COMMUN. MATH. SCI.
Vol. 15, No. 1, pp. 97–121

GLOBAL WELL-POSEDNESS AND PULLBACK ATTRACTORS FOR A TWO-DIMENSIONAL NON-AUTONOMOUS MICROPOLAR FLUID FLOWS WITH INFINITE DELAYS*

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Abstract. This paper studies the non-autonomous micropolar fluid flows in two-dimensional bounded domains with external forces containing infinite delay effects. The authors first prove the global well-posedness of the weak solutions and then establish the existence of the pullback attractors for the associated process.

Key words. Micropolar fluid flows, infinite delays, pullback attractor.

AMS subject classifications. 35B40, 35Q30, 35B41.

1. Introduction

The micropolar fluid flows were first formulated by Eringen [15] in 1966, which describe fluids consisting of randomly oriented particles suspended in a viscous medium. According to [26], the motion of the micropolar fluid flows can be described by the following equations:

$$\frac{\partial u}{\partial t} - (\nu + \nu_r)\Delta u - 2\nu_r \nabla \times \omega + (u \cdot \nabla)u + \nabla p = f, \tag{1.1}$$

$$\nabla \cdot u = 0, \tag{1.2}$$

$$\frac{\partial \omega}{\partial t} - (c_a + c_d)\Delta \omega + 4\nu_r \omega + (u \cdot \nabla)\omega - (c_0 + c_d - c_a)\nabla(\nabla \cdot \omega) - 2\nu_r \nabla \times u = \tilde{f}, \tag{1.3}$$

where $u = (u_1, u_2, u_3)$ is the velocity, p represents the pressure, $\omega = (\omega_1, \omega_2, \omega_3)$ is the microrotation field interpreted as the angular velocity field of rotation of particles. $f = (f_1, f_2, f_3)$ and $\tilde{f} = (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3)$ are external force and moments, respectively. The positive parameters $\nu, \nu_r, c_0, c_a, c_d$ denote the viscosity coefficients. In fact, ν is the usual Newtonian viscosity and ν_r the microrotation viscosity. From [15,26], we see that Equations (1.1)–(1.3) express the balance of momentum, mass and moment of momentum, accordingly. If $\nu_r = 0$ and $\omega = (0,0,0)$, then Equations (1.1)–(1.3) reduce to the incompressible Navier–Stokes equations. Therefore, the equations of micropolar fluid flows can be regarded as a generalization of the Navier–Stokes equations in the sense that they take into account the microstructure of the fluid. One can see, e.g. [26, 27], for physical background.

Due to the wide applications in the real world, the micropolar fluid flows have been well studied by some mathematicians and physicists. First, we must mention that Łukaszewicz has obtained fruitful results in his monograph [26]. Also, a series of papers are devoted to the existence and uniqueness of solutions (see, e.g. [14, 16, 17, 23–28]). At the same time, the long time behavior of solutions has been investigated from various aspects. For example, the existence and estimation of Hausdorff and fractal

*Received: September 29, 2015; accepted (in revised form): April 16, 2016. Communicated by Alexis Vasseur.
Supported by NSFC of P.R. China (No. 11271290, 10901121).
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dimension of the L^2-global attractor was studied in [27]; the existence of H^2-compact global attractor was proved in [9]; the global and uniform attractor on unbounded domain was verified in [13] and [29, 37, 41], respectively; the uniform attractor of non-homogeneous micropolar fluid flows in non-smooth domains was obtained in [10]; the H^1-pullback attractor was obtained in [11,30]; the existence of L^2-pullback attractor in Lipschitz bounded domain with non-homogeneous boundary conditions was established in [12]. The pullback asymptotic behaviors of solutions for non-autonomous micropolar fluid flows in two-dimensional bounded domains was investigated in [44].

However, to our knowledge, there is not many references discussing the micropolar fluid flows with infinite delays so far. As we know, delay terms appear naturally, for instance as effects in wind tunnel experiments. Also, the delay situations may occur as well, when we want to control the system via applying a force which considers not only the present state but also the history state of the system.

In this paper, we consider the situation that the velocity component in the x_3-direction is zero and the axes of rotation of particles are parallel to the x_3-axis. That is, $u=(u_1,u_2,0)$, $\omega=(0,0,\omega_3)$, $f=(f_1,f_2,0)$ and $\tilde{f}=(0,0,\tilde{f}_3)$. Then we discuss the following equations of two-dimensional non-autonomous incompressible micropolar fluid flows with infinite delays:

$$\frac{\partial u}{\partial t}-(\nu+\nu_r)\Delta u-2\nu_r\nabla^{\perp}\omega+(u\cdot\nabla)u+\nabla p=f(t,x)+g(t,u_t),\ t>\tau,\ x\in\Omega, \tag{1.4}$$

$$\frac{\partial \omega}{\partial t}-\alpha\Delta\omega+4\nu_r\omega-2\nu_r\nabla\times u+(u\cdot\nabla)\omega=\tilde{f}(t,x)+\tilde{g}(t,\omega_t),\ t>\tau,\ x\in\Omega, \tag{1.5}$$

$$\nabla\cdot u=0,\ \text{in}\ (\tau,+\infty)\times\Omega, \tag{1.6}$$

$$u=0,\quad \omega=0,\ \text{on}\ (\tau,+\infty)\times\partial\Omega, \tag{1.7}$$

$$(u(\tau+s,x),\omega(\tau+s,x))=\phi(s,x),\ s\in(-\infty,0],\ \tau\in\mathbb{R},\ x\in\Omega, \tag{1.8}$$

where $\alpha:=(c_a+c_d)$, $x:=(x_1,x_2)\in\Omega$, $t>\tau$ for some $\tau\in\mathbb{R}$, and $\Omega\subset\mathbb{R}^2$ is an open and bounded domain with smooth boundary $\partial\Omega$, such that the following Poincaré inequality holds:

$$\lambda_1\|\varphi\|^2_{L^2(\Omega)}\le\|\nabla\varphi\|^2_{L^2(\Omega)},\ \forall\varphi(x)\in H^1_0(\Omega), \tag{1.9}$$

where $\lambda_1>0$ denotes the first eigenvalue of the operator $-\Delta$ in $L^2(\Omega)$ with domain $H^1_0(\Omega)\cap H^2(\Omega)$ and with the Dirichlet boundary condition. Note that λ_1 is a constant depending only on Ω.

In Equations (1.4)–(1.8), the unknown vector function $u:=(u_1,u_2)$ is the velocity field of the fluid, and the unknown scalar functions p and ω are its pressure and angular velocity, respectively. $f(t,x):=(f_1,f_2)$ is the external force and $\tilde{f}(t,x)$ is the scalar moments, respectively. The vector function $g(t,u_t):=(g_1,g_2)$ and scalar function $\tilde{g}(t,\omega_t)$ are additional external forces containing some hereditary characteristics u_t and ω_t, which are defined on $(-\infty,0]$ as follows:

$$u_t=u_t(\cdot):=u(t+\cdot),\quad \omega_t=\omega_t(\cdot):=\omega(t+\cdot),\ t\ge\tau. \tag{1.10}$$

In addition, $\phi(s,x)=(u(\tau+s,x),\omega(\tau+s,x))=(u_\tau,\omega_\tau)$ is the initial datum in the interval of delay time $(-\infty,0]$, and

$$\nabla\times u:=\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\quad\text{and}\quad\nabla^{\perp}\omega:=(\frac{\partial\omega}{\partial x_2},-\frac{\partial\omega}{\partial x_1}).$$

The first purpose of the present paper is to prove the global well-posedness of the weak solutions for Equations (1.4)–(1.8). The main arguments used here are the Faedo–Galerkin approximation and energy equality, as well as the compact embedding between the Sobolev spaces. The second goal of this paper is to establish the existence of pullback attractors for fixed bounded sets. To this end, we verify some useful estimates for the solutions, and by so we prove the existence of the pullback absorbing set for the associated process $\{U(t,\tau)\}_{t\geq\tau}$. Then we establish the asymptotically compactness of the process $\{U(t,\tau)\}_{t\geq\tau}$ and obtain the existence of pullback attractors.

It is worth mentioning that Caraballo and Real have studied the asymptotic behavior of the Navier–Stokes equations with finite delays in [6, 7]. Later, Marín-Rubio, Real, and Valero in [34] extended the results of [7] to unbounded domains. Recently, Marín-Rubio, Real, and Valero proved the global well-posedness of the weak solutions for the two-dimensional Navier–Stokes equations with an external force containing infinite delays in [36]. Also, they obtained the existence of pullback attractors. We want to point out that the idea of this paper originates from paper [36]. Compared with the Navier–Stokes equations studied in [36], the equations of micropolar fluid flows contain the angular velocity field ω of the micropolar particles, which leads to a different nonlinear term $B(u,w)$ and an additional term $N(w)$ in the abstract equation (see Equation (3.1)). Due to these differences, more delicate estimates and analysis are required in our studies.

The rest of the paper is organized as follows. Section 2 is preliminaries. Section 3 is devoted to the proof of the global well-posedness of the weak solutions for Equations (1.4)–(1.8). In Section 4, we first concentrate on proving the existence of the pullback attractors. Then we end the paper with two remarks on the extensions of our studies.

2. Preliminaries

In this section, we first introduce some notations and operators. Then we put the Equations (1.4)–(1.8) into an abstract form and specify the definition of its weak solutions.

Let's denote by $L^p(\Omega)$ and $W^{m,p}(\Omega)$ respectively the usual Lebesgue space and Sobolev space (see [1, 3, 4]) endowed with norms $\|\cdot\|_p$ and $\|\cdot\|_{m,p}$ as

$$\|\varphi\|_p := \Big(\int_\Omega |\varphi|^p \mathrm{d}x\Big)^{1/p} \quad \text{and} \quad \|\varphi\|_{m,p} := \Big(\sum_{|\beta|\leq tm}\int_\Omega |\partial^\beta\varphi|^p \mathrm{d}x\Big)^{1/p}.$$

Especially, we denote $\|\cdot\| := \|\cdot\|_2$, $H^m(\Omega) := W^{m,2}(\Omega)$ and $H_0^1(\Omega)$ the closure of $\{\varphi\in \mathcal{C}_0^\infty(\Omega)\}$ with respect to $H^1(\Omega)$ norm. Then we introduce the following function spaces:

$\mathcal{V} := \{\varphi\in\mathcal{C}_0^\infty(\Omega)\times\mathcal{C}_0^\infty(\Omega) \,|\, \varphi=(\varphi_1,\varphi_2), \nabla\cdot\varphi=0\}$,

$H :=$ "completion of $\mathcal{V}$ in $L^2(\Omega)\times L^2(\Omega)$ norm with norm $\|\cdot\|_H$ and dual space H^*,"

$V :=$ "completion of $\mathcal{V}$ in $H^1(\Omega)\times H^1(\Omega)$ norm with norm $\|\cdot\|_V$ and dual space V^*,"

$\widehat{H} :=$ "$H\times L^2(\Omega)$ with norm $\|\cdot\|_{\widehat{H}}$ and dual space $\widehat{H}^*$,"

$\widehat{V} :=$ "$V\times H_0^1(\Omega)$ with norm $\|\cdot\|_{\widehat{V}}$ and dual space $\widehat{V}^*$."

Note that $\|\cdot\|_{\widehat{H}}$ and $\|\cdot\|_{\widehat{V}}$ are defined as

$$\|(u,v)\|_{\widehat{H}} := (\|u\|_H^2 + \|v\|^2)^{1/2}, \quad \|(u,v)\|_{\widehat{V}} := (\|u\|_V^2 + \|v\|_{1,2}^2)^{1/2}.$$

Throughout this article, we simplify the notations $\|\cdot\|_H$ and $\|\cdot\|_{\widehat{H}}$ by the same notation $\|\cdot\|$ if there is no confusion occurs. According to the above notations, we further denote

$L^p(I;X) :=$ "space of strongly measurable functions on the interval I, with values in a Banach space X, endowed with norm

$$\|\varphi\|_{L^p(I;X)} := \Big(\int_I \|\varphi\|_X^p \mathrm{d}t\Big)^{1/p}, \quad \text{for } 1 \le p < \infty;\text{”}$$

$\mathcal{C}(I;X) :=$ “space of continuous functions on the interval I, with values in the Banach space X, endowed with the usual norm;”

$L^2_{loc}(I;\widehat{H}) :=$ “space of square locally integrable functions from the interval I to $\widehat{H}$;”

$W^{1,2}_{loc}(I;\widehat{H}) :=$ $\{G \mid G \in L^2_{loc}(I;\widehat{H}) \text{ and } G' \in L^2_{loc}(I;\widehat{H})\}$, here “ $'$ ” means the generalized derivative with respect to time variable.

In addition, we denote by $(\cdot,\cdot)$ the inner product in $L^2(\Omega)$, H, or $\widehat{H}$, and by $\langle\cdot,\cdot\rangle$ the dual pairing between V and V^* or between $\widehat{V}$ and $\widehat{V}^*$.

To write Equations (1.4)–(1.8) into the abstract form, we further introduce three operators. First, the operator A is defined as

$$\begin{aligned}\langle Aw,\phi\rangle &:= (\nu+\nu_r)(\nabla u,\nabla\Phi) + \alpha(\nabla\omega,\nabla\phi_3)\\ &= (\nu+\nu_r)\sum_{i,j=1}^{2}\int_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial\phi_i}{\partial x_j}\mathrm{d}x + \alpha\sum_{i=1}^{2}\int_\Omega \frac{\partial\omega}{\partial x_i}\frac{\partial\phi_3}{\partial x_i}\mathrm{d}x,\end{aligned} \tag{2.1}$$

for any $w=(u,\omega)$ and $\phi=(\Phi,\phi_3)$ belonging to $\widehat{V}$, where $u=(u_1,u_2)\in V$ and $\Phi = (\phi_1,\phi_2)\in V$. In fact, $D(A)=\widehat{V}\cap\big(H^2(\Omega)\big)^3$. Secondly, the operator $B(\cdot,\cdot)$ is defined via

$$\langle B(u,\psi),\phi\rangle := ((u\cdot\nabla)\psi,\phi) = \sum_{j=1}^{3}\sum_{i=1}^{2}\int_\Omega u_i\frac{\partial\psi_j}{\partial x_i}\phi_j\mathrm{d}x,$$

for any $u=(u_1,u_2)\in V$, $\psi=(\psi_1,\psi_2,\psi_3)\in\widehat{V}$ and any $\phi=(\phi_1,\phi_2,\phi_3)\in\widehat{V}$. Thirdly, the operator $N(\cdot)$ is defined by

$$N(w) := (-2\nu_r\nabla^\perp\omega, -2\nu_r\nabla\times u + 4\nu_r\omega), \forall w=(u,\omega)\in\widehat{V} \text{ with } u=(u_1,u_2)\in V.$$

Some useful estimations for the operators A, $B(\cdot,\cdot)$, and $N(\cdot)$ have been established in [27, 29]. For completeness, we recall them as follows.

Lemma 2.1. (see [27, 29])

(1) *There are two positive constants c_1 and c_2 such that*

$$c_1\langle Aw,w\rangle \le \|w\|^2_{\widehat{V}} \le c_2\langle Aw,w\rangle, \ \forall w\in\widehat{V}. \tag{2.2}$$

Furthermore, for any $w\in D(A)$, there holds

$$\min\{\nu+\nu_r,\alpha\}\|\nabla w\|^2 \le \langle Aw,w\rangle \le \|w\|\|Aw\| \le \lambda_1^{-\frac{1}{2}}\|\nabla w\|\|Aw\|. \tag{2.3}$$

(2) *There exists some positive constant λ which depends only on Ω, such that for any $(u,\psi,\varphi)\in V\times\widehat{V}\times\widehat{V}$ there holds*

$$|\langle B(u,\psi),\varphi\rangle| \le \begin{cases}\lambda\|u\|^{\frac12}\|\nabla u\|^{\frac12}\|\varphi\|^{\frac12}\|\nabla\varphi\|^{\frac12}\|\nabla\psi\|,\\ \lambda\|u\|^{\frac12}\|\nabla u\|^{\frac12}\|\psi\|^{\frac12}\|\nabla\psi\|^{\frac12}\|\nabla\varphi\|.\end{cases} \tag{2.4}$$

Moreover, if $(u,\psi,\varphi)\in V\times D(A)\times D(A)$, then

$$|\langle B(u,\psi),A\varphi\rangle| \le \lambda\|u\|^{\frac12}\|\nabla u\|^{\frac12}\|\nabla\psi\|^{\frac12}\|A\psi\|^{\frac12}\|A\varphi\|. \tag{2.5}$$

(3) *There exists a positive constant $c(\nu_r)$ such that*

$$\|N(\psi)\| \le c(\nu_r)\|\psi\|_{\widehat{V}}, \quad \forall \psi \in \widehat{V}. \tag{2.6}$$

In addition,

$$-\langle N(\psi), A\psi\rangle \le \frac{1}{4}\|A\psi\|^2 + c^2(\nu_r)\|\psi\|_{\widehat{V}}^2, \forall \psi \in D(A), \tag{2.7}$$

$$\delta_1\|\psi\|_{\widehat{V}}^2 \le \langle A\psi,\psi\rangle + \langle N(\psi),\psi\rangle, \quad \forall \psi \in \widehat{V}, \tag{2.8}$$

hereinafter $\delta_1 := \min\{\nu,\alpha\}$.

On the base of Lemma 2.1, we further have

Lemma 2.2.

(1) *The operator A is linear continuous both from $\widehat{V}$ to $\widehat{V}^*$ and from $D(A)$ to $\widehat{H}$, and so is for the operator $N(\cdot)$ from $\widehat{V}$ to $\widehat{H}$.*

(2) *The operator $B(\cdot,\cdot)$ is continuous from $V \times \widehat{V}$ to $\widehat{V}^*$. Moreover, for any $u \in V$ and $w \in \widehat{V}$, there holds*

$$\langle B(u,\psi),\varphi\rangle = -\langle B(u,\varphi),\psi\rangle, \quad \forall u \in V, \ \forall \psi \in \widehat{V}, \ \forall \varphi \in \widehat{V}. \tag{2.9}$$

Proof.

(1). The continuity of the operators A and $N(\cdot)$ follows directly from Equations (2.2)–(2.3) and (2.6), respectively. The linearity of the operator A is evident. So we only need check the linearity of the operator $N(\cdot)$. Indeed, for any $\phi = (\Phi,\phi_3) \in \widehat{V}$ with $\Phi = (\phi_1,\phi_2)$ and $\psi = (\Psi,\psi_3) \in \widehat{V}$ with $\Psi = (\psi_1,\psi_2)$, we have

$$\begin{aligned}
&N(\phi) - N(\psi)\\
&= \big(-2\nu_r(\nabla^\perp\phi_3 - \nabla^\perp\psi_3), -2\nu_r(\nabla\times\Phi - \nabla\times\Psi) + 4\nu_r(\phi_3-\psi_3)\big)\\
&= \big(-2\nu_r(\frac{\partial\phi_3}{\partial x_2} - \frac{\partial\psi_3}{\partial x_2}, -\frac{\partial\phi_3}{\partial x_1} + \frac{\partial\psi_3}{\partial x_1}), -2\nu_r(\frac{\partial\phi_2}{\partial x_1} - \frac{\partial\phi_1}{\partial x_2} - \frac{\partial\psi_2}{\partial x_1} + \frac{\partial\psi_1}{\partial x_2})\\
&\quad + 4\nu_r(\phi_3-\psi_3)\big)\\
&= \big(-2\nu_r(\frac{\partial(\phi_3-\psi_3)}{\partial x_2}, -\frac{\partial(\phi_3-\psi_3)}{\partial x_1}), -2\nu_r(\frac{\partial(\phi_2-\psi_2)}{\partial x_1} - \frac{\partial(\phi_1-\psi_1)}{\partial x_2})\\
&\quad + 4\nu_r(\phi_3-\psi_3)\big)\\
&= \big(-2\nu_r\nabla^\perp(\phi_3-\psi_3), -2\nu_r\nabla\times(\Phi-\Psi) + 4\nu_r(\phi_3-\psi_3)\big)\\
&= N(\phi-\psi).
\end{aligned} \tag{2.10}$$

(2). The continuity of the operators $B(\cdot,\cdot)$ follows directly from Equation (2.4). We next verify Equation (2.9). In fact, for any $u \in V$ and $\psi \in \widehat{V}$, we have by direct computation that

$$\begin{aligned}
\langle B(u,\psi),\psi\rangle = ((u\cdot\nabla)\psi,\psi) &= \sum_{j=1}^{3}\sum_{i=1}^{2}\int_\Omega u_i\frac{\partial\psi_j}{\partial x_i}\psi_j \mathrm{d}x = \sum_{j=1}^{3}\sum_{i=1}^{2}\frac{1}{2}\int_\Omega u_i\frac{\partial\psi_j^2}{\partial x_i}\mathrm{d}x\\
&= -\frac{1}{2}\sum_{j=1}^{3}\sum_{i=1}^{2}\int_\Omega \psi_j^2\frac{\partial u_i}{\partial x_i}\mathrm{d}x = -\frac{1}{2}\sum_{j=1}^{3}\int_\Omega \psi_j^2(\nabla\cdot u)\mathrm{d}x = 0.
\end{aligned} \tag{2.11}$$

Hence, Equation (2.9) is valid as a consequence of Equation (2.11). ∎

3. Global well-posedness of the weak solutions

The task of this section is to establish the global well-posedness of the weak solutions to Equations (1.4)–(1.8).

First, according to the notations and operators introduced in Section 2, we can formulate the weak version of Equations (1.4)–(1.8) as follows:

$$\frac{\partial w}{\partial t}+Aw+B(u,w)+N(w)=F(t)+G(t,w_t),\ \ t>\tau, \tag{3.1}$$

$$w|_{t=\tau}=w_\tau=w(\tau+s)=(u(\tau+s),\omega(\tau+s)):=\phi(s),\ \ s\in(-\infty,0], \tag{3.2}$$

where

$$w=(u,\omega),\ \ F(t)=F(t,x):=(f(t,x),\tilde{f}(t,x)),\ \ G(t,w_t):=(g(t,u_t),\tilde{g}(t,w_t)), \tag{3.3}$$

and the functions u_t and w_t are defined by Equation (1.10). Note that Equation (3.1) is understood in the distribution sense of $\mathcal{D}'(\tau,T;\widehat{V}^*)$.

In order to deal with the infinite delays, we introduce the space $\mathcal{C}_\gamma(\widehat{H})$ with some suitable $\gamma>0$ as follows:

$$\mathcal{C}_\gamma(\widehat{H}):=\{\varphi\in\mathcal{C}((-\infty,0];\widehat{H})|\ \exists \lim_{s\to-\infty} e^{\gamma s}\varphi(s)\in\widehat{H}\}, \tag{3.4}$$

which is a Banach space with the norm

$$\|\varphi\|_\gamma:=\sup_{s\in(-\infty,0]} e^{\gamma s}\|\varphi(s)\|.$$

We now specify the definition of weak solutions to problem (3.1)–(3.2).

Definition 3.1. *For each $T>\tau$, if a function $w\in\mathcal{C}((-\infty,T];\widehat{H})\cap L^2(\tau,T;\widehat{V})$ with $w_\tau=\phi(s)\in\mathcal{C}_\gamma(\widehat{H})$ such that for $t\in(\tau,T)$ and for any $\varphi\in\widehat{V}$ the equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}(w,\varphi)+\langle Aw,\varphi\rangle+\langle B(u,w),\varphi\rangle+\langle N(w),\varphi\rangle=\langle F(t),\varphi\rangle+(G(t,w_t),\varphi) \tag{3.5}$$

holds in the distribution sense of $\mathcal{D}'(\tau,T)$. Then w is called a weak solution of problem (3.1)–(3.2) in the interval $(-\infty,T]$.

To prove the global well-posedness of the weak solutions to problem (3.1)–(3.2), we need formulate some assumptions on the functions F and G.

(I) Assume that for each $T>\tau$, $F(\cdot,x)=(f(\cdot,x),\tilde{f}(\cdot,x))\in L^2(\tau,T;\widehat{V}^*)$.

(II) Assume that $G:[\tau,T]\times\mathcal{C}_\gamma(\widehat{H})\mapsto G(t,\xi)\in(L^2(\Omega))^3$ satisfies:

(i) For any $\xi\in\mathcal{C}_\gamma(\widehat{H})$, the mapping $[\tau,T]\ni t\mapsto G(t,\xi)\in(L^2(\Omega))^3$ is measurable;

(ii) $G(\cdot,0)=(0,0,0)$;

(iii) There exists a constant $L_G>0$ such that for any $t\in[\tau,T]$ and any $\xi,\eta\in\mathcal{C}_\gamma(\widehat{H})$,

$$\|G(t,\xi)-G(t,\eta)\|\le L_G\|\xi-\eta\|_\gamma.$$

Note that the above conditions (ii) and (iii) imply

$$\|G(t,\xi)\|\le L_G\|\xi\|_\gamma,\ \ \forall\xi\in\mathcal{C}_\gamma(\widehat{H}). \tag{3.6}$$

We begin with the existence of the weak solutions.

THEOREM 3.1 (Existence). *Let assumptions* (I) *and* (II) *hold, and suppose that* γ *satisfies* $\delta_1\lambda_1 < 2\gamma$. *Then for any given initial datum* $\phi(s)\in\mathcal{C}_\gamma(\widehat{H})$ *and for any* $T>\tau$, *there corresponds at least one weak solution to problem (3.1)–(3.2) in the interval* $(-\infty,T]$.

Proof. We will use three steps to prove Theorem 3.1.

Step One: *Local existence and uniqueness of the Galerkin approximate solutions.* We first recall some properties of the operator A defined by Equation (2.1). According to the classical spectral theory of elliptic operators (see [4,33]), there exists a sequence $\{\lambda_n\}_{n=1}^{\infty}$ satisfying

$$0<\lambda_1\le\lambda_2\le\cdots\le\lambda_n\le\cdots,\ \ \lambda_n\to+\infty\ \text{ as}\, n\to\infty,$$

and a sequence of elements $\{v_n\}_{n=1}^{\infty}\subseteq D(A)$, which forms a Hilbert basis of $\widehat{H}$ and span $\{v_1,v_2,\dots,v_n,\dots\}$ is dense in $\widehat{V}$, such that

$$Av_n=\lambda_n v_n,\ \ \forall n\in\mathbb{N}. \tag{3.7}$$

Denote $V_m:=\mathrm{span}\{v_1,v_2,\dots,v_m\}$ and consider the projector

$$P_m w:=\sum_{j=1}^{m}(w,v_j)v_j,\ \ w\in\widehat{H}\ \text{or}\ \widehat{V}.$$

For each $T>\tau$, define the Galerkin approximate solutions of problem (3.1)–(3.2) as

$$w^{(m)}(t):=\sum_{j=1}^{m}\beta_{m,j}(t)v_j,$$

where the coefficients $\beta_{m,j}(t)$ are desired to satisfy the following Cauchy problem of ordinary differential equations:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(w^{(m)}(t),v_j)+\langle Aw^{(m)}(t),v_j\rangle+\langle B(u^{(m)}(t),w^{(m)}(t)),v_j\rangle+\langle N(w^{(m)}(t)),v_j\rangle\\ &=\langle F(t),v_j\rangle+(G(t,w_t^{(m)}),v_j),\ \ 1\le j\le m,\ \ t\in(\tau,T), &(3.8)\\ &\ \ w^{(m)}(\tau+s)=P_m\phi(s),\ \ s\in(-\infty,0]. &(3.9)\end{aligned}$$

The above Cauchy problem of ordinary differential equations with infinite delays fulfills the conditions for the existence and uniqueness of a local solution in [21, Thereom 1.1]. So we get the local existence and uniqueness of the Galerkin approximate solutions.

Step Two: *A priori estimates of the Galerkin approximate solutions.* We now deduce a priori estimates to obtain the global existence of the Galerkin approximate solutions. Multiplying Equation (3.8) by $\beta_{m,j}(t)$, summing up for j from 1 to m and then using Equations (1.9), (2.8), and (2.11), we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w^{(m)}(t)\|^2+\frac{\delta_1\lambda_1}{2}\|w^{(m)}(t)\|^2+\frac{\delta_1}{2}\|w^{(m)}(t)\|_{\widehat{V}}^2\\ &\le\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w^{(m)}(t)\|^2+\delta_1\|w^{(m)}(t)\|_{\widehat{V}}^2\\ &\le\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w^{(m)}(t)\|^2+\langle Aw^{(m)}(t),w^{(m)}(t)\rangle+\langle N(w^{(m)}(t)),w^{(m)}(t)\rangle\end{aligned}$$

$$=\langle F(t),w^{(m)}(t)\rangle+\big(G(t,w_t^{(m)}),w^{(m)}(t)\big). \tag{3.10}$$

Since

$$\|w_t^{(m)}\|_\gamma=\sup_{s\le 0}e^{\gamma s}\|w^{(m)}(t+s)\|\ge\|w^{(m)}(t)\|,\ \ \tau\le t\le T, \tag{3.11}$$

by Equations (3.6), (3.10)–(3.11), and Cauchy's inequality,

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w^{(m)}(t)\|^2+\frac{\delta_1\lambda_1}{2}\|w^{(m)}(t)\|^2+\frac{\delta_1}{2}\|w^{(m)}(t)\|_{\widehat{V}}^2\\ &\le\langle F(t),w^{(m)}(t)\rangle+(G(t,w_t^{(m)}),w^{(m)}(t))\\ &\le\|F(t)\|_{\widehat{V}^*}\|w^{(m)}(t)\|_{\widehat{V}}+L_G\|w_t^{(m)}\|_\gamma\|w^{(m)}(t)\|\\ &\le\frac{\delta_1}{4}\|w^{(m)}(t)\|_{\widehat{V}}^2+\frac{\|F(t)\|_{\widehat{V}^*}^2}{\delta_1}+L_G\|w_t^{(m)}\|_\gamma^2,\end{aligned} \tag{3.12}$$

hence,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|w^{(m)}(t)\|^2+\delta_1\lambda_1\|w^{(m)}(t)\|^2+\frac{\delta_1}{2}\|w^{(m)}(t)\|_{\widehat{V}}^2\\ &\le\frac{2\|F(t)\|_{\widehat{V}^*}^2}{\delta_1}+2L_G\|w_t^{(m)}\|_\gamma^2,\quad t\in(\tau,T).\end{aligned} \tag{3.13}$$

Let $\tau\le\theta\le t\le T$. Changing the time variable t by θ, we rewrite Equation (3.13) as

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\theta}\|w^{(m)}(\theta)\|^2+\delta_1\lambda_1\|w^{(m)}(\theta)\|^2+\frac{\delta_1}{2}\|w^{(m)}(\theta)\|_{\widehat{V}}^2\\ &\le\frac{2\|F(\theta)\|_{\widehat{V}^*}^2}{\delta_1}+2L_G\|w_\theta^{(m)}\|_\gamma^2,\quad \theta\in(\tau,T).\end{aligned} \tag{3.14}$$

Multiplying Equation (3.14) by $e^{-\delta_1\lambda_1(t-\theta)}$ and then integrating it for θ over $[\tau,t]$ give

$$\begin{aligned}&\|w^{(m)}(t)\|^2+\frac{\delta_1}{2}\int_\tau^t e^{-\delta_1\lambda_1(t-\theta)}\|w^{(m)}(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta\\ &\le e^{-\delta_1\lambda_1(t-\tau)}\|w^{(m)}(\tau)\|^2+2\int_\tau^t e^{-\delta_1\lambda_1(t-\theta)}\Big(\frac{\|F(\theta)\|_{\widehat{V}^*}^2}{\delta_1}+L_G\|w_\theta^{(m)}\|_\gamma^2\Big)\mathrm{d}\theta,\end{aligned} \tag{3.15}$$

by which we get

$$\begin{aligned}\|w_t^{(m)}\|_\gamma^2&=\Big(\sup_{s\in(-\infty,0]}e^{\gamma s}\|w^{(m)}(t+s)\|\Big)^2\\ &\le\max\Big\{\sup_{s\in(-\infty,\tau-t]}e^{2\gamma s}\|w^{(m)}(t+s)\|^2,\ \sup_{s\in(\tau-t,0]}e^{2\gamma s}\|w^{(m)}(t+s)\|^2\Big\}\\ &=:\max\{I_1,I_2+I_3\},\end{aligned} \tag{3.16}$$

where

$$\begin{aligned}I_1&:=\sup_{s\in(-\infty,\tau-t]}e^{2\gamma s}\|w^{(m)}(t+s)\|^2,\\ I_2&:=\sup_{s\in(\tau-t,0]}\Big[e^{2\gamma s-\delta_1\lambda_1(t+s-\tau)}\|w^{(m)}(\tau)\|^2\Big],\end{aligned}$$

$$I_3:=2\sup_{s\in(\tau-t,0]}\Big[e^{2\gamma s}\int_\tau^{t+s}e^{-\delta_1\lambda_1(t+s-\theta)}\big(\frac{\|F(\theta)\|^2_{\widehat{V}^*}}{\delta_1}+L_G\|w_\theta^{(m)}\|^2_\gamma\big)\mathrm{d}\theta\Big].$$

We next estimate I_1, I_2 and I_3. On one hand, since $\delta_1\lambda_1<2\gamma$,

$$\begin{aligned}I_1&=\sup_{s\in(-\infty,\tau-t]}e^{2\gamma s}\|P_m\phi(t+s-\tau)\|^2\le\sup_{s\in(-\infty,\tau-t]}e^{2\gamma s}\|\phi(t+s-\tau)\|^2\\&=\sup_{s\in(-\infty,0]}e^{2\gamma[s-(t-\tau)]}\|\phi(s)\|^2=e^{2\gamma(\tau-t)}\|\phi(s)\|^2_\gamma\le e^{-\delta_1\lambda_1(t-\tau)}\|\phi(s)\|^2_\gamma.\end{aligned}\tag{3.17}$$

On the other hand, by direct computation,

$$I_2\le e^{-\delta_1\lambda_1(t-\tau)}\|w^{(m)}(\tau)\|^2=e^{-\delta_1\lambda_1(t-\tau)}\|\phi(0)\|^2\le e^{-\delta_1\lambda_1(t-\tau)}\|\phi(s)\|^2_\gamma,\tag{3.18}$$

$$I_3\le2\int_\tau^t e^{-\delta_1\lambda_1(t-\theta)}\big(\frac{\|F(\theta)\|^2_{\widehat{V}^*}}{\delta_1}+L_G\|w_\theta^{(m)}\|^2_\gamma\big)\mathrm{d}\theta.\tag{3.19}$$

Then Equations (3.16)–(3.19) gives

$$\|w_t^{(m)}\|^2_\gamma\le e^{-\delta_1\lambda_1(t-\tau)}\|\phi(s)\|^2_\gamma+2\int_\tau^t e^{-\delta_1\lambda_1(t-\theta)}\big(\frac{\|F(\theta)\|^2_{\widehat{V}^*}}{\delta_1}+L_G\|w_\theta^{(m)}\|^2_\gamma\big)\mathrm{d}\theta.\tag{3.20}$$

Using Gronwall's inequality on Equation (3.20) yields

$$\|w_t^{(m)}\|^2_\gamma\le e^{-(\delta_1\lambda_1-2L_G)(t-\tau)}\|\phi(s)\|^2_\gamma+\frac{2}{\delta_1}\int_\tau^t e^{-(\delta_1\lambda_1-2L_G)(t-\theta)}\|F(\theta)\|^2_{\widehat{V}^*}\mathrm{d}\theta,\tag{3.21}$$

which implies that for any given $\mathcal{R}>0$, there corresponds a constant $C_1(\tau,T,\mathcal{R})$ (depending on the quantities τ, T, $\mathcal{R}$, as well as on the constants δ_1, λ_1, L_G and the given function F) such that

$$\|w_t^{(m)}\|^2_\gamma\le C_1(\tau,T,\mathcal{R}),\ \ \forall t\in[\tau,T],\ \ \|\phi(s)\|_\gamma\le\mathcal{R},\ \ \forall m\ge1.\tag{3.22}$$

Together, Equations (3.11) and (3.22) imply that

$$\{w^{(m)}(\cdot)\}_{m\ge1}\text{is bounded in}\,L^\infty(\tau,T;\widehat{H}).\tag{3.23}$$

We also can get from Equations (3.15) and (3.22) that

$$\begin{aligned}&\frac{\delta_1}{2}e^{-\delta_1\lambda_1(t-\tau)}\int_\tau^t\|w^{(m)}(\theta)\|^2_{\widehat{V}}\mathrm{d}\theta\\&\le\|w^{(m)}(\tau)\|^2+2\int_\tau^t e^{-\delta_1\lambda_1(t-\theta)}\big(\frac{\|F(\theta)\|^2_{\widehat{V}^*}}{\delta_1}+L_G\|w_\theta^{(m)}\|^2_\gamma\big)\mathrm{d}\theta\\&\le\mathcal{R}^2+2\int_\tau^t e^{-\delta_1\lambda_1(t-\theta)}\big(\frac{\|F(\theta)\|^2_{\widehat{V}^*}}{\delta_1}+L_GC_1(\tau,T,\mathcal{R})\big)\mathrm{d}\theta.\end{aligned}\tag{3.24}$$

Similar to Equation (3.22), we see from Equation (3.24) that there exists another constant $C_2(\tau,T,\mathcal{R})$ such that

$$\|w^{(m)}(\cdot)\|^2_{L^2(\tau,T;\widehat{V})}\le C_2(\tau,T,\mathcal{R}),\forall m\ge1.\tag{3.25}$$

Now, using Equation (3.8) and the argument of denseness, we get for any $\psi\in\widehat{V}$ that

$$\begin{aligned}&\langle (w^{(m)})'(t),\psi\rangle+\langle Aw^{(m)}(t),\psi\rangle+\langle B(u^{(m)}(t),w^{(m)}(t)),\psi\rangle+\langle N(w^{(m)}(t)),\psi\rangle\\ =&\langle F(t),\psi\rangle+(G(t,w_t^{(m)}),\psi),\end{aligned}$$

which, together with Equations (2.2), (2.4), and (2.6), gives

$$\begin{aligned}&|\langle (w^{(m)})'(t),\psi\rangle|\\ \le&|\langle Aw^{(m)}(t),\psi\rangle|+|\langle B(u^{(m)}(t),w^{(m)}(t)),\psi\rangle|+|\langle N(w^{(m)}(t)),\psi\rangle|\\ &\quad+|\langle F(t),\psi\rangle|+|(G(t,w_t^{(m)}),\psi)|\\ \le& C_3(\lambda,\nu_r)\big(\|w^{(m)}(t)\|_{\widehat{V}}+\|u^{(m)}(t)\|^{\frac12}\|\nabla u^{(m)}(t)\|^{\frac12}\|w^{(m)}(t)\|^{\frac12}\|\nabla w^{(m)}(t)\|^{\frac12}\\ &\quad+\|F(t)\|_{\widehat{V}^*}+\|G(t,w_t^{(m)})\|\big)\|\psi\|_{\widehat{V}}\\ \le& C_3(\lambda,\nu_r)\big(\|w^{(m)}(t)\|_{\widehat{V}}+\|w^{(m)}(t)\|\|w^{(m)}(t)\|_{\widehat{V}}+\|F(t)\|_{\widehat{V}^*}\\ &\quad+\|G(t,w_t^{(m)})\|\big)\|\psi\|_{\widehat{V}},\end{aligned}\tag{3.26}$$

where the positive constant $C_3(\lambda,\nu_r)$ depends only on λ and $c(\nu_r)$, which are as in Lemma 2.1. Then (3.26) implies

$$\begin{aligned}&\|(w^{(m)})'(t)\|_{\widehat{V}^*}\\ \le& C_3(\lambda,\nu_r)\big(\|w^{(m)}(t)\|_{\widehat{V}}+\|w^{(m)}(t)\|\|w^{(m)}(t)\|_{\widehat{V}}+\|F(t)\|_{\widehat{V}^*}+\|G(t,w_t^{(m)})\|\big).\end{aligned}\tag{3.27}$$

Hence, it follows from Equations (3.6), (3.23), (3.25), and (3.27) that there exists a constant $C_4(\tau,T,\mathcal{R})$ such that

$$\int_\tau^T\|(w^{(m)})'(\theta)\|^2_{\widehat{V}^*}\mathrm{d}\theta\le C_4(\tau,T,\mathcal{R}),\ \forall m\ge 1,\tag{3.28}$$

and consequently,

$$\{(w^{(m)})'(\cdot)\}_{m\ge1}\ \text{is bounded in}\,L^2(\tau,T;\widehat{V}^*).\tag{3.29}$$

By (Equations 3.22)–(3.23), (3.25), (3.29), and the local existence obtained in step one, we get the global existence of the Galerkin approximate solutions for all time $t\in[\tau,T]$.

Step Three: *Existence of the global weak solutions.* We will prove that the limit function of the Galerkin approximate solutions is a weak solution of problem (3.1)–(3.2). Using the diagonal procedure, we deduce from Equations (3.22), (3.23), (3.25) and (3.29) that there exists a subsequence (which is still denoted by) $\{w^{(m)}\}$, an element $w\in L^\infty(\tau,T;\widehat{H})\cap L^2(\tau,T;\widehat{V})$ with $w'\in L^2(\tau,T;\widehat{V}^*)$, and some $\xi(t)\in L^2(\tau,T;(L^2(\Omega))^3)$, such that

$$w^{(m)}\rightharpoonup^* w\ \text{weakly star in}\ L^\infty(\tau,T;\widehat{H}),\ m\to\infty,\tag{3.30}$$

$$w^{(m)}\rightharpoonup w\ \text{weakly in}\ L^2(\tau,T;\widehat{V}),\ m\to\infty,\tag{3.31}$$

$$(w^{(m)})'\rightharpoonup w'\ \text{weakly in}\ L^2(\tau,T;\widehat{V}^*),\ m\to\infty,\tag{3.32}$$

$$w^{(m)}\longrightarrow w\ \text{strongly in}\ L^2(\tau,T;\widehat{H}),\ m\to\infty,\tag{3.33}$$

$$G(t,w_t^{(m)})\rightharpoonup\xi(t)\ \text{weakly in}\ L^2(\tau,T;(L^2(\Omega)^3),\ m\to\infty.\tag{3.34}$$

By Equations (3.31)–(3.32) and the compact embedding theorem (see, e.g., [5, 22, 39]), we have

$$w^{(m)}\in\mathcal{C}([\tau,T];\widehat{H}),\ w\in\mathcal{C}([\tau,T];\widehat{H}).\tag{3.35}$$

In order to pass to the limit in Equation (3.8) to obtain a weak solution, it is sufficient to prove the following convergent relations of the nonlinear terms:

$$\lim_{m\to\infty}\int_\tau^T \langle B(u^{(m)}(t),w^{(m)}(t)),\psi\rangle \mathrm{d}t = \int_\tau^T \langle B(u(t),w(t)),\psi\rangle \mathrm{d}t,\ \ \forall\psi\in\widehat{V}, \tag{3.36}$$

$$\lim_{m\to\infty}\int_\tau^T \big(G(t,w_t^{(m)}),\psi\big) = \int_\tau^T \big(G(t,w_t),\psi\big)\mathrm{d}t,\ \ \forall\psi\in\widehat{V}. \tag{3.37}$$

Proof. **(Proof of Equation (3.36).)** By the definition of the operator $B(\cdot,\cdot)$ and Equation (2.4), we have

$$\begin{aligned}
&|\langle B(u^{(m)},w^{(m)}),\psi\rangle - \langle B(u,w),\psi\rangle|\\
&= |\langle B(u^{(m)}-u,w^{(m)}),\psi\rangle + \langle B(u,w^{(m)}-w),\psi\rangle|\\
&\le \lambda\|u^{(m)}-u\|^{\frac12}\|\nabla(u^{(m)}-u)\|^{\frac12}\|w^{(m)}\|^{\frac12}\|\nabla w^{(m)}\|^{\frac12}\|\nabla\psi\|\\
&\quad +\lambda\|u\|^{\frac12}\|\nabla u\|^{\frac12}\|w^{(m)}-w\|^{\frac12}\|\nabla(w^{(m)}-w)\|^{\frac12}\|\nabla\psi\|,\ \ \forall\psi\in\widehat{V}.
\end{aligned} \tag{3.38}$$

Therefore, by Equations (3.25), (3.33), and (3.38), we obtain

$$\begin{aligned}
&\Big|\int_\tau^T (\langle B(u^{(m)},w^{(m)}),\psi\rangle - \langle B(u,w),\psi\rangle)\mathrm{d}t\Big|\\
&\le \lambda\|\nabla\psi\|\int_\tau^t \|u^{(m)}-u\|^{\frac12}\|\nabla(u^{(m)}-u)\|^{\frac12}\|w^{(m)}\|^{\frac12}\|\nabla w^{(m)}\|^{\frac12}\mathrm{d}t\\
&\quad +\lambda\|\nabla\psi\|\int_\tau^t \|u\|^{\frac12}\|\nabla u\|^{\frac12}\|w^{(m)}-w\|^{\frac12}\|\nabla(w^{(m)}-w)\|^{\frac12}\mathrm{d}t\\
&\le \lambda\|u^{(m)}-u\|^{\frac14}_{L^2(\tau,t;H)}\|w^{(m)}\|^{\frac12}_{L^2(\tau,t;\widehat{H})}\|(u^{(m)}-u)\|^{\frac14}_{L^2(\tau,t;V)}\|w^{(m)}\|^{\frac14}_{L^2(\tau,t;\widehat{V})}\|\psi\|_{\widehat{V}}\\
&\quad +\lambda\|u\|^{\frac14}_{L^2(\tau,t;H)}\|w^{(m)}-w\|^{\frac14}_{L^2(\tau,t;\widehat{H})}\|u\|^{\frac14}_{L^2(\tau,t;V)}\|(w^{(m)}-w)\|^{\frac14}_{L^2(\tau,t;\widehat{V})}\|\psi\|_{\widehat{V}}\\
&\quad \longrightarrow 0,\ \ \text{as}\ \ m\to\infty,
\end{aligned}$$

hence Equation (3.36) follows. ∎

Proof. **(Proof of Equation (3.37)).** From Equations (3.30) and (3.33), we get

$$w^{(m)}(t)\longrightarrow w(t)\ \ \text{strongly in}\ \ \widehat{H},\quad \text{a.e. } t\in[\tau,T]. \tag{3.39}$$

Since

$$w^{(m)}(s_2)-w^{(m)}(s_1) = \int_{s_1}^{s_2}(w^{(m)})'(r)\mathrm{d}r\ \ \text{in}\ \ \widehat{V}^*,\ \ \forall s_1,s_2\in[\tau,T], \tag{3.40}$$

we conclude from Equations (3.29) and (3.40) that $\{w^{(m)}\}$ is equicontinuous on the interval $[\tau,T]$ with values in $\widehat{V}^*$. Note that the injections $\widehat{V}\hookrightarrow\widehat{H}\hookrightarrow\widehat{V}^*$ are dense and compact. Thus, Equation (3.39), the equicontinuity in $\widehat{V}^*$ and the Ascoli–Arzelá theorem give

$$w^{(m)}(\cdot)\longrightarrow w(\cdot)\ \ \text{strongly in}\ \ \mathcal{C}([\tau,T];\widehat{V}^*). \tag{3.41}$$

It then follows from Equations (3.11), (3.22), (3.35), and (3.41) that for any sequence $\{t_m\}\subseteq[\tau,T]$ with $t_m\to t$ as $m\to\infty$, it holds that

$$w^{(m)}(t_m)\rightharpoonup w(t) \text{ weakly in } \widehat{H}. \tag{3.42}$$

In the sequel, we still denote w the function in $(-\infty,\tau]$ concatenated with the above limit in $[\tau,T]$. We next prove

$$w^{(m)}(\cdot)\longrightarrow w(\cdot) \text{ strongly in } \mathcal{C}([\tau,T];\widehat{H}). \tag{3.43}$$

If Equation (3.43) is not true, then, by Equation (3.35), there would exists an ϵ_0, a value $t_*\in[\tau,T]$ and subsequences (denoted by the same) $\{w^{(m)}\}$ and $\{t_m\}\subseteq[\tau,T]$ with $t_m\to t_*$, such that

$$\|w^{(m)}(t_m)-w(t_*)\|\geq\epsilon_0,\ \ \forall m\geq 1. \tag{3.44}$$

We shall prove that Equation (3.44) does not hold true. To this end, we shall prove

$$w^{(m)}(t_m)\longrightarrow w(t_*) \text{ strongly in } \widehat{H} \text{ as } m\to\infty. \tag{3.45}$$

Since Equation (3.42) and the lower semicontinuity of the norm yield

$$\|w(t_*)\|\leq\liminf_{m\to\infty}\|w^{(m)}(t_m)\|, \tag{3.46}$$

and notice that $\widehat{H}$ is a Hilbert space, it is sufficient to prove that

$$\|w(t_*)\|\geq\limsup_{m\to\infty}\|w^{(m)}(t_m)\|. \tag{3.47}$$

In fact, Equations (3.30)–(3.33) give

$$\int_\tau^{t_m}\langle F(r),w^{(m)}(r)\rangle \mathrm{d}r\longrightarrow\int_\tau^{t_*}\langle F(r),w(r)\rangle \mathrm{d}r. \tag{3.48}$$

Also, from Equations (3.6), (3.22), and (3.23), we see that for any $s\leq t$ with $s,t\in[\tau,T]$ there corresponds a constant C_5 depending on L_G,δ_1,λ_1 and $C_1(\tau,T,\mathcal{R})$ such that

$$\begin{aligned}\Big|\int_s^t\big(G(r,w_r^{(m)}),w^{(m)}(r)\big)\mathrm{d}r\Big|&\leq\int_s^t\Big(\frac{1}{2\delta_1\lambda_1}\|G(r,w_r^{(m)})\|^2+\frac{\delta_1\lambda_1}{2}\|w^{(m)}(r)\|^2\Big)\mathrm{d}r\\&\leq\int_s^t\Big(\frac{L_G^2}{2\delta_1\lambda_1}\|w_r^{(m)}\|_\gamma^2+\frac{\delta_1\lambda_1}{2}\|w^{(m)}(r)\|^2\Big)\mathrm{d}r\\&\leq C_5(t-s)+\frac{\delta_1\lambda_1}{2}\int_s^t\|w^{(m)}(r)\|^2\mathrm{d}r.\end{aligned} \tag{3.49}$$

Now Equations (3.10) and (3.49) yield

$$\begin{aligned}&\frac{1}{2}\|w^{(m)}(t)\|^2+\frac{\delta_1}{2}\int_s^t\|w^{(m)}(r)\|_{\widehat{V}}^2\mathrm{d}r\\&\leq\frac{1}{2}\|w^{(m)}(s)\|^2+\int_s^t\langle F(r),w^{(m)}(r)\rangle \mathrm{d}r+C_5(t-s),\ \ \forall\tau\leq s\leq t\leq T.\end{aligned} \tag{3.50}$$

By Equations (3.30)–(3.34), (3.36), and the convergent relation (see [36, (11)])

$$P_m\phi(s)\longrightarrow\phi(s) \text{ strongly in } \mathcal{C}_\gamma(\widehat{H}), \tag{3.51}$$

we conclude that $w\in\mathcal{C}([\tau,T];\widehat{H})$ is a solution of the following problem

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(w(t),v)+\langle Aw(t),v\rangle+\langle B(u(t),w(t)),v\rangle+\langle N(w(t)),v\rangle\\=&\langle F(t),v\rangle+(\xi(t),v),\ \ \forall v\in\widehat{V},\\&w(\tau)=\phi(0).\end{aligned}$$

Also, one can check that $w(t)$ satisfies the inequality for any $s,t\in[\tau,T]$

$$\begin{aligned}&\frac{1}{2}\|w(t)\|^2+\delta_1\int_s^t\|w(r)\|_{\widehat{V}}^2\mathrm{d}r\\\leq&\frac{1}{2}\|w(s)\|^2+\int_s^t\langle F(r),w(r)\rangle\mathrm{d}r+\frac{1}{2\delta_1\lambda_1}\int_s^t\|\xi(r)\|^2\mathrm{d}r+\frac{\delta_1\lambda_1}{2}\int_s^t\|w(r)\|^2\mathrm{d}r\\\leq&\frac{1}{2}\|w(s)\|^2+\int_s^t\langle F(r),w(r)\rangle\mathrm{d}r+\frac{1}{2\delta_1\lambda_1}\int_s^t\|\xi(r)\|^2\mathrm{d}r+\frac{\delta_1}{2}\int_s^t\|w(r)\|_{\widehat{V}}^2\mathrm{d}r.\end{aligned}\tag{3.52}$$

By Equatins (3.34) and (3.49), and the lower semi-continuity of the norm, we obtain that

$$\int_s^t\|\xi(r)\|^2\mathrm{d}r\leq\liminf_{m\to\infty}\int_s^t\|G(r,w_r^{(m)})\|^2\mathrm{d}r\leq2\delta_1\lambda_1C_5(t-s),\ \ \forall s,t\in[\tau,T].\tag{3.53}$$

So w also satisfies Equation (3.50).

Now, let's define two functions $\mathcal{H}_m(t),\mathcal{H}(t):[\tau,T]\mapsto\mathbb{R}$ as

$$\mathcal{H}_m(t):=\frac{1}{2}\|w^{(m)}(t)\|^2-\int_\tau^t\langle F,w^{(m)}(r)\rangle\mathrm{d}r-C_5t,\tag{3.54}$$

$$\mathcal{H}(t):=\frac{1}{2}\|w(t)\|^2-\int_\tau^t\langle F,w(r)\rangle\mathrm{d}r-C_5t,\tag{3.55}$$

where C_5 comes from Equation (3.49). From Equation (3.50), we deduce for any $t',t''\in[\tau,T]$ with $t'\leq t''$ that

$$\frac{1}{2}\|w^{(m)}(t'')\|^2\leq\frac{1}{2}\|w^{(m)}(t')\|^2+\int_{t'}^{t''}\langle F(r),w^{(m)}(r)\rangle\mathrm{d}r+C_5(t''-t'),\tag{3.56}$$

which implies $\mathcal{H}_m(t)$ is a non-increasing function. Moreover, Equation (3.35) implies $\mathcal{H}_m(t)$ is a continuous function for $t\in[\tau,T]$. Similarly, $\mathcal{H}(t)$ is also a non-increasing and continuous function with respect to $t\in[\tau,T]$. By these facts and Equation (3.39), we can conclude that

$$\mathcal{H}_m(t)\longrightarrow\mathcal{H}(t),\ \ \text{a.e. }t\in[\tau,T].\tag{3.57}$$

If $t_*=\tau$, Equation (3.47) can be trivially obtained from Equations (3.50) and (3.51). Thus, we can suppose that $t_*>\tau$, and there exists a monotonous increasing sequence $\{\widetilde{t_k}\}\subseteq[\tau,T]$ such that $\lim\limits_{k\to\infty}\widetilde{t_k}=t_*$ and

$$\lim_{m\to\infty}\mathcal{H}_m(\widetilde{t_k})=\mathcal{H}(\widetilde{t_k})\ \text{ for all }\ k\in\mathbb{N}.\tag{3.58}$$

The continuity of $\mathcal{H}$ shows that for any $\epsilon>0$ there corresponds some k_ϵ such that

$$|\mathcal{H}(\widetilde{t_k})-\mathcal{H}(t_*)|<\frac{\epsilon}{2},\ \text{ for all }\ k\geq k_\epsilon.\tag{3.59}$$

Note that $\mathcal{H}_m$ is non-increasing and the convergent relation Equation (3.58) holds for all $\widetilde{t_k}$, we can take $m > m(k_\epsilon)$ such that $t_m > \widetilde{t_k}$ and have

$$\begin{aligned}\mathcal{H}_m(t_m) - \mathcal{H}(t_*) &\leq \mathcal{H}_m(\widetilde{t_k}) - \mathcal{H}(t_*) \leq |\mathcal{H}_m(\widetilde{t_k}) - \mathcal{H}(\widetilde{t_k})| + |\mathcal{H}(\widetilde{t_k}) - \mathcal{H}(t_*)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned} \tag{3.60}$$

Therefore, Equations (3.48) and (3.60) yield

$$\begin{aligned}\limsup_{m\to\infty} \mathcal{H}_m(t_m) &= \limsup_{m\to\infty}\left(\frac{1}{2}\|w^{(m)}(t_m)\|^2 - \int_\tau^{t_m} \langle F, w^{(m)}(r)\rangle \mathrm{d}r - C_5 t_m\right) \\ &= \frac{1}{2}\limsup_{m\to\infty}\|w^{(m)}(t_m)\|^2 - \int_\tau^{t_*}\langle F, w(r)\rangle \mathrm{d}r - C_5 t_* \\ &\leq \mathcal{H}(t_*) = \frac{1}{2}\|w(t_*)\|^2 - \int_\tau^{t_*}\langle F, w(r)\rangle \mathrm{d}r - C_5 t_*, \end{aligned} \tag{3.61}$$

which shows that Equation (3.47) holds. Thus Equation (3.45) follows and Equation (3.44) does not hold true. Consequently, Equation (3.43) is proved.

Now, from Equations (3.9), (3.43), and (3.51), we deduce that

$$\begin{aligned}&\sup_{s\leq 0} e^{\gamma s}\|w^{(m)}(t+s) - w(t+s)\| \\ &= \max\Big\{\sup_{s\in(-\infty,\tau-t]} e^{\gamma s}\|P_m\phi(s+t-\tau) - \phi(s+t-\tau)\|, \\ &\qquad \sup_{s\in[\tau-t,0]} e^{\gamma s}\|w^{(m)}(t+s) - w(t+s)\|\Big\} \\ &\leq \max\Big\{e^{\gamma(\tau-t)}\|P_m\phi-\phi\|_\gamma, \max_{s\in[\tau,t]}\|w^{(m)}(s) - w(s)\|\Big\} \longrightarrow 0 \text{ as } m\to\infty,\end{aligned}$$

which implies

$$w_t^{(m)} \longrightarrow w_t \text{ in } \mathcal{C}_\gamma(\widehat{H}), \ \forall t \leq T. \tag{3.62}$$

Then Equation (3.62) and Assumption (II)(iii) show that

$$G(t, w_t^{(m)}) \longrightarrow G(t, w_t) \text{ strongly in } L^2(\tau, T; (L^2(\Omega))^3). \tag{3.63}$$

So Equation (3.37) follows immediately from Equation (3.63). ∎

Note that all terms in Equation (3.8) are linear with respect to $w^{(m)}$ or $w_t^{(m)}$ except the terms $\langle B(u^{(m)}(t), w^{(m)}(t)), v_j\rangle$ and $(G(t, w_t^{(m)}), v_j)$. Fortunately, we have proved Equations (3.36) and (3.37). Therefore, taking Equations (3.30)–(3.33) and Equations (3.36)–(3.37) into account, we can pass to the limit in Equation (3.8), concluding that $w \in \mathcal{C}([\tau, T]; \widehat{H})$ is a weak solution of problem (3.1)–(3.2). The proof of Theorem 3.1 is complete. ∎

We are going to investigate the uniqueness of the weak solution.

THEOREM 3.2 (Uniqueness). *Let the conditions of Theorem 3.1 hold, then for any given initial datum $\phi(s) \in \mathcal{C}_\gamma(\widehat{H})$ and for any $T > \tau$, there corresponds at most one weak solution to problem (3.1)–(3.2) in the interval* $(-\infty, T]$.

Proof. Let $w^{(1)}=(u^{(1)},\omega^{(1)})$ and $w^{(2)}=(u^{(2)},\omega^{(2)})$ be two solutions in the interval $(-\infty,T]$ of problem (3.1)–(3.2), with the same initial datum $w^{(1)}(\tau)=w^{(2)}(\tau)=\phi(s)$. Denote $w=(u,\omega)=w^{(1)}-w^{(2)}$. Then w satisfies

$$\frac{\partial w}{\partial t}+Aw+B(u^{(1)},w^{(1)})-B(u^{(2)},w^{(2)})+N(w)=G(t,w_t^{(1)})-G(t,w_t^{(2)}) \tag{3.64}$$

for $t\in(\tau,T]$, and

$$w(\theta)=0,\ \ \forall\theta\leq\tau. \tag{3.65}$$

Multiplying Equation (3.64) by $w(t)$, we can get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2&+\langle Aw(t),w(t)\rangle+\langle B(u^{(1)}(t),w^{(1)}(t))-B(u^{(2)}(t),w^{(2)}(t)),w(t)\rangle\\ &+\langle N(w(t)),w(t)\rangle=\big(G(t,w_t^{(1)})-G(t,w_t^{(2)}),w(t)\big),\ \ t\in(\tau,T].\end{aligned} \tag{3.66}$$

By Equations (2.4) and (2.11) and the obvious facts that

$$\|u(t)\|\leq\|w(t)\|\ \ \text{and}\ \ \|\nabla u(t)\|\leq\|\nabla w(t)\|,$$

we see

$$\begin{aligned}&|\langle B(u^{(1)}(t),w^{(1)}(t))-B(u^{(2)}(t),w^{(2)}(t)),w(t)\rangle|\\ =&|\langle B(u^{(1)}(t)-u^{(2)}(t),w^{(1)}(t)),w(t)\rangle+\langle B(u^{(2)}(t),w(t)),w(t)\rangle|\\ =&|\langle B(u(t),w^{(1)}(t)),w(t)\rangle|\leq\lambda\|u(t)\|^{\frac{1}{2}}\|\nabla u(t)\|^{\frac{1}{2}}\|w(t)\|^{\frac{1}{2}}\|\nabla w(t)\|^{\frac{1}{2}}\|\nabla w^{(1)}(t)\|\\ \leq&\lambda\|w(t)\|\|w(t)\|_{\widehat{V}}\|w^{(1)}(t)\|_{\widehat{V}}.\end{aligned} \tag{3.67}$$

Combining Equations (2.8) and (3.67), and Assumption (II)(iii), we get from Equation (3.66) that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2+\delta_1\|w(t)\|_{\widehat{V}}^2\\ \leq&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2+\langle Aw(t),w(t)\rangle+\langle N(w(t)),w(t)\rangle\\ \leq&(G(t,w_t^{(1)})-G(t,w_t^{(2)}),w(t))+|\langle B(u^{(1)}(t),w^{(1)}(t))-B(u^{(2)}(t),w^{(2)}(t)),w(t)\rangle|\\ \leq&L_G\|w_t\|_\gamma\|w(t)\|+\lambda\|w(t)\|\|w(t)\|_{\widehat{V}}\|w^{(1)}(t)\|_{\widehat{V}}.\end{aligned} \tag{3.68}$$

Now Equation (3.65) implies

$$\|w_\theta\|_\gamma=\sup_{s\leq0}e^{\gamma s}\|w(\theta+s)\|\leq\sup_{s\in[\tau,\theta]}\|w(s)\|,\ \ \tau\leq\theta\leq T. \tag{3.69}$$

Hence, integrating Equation (3.68) gives

$$\begin{aligned}&\|w(t)\|^2+2\delta_1\int_\tau^t\|w(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta-\|w(\tau)\|^2\\ =&\|w(t)\|^2+2\delta_1\int_\tau^t\|w(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta\\ \leq&2L_G\int_\tau^t\|w_\theta\|_\gamma\|w(\theta)\|\mathrm{d}\theta+2\lambda\int_\tau^t\|w(\theta)\|\|w(\theta)\|_{\widehat{V}}\|w^{(1)}(\theta)\|_{\widehat{V}}\mathrm{d}\theta\end{aligned}$$

$$
\begin{aligned}
&\le 2L_G\int_\tau^t \sup_{r\in[\tau,\theta]}\|w(r)\|\|w(\theta)\|\mathrm{d}\theta+2\lambda\int_\tau^t\|w(\theta)\|\|w(\theta)\|_{\widehat{V}}\|w^{(1)}(\theta)\|_{\widehat{V}}\mathrm{d}\theta\\
&\le 2L_G\int_\tau^t \sup_{r\in[\tau,\theta]}\|w(r)\|^2\mathrm{d}\theta+2\delta_1\int_\tau^t\|w(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta+\frac{\lambda^2}{2\delta_1}\int_\tau^t\|w(\theta)\|^2\|w^{(1)}(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta\\
&\le (2L_G+\frac{\lambda^2}{2\delta_1})\int_\tau^t(1+\|w^{(1)}(\theta)\|_{\widehat{V}}^2)\sup_{r\in[\tau,\theta]}\|w(r)\|^2\mathrm{d}\theta+2\delta_1\int_\tau^t\|w(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta, \qquad (3.70)
\end{aligned}
$$

which implies

$$
\sup_{r\in[\tau,t]}\|w(r)\|^2\le(2L_G+\frac{\lambda^2}{2\delta_1})\int_\tau^t(1+\|w^{(1)}(\theta)\|_{\widehat{V}}^2)\sup_{r\in[\tau,\theta]}\|w(r)\|^2\mathrm{d}\theta. \qquad (3.71)
$$

Using Gronwall's inequality on Equation (3.71) yields

$$
\sup_{r\in[\tau,t]}\|w(r)\|=0,\ \ \forall t\in[\tau,T], \qquad (3.72)
$$

which finishes the proof of Theorem 3.2. □

We now verify the stability of the weak solutions with respect to the initial data.

Theorem 3.3 (Stability). *Assume that the conditions of Theorem 3.1 hold. Let $w^{(i)}$ with $i=1,2$ be two solutions of problem (3.1)–(3.2) in the interval $(-\infty,T]$ with initial data $\phi^{(i)}(s)\in\mathcal{C}_\gamma(\widehat{H})$, respectively. Then*

$$
\begin{aligned}
\max_{r\in[\tau,t]}\|w^{(1)}(r)-w^{(2)}(r)\|^2&\le\big(\|\phi^{(1)}(\tau)-\phi^{(2)}(\tau)\|^2+\frac{L_G}{2\gamma}\|\phi^{(1)}(s)-\phi^{(2)}(s)\|_\gamma^2\big)\\
&\quad\cdot\exp\big[C_6\int_\tau^t(2+\|w^{(1)}(\theta)\|_{\widehat{V}}^2)\mathrm{d}\theta\big],\ \ \forall t\in[\tau,T], \qquad (3.73)\\
\|w_t^{(1)}-w_t^{(2)}\|_\gamma^2&\le(2+\frac{L_G}{2\gamma})\|\phi^{(1)}(s)-\phi^{(2)}(s)\|_\gamma^2\\
&\quad\cdot\exp\big[C_6\int_\tau^t(2+\|w^{(1)}(\theta)\|_{\widehat{V}}^2)\mathrm{d}\theta\big],\ \ \forall t\in[\tau,T], \qquad (3.74)
\end{aligned}
$$

where

$$
C_6:=\max\big\{\lambda^2/2\delta_1,\ 2L_G\big\}. \qquad (3.75)
$$

Proof. Let $w^{(i)}=(u^{(i)},\omega^{(i)})$ $(i=1,2)$ be two solutions in the interval $(-\infty,T]$ of problem (3.1)–(3.2) with the initial data $w^{(i)}(\tau)=\phi^{(i)}(s)$ respectively. If we denote

$$
u=u^{(1)}-u^{(2)},\ \ \omega=\omega^{(1)}-\omega^{(2)},\ \ w=(u,\omega)=w^{(1)}-w^{(2)},\ \ \phi(\cdot)=\phi^{(1)}(\cdot)-\phi^{(2)}(\cdot),
$$

then it holds that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2+\langle Aw(t),w(t)\rangle+\langle N(w(t)),w(t)\rangle\\
&\quad+\langle B(u^{(1)}(t),w^{(1)}(t))-B(u^{(2)}(t),w^{(2)}(t)),w(t)\rangle\\
&=\big(G(t,w_t^{(1)})-G(t,w_t^{(2)}),w(t)\big). \qquad (3.76)
\end{aligned}
$$

Similar to Equation (3.67), one can obtain

$$\begin{aligned}
&|\langle B(u^{(1)}(t),w^{(1)}(t))-B(u^{(2)}(t),w^{(2)}(t)),w(t)\rangle|\\
&=|\langle B(u(t),w^{(1)}(t)),w(t)\rangle|\le\lambda\|w(t)\|\|w(t)\|_{\widehat{V}}\|w^{(1)}(t)\|_{\widehat{V}}\\
&\le\delta_1\|w(t)\|_{\widehat{V}}^2+\frac{\lambda^2}{4\delta_1}\|w(t)\|^2\|w^{(1)}(t)\|_{\widehat{V}}^2.
\end{aligned}\tag{3.77}$$

We note that

$$\begin{aligned}
\|w_\theta\|_\gamma&=\sup_{s\le 0}e^{\gamma s}\|w(\theta+s)\|\\
&=\max\Big\{\sup_{s\in(-\infty,\tau-\theta]}e^{\gamma s}\|\phi(\theta-\tau+s)\|,\ \sup_{s\in(\tau-\theta,0]}e^{\gamma s}\|w(\theta+s)\|\Big\}\\
&\le\max\Big\{e^{\gamma(\tau-\theta)}\|\phi(s)\|_\gamma,\ \max_{s\in[\tau,\theta]}\|w(s)\|\Big\},
\end{aligned}\tag{3.78}$$

which together with Assumption (II)(iii) gives

$$\begin{aligned}
&\int_\tau^t\big(G(\theta,w_\theta^{(1)})-G(\theta,w_\theta^{(2)}),w(\theta)\big)\mathrm{d}\theta\le\int_\tau^t L_G\|w_\theta\|_\gamma\|w(\theta)\|\mathrm{d}\theta\\
&\le\int_\tau^t L_Ge^{\gamma(\tau-\theta)}\|\phi(s)\|_\gamma\|w(\theta)\|\mathrm{d}\theta+\int_\tau^t L_G\|w(\theta)\|\cdot\max_{s\in[\tau,\theta]}\|w(s)\|\mathrm{d}\theta.
\end{aligned}\tag{3.79}$$

Integrating Equation (3.76), using Equations (2.8), (3.77), and (3.79), we see for any $t\in[\tau,T]$ that

$$\begin{aligned}
&\frac{1}{2}\|w(t)\|^2+\delta_1\int_\tau^t\|w(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta\\
&\le\frac{1}{2}\|w(t)\|^2+\int_\tau^t\langle Aw(\theta),w(\theta)\rangle\mathrm{d}\theta+\int_\tau^t\langle N(w(\theta),w(\theta)\rangle\mathrm{d}\theta\\
&\le\frac{1}{2}\|\phi(\tau)\|^2+\int_\tau^t L_Ge^{\gamma(\tau-\theta)}\|\phi(s)\|_\gamma\|w(\theta)\|\mathrm{d}\theta+\int_\tau^t L_G\|w(\theta)\|\cdot\max_{s\in[\tau,\theta]}\|w(s)\|\mathrm{d}\theta\\
&\quad+\delta_1\int_\tau^t\|w(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta+\frac{\lambda^2}{4\delta_1}\int_\tau^t\|w(\theta)\|^2\|w^{(1)}(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta,
\end{aligned}$$

which obviously yields

$$\begin{aligned}
\|w(t)\|^2\le{}&\|\phi(\tau)\|^2+\frac{\lambda^2}{2\delta_1}\int_\tau^t\|w(\theta)\|^2\|w^{(1)}(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta+2L_G\|\phi(s)\|_\gamma\int_\tau^t e^{\gamma(\tau-\theta)}\|w(\theta)\|\mathrm{d}\theta\\
&+2L_G\int_\tau^t\|w(\theta)\|\cdot\max_{s\in[\tau,\theta]}\|w(s)\|\mathrm{d}\theta.
\end{aligned}\tag{3.80}$$

Observe that

$$\begin{aligned}
2L_G\|\phi(s)\|_\gamma\int_\tau^t e^{\gamma(\tau-\theta)}\|w(\theta)\|\mathrm{d}\theta&\le\frac{L_G}{2\gamma}\|\phi(s)\|_\gamma^2+2\gamma L_G\Big(\int_\tau^t e^{\gamma(\tau-\theta)}\|w(\theta)\|\mathrm{d}\theta\Big)^2\\
&\le\frac{L_G}{2\gamma}\|\phi(s)\|_\gamma^2+2\gamma L_G\int_\tau^t e^{2\gamma(\tau-\theta)}\mathrm{d}\theta\cdot\int_\tau^t\|w(\theta)\|^2\mathrm{d}\theta\\
&\le\frac{L_G}{2\gamma}\|\phi(s)\|_\gamma^2+L_G\int_\tau^t\max_{r\in[\tau,\theta]}\|w(r)\|^2\mathrm{d}\theta.
\end{aligned}\tag{3.81}$$

Replacing t with r of Equation (3.80), then taking the maximum for $r \in [\tau,t]$ and using Equation (3.81), we have

$$\begin{aligned}
&\max_{r\in[\tau,t]}\|w(r)\|^2 \\
&\le \|\phi(\tau)\|^2 + \frac{\lambda^2}{2\delta_1}\int_\tau^t \|w(\theta)\|^2\|w^{(1)}(\theta)\|_{\widehat{V}}^2 \mathrm{d}\theta \\
&\quad + \frac{L_G}{2\gamma}\|\phi(s)\|_\gamma^2 + L_G\int_\tau^t \max_{r\in[\tau,\theta]}\|w(r)\|^2\mathrm{d}\theta + 2L_G\int_\tau^t \|w(\theta)\|\cdot\max_{s\in[\tau,\theta]}\|w(s)\|\mathrm{d}\theta \\
&\le \|\phi(\tau)\|^2 + \frac{\lambda^2}{2\delta_1}\int_\tau^t \max_{r\in[\tau,\theta]}\|w(r)\|^2\|w^{(1)}(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta + \frac{L_G}{2\gamma}\|\phi(s)\|_\gamma^2 \\
&\quad + L_G\int_\tau^t \max_{r\in[\tau,\theta]}\|w(r)\|^2\mathrm{d}\theta + 2L_G\int_\tau^t \max_{r\in[\tau,\theta]}\|w(r)\|^2\mathrm{d}\theta \\
&\le \|\phi(\tau)\|^2 + \frac{L_G}{2\gamma}\|\phi(s)\|_\gamma^2 + C_6\int_\tau^t (2+\|w^{(1)}(\theta)\|_{\widehat{V}}^2)\cdot\max_{r\in[\tau,\theta]}\|w(r)\|^2\mathrm{d}\theta, \qquad (3.82)
\end{aligned}$$

where C_6 is as in Equation (3.75). Obviously, Equation (3.82) gives

$$\max_{r\in[\tau,t]}\|w(r)\|^2 \le \|\phi(\tau)\|^2 + \frac{L_G}{2\gamma}\|\phi(s)\|_\gamma^2 + C_6\int_\tau^t (2+\|w^{(1)}(\theta)\|_{\widehat{V}}^2)\cdot\max_{r\in[\tau,\theta]}\|w(r)\|^2\mathrm{d}\theta. \tag{3.83}$$

Using Gronwall's inequality on Equation (3.83) yields Equation (3.73). Then, Equation (3.74) follows easily from Equations (3.78) and (3.83). The proof of Theorem 3.3 is complete. □

4. Existence of the pullback attractors

Our goal in this section is to prove the existence of pullback attractor for the process associated to problem (3.1)–(3.2). By Theorem 3.1, we see that the biparametric family of maps of solutions operators $\{U(t,\tau)\}_{t\ge\tau} : \mathcal{C}_\gamma(\widehat{H}) \mapsto \mathcal{C}_\gamma(\widehat{H})$ defined by

$$U(t,\tau):\ \phi(s)\mapsto U(t,\tau)\phi(s) = w_t(s),\ \ t\ge\tau,\ \ s\in(-\infty,0], \tag{4.1}$$

generates a continuous process in $\mathcal{C}_\gamma(\widehat{H})$, where w is the solution of problem (3.1)–(3.2) corresponding to the initial datum $\phi(s)\in\mathcal{C}_\gamma(\widehat{H})$, and $w_t(s)$ is defined as in Equation (1.10).

We first introduce some concepts related to the pullback attractors. For the abstract concepts and results about these aspects, as well as the applications to some concrete partial differential equations, one can refer to [8, 18–20, 31, 32, 35, 38, 40, 42, 43].

Definition 4.1.

(1) *A family $\widehat{\mathcal{B}_0} = \{\mathcal{B}_0(t) | t\in\mathbb{R}\}$ of subsets of $\mathcal{C}_\gamma(\widehat{H})$ is called pullback absorbing for bounded sets if, for any bounded set $\mathcal{B}$ of $\mathcal{C}_\gamma(\widehat{H})$ and each t, there corresponds a time $\tau(\mathcal{B},t)$ such that $U(t,\tau)\mathcal{B}\subseteq\mathcal{B}_0(t)$ for all $\tau\le\tau(\mathcal{B},t)$.*

(2) *The process $\{U(t,\tau)\}_{t\ge t\tau}$ is said to be pullback $\widehat{\mathcal{B}_0}$-asymptotically compact if, for any $t\in\mathbb{R}$, any sequences $\{\tau_n\}\subseteq(-\infty,t]$ with $\tau_n\to-\infty$ as $n\to\infty$, and any $x_n\in\mathcal{B}_0(\tau_n)$, the sequence $\{U(t,\tau_n)x_n\}$ is relatively compact in $\mathcal{C}_\gamma(\widehat{H})$.*

(3) *A family* $\widehat{\mathcal{A}}=\{\mathcal{A}(t)|t\in\mathbb{R}\}$ *is said to be a pullback attractor for the process* $\{U(t,\tau)\}_{t\ge\tau}$ *in* $\mathcal{C}_\gamma(\widehat{H})$ *if it has the following properties:*

- *Compactness: for any* $t\in\mathbb{R},\mathcal{A}(t)$ *is a nonempty compact subset of* $\mathcal{C}_\gamma(\widehat{H})$*;*
- *Invariance:* $U(t,\tau)\mathcal{A}(\tau)=\mathcal{A}(t),\forall t\ge\tau$*;*
- *Pullback attracting: for any bounded set* $\mathcal{B}$ *of* $\mathcal{C}_\gamma(\widehat{H})$*, there holds*

$$\lim_{\tau\to-\infty}\operatorname{dist}_{\mathcal{C}_\gamma(\widehat{H})}(U(t,\tau)\mathcal{B},\mathcal{A}(t))=0,\forall t\in\mathbb{R},$$

where $\operatorname{dist}_X(Y,Z):=\sup_{y\in Y}\inf_{z\in Z}\operatorname{dist}_X(y,z)$ *means the Hausdorff semi-distance from* $Y\subseteq X$ *to* $Z\subseteq X$ *in the metric space* X*.*

We next prove some estimates of the solutions, which will be used when proving the existence of the pullback attractors.

Lemma 4.1. *Assume that the conditions of Theorem 3.1 hold. Let* w *be the solution of problem (3.1)–(3.2) with initial datum* $\phi(s)\in\mathcal{C}_\gamma(\widehat{H})$*, then for all* $t\ge\tau$*,*

$$\|w_t\|_\gamma^2\le e^{(-\delta_1\lambda_1+2L_G)(t-\tau)}\|\phi(s)\|_\gamma^2+\frac{2}{\delta_1}\int_\tau^t e^{(-\delta_1\lambda_1+2L_G)(t-\theta)}\|F(\theta)\|_{\widehat{V}^*}^2\mathrm{d}\theta, \tag{4.2}$$

$$\begin{aligned}\delta_1\int_\tau^t\|w(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta\le{}&2e^{\delta_1\lambda_1(t-\tau)}\|w(\tau)\|^2+5\delta_1^{-1}e^{-\delta_1\lambda_1\tau}\int_\tau^t e^{\delta_1\lambda_1\theta}\|F(\theta)\|_{\widehat{V}^*}^2\mathrm{d}\theta\\&+2e^{2L_G(t-\tau)}\|\phi(s)\|_\gamma^2+4\delta_1^{-1}e^{2L_Gt-\delta_1\lambda_1\tau}\int_\tau^t e^{(\delta_1\lambda_1-2L_G)\theta}\|F(\theta)\|_{\widehat{V}^*}^2\mathrm{d}\theta.\end{aligned} \tag{4.3}$$

Proof. The proof of Equation (4.2) is almost the same as that of Equation (3.21), so we omit the details here. We next prove Equation (4.3). Similar to Equation (3.24), one can obtain

$$\begin{aligned}&\delta_1\int_\tau^t\|w(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta\\&\le2e^{\delta_1\lambda_1(t-\tau)}\|w(\tau)\|^2+4e^{\delta_1\lambda_1(t-\tau)}\int_\tau^t e^{-\delta_1\lambda_1(t-\theta)}\Big(\frac{\|F(\theta)\|_{\widehat{V}^*}^2}{\delta_1}+L_G\|w_\theta\|_\gamma^2\Big)\mathrm{d}\theta\\&=2e^{\delta_1\lambda_1(t-\tau)}\|w(\tau)\|^2+4\delta_1^{-1}e^{-\delta_1\lambda_1\tau}\int_\tau^t e^{\delta_1\lambda_1\theta}\|F(\theta)\|_{\widehat{V}^*}^2\mathrm{d}\theta\\&\quad+4L_Ge^{-\delta_1\lambda_1\tau}\int_\tau^t e^{\delta_1\lambda_1\theta}\|w_\theta\|_\gamma^2\mathrm{d}\theta.\end{aligned} \tag{4.4}$$

Now, by Equation (4.2),

$$\begin{aligned}&\int_\tau^t e^{\delta_1\lambda_1\theta}\|w_\theta\|_\gamma^2\mathrm{d}\theta\\&\le\int_\tau^t e^{\delta_1\lambda_1\theta}\Big[e^{-(\delta_1\lambda_1-2L_G)(\theta-\tau)}\|\phi(s)\|_\gamma^2+\frac{2}{\delta_1}\int_\tau^\theta e^{-(\delta_1\lambda_1-2L_G)(\theta-s)}\|F(s)\|_{\widehat{V}^*}^2\mathrm{d}s\Big]\mathrm{d}\theta\\&=e^{(\delta_1\lambda_1-2L_G)\tau}\|\phi(s)\|_\gamma^2\int_\tau^t e^{2L_G\theta}\mathrm{d}\theta+\frac{2}{\delta_1}\int_\tau^t e^{2L_G\theta}\mathrm{d}\theta\cdot\int_\tau^\theta e^{(\delta_1\lambda_1-2L_G)s}\|F(s)\|_{\widehat{V}^*}^2\mathrm{d}s\end{aligned}$$

$$\leq \frac{e^{(\delta_1\lambda_1-2L_G)\tau+2L_G t}}{2L_G}\|\phi(s)\|_\gamma^2+\frac{e^{2L_G t}}{L_G\delta_1}\int_\tau^t e^{(\delta_1\lambda_1-2L_G)\theta}\|F(\theta)\|_{\widehat{V}^*}^2\mathrm{d}\theta. \tag{4.5}$$

Substituting Equation (4.5) into Equation (4.4) gives Equation (4.3). The proof is complete. ∎

To obtain the existence of the pullback absorbing set, we need some additional assumptions. For the sake of brevity, we denote

$$\rho := \delta_1\lambda_1 - 2L_G.$$

Notice that we have assumed $2\gamma > \delta_1\lambda_1$ in Theorem 3.1, so $2\gamma > \rho$.

Assumption (III): Assume $2L_G < \delta_1\lambda_1$ and

$$\int_{-\infty}^0 e^{\rho\theta}\|F(\theta)\|_{\widehat{V}^*}^2\mathrm{d}\theta < +\infty. \tag{4.6}$$

Note that if $F(\cdot)\in L^2_{loc}(\mathbb{R};\widehat{V}^*)$, then the condition (4.6) is equivalent to (see, e.g., [43])

$$\int_{-\infty}^t e^{-\rho(t-\theta)}\|F(\theta)\|_{\widehat{V}^*}^2\mathrm{d}\theta < +\infty, \ \ \forall t\in\mathbb{R}. \tag{4.7}$$

LEMMA 4.2. *Let Assumption* (III) *and the conditions of Theorem 3.1 hold, then the family* $\widehat{\mathcal{B}}_0 = \{\mathcal{B}_0(t)|t\in\mathbb{R}\}$ *with* $\mathcal{B}_0(t) = \mathcal{B}_{\mathcal{C}_\gamma(\widehat{H})}(0,\mathcal{R}(t))$ *is pullback absorbing for the process* $\{U(t,\tau)\}_{t\geq\tau}$ *in* $\mathcal{C}_\gamma(\widehat{H})$, *where* $\mathcal{B}_{\mathcal{C}_\gamma(\widehat{H})}(0,\mathcal{R}(t))$ *is the closed ball in* $\mathcal{C}_\gamma(\widehat{H})$ *with center zero and radius* $\mathcal{R}(t)$ *given by*

$$\mathcal{R}^2(t) := 1+\frac{2}{\delta_1}\int_{-\infty}^t e^{-\rho(t-\theta)}\|F(\theta)\|_{\widehat{V}^*}^2\mathrm{d}\theta < +\infty, \tag{4.8}$$

Proof. The result is a direct consequence of Assumption (III) and Equation (4.2). ∎

LEMMA 4.3. *Let the conditions of Lemma 4.2 hold, then the process* $\{U(t,\tau)\}_{t\geq\tau}$ *is pullback* $\widehat{\mathcal{B}}_0$*-asymptotically compact in* $\mathcal{C}_\gamma(\widehat{H})$.

Proof. Let's fix some $t^*\in\mathbb{R}$, and $w^{(n)}(\cdot)$ be a sequence of solutions with initial time τ_n and with initial data $\phi^{(n)}(s)\in\mathcal{B}_0(\tau_n) := \mathcal{B}_{\mathcal{C}_\gamma(\widehat{H})}(0,\mathcal{R}(\tau_n))$, where $\tau_n\subseteq(-\infty,t^*]$ satisfying $\tau_n\to-\infty$ as $n\to+\infty$. The task is to prove that the sequence $\{w_{t^*}^{(n)}(\cdot)\}$ defined by

$$w_{t^*}^{(n)}(\cdot) := w_{t^*}^{(n)}(\cdot;\tau_n,w_{\tau_n}) = U(t^*,\tau_n)w_{\tau_n}, \tag{4.9}$$

is relatively compact in $\mathcal{C}_\gamma(\widehat{H})$. We next divide the proofs into two steps.

Step one: *We verify that* $\{w^{(n)}(t^*+\cdot)\}$ *is relatively compact in* $\mathcal{C}([-T,0];\widehat{H})$, *where* $T>0$ *is an arbitrary time value.* In fact, it follows from Equations (4.2) and (4.8) that there exists an $n(t^*,T)$ such that

$$\tau_n < t^*-T \ \text{ and } \ \|w_t^{(n)}(s)\|_\gamma^2 \leq C_7(t^*,T), \ \ \forall t\in[t^*-T,t^*], \ \ n>n(t^*,T), \tag{4.10}$$

where

$$C_7(t^*,T):=1+\frac{2}{\delta_1}e^{-\rho(t^*-T)}\int_{-\infty}^{t^*}e^{\rho\theta}\|F(\theta)\|^2_{\widehat{V}^*}\mathrm{d}\theta. \tag{4.11}$$

Then, analogously as we did in the proof of Equation (3.43), we can use the energy method and compact embedding to establish

$$w^{(n)}(\cdot)\longrightarrow w(\cdot)\ \text{ strongly in }\ \mathcal{C}([t^*-T,t^*];\widehat{H})\ \text{ as }\ n\to\infty. \tag{4.12}$$

Evidently, Equation (4.12) gives

$$w^{(n)}(t^*+s)\longrightarrow w(t^*+s)\ \text{ strongly in }\ \mathcal{C}([-T,0];\widehat{H})\ \text{ as }\ n\to\infty. \tag{4.13}$$

Step Two: *We prove that the sequence* $\{w_{t^*}^{(n)}(\cdot)\}$ *converges strongly to* $w_{t^*}(\cdot)$ *in* $\mathcal{C}_\gamma(\widehat{H})$. To this end, we prove that for any $\varepsilon>0$ there corresponds an n_ε such that

$$\|w_{t^*}^{(n)}(s)-w_{t^*}(s)\|_\gamma=\sup_{s\in(-\infty,0]}e^{\gamma s}\|w^{(n)}(t^*+s)-w(t^*+s)\|\le\varepsilon,\ \forall n\ge n_\varepsilon. \tag{4.14}$$

By Equations (3.11) and (4.10), we see for $n\ge n(t^*,T)$ that

$$\|w^{(n)}(t)\|^2\le\|w^{(n)}(t)\|^2_\gamma\le C_7(t^*,T),\ \forall t\in[t^*-T,t^*]. \tag{4.15}$$

Since for any fixed $T>0$, $w_{t^*}^{(n)}(s)=w^{(n)}(t^*+s)$ with $s\in[-T,0]$ satisfies the estimate (4.15) for $n\ge n(t^*,T)$, so we also have from Equation (4.13) that

$$\|w(t^*+s)\|^2\le 1+Me^{\rho T},\ \forall s\in[-T,0], \tag{4.16}$$

where

$$M:=\frac{2e^{-\rho t^*}}{\delta_1}\int_{-\infty}^{t^*}e^{\rho\theta}\|F(\theta)\|^2_{\widehat{V}^*}\mathrm{d}\theta.$$

Notice that $\rho<2\gamma$. So, we can fix some $T_\varepsilon>0$ such that

$$\max\{e^{-\gamma T_\varepsilon},M^{1/2}e^{\rho/2}e^{(\rho-2\gamma)T_\varepsilon/2}\}\le\frac{\varepsilon}{4}. \tag{4.17}$$

Now Equation (4.13) shows that there exists an $n_\varepsilon\ge n(t^*,T_\varepsilon)$ such that

$$e^{\gamma s}\|w^{(n)}(t^*+s)-w(t^*+s)\|\le\varepsilon,\ \forall s\in[-T_\varepsilon,0]\ \text{ and }\ \tau_n\le t^*-T_\varepsilon,\ \forall n\ge n_\varepsilon.$$

Hence, in order to prove Equation (4.14) we only need verify

$$\sup_{s\in(-\infty,-T_\varepsilon]}e^{\gamma s}\|w^{(n)}(t^*+s)-w(t^*+s)\|\le\varepsilon,\ \forall n\ge n_\varepsilon. \tag{4.18}$$

Observe that,

$$\begin{aligned}&\sup_{s\in(-\infty,-T_\varepsilon]}e^{\gamma s}\|w^{(n)}(t^*+s)-w(t^*+s)\|\\&\le\sup_{s\in(-\infty,-T_\varepsilon]}e^{\gamma s}\|w(t^*+s)\|+\sup_{s\in(-\infty,-T_\varepsilon]}e^{\gamma s}\|w^{(n)}(t^*+s)\|.\end{aligned} \tag{4.19}$$

On the one hand, by Equation (4.16) and the choice of T_ε in Equation (4.17), it is easy to check that

$$\begin{aligned} e^{\gamma s}\|w(t^*+s)\| &\le e^{-\gamma(T_\varepsilon+k)}\left[1+Me^{\rho(T_\varepsilon+k+1)}\right]^{1/2} \\ &\le e^{-\gamma T_\varepsilon}+M^{1/2}e^{\rho/2}e^{(\rho-2\gamma)T_\varepsilon/2} \\ &\le \frac{\varepsilon}{4}+\frac{\varepsilon}{4}=\frac{\varepsilon}{2}, \text{ for } s\in[-(T_\varepsilon+k+1),-(T_\varepsilon+k)],\ \forall k\in\mathbb{N}. \end{aligned} \tag{4.20}$$

On the other hand, according to Lemma 4.2 and the facts

$$w^{(n)}(t^*+s)=\begin{cases}\phi^{(n)}(t^*-\tau_n+s), & s\in(-\infty,\tau_n-t^*) \\ w^{(n)}(t^*+s), & s\in[\tau_n-t^*,0],\end{cases} \tag{4.21}$$

we get for $n\ge n_\varepsilon$ that

$$\begin{aligned} \sup_{s\le\tau_n-t^*} e^{\gamma s}\|\phi^{(n)}(t^*-\tau_n+s)\| &= \sup_{s\le\tau_n-t^*} e^{\gamma(t^*-\tau_n+s)}e^{\gamma(\tau_n-t^*)}\|\phi^{(n)}(t^*-\tau_n+s)\| \\ &= e^{\gamma(\tau_n-t^*)}\|\phi^{(n)}(s)\|_\gamma \\ &\le e^{\gamma(\tau_n-t^*)}\mathcal{R}(\tau_n)\le e^{\gamma(\tau_n-t^*)}+M^{1/2}e^{(2\gamma-\rho)(\tau_n-t^*)/2} \\ &\le e^{-\gamma T_\varepsilon}+M^{1/2}e^{(\rho-2\gamma)T_\varepsilon/2}\le\frac{\varepsilon}{4}+\frac{\varepsilon}{4}=\frac{\varepsilon}{2}. \end{aligned} \tag{4.22}$$

Also, by the choice of n_ε, there holds

$$\begin{aligned} \sup_{s\in[\tau_n-t^*,-T_\varepsilon]} e^{\gamma s}\|w^{(n)}(t^*+s)\| &= \sup_{\theta\in[\tau_n-t^*+T_\varepsilon,0]} e^{\gamma(\theta-T_\varepsilon)}\|w^{(n)}(t^*-T_\varepsilon+\theta)\| \\ &\le e^{-\gamma T_\varepsilon}\|w^{(n)}_{t^*-T_\varepsilon}(s)\|_\gamma\le e^{-\gamma T_\varepsilon}[C_7(t^*,T_\varepsilon)]^{1/2} \\ &= e^{-\gamma T_\varepsilon}+M^{1/2}e^{(\rho-2\gamma)T_\varepsilon/2}\le\frac{\varepsilon}{2}, \end{aligned} \tag{4.23}$$

where we have also used Equation (4.10) with $T=T_\varepsilon$. Therefore, it follows from Equations (4.21)–(4.23) that

$$\begin{aligned} &\sup_{s\in(-\infty,-T_\varepsilon]} e^{\gamma s}\|w^{(n)}(t^*+s)\| \\ &\le \max\left\{\sup_{s\in(-\infty,\tau_n-t^*)} e^{\gamma s}\|\phi^{(n)}(t^*-\tau_n+s)\|,\ \sup_{s\in[\tau_n-t^*,-T_\varepsilon]} e^{\gamma s}\|w^{(n)}(t^*+s)\|\right\} \\ &\le\frac{\varepsilon}{2},\ \forall n\ge n_\varepsilon. \end{aligned} \tag{4.24}$$

Now, Equation 4.18) follows from Equations (4.19), (4.20), and (4.24). The proof is complete. ∎

At this stage, we can state the main result of this section.

THEOREM 4.1. *Let assumptions* (I)–(III) *hold and suppose* $2\gamma>\delta_1\lambda_1$. *Then the process* $\{U(t,\tau)\}_{t\ge\tau}$ *defined by Equation (4.1) has a pullback attractor* $\widehat{\mathcal{A}}=\{\mathcal{A}(t)|t\in\mathbb{R}\}$ *satisfying the properties stated in Definition 4.1(3).*

Proof. By [8, Theorem 7] or [36, Theorem 13], and lemmas 4.1–4.3, we have the desired result. ∎

In order to illustrate the extensions of Theorem 4.1, we need introduce another definition called pullback $\mathcal{D}$-attractors for a universe which is composed of families of time-dependent sets. Let there be given $\mathcal{D}$, a nonempty class of sets parameterized in time $\widehat{D}=\{D(t)|t\in\mathbb{R}\}$ with $D(t)\subset\mathcal{C}_\gamma(\widehat{H})$ for all $t\in\mathbb{R}$.

Definition 4.2. *A family of sets $\widehat{\mathcal{A}}_\mathcal{D}=\{\mathcal{A}_\mathcal{D}(t)|t\in\mathbb{R}\}$ is called a pullback $\mathcal{D}$-attractor for the process $\{U(t,\tau)\}_{t\geq\tau}$ in $\mathcal{C}_\gamma(\widehat{H})$ if it has the following properties:*

- *Compactness: for any $t\in\mathbb{R}$, $\mathcal{A}_\mathcal{D}(t)$ is a nonempty compact subset of $\mathcal{C}_\gamma(\widehat{H})$;*
- *Invariance: $U(t,\tau)\mathcal{A}_\mathcal{D}(\tau)=\mathcal{A}_\mathcal{D}(t), \forall t\geq\tau$;*
- *Pullback attracting: $\widehat{\mathcal{A}}_\mathcal{D}$ is pullback $\mathcal{D}$-attracting in the following sense:*

$$\lim_{\tau\to-\infty}\operatorname{dist}_{\mathcal{C}_\gamma(\widehat{H})}\big(U(t,\tau)D(\tau),\mathcal{A}_\mathcal{D}(t)\big)=0, \forall\widehat{D}=\{D(t)|\ t\in\mathbb{R}\}\in\mathcal{D},\ t\in\mathbb{R}.$$

We end the paper with two remarks.

Remark 4.1. We want to point out that, using the similar derivations as those as in lemmas 4.1-4.3, one can obtain the existence of the pullback $\mathcal{D}$-attractor $\widehat{\mathcal{A}}_\mathcal{D}=\{\mathcal{A}_\mathcal{D}(t)|t\in\mathbb{R}\}$ for the process $\{U(t,\tau)\}_{t\geq\tau}$ in $\mathcal{C}_\gamma(\widehat{H})$. Furthermore, if

$$\sup_{r\leq 0}\int_{-\infty}^{r}e^{-\rho(r-\theta)}\|F(\theta)\|^2_{\widehat{V}^*}\mathrm{d}\theta<+\infty,$$

then $\mathcal{A}(t)=\mathcal{A}_\mathcal{D}(t)$ for all $t\in\mathbb{R}$.

Remark 4.2. There is still much work to be done concerning the micropolar fluid flows. For example, we could consider the regularity of pullback attractors obtained in Theorem 4.1 and Remark 4.1. Also one can investigate the well-posedness, as well as the pullback asymptotic behaviors of the solutions on unbounded domains where the usual Sobolev embedding is no longer compact. These issues will be the topics of some other papers.

REFERENCES

[1] R.A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.
[2] M. Boukrouche, G. Łukaszewicz, and J. Real, *On pullback attractors for a class of two-dimensional turbulent shear flows*, Inter. J. Engrg. Sci., 44, 830–844, 2006.
[3] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2010.
[4] F. Boyer and P. Fabrie, *Mathematical Tools for the Study of the Incompressible Navier–Stokes Equations and Related Models*, Springer, New York, 2012.
[5] V.V. Chepyzhov and M.I. Vishik, *Attractors for Equations of Mathematical Physics*, AMS, R.I., 2002.
[6] T. Caraballo and J. Real, *Asymptotic behaviour of two-dimensional Navier–Stokes equations with delay*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci., 459, 3181–3194, 2003.
[7] T. Caraballo and J. Real, *attractors 2D-Navier–Stokes modes with delays*, J. Diff. Eqs., 205, 271–497, 2004.
[8] T. Caraballo, G. Łukaszewicz, J. Real, and Pullback, *attractors for asymptotically compact nonautonomous dynamical systems*, Nonlinear Anal.-TMA, 64, 484–498, 2006.
[9] J. Chen, Z. Chen, and B. Dong, *Existence of H^2-global attractors of two-dimensional micropolar fluid flows*, J. Math. Anal. Appl., 322, 512–522, 2006.
[10] J. Chen, B. Dong, and Z. Chen, *Uniform attractors of non-homogeneous micropolar fluid flows in non-smooth domains*, Nonlinearity, 20, 1619–1635, 2007.
[11] J. Chen, B. Dong, and Z. Chen, *Pullback attractors of non-autonomous micropolar fluid flows*, J. Math. Anal. Appl., 336, 1384–1394, 2007.
[12] G. Chen, *Pullback attractor for non-homogeneous micropolar fluid flows in non-smooth domains*, Nonlinear Anal.-RWA, 10, 3018–3027, 2009.

[13] B. Dong and Z. Chen, *Global attractors of two-dimensional micropolar fluid flows in some unbounded domains*, Appl. Math. Comp., 182, 610–620, 2006.

[14] B. Dong and Z. Zhang, *Global regularity of the 2D micropolar fluid flows with zero angular viscosity*, J. Diff. Eqs., 249, 200–213, 2010.

[15] A.C. Eringen, *Theory of micropolar fluids*, J. Math. Mech., 16, 1–18, 1966.

[16] Lucas C.F. Ferreira, and Juliana C. Precioso, *Existence of solutions for the 3D-micropolar fluid system with initial data in Besov–Morrey spaces*, Z. Angew. Math. Phys., 64, 1699–1710, 2013.

[17] P. Galdi and S. Rionero, *A note on the existence and uniqueness of solusions of the micropolar fluid equations*, Int. J. Engng. Sci., 15, 105–108, 1977.

[18] J. García-Luengo, P. Marín-Rubio, and J. Real, *H^2-boundedness of the pullback attractors for non-autonomous 2D Navier–Stokes equations in bounded domains*, Nonlinear Anal.-TMA, 74, 4882–4887, 2011.

[19] J. García-Luengo, P. Marín-Rubio, and J. Real, *Pullback attractors in V for non-autonomous 2D-Navier–Stokes equations and their tempered behavior*, J. Diff. Eqs., 252, 4333–4356, 2012.

[20] J. García-Luengo, P. Marín-Rubio, and J. Real, *Pullback attractors for three-dimensional non-autonomous Navier–Stokes–Voigt equations*, Nonlinearity, 25, 905–930, 2012.

[21] Y. Hino, S. Murakami, and T. Naito, *Functional Differential Equations with Infinite Delay*, Springer-Verlag, Berlin, 1991.

[22] J.-L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Nonlinéaires*, Dunod, Paris, 1969.

[23] G. Łukaszewicz, *On nonstationary flows of asymmetric fluids*, Rend. Accnd. Naz. Memorie di Matematica, 12, 83–97, 1988.

[24] G. Łukaszewicz, *On the existence, uniqueness and asymptotic properties for solutions of flows of asymmetric fluids*, Rend. Accad. Nat. Sci., Memorie di Matematica, 13, 105–120, 1989.

[25] G. Łukaszewicz, *On nonstationary flows of incompressible asymmetric fluids*, Math. Meth. Appl. Sci., 13, 219–232, 1999.

[26] G. Łukaszewicz, *Micropolar Fluids: Theory and Applications, Modeling and Simulation in Science, Engineering and Technology*, Birkhauser, Boston, 1999.

[27] G. Łukaszewicz, *Long time behavior of 2D micropolar fluid flows*, Mathematical and Computer Modelling, 34, 487–509, 2001.

[28] G. Łukaszewicz, *Asymptotic behavior of micropolar fluid flows*, Inter. J. Eng. Sci., 41, 259–269, 2003.

[29] G. Łukaszewicz and W. Sadowski, *Uniform attractor for 2D magneto-micropolar fluid flow in some unbounded domains*, Z. Angew. Math. Phys., 55, 247–257, 2004.

[30] G. Łukaszewicz and A. Tarasińska, *On H^1-pullback attractors for nonautonomous micropolar fluid equations in a bounded domain*, Nonlinear Anal.-TMA, 71, 782–788, 2009.

[31] J.A. Langa, G. Łukaszewicz, and J. Real, *Finite fractal dimension of pullback attractors for non-autonomous 2D Navier–Stokes equations in some unbounded domains*, Nonlinear Anal.-TMA, 66, 735–749, 2007.

[32] Y. Li and C.K. Zhong, *Pullback attractors for the norm-to-weak continuous process and application to the nonautonomous reaction-diffusion equations*, Appl. Math. Comp., 190, 1020–1029, 2007.

[33] J. Málek, J. Nečas, M. Rokyta, and M. Růžička, *Weak and Measure-valued Solutions to Evolutionary PDEs*, Champman-Hall, New York, 1996.

[34] P. Marin-Rubio, and J. Real, *Attractors for 2D-Navier–Stokes equations with delays on some unbounded domains*, Nonlinear Anal., 67, 2784–2799, 2007.

[35] P. Marín-Rubio, and J. Real, *On the relation between two different concepts of pullback attractors for non-autonomous dynamical systems*, Nonlinear Anal.-TMA, 71, 3956–3963, 2009.

[36] P. Marín-Rubio, J. Real, and J. Valero, *Pullback attractors for a two-dimensional Navier–Stokes model in an infinite delay case*, Nonlinear Anal.-TMA, 74, 2012–2030, 2011.

[37] B. Nowakowski, *Long-time behavior of micropolar fluid equations in cylindrical domains*, Nonlinear Anal.-RWA, 14, 2166–2179, 2013.

[38] H. Song and H. Wu, *Pullback attractors of nonautonomous reaction-diffusion equations*, J. Math. Anal. Appl., 325, 1200–1215, 2007.

[39] R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, Springer, Berlin, 1988.

[40] C. Zhao, S. Zhou, *Pullback attractors for nonautonomous incompressible non-Newtonian fluid*, J. Diff. Eqs., 238, 394–425, 2007.

[41] C. Zhao, S. Zhou, and X. Lian, *H^1-uniform attractor and asymptotic smoothing effect of solutions for a nonautonomous micropolar fluid flow in 2D unbounded domains*, Nonlinear Anal.-RWA, 9, 608–627, 2008.

[42] C. Zhao, *Pullback asymptotic behavior of solutions fora non-autonomous non-Newtonian fluid on 2D unbounded domains*, J. Math. Phys., 12, 1–21, 2012.

[43] C. Zhao, G. Liu, and W. Wang, *Smooth pullback attractors for a non-autonomous 2D non-Newtonian fluid and their tempered behavior*, J. Math. Fluid Mech., 16, 243–262, 2014.

[44] C. Zhao and W. Sun, *Pullback dynamical behaviors of the non-autonomous micropolar fluid flows*, Dynamics of PDE, 12, 265–288, 2015.