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A TRAFFIC FLOW MODEL WITH NON-SMOOTH METRIC INTERACTION: WELL-POSEDNESS AND MICRO-MACRO LIMIT*

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Abstract. We prove existence and uniqueness of solutions to a transport equation modelling vehicular traffic in which the velocity field depends non-locally on the downstream traffic density via a discontinuous anisotropic kernel. The result is obtained recasting the problem in the space of probability measures equipped with the ∞-Wasserstein distance. We also show convergence of solutions of a finite dimensional system, which provide a particle method to approximate the solutions to the original problem.

Keywords. Transport equations, non-local velocity, Wasserstein distance, macroscopic traffic flow models, micro-macro limits.

AMS subject classifications. Primary: 35F25, 35L65; Secondary: 65M12, 90B20.

1. Introduction

In this paper, we are interested in studying the macroscopic traffic flow model introduced in [7] from the point of view of measure transport equations in Wasserstein spaces.

Transport equations with non-local velocities have drawn a growing attention in the mathematical community, starting from the Vlasov equation and other models in kinetic theory, see e.g. [9,16,33]. In this context, *non-local* means that the velocity at a given point of the space depends not only on the density at that point, but on the density in a whole neighborhood. The first general results of existence and uniqueness for such equations are given by Ambrosio–Ganbo [3]. There, the authors show that Wasserstein distances are key tools to deal with these equations, since vector fields resulting from non-local interactions are Lipschitz continuous with respect to such distances. Several extensions have been proposed since then, including definition of gradient flows [4], numerical schemes [28,30], generalizations to domains with boundary [18] and to transport equations with sources [27,29].

Non-local conservation laws have been introduced recently to model a variety of evolution dynamics: besides road traffic models [7,19,21,23,32] and crowd motion models [12,14,25,26,34], they are used to describe granular flows [1], sedimenatation [5], conveyor belts [20], and aggregation phenomena [22].

As far as road traffic is concerned, an advantage in considering non-local mean velocity depending on a weighted mean of the downstream traffic density is represented by the consequent finite acceleration, whose unboundedness is one of the drawbacks of Lighthill–Whitham–Richards (LWR) [24,31] model and other classical macroscopic models, which allow for speed jumps. This limits their application in connection with consumption and pollution models, which heavily rely on acceleration estimation (see, for example, comments in [8]). The study performed in [7,19] also shows the impact of

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the monotonicity and the support location of the kernel in the appearance of oscillations of solutions. This may be of interest for applications to connected vehicles and vehicle-to-vehicle communication [6, 37], giving an insight of how this information should be used to regularize traffic flow. On the other side, numerical experiments performed in [19] show that the solution of the non-local equation converges to the solution of the classical LWR model as the support of the convolution kernel reduces to a point.

In our case, the evolution equation for the density $\rho=\rho(t,x)$ of cars on a (infinite) road is given by the following transport PDE:

$$\begin{cases} \partial_t\rho+\partial_x\left(\rho v\left(\int_x^{x+\eta}\rho(t,y)w(y-x)\,dy\right)\right)=0, & x\in\mathbb{R},\ t>0,\\ \rho(0,x)=\rho_0(x) & x\in\mathbb{R}, \end{cases} \tag{1.1}$$

where v is the mean traffic velocity, $\eta>0$ a given parameter, and $w:[0,\eta]\to\mathbb{R}^+$ is a non-increasing Lipschitz weight with $\int_0^\eta w(x)\,dx=1$. This is intended to model the fact that drivers adapt their velocity depending on the downstream traffic condition, eventually giving more attention to what happens close to them than to cars far beyond. In this respect we speak of *metric* interaction, opposite to *topological* interaction which take into account the ordering of vehicles: in this case the influence of preceding vehicles takes into account the presence of other vehicles in between [14, Section 1.1.1.7].

Solutions of ȇ-cauchy will be defined in the space $\mathcal{M}(\mathbb{R})$ of non-negative measures equipped with the Wasserstein distance, and they are to be intended in the weak sense

DEFINITION 1.1. *A measure $\mu\in C^0([0,T];\mathcal{M}(\mathbb{R}))$ is a weak solution of ȇ-cauchy if for all $\varphi\in C_c^\infty((0,T)\times\mathbb{R})$ it holds that*

$$\int_0^T\int_{\mathbb{R}}\left(\partial_t\varphi+\partial_x\phi v\left(\int_x^{x+\eta}w(y-x)\,d\mu(t,y)\right)\right)d\mu(t,x)=0.$$

The main result of the paper guarantees existence and uniqueness of weak solutions.

THEOREM 1.1. *Let $v:[0,1]\to\mathbb{R}^+$ be a Lipschitz and non-increasing function with $v(1)=0$ and $\rho_0\in L^\infty\cap\mathcal{P}_c^{ac}(\mathbb{R};[0,1])$. Then the Cauchy problem ȇ-cauchy admits a unique weak solution for all $t>0$, which satisfies*

$$0\le\rho(t,x)\le\sup\{\rho_0\}\qquad \text{for a.e. } x\in\mathbb{R},t>0.$$

Above, $\mathcal{P}_c^{ac}$ denotes the set of absolutely continuous measures with respect to the Lebesgue measure and with compact support. The proof will be given in Section 4.2. Note that, unlike [7,19], the proof of uniqueness of solutions does not rely on any entropy condition. On the other side, we must restrict the problem to initial data with compact support.

The proof of Theorem 1.1 will be decomposed in the following steps. We will first prove existence and uniqueness of the solution of ȇ-cauchy for small times in Proposition 3.1, under mild hypotheses. Moreover, in Section 4, we prove that if v and w are non-increasing, then we have existence of the solution for all times, together with the maximum principle $\rho(t,x)\le\sup\{\rho_0\}$. To this goal, we introduce a finite-dimensional approximation of densities and prove two results: on one side, a discrete version of the maximal principle for the approximate solution; on the other side, convergence of such approximate solution to the solution of ȇ-cauchy. These two results prove Theorem 1.1. Moreover, the finite-dimensional approximations also provide a particle type numerical scheme to compute solutions of e̊-cauchy. In this perspective, this study extends to non-local equations previous results on micro-to-macro limits for the classical LWR model [13,15].

Observe that the result is based on the study of e̊-cauchy in a more general setting, with v not necessarily decreasing. Assuming that v is Lipschitz and w is a (possibly discontinuous) BV interaction kernel of bounded variation, we prove in Proposition 3.1 existence and uniqueness of the solution of e̊-cauchy for small times. It is interesting to observe that the density can blow-up in a finite time T, leading to non-existence of the solution for times larger than T. Such phenomenon cannot appear for more smooth interaction kernels w (see Proposition 2.10 and Remark 3.2).

The structure of the article is the following. A short review on Wasserstein distances and general transport equations properties is given in Section 2. Transport equations with BV interaction kernels are studied in Section 3, where we prove existence and uniqueness of solutions locally in time in Proposition 3.1. In Section 4, we also define a particle approximation for the density, that provides a finite-dimensional numerical scheme for the solution of the transport PDE, and we prove convergence of the micro-macro limit.

2. Wasserstein distances and transport equations

In this section, we recall the main definitions related to Wasserstein distances, and in particular the definition and properties of the ∞-Wasserstein distance. We then recall results about transport equations with non-local velocities, in the case of smooth interaction kernels. For more details, see the monographs [35,36] and the articles [27–29].

We consider non-negative measures with a given mass $m>0$, which is conserved by the solutions of e̊-cauchy. Therefore, without loss of generality, in the following we will deal with probability measures, i.e. $m=1$.

We denote by $\mathcal{P}(\mathbb{R}^d)$ the set of probability measures on $\mathbb{R}^d$ and with $\mathcal{P}_c(\mathbb{R}^d)$ the subset of probability measures with compact support. We also denote with $\mathcal{P}^{ac}(\mathbb{R}^d)$ the subset of probability measures that are absolutely continuous with respect to the Lebesgue measure, and we identify the measure with its density with respect to the Lebesgue measure, e.g. by writing both $\int f(x)\,d\rho(x)$ and $\int f(x)\rho(x)\,dx$ for $\rho\in\mathcal{P}^{ac}(\mathbb{R}^d)$. We use the letters μ, ν, etc. for general measures in $\mathcal{P}(\mathbb{R}^d)$, keeping the notation ρ, ρ', etc. for measures in $\mathcal{P}^{ac}(\mathbb{R}^d)$. When not specified, we consider $\mathcal{P}(\mathbb{R}^d)$, $\mathcal{P}_c(\mathbb{R}^d)$, $\mathcal{P}^{ac}(\mathbb{R}^d)$, and $\mathcal{P}_c^{ac}(\mathbb{R}^d):=\mathcal{P}_c(\mathbb{R}^d)\cap\mathcal{P}^{ac}(\mathbb{R}^d)$ endowed with the ∞-Wasserstein distance, defined below.

Given a probability measure π on $\mathbb{R}^d\times\mathbb{R}^d$, one can interpret π as a path to transfer a probability measure μ on $\mathbb{R}^d$ to another probability measure ν on $\mathbb{R}^d$ as follows: each infinitesimal mass on a location x is sent to a location y with a probability given by $\pi(x,y)$. Formally, μ is sent to ν if the following property holds:

$$\int_{\mathbb{R}^d} d\pi(x,\cdot)=d\mu(x), \qquad \int_{\mathbb{R}^d} d\pi(\cdot,y)=d\nu(y),$$

or, equivantly, for all $f,g\in C_c^\infty(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}(f(x)+g(y))\,d\pi(x,y)=\int_{\mathbb{R}^d} f(x)\,d\mu(x)+\int_{\mathbb{R}^d} g(y)\,d\nu(y).$$

In this case, π is called a transference plan from μ to ν. We denote by $\Pi(\mu,\nu)$ the set of such transference plans.

Fix now $p\in[1,+\infty)$. One can define a cost for π as follows:

$$J[\pi]:=\int_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|^p\,d\pi(x,y)$$

and look for a minimizer of J in $\Pi(\mu,\nu)$. Such problem is called the Monge–Kantorovich problem. A minimizer of J in $\Pi(\mu,\nu)$ always exists, see [35]. A natural space on which J is finite is the space of Borel probability measures with finite p-moment, that is

$$\mathcal{P}_p(\mathbb{R}^d):=\left\{\mu\in\mathcal{P}(\mathbb{R}^d)\mid\int|x|^p\,d\mu(x)<\infty\right\}. \tag{2.1}$$

The p-Wasserstein distance is defined on $\mathcal{P}_p(\mathbb{R}^n)\times\mathcal{P}_p(\mathbb{R}^n)$ as

$$W_p(\mu,\nu):=\min_{\pi\in\Pi(\mu,\nu)}J[\pi]^{1/p}.$$

It is indeed a distance on $\mathcal{P}_p(\mathbb{R}^d)$, see [35].

REMARK 2.1. The Wasserstein distances can be interpreted in terms of distance between two densities of particles or of vehicles. Consider a set of N initial positions for particles of mass $\frac{1}{N}$ and a set of N final positions. A transference plan π is then a choice of sending each initial position to a final one, via a bijection. The corresponding cost is then the cost of the transportation, as the sum of the pth power of distances. The Wasserstein distance corresponds to the minimizer of such cost.

The continuous version of the Wasserstein distance is then recovered when considering the number of particles growing to infinity. Such identification will be fully exploited in Section 4, by considering finite-dimensional approximation of densities.

Several topological properties are of interest for the space $\mathcal{P}_p(\mathbb{R}^d)$ endowed with the Wasserstein distance W_p, see [35,36]. For future use, here we recall the following.

PROPOSITION 2.1. *The Wasserstein distance metrizes weak convergence in $\mathcal{P}_p(\mathbb{R}^d)$, i.e.*

$$W_p(\mu_n,\mu)\to 0$$

if and only if

$$\mu_n\rightharpoonup\mu\qquad\text{and}\qquad\lim_{R\to+\infty}\limsup_{n\to\infty}\int_{|x|>R}|x|^p\,d\mu_n(x)=0.$$

In particular, $W_p(\mu_n,\mu)\to 0$ if and only if $\mu_n\rightharpoonup\mu$ in $\mathcal{P}(K)$ with K compact.

PROPOSITION 2.2. *The space $\mathcal{P}_p(\mathbb{R}^d)$ endowed with the Wasserstein distance W_p is complete.*

PROPOSITION 2.3. *The Wasserstein distances are ordered, i.e. $p\le q$ implies*

$$W_p(\mu,\nu)\le W_q(\mu,\nu).$$

We finally recall the Kantorovich–Rubinstein duality formula for the 1-Wasserstein distance, see e.g. [35, Ch. 1].

PROPOSITION 2.4. *Let $\mu,\nu\in\mathcal{P}_1(\mathbb{R}^d)$. Then it holds that*

$$W_1(\mu,\nu)=\sup\left\{\int f(x)\,d(\mu-\nu)(x)\mid\mathrm{Lip}(f)\le 1\right\}. \tag{2.2}$$

2.1. The ∞-Wassersein distance. In this section, we recall the definition of the ∞-Wasserstein distance and prove some useful properties. We remark that the use of the ∞-Wasserstein distance allows to recover the necessary estimates for the convolution $\int \rho(y)w(y-x)\,dy$ even for BV kernels w, see e.g. Proposition 3.3 below.

Given two probability measures μ,ν and the space of transference plans $\Pi(\mu,\nu)$ with marginal probabilities μ,ν, we denote with $C_\infty(\pi)$ the following cost of a transference plan $\pi\in\Pi(\mu,\nu)$:

$$C_\infty(\pi):=\{\pi-\operatorname{ess\,sup}(|x-y|)\}.$$

The ∞-Wasserstein distance is then defined as

$$W_\infty(\mu,\nu):=\inf\{C_\infty(\pi)\mid \pi\in\Pi(\mu,\nu)\}.$$

We first recall the existence of an optimal transference plan for probability measures with compact support, see [11, Prop. 2.1]. Observe that, in analogy with e̊-moment, we have $\mathcal{P}_\infty(\mathbb{R}^d)=\mathcal{P}_c(\mathbb{R}^d)$.

Proposition 2.5. *Let $\mu,\nu\in\mathcal{P}_c(\mathbb{R}^d)$. Then there exists $\pi_*\in\Pi(\mu,\nu)$ realizing W_∞, i.e.*

$$W_\infty(\mu,\nu)=C_\infty(\pi_*).$$

Remark 2.2. Similarly to the standard Wasserstein distance, the ∞-Wasserstein distance can be interpreted in terms of distance between two densities of particles or vehicles. Given N initial positions for particles of mass $\frac{1}{N}$ and a set of N final positions, a transference plan π is the choice of sending each initial position to a final one via a bijection. The corresponding cost $C_\infty(\pi)$ is the maximal distance between an initial and a final position, hence the ∞-Wasserstein distance is the minimizer of such cost among all transference plans.

Such interpretation is even easier in the case of 1D models, such as for densities of vehicles. Consider a platoon of N vehicles, with initial and final *ordered* positions $x_1,\dots,x_N$ and $y_1,\dots,y_N$, respectively. Then, an optimal transference plan π_* preserves the order, i.e. it sends x_i to y_i (see, e.g., [35]). As a consequence, the ∞-Wasserstein distance coincides with $\max_i|x_i-y_i|$.

It is easy to prove that ordering of the Wasserstein distances is preserved even with the ∞-Wasserstein distance.

Proposition 2.6. *For any $p\in[1,+\infty)$ it holds that*

$$W_p(\mu,\nu)\le W_\infty(\mu,\nu).$$

Proof. Let π be a transference plan realizing $W_\infty(\mu,\nu)$. Then it holds that

$$W_p^p(\mu,\nu)\le\int|x-y|^p\,d\pi(x,y)\le\int C_\infty(\pi)^p\,d\pi(x,y)=W_\infty^p(\mu,\nu).$$

□

We now prove lower semicontinuity of W_∞ with respect to the weak convergence of measures in a compact space.

Proposition 2.7. *Let K be a compact set in $\mathbb{R}^d$, and μ_n a sequence in $\mathcal{P}(K)$. If $\mu_n\rightharpoonup\mu$ and $\nu\in\mathcal{P}_c(\mathbb{R}^d)$, then*

$$W_\infty(\nu,\mu)\le\liminf_{n\to\infty}W_\infty(\nu,\mu_n).$$

Proof. First observe that $\mathrm{supp}(\mu)\subset K$. Since $\mathrm{supp}(\nu)$ is compact, eventually replacing K by $K\cup\mathrm{supp}(\nu)$, one has that all measures have compact support in K.

We prove the result by passing to a minimizing subsequence (that we do not relabel) and prove that $W_\infty(\nu,\mu)\leq\lim_{n\to\infty}W_\infty(\nu,\mu_n)$. By Proposition 2.5, for each n there exists a transference plan π_n realizing $W_\infty(\nu,\mu_n)$. Since $\pi_n\in\mathcal{P}(K\times K)$ and $K\times K$ is compact, then Prokhorov's theorem ensures the existence of a subsequence (that we do not relabel) for which it holds that $\pi_n\rightharpoonup\pi_*$ for some $\pi_*\in\mathcal{P}(K\times K)$. It is easy to prove that $\pi_*\in\Pi(\nu,\mu)$. Since C_∞ is lower semicontinuous with respect to the weak topology (see e.g. [11, Lemma 2.3]), then $C_\infty(\pi_*)\leq\lim_n C_\infty(\pi_n)$. By recalling that $W_\infty(\nu,\mu)\leq C_\infty(\pi_*)$, the result is proved. □

We now prove results related to the topology of $\mathcal{P}_c(\mathbb{R}^d)$ endowed with the ∞-Wasserstein distance.

Proposition 2.8. *The space $\mathcal{P}_c(\mathbb{R}^d)$ is complete with respect to the metric W_∞. Moreover, let μ_n a sequence in $\mathcal{P}_c(\mathbb{R}^d)$. If $W_\infty(\mu_n,\mu)\to 0$ for some $\mu\in\mathcal{P}_c(\mathbb{R}^d)$, then $\mu_n\rightharpoonup\mu$.*

Proof. Let μ_n be a Cauchy sequence in $\mathcal{P}_c(\mathbb{R}^d)$ endowed with the metric W_∞. For a given $\varepsilon>0$, consider N such that for all $n\geq N$, $k>0$ it holds that $W_\infty(\mu_n,\mu_{n+k})<\varepsilon$. In particular it holds that $W_\infty(\mu_N,\mu_n)<\varepsilon$, which in turn implies that $\mathrm{supp}(\mu_n)\subset\cup\{B(x,\varepsilon)\mid x\in\mathrm{supp}(\mu_N)\}$ for all $n\geq N$. Since such set is bounded, and the supports of μ_k for $k<N$ are bounded too, then there exists a compact set K containing the support of all μ_n.

Since the μ_n have uniformly bounded support, then they also have uniformly bounded pth moment for each $p\in[1,\infty)$. Recall that each space $\mathcal{P}_p(\mathbb{R}^d)$ is complete with respect to W_p, see Proposition 2.2. Observe that it holds that $W_p(\mu_n,\mu_m)\leq W_\infty(\mu_n,\mu_m)$, then μ_n is a Cauchy sequence in $\mathcal{P}_p(\mathbb{R}^d)$, hence there exists μ_* for which $W_p(\mu_n,\mu_*)\to 0$ for all $p\in[1,\infty)$. Then, by Proposition 2.1, we have that $\mu_n\rightharpoonup\mu_*$.

We are now left to prove that $W_\infty(\mu_n,\mu_*)\to 0$, that is a direct consequence of lower semicontinuity of W_∞ with respect to the weak topology of measures, proved in Proposition 2.7. □

Remark 2.3. It is false that weak convergence of measures implies convergence with respect to the metric W_∞, even in $\mathcal{P}(K)$ with K compact. For example, consider the following sequence $\mu_n:=\frac{n-1}{n}\delta_0+\frac{1}{n}\delta_1$, that weakly converges to $\mu_*:=\delta_0$, but for which it holds that $W_\infty(\mu_n,\mu_*)=1$ for all n. This is in sharp contrast with the W_p metric with $p\in[1,+\infty)$, as recalled in Proposition 2.1.

2.2. Transport equations with smooth non-local interactions. In this section, we study the transport equation with non-local interactions, i.e. the following Cauchy problem:

$$\begin{cases}\partial_t\mu+\nabla_x\cdot(V[\mu]\mu)=0, & x\in\mathbb{R}^d,\ t>0,\\ \mu(0,x)=\mu_0(x), & x\in\mathbb{R}^d,\end{cases}\tag{2.3}$$

where V is a function that associates to each measure μ a vector field $V[\mu]$. For the simplest case of V actually not depending on μ, the solution of e̊-generale is given by the push-forward of the flow of V. We recall its definition here.

Definition 2.1. *Given a Borel map $\gamma:\mathbb{R}^d\to\mathbb{R}^d$, the push-forward of a probability measure $\mu\in\mathcal{P}(\mathbb{R}^d)$ is defined by*

$$\gamma\#\mu(A):=\mu(\gamma^{-1}(A))$$

for every subset A such that $\gamma^{-1}(A)$ is μ-measurable.

If $v:\mathbb{R}^d\times[0,T]\to\mathbb{R}^d$ is a vector field uniformly Lipschitz with respect to the space variable and continuous with respect to time, then we denote by $\gamma_t=\phi_t^v$ the flow generated by v, where $\phi_t^v(x_0)$ is the unique solution at time t of

$$\begin{cases}\dot{x}=v(t,x),\\ x(0)=x_0.\end{cases}$$

Then, we have the following result, see [35, Thm. 5.34] and [28][1].

Proposition 2.9. *Let $v:\mathbb{R}^d\times[0,T]\to\mathbb{R}^d$ be a uniformly Lipschitz vector field. Then the equation*

$$\begin{cases}\partial_t\mu+\nabla_x\cdot(v\mu)=0, & x\in\mathbb{R}^d,\ t>0,\\ \mu(0,x)=\mu_0(x), & x\in\mathbb{R}^d,\end{cases}$$

with $\mu_0\in\mathcal{P}_c(\mathbb{R}^d)$ admits a unique solution $\mu\in C^0([0,T];\mathcal{P}_c(\mathbb{R}^d))$. Such solution satisfies $\mu(t)=\phi_t^v\#\mu_0$. In particular, if $\mu_0\in\mathcal{P}^{ac}(\mathbb{R}^d)$, then $\mu(t)\in\mathcal{P}^{ac}(\mathbb{R}^d)$ for all times $t>0$.

Moreover, let v,w be two Lipschitz vector fields with Lipschitz constant L and bounded, and ϕ_t^v,ϕ_t^w the corresponding flows. Let $\mu,\nu\in\mathcal{P}_c(\mathbb{R}^d)$ be two probability measures. Then, for $p=[1,+\infty]$ it holds that

$$W_p(\phi_t^v\#\mu,\phi_t^w\#\nu)\le e^{\frac{p+1}{p}Lt}W_p(\mu,\nu)+\frac{e^{Lt/p}(e^{Lt}-1)}{L}\sup_{t\in[0,T]}\|v(.,t)-w(.,t)\|_{C^0}. \quad (2.4)$$

For V actually depending on μ, we have the following theorem, generalizing the results in [3, 27–29].

Theorem 2.1. *Let $p\in[1,+\infty]$. Let the function*

$$V[\mu]:\begin{cases}\mathcal{P}_c(\mathbb{R}) \to (C^1\cap L^\infty)(\mathbb{R}^d;\mathbb{R}^d)\\ \quad\mu \mapsto V[\mu]\end{cases}$$

satisfy

- *$V[\mu]$ is uniformly Lipschitz and uniformly bounded, i.e. there exist L, M not depending on μ, such that for all $\mu\in\mathcal{P}_c(\mathbb{R}^d),x,y\in\mathbb{R}^d$,*

$$|V[\mu](x)-V[\mu](y)|\le L|x-y|,\qquad |V[\mu](x)|\le M.$$

- *V is a Lipschitz function, i.e. there exists K such that*

$$\|V[\mu]-V[\nu]\|_{C^0}\le KW_p(\mu,\nu).$$

Then the Cauchy problem ė-generale admits a unique solution $\mu\in C^0([0,T];\mathcal{P}_c(\mathbb{R}^d))$ for all times $T>0$. Moreover, if the initial data μ_0 satisfies $\mu_0\in\mathcal{P}^{ac}(\mathbb{R}^d)$, then $\mu(t)\in\mathcal{P}^{ac}(\mathbb{R}^d)$ for all times $t\in[0,T]$.

Finally, if μ,μ' are solutions of ė-generale with initial data μ_0,ν_0, respectively, then it holds that

$$W_p(\mu(t),\nu(t))\le e^{(4L+4K)t}W_p(\mu_0,\nu_0). \quad (2.5)$$

[1]The proof of ė-stime4 is given in [28] for $p<+\infty$, but it can be easily adapted to $p=+\infty$.

Proof. The proof is given for $p<+\infty$ in [28, Prop. 4 and Thm. 2] and in [27, 29]. The key estimate is è-stime4, that holds also for $p=+\infty$. Then, the original proofs can be easily adapted to $p=+\infty$. ∎

We now adapt such result to our setting, in which $d=1$ and $V[\mu](x):=v(\int_{\mathbb{R}} w(y-x)\,d\mu(y))$ with w Lipschitz.

PROPOSITION 2.10. *Let $v:\mathbb{R}\to\mathbb{R}$ be a Lipschitz and bounded function, and $w:\mathbb{R}\to\mathbb{R}$ be a Lipschitz function with bounded support. If $\mu_0\in\mathcal{P}_c(\mathbb{R})$, then the Cauchy problem*

$$\begin{cases}\partial_t\mu+\partial_x\left(\mu v\left(\int_{\mathbb{R}} w(y-x)\,d\mu(t,y)\right)\right)=0, & x\in\mathbb{R},\ t>0,\\ \mu(0,x)=\mu_0(x), & x\in\mathbb{R},\end{cases}\tag{2.6}$$

admits a unique solution $\mu\in C^0([0,T];\mathcal{P}_c(\mathbb{R}))$ for all times $T>0$.

Moreover, if μ,ν are solutions of è-cauchysmooth with initial data μ_0,ν_0 respectively, there holds

$$W_p(\mu(t),\nu(t))\le e^{8\mathrm{Lip}(v)\mathrm{Lip}(w)t}W_p(\mu_0,\nu_0),\tag{2.7}$$

for any $p\in[1,+\infty]$.

Proof. It is sufficient to prove that $V[\mu]$ defined by $V[\mu](x):=v(\int_{\mathbb{R}} w(z-x)\,d\mu(z))$ satisfies the hypotheses of Theorem 2.1. We have

$$\begin{aligned}|V[\mu](x)-V[\mu](y)|&\le \mathrm{Lip}(v)\int |w(z-x)-w(z-y)|\,d\mu(z)\le \mathrm{Lip}(v)\mathrm{Lip}(w)|x-y|,\\ |V[\mu](x)|&\le \sup(|v|),\\ |V[\mu](x)-V[\nu](x)|&\le \mathrm{Lip}(v)\left|\int w(z-x)\,d(\mu-\nu)(z)\right|\le \mathrm{Lip}(v)\mathrm{Lip}(w)W_1(\mu,\nu)\le\\ &\le \mathrm{Lip}(v)\mathrm{Lip}(w)W_p(\mu,\nu),\end{aligned}$$

where the last estimate is based on the Kantorovich–Rubinstein duality è-KR and ordering of Wasserstein distances in Propositions 2.3 and 2.6. By identifying L,K, we find è-contdip. ∎

3. Transport equations with BV interactions kernels

In this section, we study transport equations in one space dimension with non-local velocities given by interaction kernels of the form è-cauchy. Unlike Proposition 2.10, here we do not assume that the kernel interaction w is Lipschitz continuous, but only BV. This prevents to use the results in Section 2.2. Moreover, the interaction $\int w(z-x)\,d\mu(z)$ itself is not well-defined for general probability measures, but for measures that are absolutely continuous with respect to the Lebesgue measure, i.e. for $\mu\in\mathcal{P}^{ac}(\mathbb{R})$ only.

In particular, we will often deal with measures in the space $L^\infty\cap\mathcal{P}^{ac}_c(\mathbb{R})$, endowed with the W_∞ distance. As stated above, the space $\mathcal{P}_c(\mathbb{R})$ is complete with respect to the W_∞ distance, but this does not hold anymore for $L^\infty\cap\mathcal{P}^{ac}_c(\mathbb{R})$. Nevertheless, we have convergence if both the support and the L^∞ norm are uniformly bounded, as proved in the following proposition.

LEMMA 3.1. *Let $K\subset\mathbb{R}$ be a compact set, and $\mu_n\in L^\infty\cap\mathcal{P}^{ac}(K)$ be a sequence of measures with uniformly bounded L^∞ norm and weakly converging to μ. Then $\mu\in L^\infty\cap\mathcal{P}^{ac}(K)$ and $\|\mu\|_{L^\infty}\le\limsup_{n\to\infty}\|\mu_n\|_{L^\infty}$.*

Proof. It is clear that $\mu\in\mathcal{P}(K)$. We now prove that μ is absolutely continuous with respect to the Lebesgue measure λ, by proving that, for each set A that is λ-measurable with $\lambda(A)=0$, it holds that $\mu(A)=0$.

We first prove $\mu(A)=0$ in the case of A open. Fix $(a,b)\supset K$ and define $A_n:=\left\{x\in[a,b]\mid d(x,\mathbb{R}\setminus A)\geq\frac{1}{n}\right\}$ with $d(x,B)=\inf_{y\in B}|x-y|$. Observe that the A_n are compact, they satisfy $A_n\subset A_{n+1}$ and it holds that $\lim_n A_n=A$. Define a sequence of continuous functions f_n with support in $[a-1,b+1]$ satisfying $\chi_{A_n}\leq f_n\leq\chi_A$, increasing in the sense that $f_n\leq f_{n+1}$. This implies $f_n\rightharpoonup\chi_A$. Hence, by the monotone convergence theorem and weak convergence, we have

$$\mu(A)=\sup_{n\in\mathbb{N}}\int f_n\,d\mu\leq\sup_{n\in\mathbb{N}}\limsup_{k\to\infty}\int f_n\,d\mu_k\leq\sup_{n\in\mathbb{N}}\int f_n\,dM\lambda\leq M\lambda(A)=0 \quad (3.1)$$

with $M\geq\sup_{n\in\mathbb{N}}\|\mu_n\|_{L^\infty}$.

By regularity of finite measures (see [17]), the result holds for any λ-measurable set A, then $\mu\in\mathcal{P}^{ac}(\mathbb{R})$. The estimate $\|\mu\|_{L^\infty}\leq\limsup_{n\to\infty}\|\mu_n\|_{L^\infty}$ is again a consequence of ẻ-utile. ∎

The goal of this section is to prove the following result of existence and uniqueness for small times.

PROPOSITION 3.1. *Let the following hypothesis hold:*

> **(H):** *The function $v:[0,1]\to\mathbb{R}$ is Lipschitz and bounded. The interaction kernel w satisfies $w\in BV([\alpha,\beta];\mathbb{R}^+)$ for fixed $\alpha,\beta\in\mathbb{R}$, extended with zero in $\mathbb{R}\setminus[\alpha,\beta]$, and $\int_\alpha^\beta w(x)\,dx=1$. The initial density ρ_0 satisfies $\rho_0\in L^\infty\cap\mathcal{P}_c^{ac}(\mathbb{R})$.*

Then, there exists $T>0$ such that for all $t\in(0,T)$ there exists a unique weak solution $\rho\in C^0([0,t];L^\infty\cap\mathcal{P}_c^{ac}(\mathbb{R}))$ of the Cauchy problem

$$\begin{cases}\partial_t\rho(t,x)+\partial_x(V[\rho(t)](x)\rho(t,x))=0, & x\in\mathbb{R},\ t>0,\\ \rho(0,x)=\rho_0(x), & x\in\mathbb{R},\end{cases} \quad (3.2)$$

where

$$V[\rho(t)](x)=v\left(\int\rho(t,y)w(y-x)\,dy\right).$$

To prove this proposition, we first study in the following sections a set of useful technical lemmas related to BV kernels and the corresponding transport equations. The proof of Proposition 3.1 will be then given in Section 3.3.

3.1. Estimates for convolutions with BV kernels. In this section, we prove some useful estimates for BV functions and functions defined by convolutions with BV kernels. Estimates will be based on the Total Variation norm, for which we recall the definition for real functions below. For more details see, e.g., [2].

DEFINITION 3.1. *Let $f:[\alpha,\beta]\to\mathbb{R}$ be a real function. The total variation of f is*

$$TV(f):=\sup_{p_N\in\mathcal{P}}\sum_{i=1}^{N-1}|f(x_{i+1})-f(x_i)|,$$

where $p_N\in\mathcal{P}$ is a partition $\{x_1,x_2,\dots,x_N\}$ of the interval $[\alpha,\beta]$.
We denote with $BV([\alpha,\beta];\mathbb{R})$ the set of functions with bounded total variation.

For BV functions, the following properties hold.

LEMMA 3.2. *Let $f\in BV([\alpha,\beta];\mathbb{R})$. Let $[a,b]\subset[\alpha,\beta]$ and h such that $\alpha\le a-h\le b+h\le\beta$. Then it holds that*

$$\int_a^b |f(x)-f(x-h)|\,dx\le |h|TV(f).$$

Proof. See also [2, Remark 3.25]. Assume that f is non-decreasing and $h>0$. It holds that

$$\begin{aligned}
0&\le\int_a^b f(x)-f(x-h)\,dx\\
&=\int_a^b (f(x)-f(a-h))\,dx-\int_a^b (f(x-h)-f(a-h))\,dx\\
&=\int_a^b (f(x)-f(a-h))\,dx-\int_{a-h}^{b-h} (f(x)-f(a-h))\,dx\\
&\le\int_{b-h}^b (f(x)-f(a-h))\,dx\\
&\le\int_{b-h}^b (f(b)-f(a-h))\,dx=h(f(b)-f(a-h))\\
&\le hTV(f).
\end{aligned}$$

The proof for $h<0$ is identical.

Let $f=g-l$ with g,l non-decreasing and $TV(f)=TV(g)+TV(l)$. Then it holds that

$$\begin{aligned}
\int_a^b |f(x)-f(x-h)|\,dx&=\int_a^b |(g(x)-g(x-h))-(l(x)-l(x-h))|\,dx\\
&\le\int_a^b |g(x)-g(x-h)|\,dx+\int_a^b |l(x)-l(x-h)|\,dx\\
&\le|h|(TV(g)+TV(l))\\
&=|h|TV(f).
\end{aligned}$$

□

LEMMA 3.3. *Let $\rho,\rho'\in L^\infty\cap\mathcal{P}_c^{ac}(\mathbb{R})$ and $w\in BV([\alpha,\beta];\mathbb{R})$. Then it holds that*

$$\left|\int w(x)\,d(\rho(x)-\rho'(x))\right|\le W_\infty(\rho,\rho')TV(w)\min\{\|\rho\|_{L^\infty},\|\rho'\|_{L^\infty}\}. \tag{3.3}$$

Proof. Let $\pi\in\Pi(\rho,\rho')$ be a transference plan realizing $h:=W_\infty(\rho,\rho')$, that exists due to [11, Prop. 2.1]. Decompose $w=f-g$ with f,g non-decreasing functions via the Jordan decomposition on the interval $[\alpha,\beta]$. We have

$$\begin{aligned}
\int w(x)\,d(\rho(x)-\rho'(x))&=\int (f(x)-g(x))\,d\pi(x,y)-\int (f(y)-g(y))\,d\pi(x,y)\\
&=\int (f(x)-f(y))\,d\pi(x,y)+\int (g(y)-g(x))\,d\pi(x,y)\\
&\le\int (f(x)-f(x-h))\,d\pi(x,y)+\int (g(x+h)-g(x))\,d\pi(x,y)
\end{aligned}$$

$$
\begin{aligned}
&= \int (f(x)-f(x-h))\,d\rho(x) + \int (g(x+h)-g(x))\,d\rho(x) \\
&\le \|\rho\|_{L^\infty} \left(\int (f(x)-f(x-h))\,dx + \int (g(x+h)-g(x))\,dx \right) \\
&\le \|\rho\|_{L^\infty} h TV(w),
\end{aligned}
$$

where we used that points (x,y) in the support of π satisfy $|x-y|\le h$ except for a set of zero measure. By replacing w with $-w$, we have the absolute value on the left-hand side of ê-eq1. Since the estimate is symmetric with respect to ρ,ρ', one has the result. □

Remark 3.1. The main reason for which Lemma 3.3 holds only for measures ρ,ρ' that are absolutely continuous with respect to the Lebesgue measure is that one needs to make sense of the integral $\int w(x)\,d\rho(x)$ when w is a BV function. The proposition holds for real measures since we need to use the Jordan decomposition of the BV functions.

Proposition 3.2. *Let $w\in BV([\alpha,\beta];\mathbb{R})$ and $\rho\in L^\infty(\mathbb{R})$. Then the function $f(x):=\int \rho(y)w(y-x)\,dy$ is Lipschitz, with Lipschitz constant $L\le \|\rho\|_{L^\infty}TV(w)$.*

Proof. By using Lemma 3.2, we have

$$
\begin{aligned}
|f(x_1)-f(x_2)| &\le \int \rho(y)|w(y-x_1)-w(y-x_2)|\,dy \\
&\le \|\rho\|_{L^\infty} \int |w(y-x_1)-w(y-x_1+(x_1-x_2))|\,dy \\
&\le \|\rho\|_{L^\infty} |x_1-x_2| TV(w).
\end{aligned}
$$

□

Proposition 3.3. *Let $w,w'\in BV([\alpha,\beta];\mathbb{R})$ and $\rho,\rho'\in L^\infty\cap\mathcal{P}_c^{ac}(\mathbb{R})$. Then the functions*

$$
f(x):=\int \rho(y)w(y-x)\,dy, \qquad f'(x):=\int \rho'(y)w'(y-x)\,dy
$$

satisfy

$$
\|f-f'\|_{C^0} \le W_\infty(\rho,\rho')TV(w)\min\{\|\rho\|_{L^\infty},\|\rho'\|_{L^\infty}\} + \|\rho'\|_{L^\infty}\|w-w'\|_{L^1}.
$$

Proof. We have

$$
\begin{aligned}
|f(x)-f'(x)| &\le \left| \int \rho(y)w(y-x)-\rho'(y)w(y-x)\,dy \right| + \left| \int \rho'(y)w(y-x)-\rho'(y)w'(y-x)\,dy \right| \\
&\le W_\infty(\rho,\rho')TV(w)\min\{\|\rho\|_{L^\infty},\|\rho'\|_{L^\infty}\} + \|\rho'\|_{L^\infty}\|w-w'\|_{L^1},
\end{aligned}
$$

where we used Lemma 3.3 for the first integral and the L^1-L^∞ duality for the second integral. □

3.2. Solution of transport equations with time-dependent interactions. In this section, we give estimates for solutions of the transport equations in which the vector field is time-dependent, but not depending on the solution itself.

Proposition 3.4. *Let* **(H)** *hold. Fix $\bar\rho\in C^0([0,T];L^\infty\cap\mathcal{P}_c^{ac}(\mathbb{R}))$ and define the time-dependent vector field V_t as follows:*

$$
V_t(x):=v\left(\int \bar\rho(t,y)w(y-x)\,dy \right).
$$

Let $\rho(t) \in C^0([0,T]; L^\infty \cap \mathcal{P}_c^{ac}(\mathbb{R}))$ *be the unique solution of*

$$\partial_t \rho(t,x) + \partial_x (V_t(x)\rho(t,x)) = 0 \tag{3.4}$$

corresponding to a given initial datum $\rho_0 \in L^\infty \cap \mathcal{P}_c^{ac}(\mathbb{R})$. *Moreover, it holds that*

$$W_\infty(\rho(t,\cdot), \rho(t+s,\cdot)) \le s\|v\|_\infty, \tag{3.5}$$

and

$$e^{-Lt}\|\rho_0\|_{L^\infty} \le \|\rho(t,\cdot)\|_{L^\infty} \le e^{Lt}\|\rho_0\|_{L^\infty}, \tag{3.6}$$

with $L := Lip(v) TV(w) \sup_{s\in[0,t]}\{\|\bar\rho_s\|_{L^\infty}\}$.

Proof. Since V_t does not depend on the solution of Equation (3.4), but on a given $\bar\rho$, existence and uniqueness of solutions of Equation (3.4) is given by the classical method of characteristics, see e.g. [10]. Let ϕ_t^V be the flow generated by V_t in the time interval $[t, t+s]$. Then the solution ρ of e̊-pde0 is unique, and given by the push-forward $\rho(t,\cdot) = \phi_t^V \# \rho_0$ (see Proposition 2.9).

We first prove e̊-LipW. Since it holds that $\rho(t+s,\cdot) = \phi_s^V \# \rho(t,\cdot)$, then the transference plan $\pi(x,y) = (\mathrm{Id}\times \phi_s^V)\#\rho(t,\cdot)$ satisfies $\pi \in \Pi(\rho(t,\cdot), \rho(t+s,\cdot))$. Then it holds that

$$W_\infty(\rho(t,\cdot),\rho(t+s,\cdot)) \le C_\infty(\pi) = \sup_{x\in \mathrm{supp}(\rho(t,\cdot))} \left\{|x - \phi_s^V(x)|\right\} \le s\sup(|v|).$$

We now prove e̊-LipL. Observe that Gronwall's lemma gives

$$e^{-L|t|}|b-a| \le |\phi_t^V(b) - \phi_t^V(a)| \le e^{L|t|}|b-a|, \tag{3.7}$$

where L is the Lipschitz constant of V_s in the interval $[0,t]$,
that is $L = Lip(v) TV(w) \sup_{s\in[0,t]}\{\|\bar\rho_s\|_{L^\infty}\}$ by Proposition 3.2. This implies that for any interval (a,b) there holds

$$\begin{aligned}\int_a^b \rho(t,x)\,dx &= \int_a^b \phi_t^V \# \rho_0(x)\,dx \\ &= \int_{\phi_t^V(a)}^{\phi_t^V(b)} \rho_0(x)\,dx \le \|\rho_0\|_{L^\infty}|\phi_t^V(b) - \phi_t^V(a)| \le e^{L|t|}\|\rho_0\|_{L^\infty}|b-a|,\end{aligned}$$

that implies $\|\rho(t)\|_{L^\infty} \le e^{Lt}\|\rho_0\|_{L^\infty}$. By reversing time, we have the reverse inequality. □

We now have the following comparison result.

PROPOSITION 3.5. *Let* $\bar\rho_t, \bar\rho'_t \in C^0([0,T]; L^\infty \cap \mathcal{P}_c^{ac}(\mathbb{R}))$ *be given and* w, w' *satisfy* **(H)**. *Define the time-dependent vector fields* V_t, V'_t *as follows:*

$$V_t(x) := v\left(\int \bar\rho(t,y) w(y-x)\,dy\right), \qquad V'_t(x) := v\left(\int \bar\rho'(t,y) w'(y-x)\,dy\right).$$

Define $\rho(t), \rho'(t)$ *as the unique solutions of*

$$\partial_t \rho(t,x) + \partial_x(V_t(x)\rho(t,x)) = 0, \qquad \partial_t \rho'(t,x) + \partial_x(V'_t(x)\rho'(t,x)) = 0,$$

with initial data ρ_0,ρ_0'*, respectively. Then it holds that*

$$W_\infty(\rho(t,\cdot),\rho'(t,\cdot)) \le e^{L|t|}W_\infty(\rho_0,\rho_0') + (e^{L|t|}-1)\left(\sup_{s\in[0,t]} W_\infty(\bar\rho(s,\cdot),\bar\rho'(s,\cdot)) + \frac{\|w-w'\|_{L^1}}{TV(w')}\right) \quad (3.8)$$

with $L := Lip(v)TV(w)\sup_{s\in[0,t]}\max\{\|\bar\rho(s,\cdot)\|_{L^\infty},\|\bar\rho'(s,\cdot)\|_{L^\infty}TV(w')\}$.

Proof. By Proposition 3.2, we have that both V_t and V_t' are Lipschitz vector fields with Lipschitz constant L. Denote with ϕ_t,ϕ_t' the flows of the two vector fields, that are defined for every t with $0\le t\le T$, since the vector fields are globally Lipschitz. Then, we have the following estimate by Gronwall's lemma:

$$|\phi_t(x)-\phi_t'(y)| \le e^{L|t|}|x-y| + \frac{e^{L|t|}-1}{L}\sup_{s\in[0,t]}\|V_s-V_s'\|_{C^0}.$$

By definition of V_s and V_s', Proposition 3.3 gives $\|V_s-V_s'\|_{C^0} \le$

$$\mathrm{Lip}(v)\left(W_\infty(\bar\rho(s,\cdot),\bar\rho'(s,\cdot))TV(w)\min\{\|\bar\rho(s,\cdot)\|_\infty,\|\bar\rho'(s,\cdot)\|_\infty\} + \|\bar\rho'(s,\cdot)\|_{L^\infty}\|w-w'\|_{L^1}\right).$$

By plugging the explicit expression of L, and with a careful choice in the min and max terms, we have

$$|\phi_t(x)-\phi_t'(y)| \le e^{L|t|}|x-y| + (e^{L|t|}-1)\left(\sup_{s\in[0,t]} W_\infty(\bar\rho(s,\cdot),\bar\rho'(s,\cdot)) + \frac{\|w-w'\|_{L^1}}{TV(w')}\right).$$

By reversing time, we also have

$$|\phi_t(x)-\phi_t'(y)| \ge e^{-L|t|}|x-y| - (1-e^{-L|t|})\left(\sup_{s\in[0,t]} W_\infty(\bar\rho(s,\cdot),\bar\rho'(s,\cdot)) + \frac{\|w-w'\|_{L^1}}{TV(w')}\right).$$

Now take $\pi_0\in\Pi(\rho_0,\rho_0')$ and observe that $\pi_t := (\phi_t\times\phi_t')\#\pi_0$ satisfies $\pi_t\in\Pi(\rho(t,\cdot),\rho'(t,\cdot))$. Consider the set $E_k := \left\{(x,y)\in\mathbb{R}^2 \mid |x-y|>k\right\}$ and observe that it holds that

$$\pi_t(E_k) = \pi_0(\{(\phi_t(x),\phi_t'(y) \mid |x-y|>k\}) \le \pi_0\left(\left\{(x,y) \mid |x-y|>\tilde k\right\}\right)$$

with $\tilde k := e^{-L|t|}k - (1-e^{-L|t|})\left(\sup_{s\in[0,t]} W_\infty(\bar\rho(s,\cdot),\bar\rho'(s,\cdot)) + \frac{\|w-w'\|_{L^1}}{TV(w')}\right)$.

If $\tilde k > C_\infty(\pi_0)$, one has $\pi_0\left(\left\{(x,y) \mid |x-y|>\tilde k\right\}\right) = 0$ by definition of $C_\infty(\pi_0)$. This implies that $\pi_t(E_k)=0$ for all k satisfying

$$k > e^{L|t|}C_\infty(\pi_0) + (e^{L|t|}-1)\left(\sup_{s\in[0,t]} W_\infty(\bar\rho(s,\cdot),\bar\rho'(s,\cdot)) + \frac{\|w-w'\|_{L^1}}{TV(w')}\right).$$

Hence,

$$C_\infty(\pi_t) \le e^{L|t|}C_\infty(\pi_0) + (e^{L|t|}-1)\left(\sup_{s\in[0,t]} W_\infty(\bar\rho(s,\cdot),\bar\rho'(s,\cdot)) + \frac{\|w-w'\|_{L^1}}{TV(w')}\right). \quad (3.9)$$

By recalling that $W_\infty(\rho(t,\cdot),\rho'(t,\cdot)) \le C_\infty(\pi_t)$ by definition, and passing to the infimum among all transference plans $\pi_0\in\Pi(\rho_0,\rho_0')$ on the right-hand side of e̊-e12, we find e̊-key2.

∎

3.3. Proof of Proposition 3.1. We first prove existence of a solution of e̊-gen for small times, via convergence of an explicit Euler scheme. We then prove uniqueness of the solution.

PROPOSITION 3.6. *Let* **(H)** *hold, and* $T>0$ *be fixed, with* $T<(e\mathrm{Lip}(v)TV(w)\|\rho_0\|_{L^\infty})^{-1}$. *For each* $n\in\mathbb{N}$*, consider the following trajectory* $\rho^n\in C^0([0,T];L^\infty\cap\mathcal{P}_c^{ac}(\mathbb{R}))$*:*

- $\rho^n(0,\cdot):=\rho_0$;
- $\rho^n(k2^{-n}T+t,\cdot):=\phi_t^{n,k}\#\rho^n(k2^{-n}T,\cdot)$ *for* $t\in[0,2^{-n}T]$ *and* $k=0,1,\dots,2^n-1$, *where* $\phi_t^{n,k}$ *is the flow generated by the vector field*

$$V_{n,k}(x):=v\left(\int\rho^n(k2^{-n}T,y)w(y-x)\,dy\right).$$

Then, there exists a subsequence converging to $\rho^*\in C^0([0,T];L^\infty\cap\mathcal{P}_c^{ac}(\mathbb{R}))$ *that satisfies* $\rho^*(0)=\rho_0$ *and is a weak solution of*

$$\partial_t\rho(t,x)+\partial_x(V[\rho(t)](x)\rho(t,x))=0, \tag{3.10}$$

where

$$V[\rho(t)](x)=v\left(\int\rho(t,y)w(y-x)\,dy\right).$$

Moreover, ρ^* *satisfies* $\|\rho^*(t)\|_{L^\infty}\le e\|\rho_0\|_{L^\infty}$ *for all* $t\in[0,T]$.

Proof. Observe that e̊-LipW together with boundedness of v implies that the sequence ρ^n is equibounded and equi-Lipschitz in $C^0([0,T];\mathcal{P}_c(\mathbb{R}))$. Then the Ascoli–Arzelà theorem implies the existence of a converging subsequence, that we denote again with ρ^n. We denote the limit with ρ^*, that satisfies $\rho^*(0)=\rho_0$ and $\rho^*\in C^0([0,T];\mathcal{P}_c(\mathbb{R}))$.

Observe that $\mathcal{P}_c(\mathbb{R})$ is complete with respect to the W_∞ distance, but this is not the case for $L^\infty\cap\mathcal{P}_c^{ac}(\mathbb{R})$. Then, we now prove that $\rho^*(t)\in L^\infty\cap\mathcal{P}_c^{ac}(\mathbb{R})$ for $t\in[0,T]$, by proving that $\sup_{t\in[0,T],n\in\mathbb{N}}\|\rho^n(t)\|_{L^\infty}<+\infty$ for $T<(e\mathrm{Lip}(v)TV(w)\|\rho_0\|_{L^\infty})^{-1}$ and applying Lemma 3.1. Denote $a_k^n:=\|\rho^n(k2^{-n}T,\cdot)\|_{L^\infty}$ and observe that e̊-LipL, together with the definition of $\rho^n((k+1)2^{-n}T,\cdot)$ as function of $\rho^n(k2^{-n}T,\cdot)$, gives the following recurrence rule

$$a_{k+1}^n\le a_k^n e^{2^{-n}T\mathrm{Lip}(v)TV(w)a_k^n},$$

where we have set $a_0^n=\|\rho_0\|_{L^\infty}$. We now prove by induction that $a_k^n\le e\|\rho_0\|_{L^\infty}$ for all $k\le 2^n$. We clearly have $a_0^n\le\|\rho_0\|_{L^\infty}$. Assume that a_j^n satisfies $a_j^n\le e\|\rho_0\|_{L^\infty}$ for all $j\le k$. Then we have

$$a_{j+1}^n\le a_j^n e^{2^{-n}T\mathrm{Lip}(v)TV(w)a_j^n}\le a_j^n e^{2^{-n}}\qquad\text{for }\ j=0,\dots,k,$$

that in turn implies $a_{k+1}^n\le a_0^n e^{k2^{-n}}\le ea_0^n$ for $k\le 2^n$.
This implies $\sup_{n\in\mathbb{N},k=\{0,1,\dots,2^k\}}\|\rho^n(k2^{-n}T,\cdot)\|_{L^\infty}\le e\|\rho_0\|_{L^\infty}$.

Due to the Lipschitzianity of the L^∞ norm given by e̊-LipL, we have

$$\|\rho^n(t,\cdot)\|_{L^\infty}\le e^{\mathrm{Lip}(v)e\|\rho_0\|_{L^\infty}TV(w)2^{-n}T}e\|\rho_0\|_{L^\infty},$$

that in turn implies $\limsup_{n\to\infty}\|\rho^n(t,\cdot)\|_{L^\infty}\le e\|\rho_0\|_{L^\infty}$ for all $t\in[0,T]$. Then, by Lemma 3.1, we have $\rho^*\in C^0([0,T];L^\infty\cap\mathcal{P}_c^{ac}(\mathbb{R}))$ and $\|\rho^*(t,\cdot)\|_{L^\infty}\le e\|\rho_0\|_{L^\infty}$ for all $t\in[0,T]$.

We now prove that the limit ρ^* is a weak solution of è-pde1. Let $\phi\in C_c^\infty((0,T)\times\mathbb{R};\mathbb{R})$: we need to prove that

$$\int_0^T\int_{\mathbb{R}}\left(\rho^*(t,x)\partial_t\phi(t,x)+v\left(\int\rho^*(t,y)w(y-x)\,dy\right)\rho^*(t,x)\partial_x\phi(t,x)\right)dx\,dt=0. \quad (3.11)$$

Since by construction of ρ^n the following identity holds

$$\int_0^T\int_{\mathbb{R}}(\rho^n(t,x)\partial_t\phi(t,x)+V_{n,k}(x)\rho^n(t,x)\partial_x\phi(t,x))\,dx\,dt=0,$$

we prove è-weak by proving the following three limits:

$$\lim_{n\to\infty}\int_0^T\int_{\mathbb{R}}(\rho^*(t,x)-\rho^n(t,x))\,\partial_t\phi(t,x)\,dx\,dt=0; \quad (3.12)$$

$$\lim_{n\to\infty}\int_0^T\int_{\mathbb{R}}\left(v\left(\int\rho^*(t,y)w(y-x)\,dy\right)-V_{n,k}(x)\right)\rho^*(t,x)\partial_x\phi(t,x)\,dx\,dt=0; \quad (3.13)$$

$$\lim_{n\to\infty}\int_0^T\int_{\mathbb{R}}V_{n,k}(x)\,(\rho^*(t,x)-\rho^n(t,x))\,\partial_x\phi(t,x)\,dx\,dt=0. \quad (3.14)$$

Observe that convergence of ρ^n to ρ^* in $C^0([0,T];L^\infty\cap\mathcal{P}_c^{ac}(\mathbb{R}))$ implies the existence of a sequence $\varepsilon_n\to 0$ such that $\sup_{t\in[0,T]}W_\infty(\rho^n(t,\cdot),\rho^*(t,\cdot))<\varepsilon_n$. Also recall that $\|\rho^*(t,.)\|_{L^\infty}<e\|\rho_0\|_{L^\infty}$ for all $t\in[0,T]$.

To prove è-1, observe that for each $t\in[0,T]$ the function $\partial_t\phi(t,.)$ is a BV function, hence there exists $C_1:=\sup_{t\in[0,T]}TV(\partial_t\phi(t,.))<+\infty$. Then, by Lemma 3.3, it holds that

$$\int_0^T\left|\int_{\mathbb{R}}(\rho^*(t,x)-\rho^n(t,x))\,\partial_t\phi(t,x)\,dx\right|dt\le\int_0^T\varepsilon_nC_1e\|\rho_0\|_{L^\infty},$$

that provides è-1. To prove è-2, observe that for each n there holds

$$\begin{aligned}
&\sup_{t\in[0,T],x\in\mathbb{R}}\left|v\left(\int\rho^*(t,y)w(y-x)\,dy\right)-V_{n,k}(x)\right|\\
&\le\mathrm{Lip}(v)\sup_{k\in\{0,\dots,2^n-1\},t\in[0,2^{-n}T],x\in\mathbb{R}}\left|\int\left(\rho^*(k2^{-n}T+t,y)-\rho^n(k2^{-n}T,y)\right)w(y-x)\,dy\right|\\
&\le\mathrm{Lip}(v)\sup_{k\in\{0,\dots,2^n-1\},t\in[0,2^{-n}T]}W_\infty\left(\rho^*(k2^{-n}T+t,\cdot),\rho^n(k2^{-n}T,\cdot)\right)TV(w)e\|\rho_0\|_{L^\infty}\\
&\le\mathrm{Lip}(v)\left(2^{-n}T\sup(|v|)+\varepsilon_n\right)TV(w)e\|\rho_0\|_{L^\infty},
\end{aligned} \quad (3.15)$$

where we used Lemma 3.3 and the triangular inequality

$$\begin{aligned}
&W_\infty\left(\rho^*(k2^{-n}T+t,\cdot),\rho^n(k2^{-n}T,\cdot)\right)\\
&\le W_\infty\left(\rho^*(k2^{-n}T+t,\cdot),\rho^*(k2^{-n}T,\cdot)\right)+W_\infty\left(\rho^n(k2^{-n}T,\cdot),\rho^n(k2^{-n}T,\cdot)\right)
\end{aligned}$$

and è-LipW. Then, going back to the left hand side of è-2, observe that $\|\rho^*(t,\cdot)\|_{L^\infty}\le e\|\rho_0\|_{L^\infty}$ and integrate on the bounded support of $\partial_x\phi$. By passing to the limit in è-2b, we have è-2.

Finally, to prove è-3, observe that Proposition 3.2 provides the Lipschitzianity of $V_{n,k}$, with uniform Lipschitz constant $\mathrm{Lip}(|v|)TV(w)e\|\rho_0\|_{L^\infty}$. Also, $\|V_{n,k}\|_{C_0}$ is uniformly bounded by $\|v\|_\infty$. Since one has the Lipschitzianity and boundedness of $\partial_x\phi$

too, we have uniform Lipschitzianity of $V_{n,k}\partial_x\phi$. This property, together with boundedness of the support, implies uniform bounded variation, i.e. the existence of C_2 such that for all n,k it holds that $TV(V_{n,k}\partial_x\phi)<C_2$. Then, Lemma 3.3 provides

$$\begin{aligned}\int_{\mathbb{R}} |V_{n,k}(x)(\rho^*(t,x)-\rho^n(t,x))\partial_x\phi(t,x)|\,dx &\le W_\infty(\rho^*(t,x),\rho^n(t,x))C_2\,e\,\|\rho_0\|_{L^\infty}\\ &\le \varepsilon_n C_2\,e\,\|\rho_0\|_{L^\infty}.\end{aligned}$$

By integrating with respect to time and passing to the limit, we have e̊-3. ∎

REMARK 3.2. It is interesting to observe that one can have blow-up of the density for solutions of e̊-pde1. A simple example is given by $w=\chi_{[0,1]}$, $v(x)=x$ and $\rho_0=\chi_{[-1,0]}$. Indeed, observe that the vector field $V[\rho_0]$ is a non-negative and non-increasing function satisfying $V[\rho_0](0)=0$. The evolution $\rho(t)$ has then support contained in $[a(t);0]$ for all times in which it is defined, with $a(0)=-1$ and $a(t)\ge -1$. Observe that $V[\rho(t)](a(t))= v(\int_{a(t)}^{a(t)+1}\rho(t)\,dx)=v(1)=1$. Then $a(t)=-1+t$. This implies that $\|\rho(t)\|_{L^\infty}\ge(1-t)^{-1}$, hence $\|\rho(t)\|_{L^\infty}\to\infty$ for $t\to 1$.

This is in sharp contrast with the case of solutions of e̊-pde1 with Lipschitz kernels w. Indeed, in this case hypotheses of Theorem 2.1 are satisfied, then one has existence and uniqueness of the solution for all times. Moreover, when the initial data μ_0 is a probability density in $\mathcal{P}_c^{ac}(\mathbb{R}^d)$, then it keeps being in $\mathcal{P}_c^{ac}(\mathbb{R}^d)$.

We are now ready to prove existence and uniqueness of the solution, i.e. Proposition 3.1.

Proof. **(Proof of Proposition 3.1.)** Choose $T<(e\mathrm{Lip}(v)TV(w)\|\rho_0\|_{L^\infty})^{-1}$ and apply Proposition 3.6 to have existence of a solution. Assume now to have two solutions $\bar\rho,\bar\rho'\in C^0([0,T];L^\infty\cap\mathcal{P}_c^{ac}(\mathbb{R}))$ and define the time-dependent vector fields V_t,V_t' as follows:

$$V_t(x):=v\left(\int\bar\rho(t,y)w(y-x)\,dy\right),\qquad V_t'(x):=v\left(\int\bar\rho'(t,y)w(y-x)\,dy\right).$$

Then $\bar\rho,\bar\rho'$ clearly coincide with the solutions ρ,ρ' of

$$\partial_t\rho(t,x)+\partial_x(V_t(x)\rho(t,x))=0,\qquad \partial_t\rho'(t,x)+\partial_x(V_t'(x)\rho'(t,x))=0,$$

respectively, with initial data ρ_0.

We prove uniqueness by contradiction. Let $t_0:=\inf_{t\in[0,T]}\{W_\infty(\rho(t,\cdot),\rho'(t,\cdot))>0\}$ be the first time in which ρ,ρ' do not coincide. By applying Proposition 3.5 starting from t_0, we have

$$W_\infty(\rho(t_0+s,\cdot),\rho'(t_0+s,\cdot))\le(e^{Ls}-1)\sup_{s'\in[0,s]}W_\infty(\bar\rho(t_0+s',\cdot),\bar\rho'(t_0+s',\cdot)),\quad(3.16)$$

that implies $\sup_{s\in[0,t]}W_\infty(\rho(t_0+s,\cdot),\rho'(t_0+s,\cdot))\le(e^{Lt}-1)\sup_{s\in[0,t]}W_\infty(\bar\rho(t_0+s,\cdot),$ $\bar\rho'(t_0+s,\cdot))$. By choosing $t<\frac{\ln(2)}{L}$ we have $W_\infty(\rho(t_0+s,\cdot),\rho'(t_0+s,\cdot))=0$ for all $s\in[0,t]$, that implies

$$\inf_{t\in[0,T]}\{W_\infty(\rho(t,\cdot),\rho'(t,\cdot))>0\}>t_0,$$

which gives a contradiction. ∎

Finally, we prove continuous dependence on the interaction kernel and on initial data for the solution of e̊-pde1.

PROPOSITION 3.7. *Let* **(H)** *hold. Let*

$$T<(e\mathrm{Lip}(v)TV(w)\max\{\|\rho_0\|_{L^\infty},\|\rho_0'\|_{L^\infty}\})^{-1}$$

and ρ,ρ' be the unique solutions of the Cauchy problems

$$\begin{cases}\partial_t\rho(t,x)+\partial_x(V[\rho(t)](x)\rho(t,x))=0,\\ \rho(0,\cdot)=\rho_0\end{cases}\qquad \begin{cases}\partial_t\rho'(t,x)+\partial_x(V'[\rho'(t)](x)\rho'(t,x))=0,\\ \rho'(0,\cdot)=\rho_0'\end{cases}$$

where

$$V[\rho(t)](x)=v\left(\int\rho(t,y)w(y-x)\,dy\right),\qquad V'[\rho'(t)](x)=v\left(\int\rho'(t,y)w'(y-x)\,dy\right).$$

Then it holds that

$$W_\infty(\rho(t,\cdot),\rho'(t,\cdot))\le e^{4eLt}W_\infty(\rho_0,\rho_0')+(e^{4eLt}-1)\frac{\|w-w'\|_{L^1}}{TV(w)},\tag{3.17}$$

with $L=\mathrm{Lip}(v)\max\{\|\rho_0\|_{L^\infty}TV(w),\|\rho_0'\|_{L^\infty}TV(w')\}$.

Proof. We apply Proposition 3.5 with $\rho=\bar\rho$ and $\rho'=\bar\rho'$. Moreover, since $T<(e\mathrm{Lip}(v)TV(w)\max\{\|\rho_0\|_{L^\infty},\|\rho_0\|_{L^\infty}\})^{-1}$, there holds $\|\rho(t,\cdot)\|_{L^\infty}\le e\|\rho_0\|_{L^\infty}$, $\|\rho'(t,\cdot)\|_{L^\infty}\le e\|\rho_0'\|_{L^\infty}$ for all $t\in[0,T]$. Then e̊-key2 reads as

$$W_\infty(\rho(t,\cdot),\rho'(t,\cdot))\le e^{eLt}W_\infty(\rho_0,\rho_0')+(e^{eLt}-1)\left(\sup_{s\in[0,t]}W_\infty(\rho(t,\cdot),\rho'(t,\cdot))+\frac{\|w-w'\|_{L^1}}{TV(w)}\right)$$

with $L=\mathrm{Lip}(v)\max\{\|\rho_0\|_{L^\infty}TV(w),\|\rho_0'\|_{L^\infty}TV(w')\}$. By applying the Gronwall lemma, we have e̊-key3. ∎

4. Finite dimensional approximation: the micro-macro limit

In this section, we describe a finite-dimensional approximation for the solution of e̊-cauchy, that may serve as a numerical scheme to compute such solution.

We first define a Lipschitz approximation w_ℓ for the interaction kernel w.

DEFINITION 4.1. *Let $\eta>0$ be fixed, and $w:[0,\eta]\to\mathbb{R}^+$ be a non-increasing C^1 function such that $\int_0^\eta w(x)\,dx=1$. We define $w_\ell:\mathbb{R}\to\mathbb{R}^+$ as follows*

$$w_\ell(x):=\begin{cases}w(0)\frac{\ell+2x}{\ell} & \text{if } x\in\left[-\frac{\ell}{2},0\right]\\ w(x) & \text{if } x\in[0,\eta]\\ w(\eta)\frac{2\eta+\ell-2x}{\ell} & \text{if } x\in\left[\eta,\eta+\frac{\ell}{2}\right]\\ 0 & \text{elsewhere.}\end{cases}\tag{4.1}$$

We also define a discretization $[\mu]^n$ of a given probability density $\mu\in\mathcal{P}^{ac}(\mathbb{R})$. As stated at the beginning, the same idea can be adapted to any density $\mu\in\mathcal{M}^{ac}(\mathbb{R})$, not necessarily of mass one.

DEFINITION 4.2. *Let $\mu\in\mathcal{P}^{ac}_c(\mathbb{R})$ and $n\in\mathbb{N}$ be fixed. Define $\{x_1,\dots,x_n\}$ as follows*

$$\begin{cases}x_1=\sup\left\{x\in\mathbb{R}\mid\int_{-\infty}^x d\mu<\frac1n\right\},\\ x_{i+1}=\sup\left\{x\in\mathbb{R}\mid\int_{x_i}^x d\mu<\frac1n\right\}, & \text{for } i=1,\dots,n-1,\end{cases}\tag{4.2}$$

and

$$[\mu]^n := \frac{1}{n}\sum_{i=1}^{n} \delta_{x_i}. \tag{4.3}$$

Observe that $[\mu]^n \notin \mathcal{P}^{ac}(\mathbb{R})$, hence convolutions with discontinuous kernels are not well defined. We now prove some simple properties of the discretization $[\mu]^n$.

Proposition 4.1. *Let $\mu \in \mathcal{P}_c^{ac}([a,b])$ and $n \in \mathbb{N}$ be fixed, and the x_i and $[\mu]^n$ defined as in e̊-defrho. Then it holds that*

$$|x_i - x_j| \geq |i-j|\,(n\|\mu\|_{L^\infty})^{-1}, \tag{4.4}$$

and

$$W_1(\mu,[\mu]^n) \leq \frac{b-a}{n}. \tag{4.5}$$

Proof. Assume $i > j$ and observe that it holds that

$$\frac{|i-j|}{n} = \int_{x_j}^{x_i} d\mu \leq (x_i - x_j)\|\mu\|_{L^\infty},$$

that gives e̊-ij. To prove e̊-W1disc, define $x_0 := a$ and divide the interval $[x_0,x_n] \subset [a,b]$ in the n intervals $[x_0,x_1],[x_1,x_2],\dots,[x_{n-1},x_n]$. Observing that each of the intervals contains a mass of $\frac{1}{n}$, we have

$$W_1(\mu,[\mu]^n) \leq \sum_{i=1}^{n} W_1\left(\mu\chi_{[x_{i-1},x_i]}, \frac{1}{n}\delta_{x_i}\right) \leq \sum_{i=1}^{n}\frac{1}{n}(x_i - x_{i-1}) = \frac{1}{n}(x_n - x_0) \leq \frac{1}{n}(b-a).$$

□

We are now ready to define the approximation $[\rho]^n$ of the solution ρ of e̊-cauchy.

Definition 4.3. *Let $v:[0,1] \to \mathbb{R}^+$ be a Lipschitz and non-increasing function with[2] $v(1) = 0$. Let $\eta > 0$ be fixed, and $w:[0,\eta] \to \mathbb{R}^+$ be a non-increasing Lipschitz function with $\int_0^\eta w(x)\,dx = 1$. Let $\rho_0 \in L^\infty \cap \mathcal{P}_c^{ac}(\mathbb{R})$ be given.*

Fix $n \in \mathbb{N}$ and choose the constant $\ell_n := (n\|\rho_0\|_{L^\infty})^{-1}$. Define $[\rho(t)]^n$ as

$$[\rho(t)]^n = \frac{1}{n}\sum_{i=1}^{n} \delta_{x_i(t)}, \tag{4.6}$$

where $\{x_1(t),\dots,x_n(t)\}$ is the unique solution of the finite-dimension dynamical system

$$\begin{cases} \dot{x}_n = v(0), & \\ \dot{x}_i = v\left(\frac{1}{n}\sum_{j=1}^{n} w_{\ell_n}(x_j - x_i)\right) & \text{for } i = 1,\dots,n-1, \\ x_i(0) = x_{i,0} & \text{for } i = 1,\dots,n, \end{cases} \tag{4.7}$$

where the $x_{i,0}$ are given by the discretization e̊-xi of ρ_0, i.e. $[\rho_0]^n = \frac{1}{n}\sum_{i=1}^{n}\delta_{x_{i,0}}$.

[2]We extend v with the convention $v(x) = 0$ for $x > 1$.

It is clear that the solution of e-findim exists and is unique for all times, due to the Lipschitzianity of w_{ℓ_n}. Also remark that $[\rho(t)]^n$ is the unique solution of the following transport equation:

$$\begin{cases} \partial_t [\rho(t)]^n (x) + \partial_x \left([\rho(t)]^n (x) \, v \left(\int w_{\ell_n}(y-x) \, d[\rho(t)]^n (y) \right) \right) = 0, \\ [\rho(0)]^n = [\rho_0]^n. \end{cases} \tag{4.8}$$

The proof of such identity is direct, by rewriting the interaction kernel $\int w_{\ell_n}(y-x)\,d\mu^n(y)$ in the case of μ^n given by empiric.

The main result of this section is the following convergence result, that will be proved in Section 4.2.

THEOREM 4.1. *Let $v:[0,1]\to\mathbb{R}^+$ be a Lipschitz and non-increasing function with $v(1)=0$. Let $\eta>0$ be fixed, and $w:[0,\eta]\to\mathbb{R}^+$ be a non-increasing Lipschitz function with $\int_0^\eta w(x)\,dx=1$. Let $\rho_0\in BV(\mathbb{R};[0,1])$ with compact support and $\int \rho_0(x)\,dx=1$.*

Fix any $T>0$. Then, for each $n\in N$, the approximation $[\rho(t)]^n$ given by empiric is defined in $C^0([0,T];\mathcal{P}_c(\mathbb{R}))$ and it satisfies $[\rho(t)]^n \rightharpoonup \rho(t)$ for $n\to\infty$, where $\rho\in C^0([0,T];\mathcal{P}_c(\mathbb{R}))$ is the unique solution of the Cauchy problem

$$\begin{cases} \partial_t \rho + \partial_x \left(\rho v \left(\int_x^{x+\eta} \rho(t,y) w(y-x)\,dy \right) \right) = 0, & x\in\mathbb{R},\ t>0, \\ \rho(0,x)=\rho_0(x), & x\in\mathbb{R}. \end{cases} \tag{4.9}$$

In particular, ρ is defined for all times $t>0$, and it satisfies

$$0\le \rho(t,x)\le \max\{\rho_0\} \qquad \text{for a.e. } x\in\mathbb{R}, t>0. \tag{4.10}$$

REMARK 4.1. It is interesting to observe that the convergence of $[\rho(t)]^n$ to the actual solution $\rho(t)$ cannot hold in the L^1 norm, since the approximations $[\rho(t)]^n$ are sums of Dirac deltas, hence not belonging to L^1.

Instead, the proof given below shows that the convergence can be metrized by the W_∞ distance. Moreover, convergence is ensured with respect to all Wasserstein distances W_p, whose definition is recalled in Section 2. Then, estimates of the error between the approximated solution $[\rho(t)]^n$ and the actual solution can be established with respect to such metrics. See [28] for examples of the use the Wasserstein metric in error estimates for numerical schemes.

Before proving Theorem 4.1, we address some properties of the finite-dimensional problem e-findim.

4.1. Preservation of the minimal and maximal distances. In this section, we prove that, when the initial minimal distance among agents for the model e-findim is larger than or equal to ℓ, then it keeps being larger than or equal to ℓ for all times.

PROPOSITION 4.2. *Let $\ell\ge\ell_n$ be fixed. Consider a sequence $x_1^0<x_2^0<\ldots<x_n^0$ and denote with $x(t)=(x_1(t),\ldots,x_n(t))$ the unique solution of e-findim. If $x_i^0-x_{i-1}^0\ge\ell$ for all $i=1,\ldots,n$, then it holds that $x_i(t)-x_{i-1}(t)\ge\ell$ for all times $t>0$.*

Proof. We prove that $x_i-x_{i-1}=\ell$ implies $\dot{x}_i-\dot{x}_{i-1}\ge 0$, that clearly gives the result.

If $i=n$, we have $\dot{x}_n=v(0)=\max_{\sigma\in[0,1]}v(\sigma)\ge\dot{x}_{n-1}$. For $i<n$, we have

$$\dot{x}_i-\dot{x}_{i-1}=v\left(\frac{1}{n}\sum_{j=1}^n w_{\ell_n}(x_j-x_i)\right)-v\left(\frac{1}{n}\sum_{j=1}^n w_{\ell_n}(x_j-x_{i-1})\right).$$

Since v is a non-increasing function, we have $\dot{x}_i-\dot{x}_{i-1}\geq 0$ if and only if

$$\frac{1}{n}\sum_{j=1}^{n}w_{\ell_n}(x_j-x_i)\leq\frac{1}{n}\sum_{j=1}^{n}w_{\ell_n}(x_j-x_{i-1})=\frac{1}{n}\sum_{j=2}^{n+1}w_\ell(x_{j-1}-x_{i-1}). \tag{4.11}$$

For $j<i$, we have both $x_j-x_i\leq -\ell$ and $x_{j-1}-x_{i-1}\leq -\ell$, thus $w_{\ell_n}(x_j-x_i)=w(x_j-x_i)=0$ and $w_{\ell_n}(x_{j-1}-x_{i-1})=w(x_{j-1}-x_{i-1})=0$.

For $j=i$ we have $w_{\ell_n}(x_i-x_i)=w_{\ell_n}(x_{i-1}-x_{i-1})=w_{\ell_n}(0)=w(0)$. Then, we can restrict ourselves to $j>i$. Condition e̊-qui is verified if for each $j=i+1,\dots,n$ we have

$$w_{\ell_n}(x_j-x_i)\leq w_{\ell_n}(x_{j-1}-x_{i-1}). \tag{4.12}$$

Since $x_j-x_i>0$ and $x_{j-1}-x_{i-1}>0$ for $j>i$ and w_{ℓ_n} is non-increasing in $[0,+\infty]$ as a consequence of the fact that w is non-increasing in $[0,\eta]$, condition e̊-condij is equivalent to $x_j-x_i\geq x_{j-1}-x_{i-1}$, i.e. $x_j-x_{j-1}\geq x_i-x_{i-1}=\ell$, which is true by hypothesis. □

The above property is the discrete counterpart of the maximum principle e̊-maxprinciple. The symmetric result is also true, with a similar proof.

PROPOSITION 4.3. *Let $L\geq \ell_n$ be fixed. Consider a sequence $x_1^0<x_2^0<\dots<x_n^0$ and denote with $x(t)=(x_1(t),\dots,x_n(t))$ the unique solution of e̊-findim. If $x_i^0-x_{i-1}^0\leq L$ for all $i=1,\dots,n$, then it holds that $x_i(t)-x_{i-1}(t)\leq L$ for all times $t>0$.*

Observe that, due to the compact support of the solution, the minimum value of the density is always zero, and Proposition 4.3 does not allow to recover a finer minimum principle as in [7, 19].

4.2. Convergence to the solution of e̊-cauchy. We now prove Theorem 4.1. We combine estimates for solutions of the finite-dimensional problem e̊-findim with estimates for solutions of the transport equation with a Lipschitz interaction kernel e̊-cauchysmooth and with estimate for solutions of the transport equation with a BV interaction kernel e̊-cauchy.

Proof. (**Proof of Theorem 4.1.**) The idea of the proof is to prove convergence of the approximate solution $[\rho(t)]^n$ to the solution ρ of e̊-cauchy by proving intermediate convergence results, divided in four steps. In the first step, we restrict ourselves to a subsequence of $[\rho(t)]^n$ admitting a limit ρ^* in $C^0([0,T];L^\infty\cap\mathcal{P}_c^{ac}(\mathbb{R}))$. In the second step, we define a finite-dimensional approximation $[\tilde{\rho}(t)]^n$ and prove that $[\rho(t)]^n$ and $[\tilde{\rho}(t)]^n$ have the same limit ρ^*. In the third step, we define an approximation ρ^n in $\mathcal{P}^{ac}(\mathbb{R})$ and prove that $[\tilde{\rho}(t)]^n$ and ρ^n have the same limit ρ^*. Finally, in the fourth step we prove that the limit of ρ^n is exactly ρ, first for small times and then for any time.

Step 1. Fix any $T>0$. We prove that the sequence $[\rho(t)]^n\in C^0([0,T];\mathcal{P}_c(\mathbb{R}))$ admits a subsequence (that we do not relabel) with a limit $\rho^*\in C^0([0,T];L^\infty\cap\mathcal{P}_c^{ac}(\mathbb{R}))$, that moreover satisfies $\|\rho^*(t)\|_{L^\infty}\leq\|\rho_0\|_{L^\infty}$.

Fix $[a,b]$ an interval containing the compact support of ρ_0. Then, by construction of $[\rho(0)]^n=[\rho_0]^n$, we have $\operatorname{supp}([\rho(0)]^n)\subset[a,b]$. Due to boundedness of v, we have both $\operatorname{supp}([\rho(t)]^n)\subset[a-T\sup(v),b+T\sup(v)]$ and $W_\infty([\rho(t+s)]^n,[\rho(t)]^n)\leq s\sup(v)$ for all $t\in[0,T]$ and $n\in\mathbb{N}$, i.e. equiboundedness and equi-Lipschitzianity of the sequence $[\rho(t)]^n$ with respect to the W_∞-distance. Then, eventually passing to a subsequence, there exists a limit $\rho^*\in C^0([0,T];\mathcal{P}_c(\mathbb{R}))$, that has both uniformly bounded support and uniform Lipschitz constant. From now on, the sequence of indexes n is always replaced by the subsequence with limit ρ^*.

We now prove that $\|\rho^*(t)\|_{L^\infty} \leq \|\rho_0\|_{L^\infty}$. This is equivalent to prove that for each interval $[\alpha,\beta]\subset\mathbb{R}$ it holds that $\int_\alpha^\beta \rho^*(t)\,dx \leq \|\rho_0\|_{L^\infty}(\beta-\alpha)$. Observe that convergence in W_∞ implies weak convergence of measures, then $[\rho(t)]^n \rightharpoonup \rho^*(t)$. For each $\varepsilon>0$, consider a function $\phi_\varepsilon \in C^\infty(\mathbb{R},\mathbb{R})$ with support in $[\alpha-\varepsilon,\beta+\varepsilon]$ such that $\phi_\varepsilon(x)\in[0,1]$ for all x and $\phi_\varepsilon(x)=1$ for $x\in[\alpha,\beta]$. Observe that, by definition of $[\rho(t)]^n$, for each fixed t it holds that

$$\int \phi_\varepsilon(x)\,d[\rho(t)]^n(x) = \frac{1}{n}\sum_{x_i(t)\in(\alpha-\varepsilon,\beta+\varepsilon)} \phi_\varepsilon(x_i(t)) \leq \frac{1}{n}\frac{\beta-\alpha+2\varepsilon}{\ell_n},$$

where we used the fact that the number of $x_i(t)$ in the interval $(\alpha-\varepsilon,\beta+\varepsilon)$ is bounded from above, by preservation of the minimal distance $\ell_n = (n\|\rho_0\|_{L^\infty})^{-1}$, see Proposition 4.2. By replacing ℓ_n in the last term, by recalling that $[\rho(t)]^n \rightharpoonup \rho^*(t)$ and observing that $\phi_\varepsilon \geq \chi_{[\alpha,\beta]}$, for each fixed t we have

$$\int_\alpha^\beta d\rho^*(t,x) \leq \int \phi_\varepsilon(x)\,d\rho^*(t,x) \leq \|\rho_0\|_{L^\infty}(\beta-\alpha+2\varepsilon).$$

By passing to the limit for $\varepsilon\to 0$, we have that $\rho^*\in L^\infty\cap\mathcal{P}_c^{ac}(\mathbb{R})$ and it satisfies

$$\|\rho^*(t)\|_{L^\infty} \leq \|\rho_0\|_{L^\infty}. \tag{4.13}$$

Step 2. We now define $[\tilde\rho(t)]^n$. For each n, consider the approximated kernel w_{m_n} with $m_n := \ln(n)^{-1}$. Then, define $[\tilde\rho(t)]^n$ similarly to $[\rho(t)]^n$, as follows:

$$[\tilde\rho(t)]^n = \frac{1}{n}\sum_{i=1}^n \delta_{y_i(t)},$$

where $\{y_1(t),\ldots,y_n(t)\}$ is the unique solution of the finite-dimension dynamical system

$$\begin{cases} \dot y_i = v\left(\frac{1}{n}\sum_{j=1}^n w_{m_n}(y_j-y_i)\right) & \text{for } i=1,\ldots,n\\ y_i(0)=x_{i,0} & \text{for } i=1,\ldots,n, \end{cases} \tag{4.14}$$

and the $x_{i,0}$ are given by the discretization ȩ-xi of ρ_0, i.e. $[\rho_0]^n = \frac{1}{n}\sum_{i=1}^n \delta_{x_{i,0}}$. Remark that, similarly to ȩ-mun, $[\tilde\rho(t)]^n$ is the unique solution of the following transport equation

$$\begin{cases} \partial_t [\tilde\rho(t)]^n(x) + \partial_x\left([\tilde\rho(t)]^n(x)\,v\left(\int w_{m_n}(x-y)\,d[\tilde\rho(t)]^n(y)\right)\right)=0,\\ [\rho(0)]^n = [\rho_0]^n. \end{cases} \tag{4.15}$$

We now prove that, for each $T>0$ it holds that $\lim_{n\to\infty}\sup_{t\in[0,T]} W_\infty([\rho(t)]^n,[\tilde\rho(t)]^n)=0$. This implies that the limit of both $[\rho(t)]^n$ and $[\tilde\rho(t)]^n$ is ρ^*. For each n, we define $\varepsilon_i^n(t) := \sup_{s\in[0,t]}|x_i(s)-y_i(s)|$ and $\varepsilon^n(t) := \max_{i=1,\ldots,n}\varepsilon_i^n(t)$. Observe that we have $W_\infty([\rho(t)]^n,[\tilde\rho(t)]^n)\leq\varepsilon^n(t)$, by choosing the transference plan sending $x_i(t)$ to $y_i(t)$. Also observe that $\varepsilon_i^n(0)=\varepsilon^n(0)=0$, since the initial data coincide.

Observe that the initial data satisfies $x_{i+1}^0 - x_i^0 \geq \ell_n$, then Proposition 4.2 for ȩ-findim with kernel w_{ℓ_n} gives $x_{i+1}(t)-x_i(t)\geq\ell_n$. The property does not hold for y_i for big n, since $m_n > \ell_n$.

We now estimate the evolution of ε_i^n. By the definition, we have $\varepsilon_n^n=0$. By boundedness of v, we have $\varepsilon_i^n(t+s)-\varepsilon_i^n(t)\leq 2\sup(v)s$, hence the ε_i^n are Lipschitz function. By definition of the dynamics ȇ-findim, we have

$$\dot{\varepsilon}_i^n \leq \mathrm{Lip}(v)\frac{1}{n}\sum_{j=1}^{n}|w_{\ell_n}(x_j(t)-x_i(t))-w_{m_n}(y_j(t)-y_i(t))|. \tag{4.16}$$

It is clear that $y_j(t)-y_i(t)=\varepsilon_j^n(t)-\varepsilon_i^n(t)+x_j(t)-x_i(t)$, hence $y_j(t)-y_i(t)\in[x_j(t)-x_i(t)-2\varepsilon^n(t),x_j(t)-x_i(t)+2\varepsilon^n(t)]$. For any fixed index i and for each time t, we divide the indexes j in the following five sets:

1. j is such that $x_j(t)-x_i(t)\in\left]-\infty,-\frac{m_n}{2}-2\varepsilon^n(t)\right]$. In this case, both $x_j(t)-x_i(t)$ and $y_j(t)-y_i(t)$ are in the interval $\left]-\infty,-\frac{m_n}{2}\right]$, for which it holds that $w_{\ell_n}(x_j(t)-x_i(t))=w_{m_n}(y_j(t)-y_i(t))=0$.
2. j is such that $x_j(t)-x_i(t)\in\left(-\frac{m_n}{2}-2\varepsilon^n(t),2\varepsilon^n(t)\right)$. In this case, we simply estimate
$$|w_{\ell_n}(x_j(t)-x_i(t))-w_{m_n}(y_j(t)-y_i(t))|\leq w(0).$$
Observe that $|x_j(t)-x_i(t)|\geq|j-i|\ell_n$, hence the number of indexes j in this set is smaller or equal than $\frac{\frac{m_n}{2}+4\varepsilon^n(t)}{\ell_n}$.
3. j is such that $x_j(t)-x_i(t)\in[2\varepsilon^n(t),\eta-2\varepsilon^n(t)]$: in this case both $x_j(t)-x_i(t)$ and $y_j(t)-y_i(t)$ belong to the interval $[0,\eta]$, on which w_{ℓ_n} and w_{m_n} coincide with w, that is Lipschitz. Hence $|w_{\ell_n}(x_j(t)-x_i(t))-w_{m_n}(y_j(t)-y_i(t))|\leq 2\mathrm{Lip}(w)\varepsilon^n(t)$. In this case the number of indexes j in this set is strictly smaller than $\frac{\eta}{\ell_n}$.
4. j is such that $x_j(t)-x_i(t)\in(\eta-2\varepsilon^n(t),\eta+\frac{m_n}{2}+2\varepsilon^n(t))$. Similarly to the second case, we estimate $|w_{\ell_n}(x_j(t)-x_i(t))-w_{m_n}(y_j(t)-y_i(t))|\leq w(0)$ and observe that the number of indexes j in this set is smaller or equal than $\frac{\frac{m_n}{2}+4\varepsilon^n(t)}{\ell_n}$.
5. j is such that $x_j(t)-x_i(t)\in\left[\eta+\frac{m_n}{2}+2\varepsilon^n(t),+\infty\right[$. Similarly to the first case, we have $w_{\ell_n}(x_j(t)-x_i(t))=w_{m_n}(y_j(t)-y_i(t))=0$.

By using the previous decomposition of the indexes in ȇ-epsi, we have

$$\dot{\varepsilon}_i^n(t)\leq\mathrm{Lip}(v)\frac{1}{n}\left(2w(0)\frac{\frac{m_n}{2}+4\varepsilon^n(t)}{\ell_n}+2\mathrm{Lip}(w)\varepsilon^n(t)\frac{\eta}{\ell_n}\right).$$

Taking the supremum over ε_i^n, and recalling that $\ell_n=(n\|\rho_0\|_{L^\infty})^{-1}$, we have

$$\dot{\varepsilon}^n(t)\leq C_0 m_n+C_1\varepsilon^n(t)$$

with $C_0:=\mathrm{Lip}(v)w(0)\|\rho_0\|_{L^\infty}$, $C_1:=\mathrm{Lip}(v)\|\rho_0\|_{L^\infty}(8w(0)+2\mathrm{Lip}(w)\eta)$. Since $\varepsilon^n(0)=0$ and the constants C_0,C_1 do not depend on t, the Gronwall estimate gives

$$\varepsilon^n(t)\leq\frac{C_0}{C_1}m_ne^{C_1t}=\frac{C_0}{C_1\ln(n)}e^{C_1t}.$$

This implies $W_\infty([\rho(t)]^n,[\tilde{\rho}(t)]^n)\leq\frac{C_0}{C_1\ln(n)}e^{C_1T}$ for all times $t\in[0,T]$. Then, the limit of $[\tilde{\rho}(t)]^n$ is ρ^*.

Step 3. Choose again $[a,b]$ any interval containing the support of the initial datum ρ_0, and fix any $T>0$. We define $\rho^n\in C^0([0,T];L^\infty\cap\mathcal{P}_c^{ac}(\mathbb{R}))$ as the solution of e̊-problema with initial data ρ_0 when replacing w with w_{m_n}. Since w_{m_n} is Lipschitz, we have existence and uniqueness of ρ^n for any $t\in[0,T]$ thanks to Proposition 2.10.

Now fix $T_1<(32\mathrm{Lip}(v)w(0))^{-1}$. We now compare $[\tilde\rho(t)]^n$ and $\rho^n(t)$ on the interval $[0,T_1]$, by observing that they are the solution of the same equation

$$\partial_t\mu+\partial_x\left(\mu v\left(\int w_{m_n}(y-x)\,d\mu(y)\right)\right)=0 \tag{4.17}$$

with different initial data, that are $[\tilde\rho(0)]^n=[\rho_0]^n$ and $\rho^n(0,\cdot)=\rho_0$. Since e̊-step21 satisfies the hypotheses of Proposition 2.10, we have

$$\begin{aligned}W_1([\tilde\rho(t)]^n,\rho^n(t))&\le e^{8\mathrm{Lip}(v)\mathrm{Lip}(w_{m_n})T_1}W_1([\rho_0]^n,\rho_0)\\&\le e^{16\mathrm{Lip}(v)w(0)\ln(n)T_1}W_1([\rho_0]^n,\rho_0)\le\sqrt{n}\frac{b-a}{n}=\frac{b-a}{\sqrt{n}}\end{aligned}$$

for a sufficiently big n, for which it holds that $\mathrm{Lip}(w_{m_n})\le 2\frac{w(0)}{m_n}\le 2w(0)\ln(n)$. We also used e̊-W1disc to estimate the initial distance. Then, $[\tilde\rho(t)]^n$ and ρ^n have the same limit in the interval $[0,T_1]$.

To prove the result for the initial interval $[0,T]$, subdivide it in intervals $[kT_1,(k+1)T_1]$ with $k=0,1,\dots,\frac{T}{T_1}-1$. Then, by induction one can prove $W_1([\tilde\rho(t)]^n,\rho^n(t))\le\frac{32\mathrm{Lip}(v)\,w(0)\,T(b-a)}{\sqrt{n}}$. As a consequence, the limit of ρ^n is ρ^* on the whole interval $[0,T]$.

Step 4. We now fix $T<(e\mathrm{Lip}(v)TV(w)\|\rho_0\|_{L^\infty})^{-1}$ and study $W_\infty(\rho^n(t),\rho(t))$. In this case, we observe that both ρ^n and ρ are solutions of an equation of the form e̊-cauchy with different interaction kernels w_{m_n} and w respectively, and the same initial data ρ_0. Observe that, by the particular structure of w and of its approximation w_{m_n}, it holds that $TV(w)=TV(w_{m_n})=2w(0)$ and $\|w_{m_n}-w\|_{L^1}\le\frac{m_n w(0)}{2}$. Then, by Proposition 3.7, we have

$$W_\infty(\rho^n(t),\rho(t))\le(e^{4eLt}-1)\frac{\|w_{m_n}-w\|_{L^1}}{TV(w)}\le(e^{4eLT}-1)\frac{m_n}{4}\le\frac{e^4-1}{4\ln(n)},$$

with $L=2w(0)\mathrm{Lip}(v)\|\rho_0\|_{L^\infty}$. This implies that $\rho=\rho^*$, i.e., convergence of $[\rho(t)]^n$ to ρ in the interval $[0,T]$.

Going back to Step 1, we recall that we chose a converging subsequence of $[\rho(t)]^n$, and we proved that the limit of such sub-sequence is ρ, which is the unique solution of e̊-cauchy. Then we have that the whole sequence $[\rho(t)]^n$ converges to ρ, in the time interval $[0,T]$ with $T<(e\mathrm{Lip}(v)TV(w)\|\rho_0\|_{L^\infty})^{-1}$.

We now prove that $\rho^*=\rho$ in the time interval $[0,T]$ for any $T>0$, and in particular that a solution for e̊-cauchy exists (and is unique) for all times. For $T>0$ fixed, divide the interval $[0,T]$ in intervals $[kT',(k+1)T']$ with $T'<(e\mathrm{Lip}(v)TV(w)\|\rho_0\|_{L^\infty})^{-1}$ and $k=0,1,\dots,\frac{T}{T'}-1$. Then, the previous results show that $\rho^*=\rho$ in $[0,T']$. This also implies

$$\|\rho(T')\|_{L^\infty}=\|\rho^*(T')\|_{L^\infty}\le\|\rho_0\|_{L^\infty}, \tag{4.18}$$

where we used estimate e̊-munLinf. Observe that T' satisfies hypotheses of Proposition 3.1 taking $\rho(T')$ as initial datum: this gives existence and uniqueness of the solution of e̊-cauchy on the time interval $[T',2T']$, that coincides with ρ^* on the same interval, and for which it holds that $\|\rho(2T')\|_{L^\infty}\le\|\rho_0\|_{L^\infty}$. By induction, we have $\rho^*=\rho$ on the whole interval $[0,T]$, that also gives $\|\rho(t)\|_{L^\infty}\le\|\rho_0\|_{L^\infty}$ for all $t\in[0,T]$, i.e., condition $\rho^*(t,x)\le\max(\rho_0)$ a.e. for $t\in[0,T],x\in\mathbb{R}$. ∎

4.3. Numerical simulations. We conclude this section about finite-dimensional approximation of e̊-cauchy with some numerical tests to illustrate the efficiency of the method. Let us consider problem e̊-cauchy with velocity function $v(y)=1-y$ and initial datum

$$\rho_0(x)=\begin{cases}0.8, & \text{if } -0.5<x<-0.1,\\ 0, & \text{otherwise.}\end{cases} \tag{4.19}$$

We consider both constant $w=1/\eta$ and linear decreasing $w=2(\eta-x)/\eta^2$ kernels, with $\eta=0.1$. We apply the discretization procedure (4.1), (4.2), (4.3), (4.6), (4.7) for increasing values of the system size $n=100,\ 500,\ 1000$. For better visualising the result, we plot the corresponding piecewise constant Lagrangian density [13, 15]

$$\hat{\rho}_n(t,\cdot):=\frac{1}{n}\sum_{i=1}^{n-1}\frac{1}{x_{i+1}(t)-x_i(t)}\chi_{[x_i(t),x_{i+1}(t)[},$$

and we compare it with the approximate solution computed by Lax–Friedrichs scheme with space step $\Delta x=0.0002$, see [7, 19] for details on the scheme.

Figure 4.1 shows the result of numerical integrations corresponding to the constant convolution kernel $w=1/\eta$. We can observe that the discontinuities are sharply captured, but the particle method suffers of spurious oscillations due to numerical errors in the solution of the stiff ODE system (4.7), related to the big Lipschitz constant of the regularised kernel (4.1) in the interval $[0,\eta+\ell/2]$ (we have used the MATLAB ODE solver `ode23tb` to solve system (4.7)). These oscillations are no more present in the case of the linear decreasing kernel $w=2(\eta-x)/\eta^2$, see Figure 4.2. Indeed, the behaviour of the kernel in the interval $[-\ell/2,0]$ has no impact on the resolution of system (4.7), since the minimal distance between particles $\ell=1/n$ is preserved.

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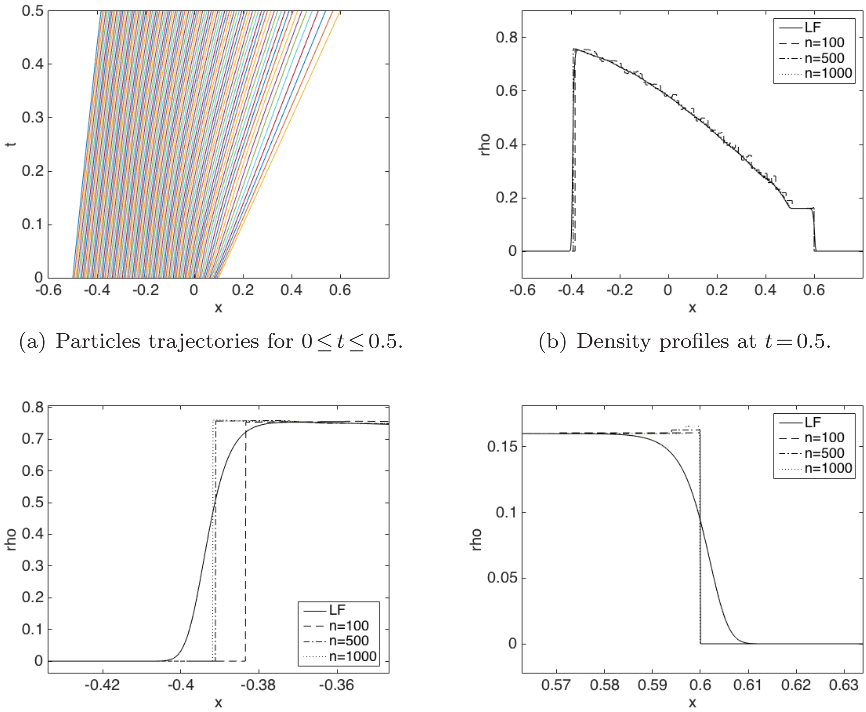

(a) Particles trajectories for $0 \leq t \leq 0.5$.

(b) Density profiles at $t = 0.5$.

(c) Zoom of upstream discontinuity in (b).

(d) Zoom of downstream jump in (b).

FIG. 4.1. *Approximate solutions* $\hat{\rho}_n$ *to* (1.1), (4.19) *with kernel* $w_\eta(x) = 1/\eta$ *obtained by the particle method with* $n = 100, 500, 1000$ *particles, and the corresponding Lax–Friedrichs approximation for* $\Delta x = 0.0002$.

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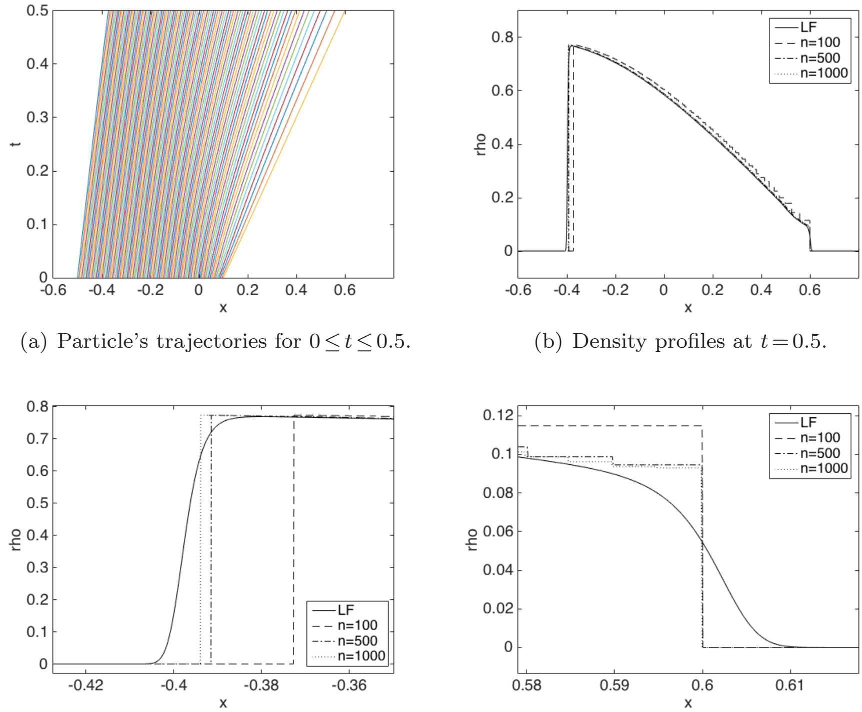

(a) Particle's trajectories for $0 \le t \le 0.5$. (b) Density profiles at $t = 0.5$.

(c) Zoom of upstream discontinuity in (b). (d) Zoom of downstream jump in (b).

FIG. 4.2. *Approximate solutions* $\hat{\rho}_n$ *to* (1.1), (4.19) *with kernel* $w = 2(\eta - x)/\eta^2$ *obtained by the particle method with* $n = 100, 500, 1000$ *particles, and the corresponding Lax–Friedrichs approximation for* $\Delta x = 0.0002$.

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