calculations.

Thus, our model for (1.2) is

$$a_k' = -(a_k - a_{k-1})^2 2^k - (\mathcal{L}^\alpha a)_k. \tag{1.7}$$

$$a_0' = -(\mathcal{L}^\alpha a)_0$$

For simplicity of notation, we will sometimes omit the α and simply write $\mathcal{L}$ instead. It should be noted that our model can also serve as discrete model for other non-local transport equations with fractional diffusion such as

$$\theta_t(x,t) = u(x,t)\theta_x(x,t) - (-\Delta)^\alpha(x,t),$$

where

$$u(x,t) = \begin{cases} \theta(x,t) - \theta(2x,t) & x \geq 0 \\ \theta(2x,t) - \theta(x,t) & x < 0 \end{cases}$$

It is unknown whether solutions to this type of equations with general initial data can blow-up or exist globally in time in the supercritical range $0 \leq \alpha < 1/2$. In [16], a related kind of "non-local" Burgers-type equation was studied and blow-up was observed in the non-viscous case in certain situations.

1.3. Outline of main results. In Section 2, we present several properties of solutions to (1.7) and show that solutions to (1.4) blow-up, strengthening the analogy between the model and (1.2). In Section 3, we prove an a priori bound on solutions akin to a global-in-time Hölder 1/2 bound in the continuous setting. In Section 4, we use this bound to prove blow-up for $0 < \alpha < 1/4$ and global regularity for $1/4 \leq \alpha < 1/2$, which can be considered the main result of this note. If this result were carried into the continuous setting, it would suggest that solutions to (1.2) in the supercritical range $1/4 \leq \alpha < 1/2$ are globally regular, contrary to natural scaling considerations.

2. Local existence and properties of solutions

Define the space

$$X^s = \{\{a_k\}_{k=0}^\infty : \|a\|_{X^s} := \sup_k |a_k| + \sup_{k\geq 1} |a_k - a_{k-1}| 2^{sk} < \infty\}.$$

One could think of X^s as being analogous to the Hölder spaces. However, the X^s are made to deal with behavior near the origin. Let $b_{k,s} = (a_k - a_{k-1})2^{sk}$.
Before we show local existence for the full system, we will state some facts about $\mathcal{L}$ and its associated semigroup.

Lemma 2.1. *Let $0 < \alpha < 1/2$.*

(a) *Suppose that for the index k, $b_{k,s} > c_s \|a\|_{X^s}$, where*

$$c_s = \frac{3}{4}(2^s - 1)^{-1}\left(1 - \frac{1}{2^{s+1} - 1}\right) < 1.$$

Then

$$((\mathcal{L}^\alpha a)_k - (\mathcal{L}^\alpha a)_{k-1})2^{sk} \geq C(\alpha)(2^{2\alpha k} - 2^{2\alpha}) b_{k,s},$$

where $C(\alpha)$ is a positive constant only dependent on α.

(b) *The operator $\mathcal{L}^\alpha$ generates a contracting semigroup $e^{-t\mathcal{L}^\alpha}$ on X^s for all $s>0$.*

(c) *The following identity, which is analogous to $\int(-\Delta)^\alpha\theta=0$, is true*

$$\sum_{k=0}^{\infty}(\mathcal{L}^\alpha a)_k 2^{-k}=0.$$

Proof.

(a) We will consider $k>1$ as the case for $k=1$ is similar. By direct computation,

$$\begin{aligned}(\mathcal{L}a)_k-(\mathcal{L}a)_{k-1} &= \sum_{n=0}^{k-2}(a_k-a_{k-1})2^{2\alpha n}+2(a_k-a_{k-1})2^{2\alpha(k-1)}\\ &\quad +2^{(1+2\alpha)k}(1-2^{-(1+2\alpha)})\sum_{n=k+1}^{\infty}(a_k-a_n)2^{-n}\\ &= (a_k-a_{k-1})\left(\frac{2^{2\alpha(k-1)}-1}{2^{2\alpha}-1}\right)+2(a_k-a_{k-1})2^{2\alpha(k-1)}\\ &\quad +2^{(1+2\alpha)k}\sum_{n=k+1}^{\infty}(a_k-a_n)(1-2^{-(1+2\alpha)})2^{-n}\\ &:=\mathrm{I}+\mathrm{II}+\mathrm{III}.\end{aligned}$$

By hypothesis, we have $c_s b_{j,s}<b_{k,s}$ for $j>k$. In terms of a, this means, for $n\geq k+1$,

$$\begin{aligned}(a_n-a_k) &= \sum_{j=k+1}^{n}(a_j-a_{j-1})\leq c_s^{-1}(a_k-a_{k-1})\sum_{j=1}^{n-k}2^{-js}\\ &= \frac{c_s^{-1}}{2^s-1}(a_k-a_{k-1})(1-2^{-(n-k)s}).\end{aligned}$$

Then

$$\begin{aligned}\mathrm{III} &\geq -\frac{c_s^{-1}}{2^s-1}2^{(1+2\alpha)k}(a_k-a_{k-1})(1-2^{-(1+2\alpha)})\sum_{n=k+1}^{\infty}(1-2^{-(n-k)s})2^{-n}\\ &= -\frac{4}{3}(1-2^{-(1+2\alpha)})(a_k-a_{k-1})2^{2\alpha k}.\end{aligned}$$

From this inequality and using that $0<\alpha<1/2$, it is easy to see that $\mathrm{II}+\mathrm{III}\geq 0$. Then

$$((\mathcal{L}a)_k-(\mathcal{L}a)_{k-1})2^{sk}\geq \mathrm{I}\cdot 2^{sk}=C(\alpha)(2^{2\alpha k}-2^{2\alpha})b_{k,s}.$$

(b) Consider the system

$$a_k'=-(\mathcal{L}a)_k.$$

Using (a), it's not hard to see that $\frac{d}{dt}\|a\|_{X^s}(t)<0$ and so $e^{-t\mathcal{L}}$ is a contracting semigroup.

(c) Follows by explicit computation. ∎

We will need the following version of Picard's theorem to prove local existence.

THEOREM 2.1 (Picard Fixed Point). *Let Y be a Banach space and let $\Gamma: Y\times Y \to Y$ be a bilinear operator such that for all $a,b\in Y$,*

$$\|\Gamma(a,b)\|_Y \le \eta \|a\|_Y \|b\|_Y.$$

Then for any $a^0\in Y$ with $4\eta\|a^0\|_Y<1$, the equation $a=a^0+\Gamma(a,a)$ has a unique solution $a\in Y$ such that $\|a\|_Y\le 1/2\eta$.

THEOREM 2.2 (Local existence). *Let $\{a_k(0)\}\in X^s$ where $s\ge 1$. Then there exists $T=T(\|a(0)\|_{X^s})$ such that there is a unique solution $\{a_k(t)\}\in C([0,T],X^s)$.*

Proof. The argument is fairly standard and we will provide a sketch. By Lemma 2.1, the operator $\mathcal{L}$ generates a contracting semigroup $e^{-t\mathcal{L}}$ on X^s. Observe that a solution will satisfy

$$a_k(t)=e^{-t\mathcal{L}}a_k^0-\int_0^t e^{-(t-s)\mathcal{L}}\{2^k(a_k-a_{k-1})^2(s)\}\,ds.$$

In writing $e^{-(t-s)\mathcal{L}}$ in the integral, we have slightly abused notation. The integrand is the kth element of $e^{-(t-s)\mathcal{L}}$ applied to the sequence $\{2^j(a_j-a_{j-1})^2(s)\}_{j=1}^{\infty}$.

Define a bilinear operator $\Gamma: X^s\times X^s\to X^s$ by

$$\Gamma(a,b)_j(t)=-\int_0^t e^{-(t-s)\mathcal{L}}\{2^k(a_k-a_{k-1})(s)(b_k-b_{k-1})(s)\}\,ds.$$

Define $\gamma(a,b)(s)$ by

$$\gamma(a,b)_k(s)=-e^{-(t-s)\mathcal{L}}\{2^k(a_k-a_{k-1})(s)(b_k-b_{k-1})(s)\}.$$

Then after doing basic estimates, one has $\|\gamma(a,b)(s)\|_{X^s}\le C\|a(s)\|_{X^s}\|b(s)\|_{X^s}$ for $s\ge 1$. From this, we have an estimate on Γ and choosing t small enough, we can apply Picard. □

Also, we have the following preservation properties:

LEMMA 2.2. *Let $\{a_k(t)\}$ be a solution of* (1.7) *in $C([0,T],X^s)$, $s>1$. Suppose $\{a_k^0\}$ is non-decreasing in k and is non-negative.*

(a) (Monotonicity and positivity.) *Then for all $t\le T$, $\{a_k(t)\}$ is non-decreasing in k and is non-negative.*

(b)(Max stays at 0.) *For $t\in[0,T]$, we have that*

$$\sup_k a_k(t)=\lim_{k\to\infty} a_k(t).$$

(c) (ℓ^∞ maximum principle.) *The supremum $\sup_k a_k(t)$ is non-increasing in t and $a_0(t)$ is increasing in t. If $\{a_k(t)\}$ is a solution to* (1.4), *then $\sup_k a_k(t)$ is constant in time.*

Proof.

(a) Set $b_k(t)=(a_k-a_{k-1})(t)2^k$. For a contradiction, suppose there exists j and a time t_0 such that $b_j(t_0)<0$. Choose a sufficiently small ϵ with $0<\epsilon<1$ such that $b_j(t_0)<-\epsilon/t_0$. Define a function f by $f(t)=\dfrac{-\epsilon}{2t_0-t}$ for $0\le t<2t_0$, so $b_j(t_0)<f(t_0)$.

Let $g(t)=\inf_k b_k(t)$. Let t_1 be the first time g crosses f so $\inf_k b_k(t_1)=f(t_1)$. Because $b_k(t_1)\to 0$ as $k\to\infty$, the infimum is actually achieved for some index. Let k_1 be the first index for which $b_{k_1}(t_1)=f(t_1)$. By minimality of t_1, $b'_{k_1}(t_1)\le f'(t_1)=-\dfrac{\epsilon}{(2t_0-t)^2}$. However, at time t_1, b_{k_1} also satisfies

$$\begin{aligned} b'_{k_1} &= -b_{k_1}^2+2b_{k_1-1}^2-((\mathcal{L}a)_{k_1}-(\mathcal{L}a)_{k_1-1})2^{k_1} \\ &\ge -\frac{\epsilon^2}{(2t_0-t)^2}-((\mathcal{L}a)_{k_1}-(\mathcal{L}a)_{k_1-1})2^{k_1}. \end{aligned}$$

By an argument analogous to Lemma 2.1, the contribution from the dissipative terms on the right side of the inequality above is negative. From this, we arrive at a contradiction to the above inequality. Because a_0 is always non-negative, a_k stays non-negative for all time by the monotonicity just proved.

(b) Follows directly from **(a)**.

(c) By **(a)**, we see that $(\mathcal{L}a)_0(t)<0$ for all time $t\le T$ so $a_0(t)$ is increasing in t. Now, for $t\in[0,T]$

$$\frac{d}{dt}\sup_{k>0} a_k(t)=\frac{d}{dt}\lim_{k\to\infty} a_k(t)=\lim_{k\to\infty}\frac{d}{dt}a_k(t)=\lim_{k\to\infty}\left(-(a_k-a_{k-1})^2(t)2^k-(\mathcal{L}^\alpha a)_k(t)\right).$$

Since we are on a compact time interval, the convergence above is uniform and so we are justified in interchanging limit and derivative. The limit of the first term on the far right is zero. Using that $\{a_k(t)\}\in X^s$ where $2\alpha<1<s$,

$$\lim_{k\to\infty}(\mathcal{L}^\alpha a)_k(t)=\lim_{k\to\infty}\left(\sum_{n=0}^{k-1}(a_k-a_n)(t)2^{2\alpha n}+\sum_{n=k+1}^{\infty}(a_k-a_n)(t)2^{2\alpha k}2^{k-n}\right)>0$$

because the limit of the first term is positive by monotonicity and the limit of the second term is zero. This completes the proof. □

In the spirit of [11], we show that solutions of (1.4) blow up in finite time. Define $a:=\lim_{k\to\infty}a_k(t)$. By Lemma 2.2(c), a is independent of t.

THEOREM 2.3. *Let $\{a_k\}_{k=0}^\infty$ be a solution of (1.4) in X^s, $s>1$ with initial data that is non-negative and non-decreasing in k. Then $\{a_k\}$ develops blow up in X^s for every $s>1$ in finite time.*

Define

$$J(t)=\sum_{k=1}^{\infty}(a-a_k(t))2^{k\delta},$$

where $0<\delta<1$. Observe that, $J\le\|a\|_{X^1}\sum_{k=1}^\infty 2^{k(\delta-1)}$. To prove we have blow up, we will show that J must become infinite in finite time. We will need the following lemma.

LEMMA 2.3. *Let $0<\delta<1$. Suppose that $\{a_k\}\in X^s$, $s>1$ and non-decreasing in k. Then*

$$\sum_{k=1}^{\infty}(a_k(t)-a_{k-1}(t))^2 2^{k(\delta+1)}\ge C_0(\delta)\sum_{k=1}^{\infty}(a-a_k(t))^2 2^{k(\delta+1)}, \tag{2.1}$$

where $C_0(\delta)$ is constant depending only on δ.

Proof. Choose $c>0$ such that $(c+1)^{-2}2^{\delta+1}>1$. We call k "good" if $a_k-a_{k-1}\geq c(a-a_k)$ and "bad" otherwise.

Claim: If the set of all good k are finite, then both sides of (2.1) are infinite.

If the set of good k is finite, then there exists K such that for $k>K$, $a_k-a_{k-1}\leq c(a-a_k)$ or equivalently $(a-a_k)\geq(c+1)^{-1}(a-a_{k-1})$. Then

$$(a-a_k)^2 2^{k(\delta+1)}\geq(c+1)^{-2(k-K+1)}(a-a_K)^2 2^{k(\delta+1)}\to\infty$$

as $k\to\infty$ by the choice of c, so the right side of (2.1) diverges.

Define c_n by $a_k-a_{k-1}=c_k(a-a_k)$. Then $(a-a_{k-1})=(1+c_k)(a-a_k)$. Since $\lim_{k\to\infty}(a-a_k)=0$, $\prod_{k=K}^{\infty}(1-\frac{c_k}{c_k+1})=\prod_{k=K}^{\infty}\frac{1}{1+c_k}=0$, so $\sum_k c_k=\infty$. For the bad k,

$$(a_k-a_{k-1})^2 2^{k(\delta+1)}=c_k^2(a-a_k)^2 2^{k(\delta+1)},$$

from which it is not hard to see that the left side of (2.1) will be infinite as well. The claim is proven.

Now suppose k_{j-1}, k_j are good such that for $k_{j-1}<k<k_j$, k is bad. Then $(a-a_{k-1})\leq(1+c)(a-a_k)$, which implies

$$\begin{aligned}\sum_{k=k_{j-1}+1}^{k_j}(a-a_k)^2 2^{k(\delta+1)}&\leq\sum_{k=k_{j-1}+1}^{k_j}(1+c)^{2(k_j-k)}(a-a_{k_j})^2 2^{k(\delta+1)}\\&=(a-a_{k_j})^2 2^{k_j(\delta+1)}\sum_{k_{j-1}+1}^{k_j}(1+c)^{2(k_j-k)}2^{(k-k_j)(\delta+1)}\\&=(a-a_{k_j})^2 2^{k_j(\delta+1)}\sum_{n=0}^{k_j-k_{j-1}+1}(1+c)^{2n}2^{-n(\delta+1)}\\&\leq C(\delta)(a-a_{k_j})^2 2^{k_j(\delta+1)},\end{aligned}$$

where the last inequality comes from the choice of c. Treating all bad k this way, the inequality (2.1) follows. ∎

Proof. (**Proof of Theorem 2.3).** Using Lemma 2.2(c) and Lemma 2.3 as well as Hölder's inequality, we have

$$\begin{aligned}\frac{d}{dt}J(t)&=\sum_{k=1}^{\infty}(a_k(t)-a_{k-1}(t))^2 2^{k(\delta+1)}\geq C_0\sum_{k=1}^{\infty}(a-a_k(t))^2 2^{k(\delta+1)}\\&\geq C_0\left(\sum_{k=1}^{\infty}2^{k(\delta-1)}\right)^{-1}\left(\sum_{k=1}^{\infty}(a-a_k(t))2^{k\delta}\right)^2=C_1(\delta)J(t)^2.\end{aligned}$$

The result follows from Gronwall's inequality. ∎

3. A priori Hölder-1/2 bound

The purpose of this section is to prove an a priori bound for solutions of (1.7) from which the results of next section will follow. If this result of this section were to be carried to the continuous setting, it would mean that solutions to (1.2) are bounded in the Hölder class $C^{1/2}$. This regularization effect has recently been conjectured in

[15] for the vanishing viscosity approximation. If such a bound were to hold, it would mean that $\alpha = 1/4$ is the true critical power for regularity and would answer the open question regarding (1.2) stated in the introduction. For the SQG Equation (1.1), the $C^{1-2\alpha}$ norm is critical [4, 9]: weak solutions that are bounded in time in $C^{1-2\alpha}$ are classical solutions.

First, we will prove bounds on (1.4), the model without dissipation, as it is more elementary. Then we will generalize to the full model (1.7) with dissipation. We rewrite (1.4) in a different form, which allows us to give a more detailed picture of how blow-up can occur. Let $b_k = (a_k - a_{k-1})2^k$. Then the b_k's satisfy

$$b_k'(t) = -b_k^2 + 2b_{k-1}^2 \tag{3.1}$$

with corresponding initial data

$$b_k(0) = -(a_k^0 - a_{k-1}^0)2^k := b_k^0.$$

By convention, we set $b_1' = -b_1^2$ and $b_0 = 0$. We will take $b_k(0) > 0$ for all k so b_k will remain non-negative by Lemma 2.2(a). The fact that the a_k's blow up in X^s, $s > 1$, means that there exists $T > 0$ such that $\lim_{t \to T} \sup_k b_k(t) 2^{k(s-1)} = \infty$. In what follows, we work with X^s, $s > 1$ solutions.

We will prove there exists an invariant region for the system of ODE's satisfied by the sequence $\{b_k(t)\}_{k=1}^\infty$. Define a sequence $\{\gamma_k\}$ by $\gamma_1 = \gamma_2 = 2$ and define γ_k, $k > 2$, to be the positive root of the polynomial

$$x^2 + \left(\frac{2}{\gamma_{k-1}^2} - 1\right)x - 2 = 0.$$

One can see that $\gamma_k \to \sqrt{2}$ as $k \to \infty$ and is decreasing in k. Hölder 1/2 control of solutions will follow from proving that the following region is invariant for $\{b_k(t)\}_{k=1}^\infty$:

$$\mathcal{I} := \{\{c_k\}_{k=1}^\infty : c_k \geq 0 \text{ for all } k \text{ and } c_k \leq \gamma_k c_{k-1} \text{ for all } k > 1\}.$$

Invariant regions have also been used to study regularity properties of dyadic models for other fluid equations [1].

Lemma 3.1. *Let $\{b_k\}_{k=1}^\infty$ be a solution of (3.1). Suppose that $\{b_k(0)\}_{k=1}^\infty \in \mathcal{I}$. Then for all $t \geq 0$, $\{b_k(t)\}_{k=1}^\infty \in \mathcal{I}$.*

Proof. The preservation of positivity of the b's follows from Lemma 2.2(a). For the other inequality, we proceed by induction. First, we will show $b_2(t) \leq 2b_1(t) = \gamma_2 b_1(t)$, for all t. It suffices to show that $(2b_1 - b_2)'(t_1) \geq 0$ at anytime t_1 such that $b_2(t) = 2b_1(t)$. This is true, since

$$(2b_1 - b_2)'(t_1) = -4b_1^2(t_1) + b_2^2(t_1) = 0.$$

Now assume $b_{k-1}(t) \leq \gamma_{k-1} b_{k-2}(t)$ for all t. We want to show $b_k(t) < \gamma_k b_{k-1}(t)$ for all t. Suppose t_k is a time for which $b_k(t_k) = \gamma_k b_{k-1}(t_k)$. Then using the inductive hypothesis

$$\begin{aligned}(\gamma_k b_{k-1} - b_k)'(t_k) &= \gamma_k(-b_{k-1}^2(t) + 2b_{k-2}^2) + b_k^2(t) - 2b_{k-1}^2(t) \\ &\geq \left[\gamma_k\left(\frac{2}{\gamma_{k-1}^2} - 1\right) + \gamma_k^2 - 2\right] b_{k-1}^2(t_k) = 0.\end{aligned}$$

This completes the proof. ∎

THEOREM 3.1 (Hölder 1/2 bound). *Suppose we have initial data lying in $\mathcal{I}$, then* $\sup_{k>0,t\geq 0} b_k(t)2^{-k/2} < \infty$.

Proof. It is easy to show that the infinite product $\prod_k \frac{\gamma_k}{\sqrt{2}}$ converges. Then by Lemma 3.1, we are done. ∎

The following additional lemma will give more insight into how blow-up occurs.

LEMMA 3.2. *Let $\{b_k\}$ be a positive solution to* (3.1). *Suppose that for $T_1 \leq t \leq T_2$ and some $k \geq 2$, $b_{k-1}(t) > b_{k-2}(t)$ (or $<$) and $b_k(T_1) > b_{k-1}(T_1)$ (or $<$). Then for $T_1 \leq t < T_2$, we have $b_k(t) > b_{k-1}(t)$ (or $<$).*

Proof. We first treat the case $k=2$ and prove the "$>$" case as the other direction will be analogous. For a contradiction, suppose the set $A_2 := \{T_2 \geq t > T_1 : b_2(t) > b_1(t)\}$ is non-empty. By continuity, A_2 has a minimum which we denote by t_2. Then $b_2(t_2) = b_1(t_2)$, which implies

$$(b_2 - b_1)'(t_2) = -b_2^2(t_2) + 3b_1^2(t_2) = 2b_1^2(t_2) > 0.$$

This leads to contradiction to the minimality of t_2 so this is case is settled.

Now, let $k > 2$. The argument is similar to the previous case. It suffices to show that $(b_k - b_{k-1})'(t) > 0$ at a time $t = t_k$ such that $b_k(t_k) = b_{k-1}(t_k)$, given that $b_{k-1}(t_k) > b_{k-2}(t_k)$. By a quick calculation, we can check that this is true. ∎

Consider initial data of the following form

$$\begin{aligned} &b_k^0 < b_{k-1}^0, \quad \text{for } k > K_0, \\ &b_k^0 > b_{k-1}^0 > 0, \quad \text{for } k \leq K_0, \end{aligned}$$

for some index $K_0 > 1$. Then by the previous lemma, we know that for all t, there exists $K_t \geq K_0$ such that

$$\begin{aligned} &b_k(t) < b_{k-1}(t), \quad \text{for } k > K_t, \\ &b_k(t) > b_{k-1}(t) > 0, \quad \text{for } k \leq K_t. \end{aligned}$$

The scenario we describe is analogous to describing the concavity of solutions in the continuous setting. We view $b_k > b_{k-1}$ as saying that the solution is concave up on the interval $(2^{-k}, 2^{-k+2})$ and vice versa, remembering the analogy $b_k(t) \approx -\theta_x(2^{-k}, t)$. The index K_t can be thought as encoding the inflection point. In order for the solution to blow-up, $K_t \to \infty$ as $t \to T$.

3.1. With dissipation. In this section, we prove the Hölder 1/2 bound for the system with fractional dissipation. We still consider initial data $\{a_k^0\}_{k=0}^\infty \in X^s$, $s > 1$, that is positive and non-decreasing in k. As before we consider the following system satisfied by $b_k = (a_k - a_{k-1})2^k$:

$$b_k'(t) = -b_k^2 + 2b_{k-1}^2 + 2^k((\mathcal{L}a)_{k-1} - (\mathcal{L}a)_k). \tag{3.2}$$

The bound will follow quickly from the following lemma about dissipation.

LEMMA 3.3. *Let $0 < \alpha \leq 1/2$. Fix a time t. Suppose $b_j(t) = \gamma_j b_{j-1}(t)$ and that $b_k(t) < \gamma_k b_{k-1}(t)$, for $k > j$. Then for $j \geq 2$,*

$$(\mathcal{L}a)_j(t) - (\mathcal{L}a)_{j-1}(t) > \frac{\gamma_j}{2}\left[(\mathcal{L}a)_{j-1}(t) - (\mathcal{L}a)_{j-2}(t)\right].$$

Proof. For simplicity of expression, all expressions are evaluated at t. By direction calculation,

$$A := (\mathcal{L}a)_j - (\mathcal{L}a)_{j-1} = \sum_{n=0}^{j-2}(a_j - a_{j-1})2^{2\alpha n} + 2(a_j - a_{j-1})2^{2\alpha(j-1)} + 2^{(1+2\alpha)j}(1-2^{-(1+2\alpha)})\sum_{n=j+1}^{\infty}(a_j - a_n)2^{-n}. \tag{3.3}$$

Also, we have

$$\begin{aligned} B := \frac{\gamma_j}{2}[(\mathcal{L}a)_{j-1} - (\mathcal{L}a)_{j-2}] &= \sum_{n=0}^{j-3}\frac{\gamma_j}{2}(a_{j-1} - a_{j-2})2^{2\alpha n} + \gamma_j(a_{j-1} - a_{j-2})2^{2\alpha(j-2)} \\ &\quad + \gamma_j 2^{(1+2\alpha)(j-1)-1}(1-2^{-(1+2\alpha)})\sum_{n=j}^{\infty}(a_{j-1} - a_n)2^{-n} \\ &= \sum_{n=0}^{j-3}(a_j - a_{j-1})2^{2\alpha n} + 2(a_j - a_{j-1})2^{2\alpha(j-2)} \\ &\quad + \gamma_j 2^{(1+2\alpha)(j-1)-1}(1-2^{-(1+2\alpha)})\sum_{n=j}^{\infty}(a_{j-1} - a_n)2^{-n}, \end{aligned}$$

where we have used $b_k = \gamma_k b_{k-1}$ in the last equality. Subtracting the two expressions above we get

$$\begin{aligned} A - B &= (2 - 2^{-2\alpha})(a_j - a_{j-1})2^{2\alpha(j-1)} + \frac{\gamma_j}{4}(1-2^{-(1+2\alpha)})(a_j - a_{j-1})2^{2\alpha(j-1)} \\ &\quad + 2^{(1+2\alpha)j}(1-2^{-(1+2\alpha)})\sum_{n=j+1}^{\infty}(a_j - a_n)2^{-n} \\ &\quad - \gamma_j 2^{(1+2\alpha)(j-1)-1}(1-2^{-(1+2\alpha)})\sum_{n=j+1}^{\infty}(a_{j-1} - a_n)2^{-n} \\ &= (2 + \frac{\gamma_j}{2} - 2^{-2\alpha} - \gamma_j 2^{-(2+2\alpha)})(a_j - a_{j-1})2^{2\alpha(j-1)} \\ &\quad + \gamma_j(1-2^{-(1+2\alpha)})2^{(1+2\alpha)(j-1)-1}\sum_{n=j+1}^{\infty}(a_j - a_{j-1})2^{-n} \\ &\quad + (1-2^{-(1+2\alpha)})(1-\gamma_j 2^{-(2+2\alpha)})2^{(1+2\alpha)j}\sum_{n=j+1}^{\infty}(a_j - a_n)2^{-n}. \end{aligned}$$

Using that $b_{k+1} < \gamma_k b_k$ for $k \geq j$ and that γ_k is decreasing, we see that

$$a_j - a_n \geq \frac{\left(\frac{\gamma_j}{2}\right)^{n-j+1} - \frac{\gamma_j}{2}}{1 - \frac{\gamma_j}{2}}(a_j - a_{j-1}) \quad \text{for } n > j.$$

From this we can get $\sum_{n=j+1}^{\infty}(a_j-a_n)2^{-n}\geq -\frac{2\gamma_j}{4-\gamma_j}(a_j-a_{j-1})2^{-j}$. Inserting this into the above estimate we get that the above expression is bounded below by

$$\begin{aligned}&\left(2+\frac{\gamma_j}{2}-\frac{\gamma_j 2^{1+2\alpha}}{4-\gamma_j}(1-\gamma_j 2^{-(2+2\alpha)})\right)(1-2^{-(1+2\alpha)})(a_j-a_{j-1})2^{2\alpha(j-1)}\\ =&\frac{2(4-2^{2\alpha}\gamma_j)}{4-\gamma_j}(1-2^{-(1+2\alpha)})(a_j-a_{j-1})2^{2\alpha(j-1)}.\end{aligned}$$

Since $\gamma_j\leq 2$, we see that this expression is positive for $0<\alpha\leq 1/2$. The case for when $j=2$ is done analogously. This completes the proof. ∎

Combining this lemma with the proof from Lemma 3.1, we have the main result of this subsection.

THEOREM 3.2. *Let* $\{b_k\}_{k=1}^{\infty}$ *be a solution of* (3.2). *Suppose that* $\{b_k(0)\}_{k=1}^{\infty}\in\mathcal{I}$. *Then for all* $t\geq 0$, $\{b_k(t)\}_{k=1}^{\infty}\in\mathcal{I}$. *In particular,* $\sup_{k>0,t\geq 0}b_k(t)2^{-k/2}<\infty$.

Proof. First, we show $b_2(t)\leq 2b_1(t)$, for all t. It suffices to show that $(2b_1-b_2)'(t_1)\geq 0$ at anytime t_1 such that $b_2(t_1)=2b_1(t_1)$ while $b_k(t_1)<\gamma_k b_{k-1}(t_1)$ for $k>2$. This is true, since

$$(2b_1-b_2)'(t_1)=-4b_1^2(t)+b_2^2(t)-4[(\mathcal{L}a)_1-(\mathcal{L}a)_0]+4[(\mathcal{L}a)_2-(\mathcal{L}a)_1]>0$$

by the previous Lemma. Now we want to show $b_k(t)<\gamma_k b_{k-1}(t)$ for all t. Suppose t_k is the first time for which $b_k(t_k)=\gamma_k b_{k-1}(t_k)$. Then $b_j(t_k)<\gamma_j b_{j-1}(t_k)$ for $j\neq k$. By direct computation we get

$$\begin{aligned}(\gamma_k b_{k-1}-b_k)'(t_k)&=\gamma_k(-b_{k-1}^2(t)+2b_{k-2}^2)+b_k^2(t)-2b_{k-1}^2(t)\\ &\quad-\gamma_k 2^{k-1}[(\mathcal{L}a)_{k-1}-(\mathcal{L}a)_{k-2}]+2^k[(\mathcal{L}a)_k-(\mathcal{L}a)_{k-1}]\\ &\geq\left[\gamma_k\left(\frac{2}{\gamma_{k-1}^2}-1\right)+\gamma_k^2-2\right]b_{k-1}^2(t_k)=0,\end{aligned}$$

where we have used the previous lemma in the inequality. ∎

4. Regularity for $1/4<\alpha\leq 1/2$ and blow-up for $\alpha<1/4$

The previous theorem will allow us to adapt the argument of Theorem 2.3 and prove blow-up for (1.7) for $0<\alpha<1/4$.

THEOREM 4.1. *Let* $0<\alpha<1/4$ *Suppose* $\{a_k^0\}$ *is increasing in* k *and is non-negative. Then there exists initial datum in* X^s, $s>1$, *such that solutions blow-up in finite time.*

Proof. As before we let

$$J(t)=\sum_{k=0}^{\infty}(a_\infty(t)-a_k(t))2^{k\delta}.$$

for $0<\delta<1-4\alpha$ where $a_\infty(t)=\lim_{k\to\infty}a_k(t)$. Then by computing and by using Lemma 2.3, we get

$$\begin{aligned}\frac{d}{dt}J(t)&=\sum_{k=1}^{\infty}(a_k(t)-a_{k-1}(t))^2 2^{k(\delta+1)}-\sum_{k=1}^{\infty}((\mathcal{L}a)_\infty-(\mathcal{L}a)_k)2^{k\delta}\\ &\geq C_0\sum_{k=1}^{\infty}(a_\infty(t)-a_k(t))^2 2^{k(\delta+1)}-\sum_{k=1}^{\infty}((\mathcal{L}a)_\infty-(\mathcal{L}a)_k)2^{k\delta},\end{aligned}$$

where

$$(\mathcal{L}a)_\infty = \lim_{k\to\infty}(\mathcal{L}a)_k = \sum_{n=0}^{\infty}(a_\infty - a_n)2^{2\alpha n}.$$

Now,

$$\begin{aligned}&(\mathcal{L}a)_\infty - (\mathcal{L}a)_k\\ &=\sum_{n=0}^{k-1}(a_\infty - a_k)2^{2\alpha n} + \sum_{n=k}^{\infty}(a_\infty - a_n)2^{2\alpha n} - \sum_{n=k+1}^{\infty}(a_k - a_n)2^{(1+2\alpha)k}2^{-n}.\end{aligned}$$

We analyze each of the above three terms separately. We note that

$$\sum_{n=0}^{k-1}(a_\infty - a_k)2^{2\alpha n} \le C(a_\infty - a_k)2^{2\alpha k}.$$

By Theorem 3.2,

$$\sum_{n=k}^{\infty}(a_\infty - a_n)2^{2\alpha n} \le C\sum_{n=k}^{\infty}2^{(2\alpha-1/2)n} \le C2^{(2\alpha-1/2)(k-1)}.$$

For the last term we have

$$\sum_{n=k+1}^{\infty}(a_n - a_k)2^{(1+2\alpha)k}2^{-n} \le (a_\infty - a_k)2^{(1+2\alpha)k}\sum_{n=k+1}^{\infty}2^{-n} \le C(a_\infty - a_k)2^{2\alpha k}.$$

Then using $\alpha < 1/4$ and $0 < \delta < 1 - 4\alpha$,

$$\sum_{k=1}^{\infty}((\mathcal{L}a)_\infty - (\mathcal{L}a)_k)2^{k\delta} \le C_1 + C_2\sum_{k=1}^{\infty}(a_\infty - a_k)2^{(2\alpha+\delta)k}.$$

For every $\epsilon > 0$, by an application of Hölder's inequality and using that $\alpha < 1/4$,

$$\sum_{k=1}^{\infty}(a_\infty - a_k)2^{(2\alpha+\delta)k} \le \epsilon\sum_{k=1}^{\infty}(a_\infty - a_k)^2 2^{(\delta+1)k} + C_\epsilon\|a\|_{\ell^\infty}.$$

Given these bounds and following the proof of Theorem 2.3 we can show

$$J'(t) \ge C(\delta)J(t)^2 - C(1 + \|a\|_{\ell^\infty}).$$

By choosing initial data appropriately, the inequality above leads to blow-up. □

Now, we will move towards proving regularity for $1/4 < \alpha < 1/2$.

THEOREM 4.2. *Let $1/4 < \alpha < 1/2$. Suppose we have the same hypotheses as in Theorem 3.2. Then solutions exist for all time, in particular, for $s > 1$*

$$\sup_{t>0}\|a(t)\|_{X^s} < \infty.$$

Proof. After a computation,

$$(b_{k,s})'(t) = -b_k \cdot b_{k,s}(t) + 2^s b_{k-1} \cdot b_{k-1,s}(t) - ((\mathcal{L}a)_k - (\mathcal{L}a)_{k-1})(t) 2^{sk}. \qquad (4.1)$$

From Theorem 3.2, $b_{k-1}(t) \leq C_1 2^{k/2}$ for all $t \in [t_0, t_1]$ where the constant C_1 is independent of the time interval. Then there exists K' such that if $k \geq K'$ and $b_{k,s}(t) > c_s \|a\|_{X^s}(t)$ where c_s is the constant from Lemma 2.1,

$$(b_{k,s})'(t) \leq b_{k,s} \left(C_1 2^{k/2+s} - C(\alpha)(2^{2\alpha k} - 2^{2\alpha}) \right) < 0.$$

In the above inequality, we use $1/4 < \alpha < 1/2$. This implies that for any $T > 0$,

$$\lim_{t \to T, k \to \infty} b_{k,s}(t) \neq \infty.$$

Therefore, the solution exists globally in time. □

REMARK 4.1. It is unclear whether the hypothesis that $\{b_k^0\}_{k=1}^{\infty} \in \mathfrak{I}$ can be weakened as it is crucial to the proof of the a priori bound. We conjecture that any reasonable initial data will eventually satisfy such a condition at least for large enough indexes.

Summary: We have shown that when $\alpha > 1/4$, under mild assumptions on the initial data, it is possible to have global solutions: the dissipation terms win over the nonlinearity. In order to have blow-up, energy initially present in the lower modes must reach the higher modes. By making use of a priori Hölder-type control and an estimate on the dissipative terms, we have shown that such an energy transfer must stop at some time.

It is important to note that our conditions on the initial data are not a condition of "smallness" with regard to some norm. Our results of global regularity can hold for large initial data. Also, one can easily find initial data for which our model blows up for $0 < \alpha < 1/4$ and is regular for $\alpha > 1/4$.

Acknowledgements. This work was completed under the guidance of Prof. Alex Kiselev and with support from NSF-DMS grants 1412023 and 1453199. The author also wishes to thank Vlad Vicol and Xiaoqian Xu for useful discussions and Hongjie Dong for useful comments on an earlier version.

REFERENCES

[1] D. Barbato, F. Morandin, and M. Romito, *Smooth solutions for the dyadic model*, Nonlinearity, 24:3083–3097, 2011.

[2] A. Cheskidov, S. Friedlander, and N. Pavlovic, *An inviscid dyadic model of turbulence: the fixed point and Onsager's conjecture*, J. Math. Phys., 48:065503, 2007.

[3] P. Constantin, A. Tarfulea, and V. Vicol, *Long time dynamics of forced critical SQG*, Comm. Math. Phys., 335(1):93–141, 2015.

[4] P. Constantin and J. Wu, *Regularity of Hölder continuous solutions of the supercritical quasi-geostrophic equation*, Ann. Inst. H. Poincare Anal. Non Lineaire, 25(6):1103–1110, 2008.

[5] A. Cordoba and D. Cordoba, *A maximum principle applied to quasi-geostrophic equations*, Comm. Math. Phys., 249:511–528, 2004.

[6] A. Cordoba, D. Cordoba, and M. Fontelos, *Formation of singularities for a transport equation with nonlocal velocity*, Ann. of Math., 162(2):1377–1389, 2005.

[7] T. Do, *On a 1D transport equation with nonlocal velocity and supercritical dissipation*, J. Diff. Eqs., 256(9):3166–3178, 2014.

[8] H. Dong, *Well-posedness for a transport equation with nonlocal velocity*, J. Funct. Anal., 255:3070–3097, 2008.

[9] H. Dong and N. Pavlovic, *Regularity criteria for the dissipative quasi-geostrophic in Hölder spaces*, Comm. Math. Phys., 290:801–812, 2009.
[10] N. Katz and N. Pavlovic, *Finite time blow-up for a dyadic model of the Euler equations*, Trans of AMS, 357:695–708, 2005.
[11] A. Kiselev, *Regularity and blow up for active scalars*, Math. Model. Math. Phenom., 5(4):225–255, 2010.
[12] A. Kiselev and A. Zlatos, *On discrete models of the Euler equation*, IMRN., 38:2315–2339, 2005.
[13] D. Li and J. Rodrigo, *Blow-up of solutions for a 1d transport equation with nonlocal velocity and supercritical dissapation*, Adv. Math., 217:2563–2568, 2008.
[14] A. Majda and A. Bertozzi, *Vorticity and Incompressible Flow*, Cambrige University Press, 2002.
[15] L. Silvestre and V. Vicol, *On a transport equation with nonlocal drift*, Trans. Amer. Math. Soc., 368(9): 6159–6188, 2016.
[16] H. Yang and S. Goodchild, *Local well-posedness of a nonlocal burgers equation*, Involve, 9(1):67–82, 2016.