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PIECEWISE SMOOTH SOLUTIONS TO THE BURGERS–HILBERT EQUATION*

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Abstract. The paper is concerned with the Burgers–Hilbert equation $u_t + (u^2/2)_x = \mathbf{H}[u]$, where the right-hand side is a Hilbert transform. Unique entropy admissible solutions are constructed locally in time, having a single shock. In a neighborhood of the shock curve, a detailed description of the solution is provided.

Key words. Piecewise smooth solutions, Burgers–Hilbert equation, shock, uniqueness.

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1. Introduction

Consider the balance law obtained from Burgers' equation by adding the Hilbert transform as a source term

$$u_t + \left(\frac{u^2}{2}\right)_x = \mathbf{H}[u]. \tag{1.1}$$

Here,

$$\mathbf{H}[f](x) \doteq \lim_{\varepsilon\to 0+} \frac{1}{\pi}\int_{|y|>\varepsilon} \frac{f(x-y)}{y}\,dy \tag{1.2}$$

denotes the Hilbert transform of a function $f \in \mathbf{L}^2(\mathbb{R})$. The above equation was derived in [1] as a model for nonlinear waves with constant frequency. For initial data

$$u(0,x) = \bar{u}(x), \tag{1.3}$$

in $H^2(\mathbb{R})$, the local existence and uniqueness of the solution to Equation (1.1) was proved in [7], together with a sharp estimate on the time interval where this solution remains regular. See also [8] for a shorter proof. For general initial data $\bar{u} \in \mathbf{L}^2(\mathbb{R})$, the global existence of entropy weak solutions was recently proved in [4] together with a partial uniqueness result. We remark that, in this general setting, the well-posedness of the Cauchy problem remains a largely open question.

In the present paper, we consider an intermediate situation. Namely, we construct solutions of Equation (1.1) which are piecewise continuous, with a single shock. Our solutions have the form

$$u(t,x) = \varphi\big(x - y(t)\big) + w\big(t, x - y(t)\big),$$

where $t \mapsto y(t)$ denotes the location of the shock. Here, $w \in H^2\big(]-\infty,0[\cup]0,+\infty[\big)$, while $\varphi(x) = \frac{2}{\pi}|x|\ln|x|$, for x near the origin.

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In Section 2 we write Equation (1.1) in an equivalent form and state an existence-uniqueness theorem, locally in time. The key a priori estimates on approximate solutions and a proof of the main theorem are then worked out in sections 3–5.

The present results can be easily extended to the case of solutions with finitely many non-interacting shocks. An interesting open problem is to describe the local behavior of a solution in a neighborhood of a point (t_0,x_0) where either (i) a new shock is formed or (ii) two shocks merge into a single one. Motivated by the analysis in [12] we conjecture that, for generic initial data

$$\bar{u} \in H^2(\mathbb{R}) \cap \mathcal{C}^3(\mathbb{R}),$$

the corresponding solution of Equation (1.1) remains piecewise smooth with finitely many shock curves on any domain of the form $[0,T]\times\mathbb{R}$. We thus regard the present results as a first step toward a description of all generic singularities. For other examples of hyperbolic equations where generic singularities have been studied, we refer to [2,3,5, 6,9]. The possible emergence of singularities, for more general dispersive perturbations of Burgers' equation, has been recently studied in [10].

2. Statement of the main result

Consider a piecewise smooth solution of Equation (1.1) with one single shock. Calling $y(t)$ the location of the shock at time t, by the Rankine–Hugoniot conditions, we have

$$\dot{y}(t) = \frac{u^-(t)+u^+(t)}{2}. \tag{2.1}$$

where u^-,u^+ denote the left and right limits of $u(t,x)$ as $x \to y(t)$. Here and in the sequel, the upper dot denotes a derivative with respect to time. It is convenient to shift the space coordinate, replacing x with $x-y(t)$, so that in the new coordinate system the shock is always located at the origin. In these new coordinates, Equation (1.1) takes the equivalent form

$$u_t + \left(\frac{u^2}{2}\right)_x - \dot{y}\, u_x = \mathbf{H}[u]. \tag{2.2}$$

We shall construct solutions to Equation (2.2) in a special form, providing a cancellation between leading order terms in the transport equation and the Hilbert transform.

Consider a smooth function with compact support $\eta \in \mathcal{C}_c^\infty(\mathbb{R})$, with $\eta(x)=\eta(-x)$, and such that

$$\begin{cases} \eta(x)=1 & \text{if} \quad |x|\le 1, \\ \eta(x)=0 & \text{if} \quad |x|\ge 2, \\ \eta'(x)\le 0 & \text{if} \quad x\in[1,2]. \end{cases} \tag{2.3}$$

Moreover, define

$$\varphi(x) \doteq \frac{2|x|\ln|x|}{\pi}\cdot \eta(x). \tag{2.4}$$

Notice that φ has support contained in the interval $[-2,2]$ and is smooth separately on the domains $\{x<0\}$ and $\{x>0\}$.

In addition, we consider the space of functions

$$\mathcal{H} \doteq H^2\big(]-\infty,0[\cup]0,+\infty[\big). \tag{2.5}$$

Every function $w \in \mathcal{H}$ is continuously differentiable outside the origin. The distributional derivative of w_x is an $\mathbf{L}^2$ function restricted to the half lines $]-\infty,0[$ and $]0,+\infty[$. However, both w and w_x can have a jump at the origin. It is clear that the traces

$$\begin{cases} u^- \doteq w(0-), \\ u^+ \doteq w(0+), \end{cases} \qquad \begin{cases} b^- \doteq w_x(0-), \\ b^+ \doteq w_x(0+) \end{cases} \tag{2.6}$$

are continuous linear functionals on $\mathcal{H}$.

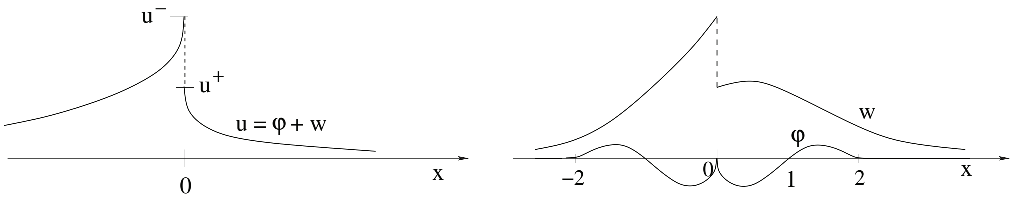

FIG. 2.1. *Decomposing a piecewise regular function $u=\varphi+w$ as a sum of the function φ defined at (2.4) and a function $w \in H^2(\mathbb{R}\setminus\{0\})$, continuously differentiable outside the origin.*

Solutions of Equation (2.2) will be constructed in the form

$$u(t,x) = \varphi(x) + w(t,x). \tag{2.7}$$

In order that the shock be entropy admissible, the function w should range in the open domain

$$\mathcal{D} \doteq \Big\{ w \in H^2\big(\mathbb{R}\setminus\{0\}\big); \quad w(0-) > w(0+) \Big\}. \tag{2.8}$$

By Equations (2.6)–(2.8), for $x \approx 0$, this solution has the asymptotic behavior

$$u(t,x) = \begin{cases} u^-(t) + b^-(t)x + \frac{2|x|\ln|x|}{\pi} + \mathcal{O}(1)\cdot|x|^{3/2} & \text{if } x<0, \\ u^+(t) + b^+(t)x + \frac{2|x|\ln|x|}{\pi} + \mathcal{O}(1)\cdot|x|^{3/2} & \text{if } x>0 \end{cases} \tag{2.9}$$

for suitable functions $u^\pm, b^\pm$. Here and throughout the sequel, the Landau symbol $\mathcal{O}(1)$ denotes a uniformly bounded quantity.

Substutiting Equation (2.7) into Equation (2.2) and recalling Equation (2.6), one obtains

$$w_t + \left(\varphi + w - \frac{u^- + u^+}{2} \right)(\varphi_x + w_x) = \mathbf{H}[\varphi] + \mathbf{H}[w]. \tag{2.10}$$

To derive estimates on the Hilbert transform, the following observation is useful. Consider a function f with compact support, continuously differentiable for $x<0$ and for $x>0$, with a jump at the origin. Then, for any $x\neq 0$, an integration by parts yields[1]

$$\mathbf{H}[f](x)=\frac{1}{\pi}\int_{-\infty}^{\infty} f'(y)\ln|x-y|\,dy+\frac{1}{\pi}\big[f(0+)-f(0-)\big]\ln|x|. \tag{2.11}$$

A similar computation shows that, to leading order, the Hilbert transform of w near the origin is given by

$$\mathbf{H}[w](x)=\frac{u^+-u^-}{\pi}\ln|x|+\mathcal{O}(1), \tag{2.12}$$

with u^-,u^+ as in Equation (2.6). On the other hand, for $x\approx 0$, one has

$$\begin{aligned}&\left(\varphi(x)+w(x)-\frac{w(0-)+w(0+)}{2}\right)\varphi_x(x)\\ &=\left(\mathrm{sign}(x)\cdot\frac{u^+-u^-}{2}+\mathcal{O}(1)\cdot|x|\ln|x|\right)\cdot\frac{2\,\mathrm{sign}(x)\cdot\big(1+\ln|x|\big)}{\pi}\\ &=\frac{u^+-u^-}{\pi}\ln|x|+\mathcal{O}(1).\end{aligned} \tag{2.13}$$

The identity between the leading terms in Equations (2.12) and (2.13) achieves a crucial cancellation between the two sides of Equation (2.10). It is thus convenient to write this equation in the equivalent form

$$w_t+\left(\varphi+w-\frac{u^-+u^+}{2}\right)w_x=\mathbf{H}[\varphi]-\varphi\varphi_x+\left(\mathbf{H}[w]-\left(w-\frac{u^-+u^+}{2}\right)\varphi_x\right). \tag{2.14}$$

Definition. *By an* **entropic solution** *to the Cauchy problem (2.10) with initial data*

$$w(0,\cdot)=\bar w\ \in\ \mathcal{D}, \tag{2.15}$$

we mean a function $w:[0,T]\times\mathbb{R}\mapsto\mathbb{R}$ such that

(i) For every $t\in[0,T]$, the norm $\|w(t,\cdot)\|_{H^2(\mathbb{R}\setminus\{0\})}$ remains uniformly bounded. As $x\to 0$, the limits satisfy

$$u^-(t)\doteq u(t,0-)\ >\ u(t,0+)\ \doteq\ u^+(t). \tag{2.16}$$

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[1] Indeed, if $f\in\mathcal{C}_c^\infty(\mathbb{R})$, then, for a suitably large constant M, we have

$$\begin{aligned}\pi\cdot\mathbf{H}[f](x)&=\lim_{\varepsilon\to 0+}\int_{|y-x|>\varepsilon}\frac{f(x-y)}{y}\,dy=-\lim_{\varepsilon\to 0+}\int_{|y-x|>\varepsilon}\frac{f(x+y)}{y}\,dy\\ &=-\lim_{\varepsilon\to 0+}\left(\int_{-M}^{-\varepsilon}+\int_{\varepsilon}^{M}\right)\frac{f(x+y)-f(x)}{y}\,dy\\ &=\lim_{\varepsilon\to 0+}\left(\int_{-M}^{-\varepsilon}+\int_{\varepsilon}^{M}\right)f'(x+y)\ln|y|\,dy-\lim_{\varepsilon\to 0+}[f(x-\varepsilon)-f(x)]\ln\varepsilon\\ &\qquad+\lim_{\varepsilon\to 0+}[f(x+\varepsilon)-f(x)]\ln\varepsilon+[f(x-M)-f(x)]\ln M-[f(x+M)-f(x)]\ln M\\ &=\int_{-\infty}^{\infty}f'(x+y)\ln|y|\,dy=\int_{-\infty}^{\infty}f'(y)\ln|x-y|\,dy.\end{aligned}$$

By approximating f with a sequence of smooth functions with compact support, we obtain Equation (2.11).

(ii) *Equation (2.14) is satisfied in integral sense. Namely, for every* $t_0 \geq 0$ *and* $x_0 \neq 0$*, calling* $t \mapsto x(t;t_0,x_0)$ *the solution to the Cauchy problem*

$$\dot{x} \doteq \varphi(x) + w(t,x) - \frac{u^-(t)+u^+(t)}{2}, \qquad x(t_0) = x_0, \tag{2.17}$$

one has

$$w(t_0,x_0) = \bar{w}(x(0;t_0,x_0)) + \int_0^{t_0} F(t,x(t;t_0,x_0))\,dt, \tag{2.18}$$

with

$$F \doteq \mathbf{H}[\varphi] - \varphi\varphi_x + \left(\mathbf{H}[w] - \left(w - \frac{u^-+u^+}{2}\right)\varphi_x\right). \tag{2.19}$$

A few remarks are in order:

(i) The bound on the norm $\|w(t,\cdot)\|_{H^2}$ implies that the limits in Equation (2.16) are well defined. By requiring that the inequality in Equation (2.16) holds, we make sure that the shock is entropy admissible.

(ii) Since $w(t,\cdot) \in H^2(\mathbb{R}\setminus\{0\})$, the right-hand side of the ODE in Equation (2.17) is continuously differentiable with respect to x. Combined with the inequalities in Equation (2.16), this implies that the backward characteristic $t \mapsto x(t;t_0,x_0)$ is well defined for all $t \in [0,t_0]$.

(iii) In [11], a function satisfying the integral Equations (2.18) was called a **broad solution**. The regularity assumption on $w(t,\cdot)$ and the fact that the source term F in Equation (2.19) is continuous outside the origin imply that $w = w(t,x)$ is continuously differentiable with respect to both variables t,x for $x \neq 0$. Therefore, the identity in Equation (2.14) is satisfied at every point (t,x), with $x \neq 0$.

The main result of this paper provides the existence and uniqueness of an entropic solution, locally in time.

THEOREM 2.1. *For every* $\bar{w} \in \mathcal{D}$*, there exists* $T > 0$ *such that the Cauchy problem (2.2), (2.15) admits a unique entropic solution, defined for* $t \in [0,T]$.

In turn, Theorem 2.1 yields the existence of a piecewise regular solution to the Burgers–Hilbert equation (1.1), locally in time, for initial data of the form

$$u(0,x) = \varphi(x) + \bar{w}(x),$$

with $\bar{w} \in \mathcal{D}$.

The solution $w = w(t,x)$ of Equation (2.14) will be obtained as a limit of a sequence of approximations. More precisely, for $n = 1$, we define

$$w_1(t,\cdot) = \bar{w} \qquad \text{for all} \quad t \geq 0. \tag{2.20}$$

Next, let the nth approximation $w_n(t,x)$ be constructed. By induction, we then define $w_{n+1}(t,x)$ to be the solution of the linear, non-homogeneous Cauchy problem

$$w_t + \left(\varphi + w_n - \frac{u_n^- + u_n^+}{2}\right) w_x = \mathbf{H}[\varphi] - \varphi\varphi_x + \left(\mathbf{H}[w] - \left(w - \frac{u^-+u^+}{2}\right)\varphi_x\right). \tag{2.21}$$

with initial data (2.15).

The induction argument requires the following three steps:

(i) Existence and uniqueness of solutions to the linear problem (2.21) with initial data (2.15).

(ii) A priori bounds on the strong norm $\|w_n(t)\|_{H^2(\mathbb{R}\setminus\{0\})}$, uniformly valid for $t \in [0,T]$ and all $n \geq 1$.

(iii) Convergence in a weak norm, which will follow from the bound

$$\sum_{n\geq 1} \|w_{n+1}(t) - w_n(t)\|_{H^1(\mathbb{R}\setminus\{0\})} < \infty.$$

In the following sections, we shall provide estimates on each term on the right-hand side of Equation (2.21) and complete the above steps (i)–(iii).

3. Estimates on the source terms

To estimate the right-hand side of Equation 2.21), we consider again the cutoff function η in Equation (2.3) and split an arbitrary function $w \in H^2(\mathbb{R}\setminus\{0\})$ as a sum:

$$w = v_1 + v_2 + v_3, \tag{3.1}$$

where

$$v_1(x) \doteq \begin{cases} w(0-)\cdot\eta(x) & \text{if } x<0, \\ w(0+)\cdot\eta(x) & \text{if } x>0, \end{cases} \qquad v_2(x) \doteq \begin{cases} w_x(0-)\cdot x\eta(x) & \text{if } x<0, \\ w_x(0+)\cdot x\eta(x) & \text{if } x>0, \end{cases} \tag{3.2}$$

$$v_3 = w - v_1 - v_2. \tag{3.3}$$

The right-hand side of Equation (2.21) can be expressed as the sum of the following terms:

$$A \doteq \mathbf{H}[\varphi], \qquad B \doteq \varphi\varphi_x, \qquad C \doteq \mathbf{H}[v_2+v_3], \qquad D \doteq \mathbf{H}[v_1] - \Big(w - \frac{u^- + u^+}{2}\Big)\varphi_x. \tag{3.4}$$

The goal of this section is to provide a priori bounds of the size of these source terms and on their first and second derivatives.

LEMMA 3.1. *There exist constants K_0, K_1 such that the following holds. For any $\delta \in]0, 1/2]$ and any $w \in H^2(\mathbb{R}\setminus\{0\})$, the source terms in (3.4) satisfy*

$$\|A\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} + \|B\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} \leq K_0 \cdot \delta^{-2/3}, \tag{3.5}$$

$$\|C\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} + \|D\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} \leq K_1 \delta^{-2/3} \cdot \|w\|_{H^2(\mathbb{R}\setminus\{0\})}. \tag{3.6}$$

Proof.

(1) We begin by observing that the function φ is continuous with compact support, smooth outside the origin. Therefore, the Hilbert transform $A = \mathbf{H}[\varphi]$ is smooth outside the origin. As $|x| \to \infty$, one has

$$A(x) = \mathcal{O}(1)\cdot x^{-1}, \qquad A_x(x) = \mathcal{O}(1)\cdot x^{-2}, \qquad A_{xx}(x) = \mathcal{O}(1)\cdot x^{-3}. \tag{3.7}$$

In addition, as $x \to 0$, we claim that

$$A(x) = \mathcal{O}(1)\cdot x\ln^2|x|, \qquad A_x(x) = \mathcal{O}(1)\cdot \ln^2|x|, \qquad A_{xx}(x) = \mathcal{O}(1)\cdot \frac{\ln|x|}{x}. \tag{3.8}$$

Indeed, to fix the ideas, let $0<x<1/2$. By Equation (2.11), we have

$$\pi\cdot\mathbf{H}[\varphi](x)=\int_{-2}^{2}\varphi'(y)\ln|x-y|\,dy=I_1+I_2+I_3, \tag{3.9}$$

where

$$I_1\doteq\left(\int_{-2}^{-1}+\int_{1}^{2}\right)\varphi'(y)\ln|x-y|\,dy=\mathcal{O}(1)\cdot x, \tag{3.10}$$

$$\frac{\pi}{2}I_2\doteq\int_{-1}^{0}-\ln|x-y|\,dy+\int_{0}^{1}\ln|x-y|\,dy=\left(\int_{-x}^{x}-\int_{1-x}^{1+x}\right)\ln|y|\,dy=O(1)\cdot x\ln x, \tag{3.11}$$

and, moreover,

$$\begin{aligned}\frac{\pi}{2}I_3&\doteq\int_{0}^{1}\ln|y|\ln|x-y|\,dy+\int_{-1}^{0}-\ln|y|\ln|x-y|dy\\&=\left(\int_{0}^{x/2}+\int_{x/2}^{x}+\int_{x-1}^{0}-\int_{-1}^{0}\right)\ln|y|\ln|x-y|\,dy\\&=\left(\int_{0}^{x/2}+\int_{x/2}^{x}\right)\ln|y|\ln|x-y|\,dy-\int_{0}^{x}\ln|y-1|\ln|x-y+1|\,dy\\&\doteq I_{31}+I_{32}+I_{33}.\end{aligned} \tag{3.12}$$

We now have

$$\begin{aligned}|I_{31}|&\le\ln\left|\frac{x}{2}\right|\cdot\int_{0}^{x/2}\ln|y|\,dy=O(1)\cdot x\ln^2|x|,\\|I_{32}|&\le\ln\left|\frac{x}{2}\right|\cdot\int_{x/2}^{x}\ln|x-y|\,dy=O(1)\cdot x\ln^2|x|,\\|I_{33}|&\le\int_{0}^{x}\ln|1-x|\ln|1+x||\,dy=O(1)\cdot x^3.\end{aligned} \tag{3.13}$$

Hence, $\mathbf{H}[\varphi]=O(1)\cdot x\ln^2|x|$. This yields the first estimate in Equation (3.8).

Next, we estimate the derivative $\pi\partial_x\mathbf{H}[\varphi]=\partial_xI_1+\partial_xI_2+\partial_xI_3$. The term $|\partial_xI_1|$ is uniformly bounded, while

$$\frac{\pi}{2}\partial_xI_2=\int_{0}^{2x}\frac{1}{x-y}\,dy+\int_{2x}^{1}\frac{1}{x-y}\,dy-\int_{-1}^{0}\frac{1}{x-y}\,dy=O(1)\cdot\ln|x|. \tag{3.14}$$

Differentiating I_3 with respect to x, we obtain

$$\begin{aligned}\frac{\pi}{2}\partial_xI_3=&\left(\int_{-1}^{-x/2}+\int_{-x/2}^{0}\right)\frac{-\ln|y|}{x-y}\,dy+\left(\int_{0}^{x/2}+\int_{3x/2}^{1}\right)\frac{\ln|y|}{x-y}\,dy\\&+\lim_{\epsilon\to0}\left(\int_{x/2}^{x-\epsilon}+\int_{x+\epsilon}^{3x/2}\right)\frac{\ln y}{x-y}\,dy.\end{aligned} \tag{3.15}$$

Assuming $0<x<1/2$, we obtain

$$\int_{-1}^{-x/2} \frac{-\ln|y|}{x-y}\,dy \le \int_{-1}^{x/2} \frac{-\ln|y|}{|y|}\,dy = O(1)\cdot \ln^2|x|,$$

$$\int_{-x/2}^{0} \frac{-\ln|y|}{x-y}\,dy \le \int_{-x/2}^{0} \frac{-\ln|y|}{x}\,dy = O(1)\cdot \ln|x|,$$

$$\int_{0}^{x/2} \frac{\ln|y|}{x-y}\,dy \le \int_{0}^{x/2} \frac{\ln|y|}{x/2}\,dy = O(1)\cdot \ln|x|,$$

$$\int_{3x/2}^{1} \frac{\ln|y|}{x-y}\,dy \le \ln\left|\frac{3x}{2}\right| \int_{3x/2}^{1} \frac{1}{x-y}\,dy = O(1)\cdot \ln^2|x|.$$

The remaining term is estimated as

$$\left(\int_{x/2}^{x-\epsilon} + \int_{x-\epsilon}^{3x/2}\right) \frac{\ln y}{x-y}\,dy = \left(\int_{x/2}^{x-\epsilon} + \int_{x-\epsilon}^{3x/2}\right) \frac{\ln y - \ln x}{x-y}\,dy \le \frac{2}{x}(x-2\epsilon) \le 2.$$

Combining the previous estimates, we obtain $\partial_x \mathbf{H}[\varphi](x) = O(1)\cdot \ln^2|x|$. This gives the second estimate in Equation (3.8).

Finally, we estimate the second derivative of the Hilbert transform $\partial_{xx}\mathbf{H}[\varphi] = \sum_{i=1}^{3} \partial_{xx}(I_i)$. By Equations (3.10) and (3.14), we obtain

$$\partial_{xx} I_1 = \mathcal{O}(1),$$

$$\frac{\pi}{2}\partial_{xx} I_2 = -\int_{2x}^{1} \frac{1}{(x-y)^2}\,dy + \int_{-1}^{0} \frac{1}{(x-y)^2}\,dy = O(1)\cdot \frac{\ln|x|}{x}. \tag{3.16}$$

$$\begin{aligned}\frac{\pi}{2}\partial_{xx} I_3 = {}& \left(\int_{-1}^{-x/2} + \int_{-x/2}^{0}\right) \frac{\ln|y|}{(x-y)^2}\,dy - \left(\int_{0}^{x/2} + \int_{3x/2}^{1}\right) \frac{\ln|y|}{(x-y)^2}\,dy \\ & + \frac{\ln|x/2|}{x} + \frac{3\ln|3x/2|}{x} + \partial_x\left(\int_{x/2}^{3x/2} \frac{\ln|y|}{x-y}\,dy\right).\end{aligned} \tag{3.17}$$

Assuming $0<x<1/2$, we obtain

$$\begin{aligned}
&\left|\int_{-1}^{-x/2} \frac{\ln|y|}{(x-y)^2}\,dy\right| \le \ln\left|\frac{x}{2}\right| \int_{-1}^{x/2} \frac{1}{(x-y)^2}\,dy = O(1)\cdot \frac{\ln|x|}{x},\\
&\left|\int_{-x/2}^{0} \frac{\ln|y|}{(x-y)^2}\,dy\right| \le \int_{-x/2}^{0} \frac{-\ln|y|}{x^2}\,dy = O(1)\cdot \frac{\ln|x|}{x},\\
&\left|\int_{0}^{x/2} \frac{\ln|y|}{(x-y)^2}\,dy\right| \le \int_{0}^{x/2} \frac{\ln|y|}{(x/2)^2}\,dy = O(1)\cdot \frac{\ln|x|}{x},\\
&\left|\int_{3x/2}^{1} \frac{\ln|y|}{(x-y)^2}\,dy\right| \le \ln\left|\frac{3x}{2}\right| \int_{3x/2}^{1} \frac{1}{(x-y)^2}\,dy = O(1)\cdot \frac{\ln|x|}{x}.
\end{aligned} \tag{3.18}$$

The remaining term is estimated by

$$\begin{aligned}\partial_x\left(\int_{x/2}^{3x/2}\frac{\ln|y|}{x-y}\right)dy &= \partial_x\left(\int_{-x/2}^{x/2}\frac{\ln|x-y|}{y}\,dy\right)\\ &= \int_{-x/2}^{x/2}\frac{1}{y(x-y)}\,dy+\frac{\ln|x/2|}{x}-\frac{\ln|3x/2|}{x},\end{aligned}\tag{3.19}$$

where

$$\left|\int_{-x/2}^{x/2}\frac{1}{y(x-y)}\,dy\right|=\left|\int_{-x/2}^{x/2}\frac{1}{y}\Big(\frac{1}{x-y}-\frac{1}{x}\Big)\,dy\right|=\left|\int_{-x/2}^{x/2}\frac{1}{x(x-y)}\,dy\right|\le\frac{2}{x}.\tag{3.20}$$

Therefore, by Equations (3.16) and (3.18)–(3.20), we have $\partial_{xx}\mathbf{H}[\varphi](x)=O(1)\cdot\dfrac{\ln|x|}{x}$.

(2) The function $B=\varphi\varphi_x$ is smooth outside the origin and vanishes for $|x|\ge 2$. As $x\to 0$, the following estimates are straightforward:

$$B(x)=\mathcal{O}(1)\cdot|x|\ln^2|x|,\qquad B_x(x)=\mathcal{O}(1)\cdot\ln^2|x|,\qquad B_{xx}(x)=\mathcal{O}(1)\cdot\frac{\ln|x|}{|x|}.\tag{3.21}$$

(3) Next, we observe that $v_3\in H^2(\mathbb{R})$. Moreover, there exists a constant C_η such that

$$\|v_3\|_{H^2(\mathbb{R})}\le C_\eta\cdot\|w\|_{H^2(\mathbb{R}\setminus\{0\})}.$$

Clearly, the Hilbert transform $\mathbf{H}[v_3]$ satisfies the same bounds. Hence,

$$\big\|\mathbf{H}[v_3]\big\|_{H^2(\mathbb{R})}=\mathcal{O}(1)\cdot\|w\|_{H^2(\mathbb{R}\setminus\{0\})}.\tag{3.22}$$

We observe that v_2 is Lipschitz continuous, has compact support, and is continuously differentiable outside the origin. Since v_2 has better regularity properties than φ, the same arguments used to estimate the Hilbert transform of φ also apply to $\mathbf{H}[v_2]$. More precisely, as in Equation (3.7), for $|x|\to\infty$, we have

$$\mathbf{H}[v_2](x)=\mathcal{O}(1)\cdot x^{-1},\qquad \mathbf{H}[v_2]_x(x)=\mathcal{O}(1)\cdot x^{-2},\qquad \mathbf{H}[v_2]_{xx}(x)=\mathcal{O}(1)\cdot x^{-3}.\tag{3.23}$$

As in (3.8), for $x\to 0$ we have

$$\begin{aligned}&\mathbf{H}[v_2](x)=\mathcal{O}(1)\cdot x\ln^2|x|,\qquad \mathbf{H}[v_2]_x(x)=\mathcal{O}(1)\cdot\ln^2|x|,\\ &\mathbf{H}[v_2]_{xx}(x)=\mathcal{O}(1)\cdot\tfrac{\ln|x|}{x}.\end{aligned}\tag{3.24}$$

The only difference is that in Equations (3.23)–(3.24) by $\mathcal{O}(1)$ we now denote a quantity such that

$$|\mathcal{O}(1)|\le C\cdot\|w\|_{H^2(\mathbb{R}\setminus\{0\})},\tag{3.25}$$

for some constant C independent of w.

(4) Finally, observing that the the function v_1 in Equation (3.2) has compact support, for $|x|\to\infty$, we have the bounds

$$D(x)=\mathbf{H}[v_1](x)=\mathcal{O}(1)\cdot x^{-1}\qquad D_x(x)=\mathcal{O}(1)\cdot x^{-2},\qquad D_{xx}(x)=\mathcal{O}(1)\cdot x^{-3}.\tag{3.26}$$

On the other hand, for $x\to 0$, we claim that

$$D(x)=\mathcal{O}(1), \qquad D_x(x)=\mathcal{O}(1)\cdot\ln|x|, \qquad D_{xx}(x)=\mathcal{O}(1)\cdot|x|^{-1}, \tag{3.27}$$

where $\mathcal{O}(1)$ is a quantity satisfying Equation (3.25). Indeed, without loss of generality, we can assume $0<x<1/2$. Recalling the construction of w and φ, we have

$$\left(w-\frac{u^-+u^+}{2}\right)\varphi_x=\frac{(u^+-u^-)\ln|x|}{\pi}+\mathcal{O}(1). \tag{3.28}$$

The Hilbert transform of v_1 is computed by

$$\begin{aligned}\pi\mathbf{H}[v_1]&=\int_{-\infty}^{+\infty}\frac{v_1(y)}{x-y}\,dy\\&=\left(\int_{-2}^{-1}+\int_1^2\right)\frac{v_1(y)}{x-y}\,dy+\int_{-1}^0\frac{u^-}{x-y}\,dy+\left(\int_0^{x/2}+\int_{3x/2}^1\right)\frac{u^+}{x-y}\,dy+\int_{x/2}^{3x/2}\frac{u^+}{x-y}\,dy.\end{aligned}$$

The first term on the right-hand side is bounded and the last term vanishes, in the principal value sense. The second term is computed by

$$\int_{-1}^0\frac{u^-}{x-y}\,dy=u^-(-\ln|x|+\ln|x+1|)=-u^-\ln|x|+\mathcal{O}(1)\cdot|x|,$$

while the remaining integrals are estimated by

$$\left(\int_0^{x/2}+\int_{3x/2}^1\right)\frac{u^+}{x-y}\,dy=u^+(\ln|x|-\ln|x-1|)=u^+\ln|x|+\mathcal{O}(1)\cdot|x|.$$

Combining the previous estimates, we obtain

$$\mathbf{H}[v_1]=\frac{(u^+-u^-)\ln|x|}{\pi}+\mathcal{O}(1). \tag{3.29}$$

Next, we estimate the derivative $D_x(x)$. We have

$$\partial_x\left(w-\frac{u^++u^-}{2}\right)\cdot\varphi_x=\mathcal{O}(1)\cdot\ln|x|, \qquad \left(w-\frac{u^++u^-}{2}\right)\varphi_{xx}=\frac{u^+-u^-}{\pi x}+\mathcal{O}(1). \tag{3.30}$$

To estimate the derivative of $\mathbf{H}[v_1]$, we write

$$\begin{aligned}\pi\cdot\partial_x\mathbf{H}[v_1]=&\left(\int_{-2}^{-1}+\int_1^2\right)\frac{-v_1(y)}{(x-y)^2}\,dy-\int_{-1}^0\frac{u^-}{(x-y)^2}\,dy\\&+\partial_x\left(\int_0^{x/2}+\int_{3x/2}^1\right)\frac{v_1(y)}{x-y}\,dy+\partial_x\int_{x/2}^{3x/2}\frac{v_1(y)}{x-y}\,dy.\end{aligned} \tag{3.31}$$

The first term on the right-hand side of Equation (3.31) is uniformly bounded. The second term is estimated by

$$-\int_{-1}^0\frac{u^-}{(x-y)^2}\,dy\;=-\;\frac{u^-}{x}+\mathcal{O}(1).$$

Furthermore, we have

$$\partial_x\left(\int_0^{x/2}+\int_{3x/2}^1\right)\frac{v_1(y)}{x-y}\,dy=\left(\int_0^{x/2}+\int_{3x/2}^1\right)\frac{-v_1(y)}{(x-y)^2}\,dy+\frac{4u^+}{x}$$
$$=\frac{-3u^+}{x}+\mathcal{O}(1)+\frac{4u^+}{x}=\frac{u^+}{x}+\mathcal{O}(1). \tag{3.32}$$

Lastly, since $v_1(x)=u^+$ for $x\in]0,1]$, we have

$$\partial_x\int_{x/2}^{3x/2}\frac{v_1(y)}{x-y}\,dy=\partial_x\int_{-x/2}^{x/2}\frac{u^+}{y}\,dy=0. \tag{3.33}$$

Combining the previous estimates, we thus obtain

$$\partial_x\mathbf{H}[v_1](x)=\frac{u^+-u^-}{\pi x}+\mathcal{O}(1).$$

Together with Equation (3.30), as $x\to 0$, this yields the asymptotic estimate

$$D_x(x)=\mathbf{H}[v_1]_x-\left[\left(w-\frac{u^-+u^+}{2}\right)\varphi_x\right]_x=\mathcal{O}(1)\cdot\ln|x|. \tag{3.34}$$

The second derivative D_{xx} is estimated in a similar way. Indeed, by Equations (3.1)–(3.3) and (3.30), we have

$$\partial_{xx}\left(w-\frac{u^-+u^+}{2}\varphi_x\right)=\partial_{xx}\left(w-\frac{u^++u^-}{2}\right)\varphi_x+\partial_x\left(w-\frac{u^++u^-}{2}\right)\varphi_{xx}$$
$$+\partial_x\left(w-\frac{u^-+u^+}{2}\varphi_x\right)\varphi_{xx}+\left(w-\frac{u^-+u^+}{2}\varphi_x\right)\varphi_{xxx}$$
$$=-\frac{u^+-u^-}{\pi x^2}+\mathcal{O}(1)\cdot\frac{1}{x}. \tag{3.35}$$

On the other hand, differentiating Equation (3.31) and recalling Equations (3.32) and (3.33), we have

$$\pi\cdot\partial_{xx}\mathbf{H}[v_1]=\left(\int_{-2}^{-1}+\int_1^2\right)\frac{2v_1(y)}{(x-y)^3}\,dy+\int_{-1}^0\frac{2u^-}{(x-y)^3}\,dy$$
$$+\partial_x\left(\int_0^{x/2}+\int_{3x/2}^1\right)\frac{-v_1(y)}{(x-y)^2}\,dy-\frac{4u^+}{x^2}+\partial_{xx}\int_{-x/2}^{x/2}\frac{u^+}{y}\,dy. \tag{3.36}$$

As before, the first term is uniformly bounded while the last term is zero. The second term is computed by

$$\int_{-1}^0\frac{2u^-}{(x-y)^3}\,dy=\frac{u^-}{x^2}+\mathcal{O}(1). \tag{3.37}$$

The third term is estimated by

$$\partial_x\left(\int_0^{x/2}+\int_{3x/2}^1\right)\frac{-v_1(y)}{(x-y)^2}\,dy=\left(\int_0^{x/2}+\int_{3x/2}^1\right)\frac{2v_1(y)}{(x-y)^3}\,dy-\frac{2u^+}{x^2}+\frac{6u^+}{x^2}$$

$$= \frac{3u^+}{x^2} + \mathcal{O}(1). \tag{3.38}$$

Combining the above estimates (3.35)–(3.38), we obtain

$$\begin{aligned} D_{xx} &= \mathbf{H}[v_1]_{xx} - \left[\left(w - \frac{u^- + u^+}{2}\right)\varphi_x\right]_{xx} \\ &= \frac{1}{\pi}\left(\frac{u^-}{x^2} + \frac{3u^+}{x^2} - \frac{4u^+}{x^2}\right) + \frac{u^+ - u^-}{\pi x^2} + \mathcal{O}(1)\cdot\frac{1}{x} = \mathcal{O}(1)\cdot\frac{1}{x}. \end{aligned} \tag{3.39}$$

(5) By the estimates (3.8) and (3.21), it follows

$$\begin{aligned} \|A+B\|_{H^2(\mathbb{R}\setminus[-\delta,\delta])} &= \mathcal{O}(1)\cdot\left(\int_\delta^1 \frac{\ln^2|x|}{x^2}\,dx\right)^{1/2} = \mathcal{O}(1)\cdot\left(\int_\delta^1 \frac{dx}{x^{7/3}}\right)^{1/2} \\ &= \mathcal{O}(1)\cdot(\delta^{-4/3})^{1/2} = \mathcal{O}(1)\cdot\delta^{-2/3}. \end{aligned} \tag{3.40}$$

Similarly, the estimates (3.6) follow from Equation (3.22) and Equations (3.26)–(3.27). □

4. Construction of approximate solutions

In this section, given an initial datum $\bar{w} \in \mathcal{D}$, we prove that all the approximate solutions w_n at (2.20)–(2.21) are well defined, on a suitably small time interval $[0,T]$.

As in Equation (2.6), we define

$$\begin{cases} \bar{u}^- \doteq \bar{w}(0-), \\ \bar{u}^+ \doteq \bar{w}(0+), \end{cases} \qquad \begin{cases} u_n^-(t) \doteq w_n(t,0-), \\ u_n^+(t) \doteq w_n(t,0+). \end{cases}$$

To fix the ideas, assume that the initial data $\bar{w} \in H^2(\mathbb{R}\setminus\{0\})$ satisfies

$$\bar{u}^- - \bar{u}^+ = 6\delta_0, \qquad \|\bar{w}\|_{H^2(\mathbb{R}\setminus\{0\})} = \frac{M_0}{2}, \tag{4.1}$$

for some (possibly large) constants $\delta_0, M_0 > 0$.

Choosing a time interval $[0,T]$ sufficiently small, we claim that, for each $n \geq 1$, the approximate solution w_n satisfies the a priori bounds

$$\begin{cases} |u_n^-(t) - \bar{u}^-| \leq \delta_0, \\ |u_n^+(t) - \bar{u}^+| \leq \delta_0, \end{cases} \qquad \|w_n(t)\|_{H^2(\mathbb{R}\setminus\{0\})} \leq M_0, \qquad \text{for all } t \in [0,T]. \tag{4.2}$$

This will be proved by induction. For $n = 1$ these bounds are a trivial consequence of the definition (2.20). In the following, we assume that the function $w_n = w_n(t,x)$ satisfies Equation (4.2), and we show that the same bounds are satisfied by w_{n+1}. We recall that w_{n+1} is defined as the solution to the linear Equation (2.21), with initial data (2.15).

A sequence of approximate solutions $w^{(k)}$ to the linear equation (2.21) will be constructed by induction on $k \in \{1,2,\ldots\}$ For notational convenience, we introduce the function

$$a(t,x) \doteq \varphi(x) + w_n(t,x) - \frac{u_n^-(t) + u_n^+(t)}{2}. \tag{4.3}$$

As in Equation (2.17), call $t \mapsto x(t;t_0,x_0)$ the solution to the Cauchy problem

$$\dot{x} \doteq a(t,x(t)), \qquad x(t_0) = x_0. \tag{4.4}$$

We begin by defining

$$w^{(1)}(t,x) \doteq \bar{w}(x). \tag{4.5}$$

By induction, if $w^{(k)}$ has been constructed, we then set

$$w^{(k+1)}(t_0,x_0) = \bar{w}(x(0;t_0,x_0)) + \int_0^{t_0} F^{(k)}(t,x(t;t_0,x_0))\,dt, \tag{4.6}$$

where $F^{(k)}$ is defined as in Equation (2.19), with w replaced by $w^{(k)}$ and $u^{\pm}(t) = w(t,0\pm)$ replaced by $w^{(k)}(t,0\pm)$, respectively.

Assuming that w_n satisfies Equation (4.2), we will show that every approximation $w^{(k)}$ to the linear Cauchy problem (2.21), (2.15) satisfies the same bounds, on a sufficiently small time interval $[0,T]$. Our first result deals with the solution to the linear transport Equation (4.7). We show that, within a sufficiently short time interval, the H^2 norm of the solution can be amplified at most by a factor of $3/2$.

Lemma 4.1. *Let $w_n = w_n(t,x)$ be a function that satisfies the bounds (4.2) for all $t > 0$, and define $a = a(t,x)$ as in Equation (4.3). Then there exists $T > 0$ small enough, depending only on δ_0, M_0, that the following holds. For any $\tau \in [0,T]$ and any solution w of the linear equation*

$$w_t + a(t,x) w_x = 0 \tag{4.7}$$

with initial datum

$$w(0) = \bar{w} \in H^2\big(\mathbb{R}\setminus[-\delta_0\tau, \delta_0\tau]\big),$$

one has

$$\|w(\tau)\|_{H^2(\mathbb{R}\setminus\{0\})} \le \frac{3}{2}\, \|\bar{w}\|_{H^2\left(\mathbb{R}\setminus[-\delta_0\tau,\delta_0\tau]\right)}. \tag{4.8}$$

Proof.

(1) Equation (4.7) can be solved by the method of characteristics, separately on the regions where $x < 0$ and $x > 0$. We observe that characteristics move toward the origin from both sides. In this first step, we prove that all characteristics starting at time $t = 0$ inside the interval $[-\delta_0\tau,\ \delta_0\tau]$ hit the origin before time τ (see Figure 4.1). Hence, the profile $w(\tau,\cdot)$ does not depend on the values of $\bar{w}$ on this interval.

We claim that there exists $\delta_1 > 0$ such that

$$\begin{cases} a(t,x) \le -\delta_0 & \text{for all } x \in]0,\delta_1], \\ a(t,x) \ge \delta_0 & \text{for all } x \in [-\delta_1, 0[. \end{cases} \tag{4.9}$$

Indeed, Equations (4.1) and (4.2) imply

$$a(t,0+) = \frac{u_n^+(t) - u_n^-(t)}{2} \le -2\delta_0. \tag{4.10}$$

Moreover, for $x > 0$, we have

$$\big|a(t,x) - a(t,0+)\big| \le \frac{2}{\pi}|x \ln x| + \int_0^x |w_{n,x}(t,y)|\,dy \le C_0\,|x|^{1/2}, \tag{4.11}$$

for some constant C_0 depending only on the norm $\|w_n(t,\cdot)\|_{H^2}$, hence only on M_0 in Equation (4.2). Choosing $\delta_1 > 0$ small enough such that $C_0\delta_1^{1/2} < \delta_0$, from Equations (4.10)–(4.11), we obtain the first inequality in Equation (4.9). The second inequality is proved in the same way. In addition, by choosing the time interval $[0,T]$ small enough, we can also assume

$$\delta_0 T \le \delta_1. \tag{4.12}$$

(2) Multiplying Equation (4.7) by $2w$, one finds

$$(w^2)_t + (aw^2)_x = a_x w^2. \tag{4.13}$$

Integrating Equation (4.13) over the domain

$$\Omega \doteq \Big\{(t,x);\ \ |x| > \delta_0(\tau - t),\ \ t \in [0,\tau]\Big\} \tag{4.14}$$

shown in Figure 4.1, we obtain

$$\int_{-\infty}^{\infty} w^2(\tau,x)\,dx \le \int_{|x|>\delta_0\tau} \bar w^2\,dx + \int_0^{\tau}\int_{|x|>\delta_0(\tau-t)} a_x w^2\,dx\,dt. \tag{4.15}$$

Indeed, by Equations (4.9) and (4.12), for every $\tau \in]0,T[$, the flow points outward along the boundary of the domain Ω. By Equation (4.3), the derivative a_x satisfies a bound of the form

$$|a_x(t,x)| \le C_a\big(1 + |\ln|x||\big), \tag{4.16}$$

where C_a is a constant depending only on the norm $\|w_n\|_{H^2}$ in Equatino (4.2). Taking the supremum of $|a_x(t,x)|$ over the set

$$\Omega_t \doteq \{x;\ \ |x| > \delta_0(\tau - t)\} \tag{4.17}$$

from Equation (4.15), we thus obtain

$$\|w(\tau)\|^2_{\mathbf{L}^2(\mathbb{R})} \le \|\bar w\|^2_{\mathbf{L}^2(\Omega_0)} + \int_0^{\tau} C_a\Big(1 + |\ln(\delta_0(\tau-t))|\Big)\,\|w(t)\|^2_{\mathbf{L}^2(\Omega_t)}\,dt. \tag{4.18}$$

By Gronwall's lemma, this yields a bound on $\|w(\tau)\|^2_{\mathbf{L}^2}$.

(3) Next, differentiating Equation (4.7) with respect to x and multiplying by $2w_x$ we obtain

$$w_{xt} + a w_{xx} = -a_x w_x, \qquad\qquad w_x(0,\cdot) = \bar w_x. \tag{4.19}$$

$$(w_x^2)_t + (a w_x^2)_x = -a_x w_x^2. \tag{4.20}$$

Integrating Equation (4.20) over the domain Ω in Equation (4.14) and using the bound (4.16) by similar computations as before, we now obtain

$$\|w_x(\tau)\|^2_{\mathbf{L}^2(\mathbb{R})} \le \|\bar w_x\|^2_{\mathbf{L}^2(\Omega_0)} + \int_0^{\tau} C_a\Big(1 + |\ln(\delta_0(\tau-t))|\Big)\,\|w_x(t)\|^2_{\mathbf{L}^2(\Omega_t)}\,dt. \tag{4.21}$$

By Gronwall's lemma, this yields a bound on $\|w_x(\tau)\|^2_{\mathbf{L}^2}$.

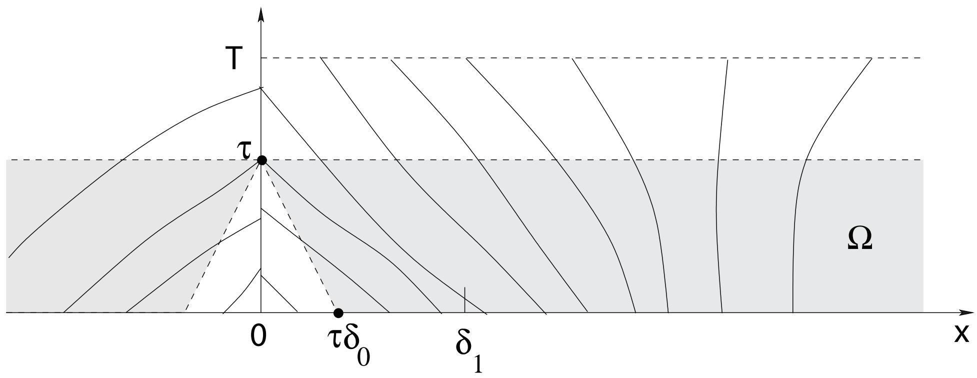

FIG. 4.1. *The norm $\|w(\tau)\|_{H^2(\mathbb{R}\setminus\{0\})}$ is estimated by using the balance laws for w^2, w_x^2, w_{xx}^2 on the shaded domain Ω. By Equation (4.9), along the boundary where $|x| = \delta_0(\tau - t)$, all characteristics move outward. Hence, no inward flux is present.*

(4) Differentiating Equation (4.19) once again and multiplying all terms by $2w_{xx}$, we find

$$w_{xxt} + aw_{xxx} = -2a_x w_{xx} - a_{xx} w_x, \qquad w_{xx}(0,\cdot) = \bar{w}_{xx}, \tag{4.22}$$

$$\left(w_{xx}^2\right)_t + \left(a w_{xx}^2\right)_x = -3 a_x w_{xx}^2 - 2a_{xx} w_x w_{xx}. \tag{4.23}$$

Integrating Equation (4.23) over the domain Ω in Equation (4.14), we obtain

$$\begin{aligned} &\int_{-\infty}^{\infty} w_{xx}^2(\tau,x)\,dx \\ &\leq \int_{|x|>\delta\tau} \bar{w}_{xx}^2(y)\,dy + \int_0^\tau \int_{|x|>\delta_0(\tau-t)} \Big(-3\,a_x w_{xx}^2 - 2a_{xx} w_x w_{xx}\Big)\,dx\,dt. \end{aligned} \tag{4.24}$$

To estimate the right-hand side of Equation (4.24), we observe that, for $|x|$ small,

$$\begin{aligned} |a_x| &= |\varphi_x + w_{n,x}| = \mathcal{O}(1)\cdot\Big(|\ln|x|| + \|w_n\|_{H^2}\Big), \\ |a_{xx}| &= |\varphi_{xx} + w_{n,xx}| = \mathcal{O}(1)\cdot \tfrac{1}{|x|} + |w_{n,xx}|. \end{aligned} \tag{4.25}$$

Recalling that $\varphi(x) = 0$ for $|x| \geq 2$, we have the bounds

$$\begin{aligned} E &\doteq |3\,a_x\,w_{xx}^2 + 2a_{xx} w_x w_{xx}| \\ &\leq \mathcal{O}(1)\cdot\big(1 + |\ln|x||\big) w_{xx}^2 + \mathcal{O}(1)\cdot\left(\frac{1}{|x|} + |w_{n,xx}|\right)\|w\|_{H^2}\, w_{xx}, \end{aligned} \tag{4.26}$$

$$\int_{\delta_0(\tau-t)}^{2} \frac{|w_{xx}(t,x)|}{x}\,dx \leq \left(\int_{\delta_0(t-s)}^{2} \frac{1}{x^2}\right)^{1/2} \|w_{xx}\|_{\mathbf{L}^2(\Omega_t)}$$

$$\leq \left(\frac{1}{\delta_0(t-s)}\right)^{1/2} \|w_{xx}\|_{\mathbf{L}^2(\Omega_t)}, \tag{4.27}$$

$$\begin{aligned}\int_0^\tau \int_{|x|>\delta_0(\tau-t)} E(t,x)\,dx dt \leq& \mathcal{O}(1)\cdot \int_0^\tau \big(1+|\ln\delta_0(\tau-t)|\big)\cdot \|w(t)\|^2_{H^2(\Omega_t)}\,dt\\ &+\mathcal{O}(1)\cdot \int_0^\tau [\delta_0(\tau-t)]^{-1/2}\cdot \|w(t)\|^2_{H^2(\Omega_t)}\,dt\\ &+\mathcal{O}(1)\cdot \int_0^\tau \|w_n(t)\|_{H^2}\cdot \|w(t)\|^2_{H^2(\Omega_t)}\,dt. \end{aligned}\tag{4.28}$$

(5) Calling $Z(t) \doteq \|w(t)\|_{H^2(\Omega_t)}$, by the estimates (4.18), (4.21), and (4.28), we obtain an integral inequality of the form

$$Z^2(\tau) \leq Z^2(0) + C_1\cdot \int_0^\tau \Big(1+|\ln\delta_0(\tau-t)| + [\delta_0(\tau-t)]^{-1/2} + M_0\Big) Z^2(t)\,dt. \tag{4.29}$$

By Gronwall's lemma, if $\tau>0$ is sufficiently small, this yields $Z(\tau)\leq \frac{3}{2}Z(0)$, proving Equation (4.8). □

The above estimate can be easily extended to the linear, non-homogeneous problem

$$w_t + a(t,x)w_x = F(t,x), \qquad w(0,x) = \bar w(x). \tag{4.30}$$

Indeed, in the same setting as Lemma 2, using Equation (4.8) and Duhamel's formula, for $\tau\in[0,T]$, we obtain

$$\|w(\tau,\cdot)\|_{H^2(\mathbb{R}\setminus\{0\})} \;\leq\; \frac{3}{2}\,\|\bar w\|_{H^2\left(\mathbb{R}\setminus[-\delta_0\tau,\,\delta_0\tau]\right)} + \frac{3}{2}\int_0^\tau \|F(t,\cdot)\|_{H^2\left(\mathbb{R}\setminus[-\delta_0(\tau-t),\,\delta_0(\tau-t)]\right)}\,dt. \tag{4.31}$$

Relying on Lemma 3.1 we now prove uniform H^2 bounds on all approximations $w^{(k)}$, on a suitably small time interval $[0,T]$.

LEMMA 4.2. *Let $w_n = w_n(t,x)$ be a function that satisfies the bounds (4.2) for all $t>0$, and define $a=a(t,x)$ as in (4.3). Then there exists $T>0$ small enough, depending only on δ_0, M_0 in Equation (4.1), so that the following holds. For every $k\geq 1$ and every $\tau\in[0,T]$, one has*

$$\|w^{(k)}(\tau)\|_{H^2(\mathbb{R}\setminus\{0\})} \leq M_0, \tag{4.32}$$

$$|w^{(k)}(\tau,0-) - \bar u^-| \leq \delta_0, \qquad |w^{(k)}(\tau,0+) - \bar u^+| \leq \delta_0. \tag{4.33}$$

Proof.

(1) Recalling the constants K_0, K_1 in Lemma 3.1, choose $T>0$ small enough so that

$$\int_0^T (\delta_0 s)^{-2/3}\,ds \;<\; \frac{M_0}{6(K_0+K_1M_0)}. \tag{4.34}$$

(2) The estimate (4.32) trivially holds for $w^{(1)}(\tau)\doteq \bar w$. Assuming that it holds for $w^{(k)}(t)$, $t\in[0,T]$, by Equation (4.31), for any $\tau\in[0,T]$, we have the estimate

$$\|w^{(k+1)}(\tau)\|_{H^2(\mathbb{R}\setminus\{0\})}$$

$$
\begin{aligned}
&\leq \frac{3}{2}\|\bar w\|_{H^2(\mathbb{R}\setminus\{0\})}+\frac{3}{2}\int_0^\tau \|A+B+C+D\|_{H^2(\mathbb{R}\setminus[-\delta_0(\tau-t),\delta_0(\tau-t)])}\,ds\\
&\leq \frac{3}{4}M_0+\frac{3}{2}\int_0^\tau K_0[\delta_0(\tau-t)]^{-2/3}\,dt+\frac{3}{2}\int_0^\tau K_1[\delta_0(\tau-t)]^{-2/3}\|w^{(k)}(t)\|_{H^2(\mathbb{R}\setminus\{0\})}\,dt\\
&\leq \frac{3}{4}M_0+\frac{3}{2}(K_0+K_1M_0)\int_0^\tau (\delta_0 s)^{-2/3}\,ds\\
&< \frac{3}{4}M_0+\frac{3}{2}(K_0+K_1M_0)\cdot\frac{M_0}{6(K_0+K_1M_0)}=M_0.
\end{aligned} \tag{4.35}
$$

By induction, this proves the bound (4.32).

(3) To prove the two estimates in Equation (4.33), we write

$$
\left|w^{(k+1)}(\tau,0+)-\bar u^+\right|\leq |\bar w(x(0;\tau,0+))-\bar u^+|+\tau\cdot \sup_{t\in[0,\tau]}\left\|(A+B+C+D)(t)\right\|_{\mathbf{L}^\infty}. \tag{4.36}
$$

The a priori bound on $\|w^{(k)}(t,\cdot)\|_{H^2(\mathbb{R}\setminus\{0\})}$ implies that the $\mathbf{L}^\infty$ norm in Equation (4.36) is uniformly bounded. By possibly choosing a smaller $T>0$, both terms on the right-hand side of Equation (4.36) will be $<\delta_0/2$. This yields the second inequality in Equation (4.33). The first inequality is proved in the same way. □

The next lemma shows that the sequence of approximations $w^{(k)}$ defined in Equations (4.5)–(4.6) converges to a solution to Equation (2.21).

LEMMA 4.3. *For some $T>0$ sufficiently small, the sequence of approximations $w^{(k)}(t,\cdot)$ converges in $H^2(\mathbb{R}\setminus\{0\})$ to a function $w=w(t,\cdot)$. The convergence is uniform for $t\in[0,T]$. This limit function provides a solution to the initial value problem (2.21) with initial data (2.15).*

Proof.

(1) By the previous bounds, the difference between two approximations can be estimated by

$$
\begin{aligned}
&\|w^{(k+1)}(\tau)-w^{(k)}(\tau)\|_{H^2(\mathbb{R}\setminus\{0\})}\\
&\leq \frac{3}{2}\int_0^\tau [\delta_0(\tau-t)]^{-2/3}K_1\|w^{(k)}(t)-w^{(k-1)}(t)\|_{H^2(\mathbb{R}\setminus[-\delta_0(\tau-t),\delta_0(\tau-t)]}\,dt.
\end{aligned} \tag{4.37}
$$

If $T>0$ is small enough, so that

$$
\frac{3}{2}\int_0^T (\delta_0 s)^{-2/3}K_1\,ds \;\leq\; \frac{1}{2},
$$

then, for every $\tau\in[0,T]$, the sequence $w^{(k)}(\tau,\cdot)$ is Cauchy in $H^2(\mathbb{R}\setminus\{0\})$, hence it converges to a unique limit function $w(\tau,\cdot)$.

(2) It remains to prove that that w provides a solution to the problem (2.21) with initial data (2.15), in the sense that the integral identities (2.18) are satisfied for all $t_0\in[0,T]$ and $x_0\neq 0$.

This is clear because, for every $\epsilon>0$ as $k\to\infty$, the source terms on the right-hand side of Equation (2.21) converge uniformly on the set $\{(t,x);\;\; t\in[0,T],\;\; |x|\geq\epsilon\}$. □

5. Convergence of the approximate solutions

By the analysis in the previous section, the sequence of approximate solutions w_n of the problem (2.21), (2.15) is well defined, on a suitably small time interval $[0,T]$. Moreover, the uniform bounds (4.2) hold.

To complete the proof of Theorem 2.1, it remains to show that the w_n converge to a limit function w, providing an entropic solution to the Cauchy problem (2.10), (2.15). Towards this goal, we prove that, on a suitably small time interval $[0,T]$, the sequence $(w_n)_{n\geq 1}$ constructed in Equation (2.21) is Cauchy with respect to the norm of $H^1(\mathbb{R}\setminus\{0\})$, hence it converges to a unique limit. This will be achieved in several steps.

Proof.

(1) For a fixed n, consider the differences

$$\begin{cases} W \doteq w_{n+1}-w_n, \\ W_n \doteq w_n - w_{n-1}, \end{cases} \qquad \begin{cases} U^- \doteq u^-_{n+1}-u^-_n, \\ U^-_n \doteq u^-_n - u^-_{n-1}, \end{cases} \qquad \begin{cases} U^+ \doteq u^-_{n+1}-u^+_n, \\ U^+_n \doteq u^+_n - u^+_{n-1}. \end{cases}$$

From Equation (2.21), we deduce

$$W_t+\left(\varphi+w_n-\tfrac{u_n^- - u_n^+}{2}\right)W_x+\left(W_n-\tfrac{U_n^-+U_n^+}{2}\right)w_{n,x}=\mathbf{H}[W]-\left(W-\tfrac{U^-+U^+}{2}\right)\varphi_x. \tag{5.1}$$

Multiplying both sides by $2W$, we obtain the balance law

$$\begin{aligned} &\left(W^2\right)_t+\left[\left(\varphi+w_n-\frac{u_n^- - u_n^+}{2}\right)W^2\right]_x \\ &=(\varphi+w_n)_x W^2-\left(W_n-\frac{U_n^-+U_n^+}{2}\right)2Ww_{n,x}+2\mathbf{H}[W]\cdot W-\left(W-\frac{U^-+U^+}{2}\right)2W\varphi_x. \end{aligned} \tag{5.2}$$

Integrating over the domain Ω in Equation (4.14) and observing that $\varphi_x(x)=\mathcal{O}(1)\big(1+|\ln|x||\big)$, we obtain

$$\begin{aligned} &\frac{1}{2}\int W^2(\tau,x)\,dx \\ &\leq -\int_0^\tau\int_{|x|>\delta_0(\tau-t)}\Big\{(\varphi+w_n)_x\cdot W^2 \\ &\qquad -\left(W_n-\frac{U_n^-+U_n^+}{2}\right)2Ww_{n,x}+2\mathbf{H}[W]\cdot W-\left(W-\frac{U^-+U^+}{2}\right)2W\varphi_x\Big\}\,dxdt \\ &=\mathcal{O}(1)\cdot\int_0^\tau\Big\{\big|\ln(\tau-t)\big|\cdot\|W(s)\|^2_{\mathbf{L}^2}+\|W_n(t)\|_{H^1}\|W(t)\|_{\mathbf{L}^2}+\|W(t)\|^2_{\mathbf{L}^2} \\ &\qquad\qquad +\big|\ln(\tau-t)\big|\cdot\|W(t)\|_{H^1}\|W(t)\|_{\mathbf{L}^2}\Big\}\,dt \\ &\leq C_3\cdot\int_0^\tau\|W(t)\|_{\mathbf{L}^2}\cdot\Big(\|W_n(t)\|_{H^1}+\big|\ln(\tau-t)\big|\,\|W(t)\|_{H^1}\Big)\,dt, \end{aligned} \tag{5.3}$$

for some constant C_3.

(2) Next, differentiating Equation (5.1) with respect to x, we obtain

$$W_{xt}+\left(\varphi+w_n-\frac{u_n^- - u_n^+}{2}\right)W_{xx}$$

$$+(\varphi_x+w_{n,x})W_x+\left(W_n-\frac{U_n^-+U_n^+}{2}\right)w_{n,xx}+W_{n,x}w_{n,x}$$
$$=\mathbf{H}[W_x]-\left(W-\frac{U^-+U^+}{2}\right)\varphi_{xx}-\varphi_x W_x. \tag{5.4}$$

Multiplying both sides by $2W_x$, we obtain the balance law

$$\begin{aligned}&(W_x^2)_t+\left[\left(\varphi+w_n-\frac{u_n^- - u_n^+}{2}\right)W_x^2\right]_x\\ =&-(\varphi_x+w_{n,x})\,W_x^2-\left(W_n-\frac{U_n^-+U_n^+}{2}\right)2W_x w_{n,xx}\\ &-2w_{n,x}W_{n,x}W_x+2\mathbf{H}[W_x]W_x-\left(W-\frac{U^-+U^+}{2}\right)2W_x\varphi_{xx}-2\varphi_x W_x^2.\end{aligned} \tag{5.5}$$

By the definition (2.4) one has

$$\|\varphi_{xx}\|_{\mathbf{L}^2(\mathbb{R}\setminus[-\delta_0(\tau-t),\delta_0(\tau-t)])}=\mathcal{O}(1)\cdot(\tau-t)^{-1/2}. \tag{5.6}$$

Integrating Equation (5.5) over the domain Ω in Equation (4.14), we obtain

$$\begin{aligned}\int_0^\infty W_x^2(t,x)\,dx=\mathcal{O}(1)\cdot\int_0^\tau\Big\{&\big|\ln(\tau-t)\big|\,\|W_x(t)\|_{\mathbf{L}^2}^2+\|W_n(t)\|_{H^1}\,\|W_x(t)\|_{\mathbf{L}^2}\\ &+\|W(t)\|_{H^1}\|W_x(t)\|_{\mathbf{L}^2}\cdot(\tau-t)^{-1/2}\Big\}\,dt.\end{aligned} \tag{5.7}$$

(3) Calling $Z(t)\doteq\|W(t)\|_{H^1(\mathbb{R}\setminus\{0\})}$, from Equations (5.3) and (5.7) we obtain an integral inequality of the form

$$Z^2(\tau)\le C_4\int_0^\tau Z(t)\cdot\Big(\|W_n(t)\|_{H^1}+Z(t)\Big)\cdot(\tau-t)^{-1/2}\,dt, \tag{5.8}$$

for some constant C_4.

We now set

$$\varepsilon_0\doteq\sup_{t\in[0,T]}\|W_n(t)\|_{H^1(\mathbb{R}\setminus\{0\})}.$$

Since $Z(0)=0$, calling τ^* the first time where $Z\ge\varepsilon_0/2$ one has

$$\frac{\varepsilon_0}{2}\le C_4\int_0^{\tau^*}\frac{\varepsilon_0}{2}\cdot\left(\epsilon_0+\frac{\varepsilon_0}{2}\right)(\tau^*-t)^{-1/2}\,dt=\frac{3}{2}C_4\varepsilon_0^2\tau^*.$$

Hence, $\tau^*\ge(3C_4)^{-1}$. Choosing $0<T<(3C_4)^{-1}$, we obtain

$$Z(t)\le\frac{\epsilon_0}{2}\qquad\qquad\text{for all}\quad t\in[0,T].$$

This establishes the desired contraction property:

$$\sup_{t\in[0,T]}\|w_{n+1}(t)-w_n(t)\|_{H^1(\mathbb{R}\setminus\{0\})}\le\frac{1}{2}\cdot\sup_{t\in[0,T]}\|w_n(t)-w_{n-1}(t)\|_{H^1(\mathbb{R}\setminus\{0\})}. \tag{5.9}$$

(4) By Equation (5.9), for every $t\in[0,T]$, the sequence of approximations $w_n(t,\cdot)$ is Cauchy in the space $H^1(\mathbb{R}\setminus\{0\})$, hence it converges to a unique limit $w(t,\cdot)$.

It remains to check that this limit function w is an entropic solution, i.e. it satisfies the integral equation (2.18). But this is clear because, for every $\epsilon > 0$, the sequence of functions

$$F_n \doteq \mathbf{H}[\varphi] - \varphi\varphi_x + \left(\mathbf{H}[w_n] - \left(w_n - \frac{u_n^- + u_n^+}{2}\right)\varphi_x\right) \tag{5.10}$$

converges to the corresponding function F in Equation (2.19), uniformly for $t \in [0,T]$ and $|x| \geq \epsilon$.

(5) Finally, to prove uniqueness, assume that $w, \tilde{w}$ are two entropic solutions. Consider the differences

$$W \doteq w - \tilde{w}, \qquad \begin{cases} U^- \doteq u^- - \tilde{u}^-, \\ U^+ \doteq u^+ - \tilde{u}^+, \end{cases}$$

and call $Z(t) \doteq \|W(t)\|_{H^1(\mathbb{R}\setminus\{0\})}$. Since $Z(0) = 0$, the same arguments used to prove Equation (5.8) now yield

$$Z^2(\tau) \leq C_4 \int_0^\tau Z(t) \cdot \big[Z(t) + Z(t)\big] \cdot (\tau - t)^{-1/2}\, dt.$$

For $\tau \in [0,T]$ sufficiently small, we thus obtain $Z(\tau) = 0$. This completes the proof of Theorem 2.1. ∎

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