Commun. Math. Sci.
Vol. 15, No. 1, pp. 55–75

ON THE APPROXIMATION OF THE PRINCIPAL EIGENVALUE FOR A CLASS OF NONLINEAR ELLIPTIC OPERATORS*

ISABEAU BIRINDELLI†, FABIO CAMILLI‡, AND ITALO CAPUZZO DOLCETTA§

Abstract. We present a finite difference method to compute the principal eigenvalue and the corresponding eigenfunction for a large class of second-order elliptic operators including notably linear operators in non-divergence form and fully nonlinear operators. The principal eigenvalue is computed by solving a finite-dimensional nonlinear min-max optimization problem. We prove the convergence of the method and discuss its implementation. Some examples where the exact solution is explicitly known show the effectiveness of the method.

Key words. Principal eigenvalue, nonlinear elliptic operators, finite difference schemes, convergence.

AMS subject classifications. 35J60, 35P30, 65M06.

1. Introduction

Consider the elliptic self-adjoint operator

$$Lu(x)=\partial_i\left(a_{ij}(x)\partial_j u(x)\right), \tag{1.1}$$

where $a_{ij}=a_{ji}$ are smooth functions in Ω, a smooth bounded open subset of $\mathbb{R}^n$, satisfying $a_{ij}\xi_i\xi_j\geq\alpha|\xi|^2$ for some $\alpha>0$. It is well-known that the minimum value λ_1 in the Rayleigh-Ritz variational formula

$$\lambda_1=\inf_{\varphi\in H_0^1(\Omega),\varphi\not\equiv 0}\frac{-\int_\Omega\varphi(x)\,L\varphi(x)\,dx}{\|\varphi\|^2_{L^2(\Omega)}}=\inf_{\varphi\in H_0^1(\Omega),\varphi\not\equiv 0}\frac{\int_\Omega a_{ij}(x)\partial_j\varphi(x)\partial_i\varphi(x)\,dx}{\|\varphi\|^2_{L^2(\Omega)}}$$

is attained at some function w_1 satisfying

$$\begin{cases} Lw_1(x)+\lambda_1 w_1(x)=0 & x\in\Omega,\\ w_1(x)=0 & x\in\partial\Omega.\end{cases}$$

The number λ_1 is usually referred to as the principal eigenvalue of L in Ω, and w_1 is the corresponding principal eigenfunction. For operators of the form (1.1) and also more general linear operators in divergence form there is a vast literature on computational methods for the principal eigenvalue (see for example [2, 10, 14, 22]).

General non-divergence type elliptic operators, namely

$$Lu(x)=a_{ij}(x)\partial_{ij}u(x)+b_i(x)\partial_i u(x)+c(x)u, \tag{1.2}$$

are not self-adjoint, and the spectral theory is then much more involved: in particular, the Rayleigh-Ritz variational formula is not available anymore. In the seminal paper [12] by M.D. Donsker and S.R.S. Varadhan, a min-max formula for the principal eigenvalue

*Received: July 8, 2015; accepted (in revised form): March 14, 2016. Communicated by Qiang Du.
†Dip. di Matematica, "Sapienza" Università di Roma, P.le Aldo Moro 2, 00185 Roma, Italy (isabeau@mat.uniroma1.it).
‡Dip. di Scienze di Base e Applicate per l'Ingegneria, "Sapienza" Università di Roma, via Scarpa 16, 00161 Roma, Italy (camilli@dmmm.uniroma1.it).
§Dip. di Matematica, "Sapienza" Università di Roma, P.le Aldo Moro 2, 00185 Roma, Italy (capuzzo@mat.uniroma1.it).

of a class of elliptic operators L, including the operator defined in (1.2), was proved, namely

$$\lambda_1 = -\inf_{\varphi\in C^2(\Omega),\varphi>0}\ \sup_{x\in\Omega} \frac{L\varphi(x)}{\varphi(x)}. \tag{1.3}$$

In that paper other representation formulas for λ_1 were also proposed in terms of large deviations and of the average long run time behavior of the positive semigroup generated by L. A further crucial step in that direction is the paper [6] by H. Berestycki, L. Nirenberg, and S.R.S. Varadhan, where the validity of formula (1.3) is proved under mild smoothness assumptions (Ω a bounded open set and $a_{ij}\in C^0(\Omega)$, b_i, $c\in L^\infty(\Omega)$). Moreover it is proved that Equation (1.3) is equivalent to

$$\lambda_1 := \sup\{\lambda\in\mathbb{R} : \exists\varphi>0 \text{ such that } L\varphi+\lambda\varphi\le 0 \quad \text{in } \Omega\}.$$

Following this path of ideas, notions of principal eigenvalue for fully nonlinear uniformly elliptic operators of the form

$$F[u] = F(x,u(x),Du(x),D^2u(x))$$

have been introduced and analyzed in [1, 5, 8, 11, 15, 20]. A now-established definition of principal eigenvalue is given by

$$\lambda_1 := \sup\{\lambda\in\mathbb{R} : \exists\varphi>0 \text{ such that } F[\varphi]+\lambda\varphi\le 0 \quad \text{in } \Omega\} \tag{1.4}$$

where the inequality in the definition (1.4) is intended in the viscosity sense. It is possible to prove under appropriate assumptions (see conditions (2.1)–(2.2)) that there exists a viscosity solution w_1 of

$$\begin{cases} F[w_1]+\lambda_1 w_1(x) = 0 & x\in\Omega, \\ w_1(x) = 0 & x\in\partial\Omega. \end{cases} \tag{1.5}$$

Moreover, the characterization (1.3) still holds in this nonlinear setting.

As it is well-known, the principal eigenvalue plays a key role in several respects, both in the existence theory and in the qualitative analysis of elliptic partial differential equations, as well in applications to large deviations [1, 12], bifurcation issues [20], ergodic and long run average cost problems in stochastic control [4]. For linear non self-adjoint operators and, a fortiori, for nonlinear ones, the principal eigenvalue can be explicitly computed only in very special cases (see e.g. [9, 21]); hence the importance of devising numerical algorithms for the problem. But, apart some specific cases (see [7] for the p-Laplace operator), approximation schemes and computational methods are not available in the literature, at least at our present knowledge.

The aim of this paper is to define a numerical scheme for the principal eigenvalue of nonlinear uniformly elliptic operators via a finite difference approximation of formula (1.3). More precisely, denoting by $\mathbb{Z}^n_h = h\mathbb{Z}^n$ the orthogonal lattice in $\mathbb{R}^n$ where $h>0$ is a discretization parameter, we consider a discrete operator F_h acting on functions defined on a discrete subset $\Omega_h\subset\mathbb{Z}^n_h$ of Ω and the corresponding approximated version of Equation (1.3), namely

$$\lambda_{1,h} = -\inf_{\varphi>0}\sup_{x\in\Omega_h} \frac{F_h[\varphi](x)}{\varphi(x)}. \tag{1.6}$$

As for the approximating operators F_h, we consider a specific class of finite difference schemes introduced in [17, 18] since they satisfy some useful properties for the convergence analysis.

We prove that, if F is uniformly elliptic and satisfies in addition some quite natural further conditions, then it is possible to define a finite difference scheme F_h such that the discrete principal eigenvalues $\lambda_{1,h}$ and the associated discrete eigenfunctions $w_{1,h}$ converge uniformly in Ω, as the mesh step h is sent to 0, respectively to the principal eigenvalue λ_1 and to the corresponding eigenfunction w_1 for the original problem (1.5). It is worth pointing out that the proof of our main convergence result, Theorem 3.2, cannot rely on standard stability results for fully nonlinear partial differential equations, (see [3]) since the limit problem does not satisfy a comparison principle (see Remark 3.1 for details).

We mention that our approach is partially inspired by the paper [13], where a similar approximation scheme is proposed for the computation of effective Hamiltonians occurring in the homogenization of Hamilton-Jacobi equations which can be characterized by a formula somewhat similar to Equation (1.3).

In Section 2, we introduce the main assumptions and investigate some issues related to the Maximum Principle for discrete operators. In Section 3, we study the approximation method for a class of finite difference schemes and prove the convergence of the scheme. In Section 4, we show that under some additional structural assumptions on F_h the inf-sup problem (1.6) can be transformed into a convex optimization problem on the nodes of the grid and we discuss its implementation. A few tests which show the efficiency of our method on some simple examples are reported in Section 4 as well.

2. The Maximum Principle for discrete operators

We start by fixing some notations and the assumptions on the operator F. Set $\Gamma=\Omega\times\mathbb{R}\times\mathbb{R}^n\times S^n$, where S^n denotes the linear space of real, symmetric $n\times n$ matrices. The function $F(x,z,p,r)$ is assumed to be continuous on Γ and locally uniformly Lipschitz continuous with respect to z,p,r for each fixed $x\in\Omega$. We will also suppose that the partial derivatives F_r, F_p, F_z satisfy the following structure conditions:

$$0<aI\le F_r\le AI, \quad |F_p|\le\mu_1, \quad -\mu_0\le F_z\le 0. \tag{2.1}$$

for some constants a, A, μ_0, μ_1. A further condition is the positive homogeneity of degree 1, that is

$$F(x,tz,tp,tr)=tF(x,z,p,r) \qquad \forall t\ge 0. \tag{2.2}$$

The principal eigenvalue of problem (1.5) is defined by

$$\lambda_1=\sup\{\lambda:\exists\varphi>0 \text{ such that } F[\varphi]+\lambda\varphi\le 0 \text{ in } \Omega\},$$

where the differential inequality $F[\varphi]+\lambda\varphi\le 0$ is meant in the viscosity sense. Under assumptions (2.1)–(2.2), there exists a viscosity solution of the problem (1.5) and the characterization (1.3) of λ_1 holds (see [8, 11]).

Remark 2.1. It is possible to define

$$\lambda_1^-=\sup\{\lambda:\exists\varphi<0 \text{ such that } F[\varphi]+\lambda\varphi\ge 0 \text{ in } \Omega\}.$$

When F is not odd in its dependence on the Hessian, then in general $\lambda_1\ne\lambda_1^-$. Of course, it is possible to see λ_1^- as λ_1 of some other operator. Hence, we will only consider in this paper λ_1. For example, for the extremal Pucci operators $\mathcal{M}^+_{a,A}(D^2u):=$

$\sup_{aI\le B\le AI}\operatorname{tr}(AD^2u)$ and $\mathcal{M}^-_{a,A}(D^2u):=\inf_{aI\le B\le AI}\operatorname{tr}(AD^2u)$, since $\mathcal{M}^+_{a,A}(-M)=-\mathcal{M}^-_{a,A}(M)$, the following holds:

$$\lambda_1^-(\mathcal{M}^+_{a,A})=\lambda_1(\mathcal{M}^-_{a,A}).$$

Remark 2.2. The assumption $F_z\le 0$, i.e. the monotonicity of the differential operator in the zero-order term, could be removed. Indeed, $\bar F:=F-c_0z$, with c_0 large, satisfies this assumption; moreover, $\bar F$ and F have the same principal eigenfunction and the eigenvalues differ by c_0.

We now describe the discrete setting that we shall consider. Given $h>0$, let $\mathbb{Z}^n_h=h\mathbb{Z}^n$ denote the orthogonal lattice in $\mathbb{R}^n$. Let F_h be a discrete operator acting on functions defined in $\Omega_h\subset\mathbb{Z}^n_h$. We shall consider an approximation of the problem (1.5) (which can be seen also as an eigenvalue problem for the discrete operator F_h). We look for a number λ and a positive function w such that

$$\begin{cases} F_h(x,w(x),[w]_x)+\lambda w(x)=0 & x\in\Omega_h,\\ w(x)=0 & x\in\partial\Omega_h,\end{cases}\tag{2.3}$$

where

- $h>0$ is the discretization parameter (h is meant to tend to 0),
- $x\in\Omega_h$ is the point where the problem (1.5) is approximated,
- w is a real valued mesh function in $\mathbb{Z}^n_h$ meant to approximate the viscosity solution of the problem (1.5),
- $[\cdot]_x$ represents the stencil of the scheme, i.e. the points in $\Omega_h\backslash\{x\}$ where the value of u is computed for writing the scheme at the point x (we assume that $[w]_x$ is independent of $w(y)$ for $|x-y|>Mh$ for some fixed $M\in\mathbb{N}$).

We denote by $\mathcal{C}_h$ the space of the mesh functions defined on $\overline{\Omega}_h$ and we introduce some basic assumptions for the scheme F_h (see [17, 18]).

(i) The operator F_h is of positive type, i.e. for all $x\in\Omega_h$, $z,\tau\in\mathbb{R}$, $u,\eta\in\mathcal{C}_h$ satisfying $0\le\eta(y)\le\tau$ for each $y\in\Omega_h$,

$$F_h(x,z,[u+\eta]_x)\ge F_h(x,z,[u]_x)\ge F_h(x,z+\tau,[u+\eta]_x).$$

(ii) The operator F_h is positively homogeneous, i.e. for all $x\in\Omega_h$, $z\in\mathbb{R}$, $u\in\mathcal{C}_h$ and $t\ge 0$,

$$F_h(x,tz,[tu]_x)=tF_h(x,z,[u]_x).$$

(iii) The family of operators $\{F_h, 0<h\le h_0\}$, where h_0 is a positive constant, is consistent with the operator F on the domain $\Omega\subset\mathbb{R}^n$, i.e. for each $u\in C^2(\Omega)$,

$$\sup_{\Omega_h}\left|F(x,u(x),Du(x),D^2u(x))-F_h(x,u(x),[u]_x)\right|\to 0 \quad \text{as } h\to 0,$$

uniformly on a compact subset of Ω.

We study below some properties related to the maximum principle and a comparison result for the operator F_h. Let us start with the following definitions.

Definition 2.1. *A function $u\in\mathcal{C}_h$ is a subsolution (respectively $v\in\mathcal{C}_h$ is a supersolution) of*

$$F_h(x,u(x),[u]_x)=f(x) \qquad x\in\Omega_h \tag{2.4}$$

if

$$F_h(x,u(x),[u]_x)\geq f(x), \qquad x\in\Omega_h$$
$$(\textit{respectively, } F_h(x,v(x),[v]_x)\leq f(x), \qquad x\in\Omega_h).$$

DEFINITION 2.2. *The Maximum Principle holds for the operator F_h in Ω_h if*

$$\begin{cases} F_h(x,u(x),[u]_x)\geq 0 & \text{in } \Omega_h,\\ u\leq 0 & \text{on } \partial\Omega_h \end{cases} \tag{2.5}$$

implies $u\leq 0$ in Ω_h.

PROPOSITION 2.1. *Assume that F_h is of positive type, is positive homogeneous, and satisfies either*

$$\begin{gathered} \text{for all } z\in\mathbb{R},\ u,\eta\in\mathcal{C}_h \text{ satisfying } 0\leq\eta(y) \text{ and } \max_{y\in[\cdot]_x}\eta(y)>0,\\ \text{then} \quad F_h(x,z,[u+\eta]_x)>F_h(x,z,[u]_x) \end{gathered} \tag{2.6}$$

or

$$\begin{gathered} \text{for all } z,\tau\in\mathbb{R},\ u,\eta\in\mathcal{C}_h \text{ satisfying } 0\leq\eta(y)\leq\tau \text{ for each } y, \text{ then}\\ F_h(x,z,[u]_x)\geq F_h(x,z+\tau,[u+\eta]_x)+c_0\tau \end{gathered} \tag{2.7}$$

for some positive constants c_0. Then the Maximum Principle holds for the operator F_h in Ω_h.

Proof. Assume by contradiction that u satisfies the problem (2.5) and $M:=\max_{\overline{\Omega}_h}u>0$. Let $\bar{x}\in\Omega_h$ be such that $u(\bar{x})=M$. Since $u\leq 0$ on $\partial\Omega_h$, it is not restrictive to assume that there exists $y\in\Omega_h$ such that $u(y)<u(\bar{x})=M$. Hence,

$$\begin{aligned} 0&\leq F_h(\bar{x},u(\bar{x}),[u]_{\bar{x}})\leq F_h(\bar{x},u(\bar{x})-M,[u-M]_{\bar{x}})\\ &<F_h(\bar{x},0,[0]_{\bar{x}})=0, \end{aligned}$$

a contradiction. A similar proof can be done with the assumption (2.7). □

REMARK 2.3. The assumptions (2.6) and (2.7) correspond to the uniform ellipticity and, respectively, to the strict monotonicity of the operator F with respect to the zero-order term.

The following proposition shows that, as it is known in the continuous case (see e.g. [6, 8]), the validity of the maximum principle for subsolutions of the operator F_h is equivalent to the positivity of the principal eigenvalue for F_h.

PROPOSITION 2.2. *Assume that the scheme F_h is of positive type and that it is positively homogeneous. Suppose that for $\lambda\in\mathbb{R}$, there exists a nonnegative grid function φ with $\varphi>0$ in Ω_h such that $F_h[\varphi]+\lambda\varphi\leq 0$. If, for $\tau<\lambda$, the function u satisfies*

$$\begin{cases} F_h(x,u(x),[u]_x)+\tau u\geq 0 & \text{in } \Omega_h\\ u\leq 0 & \text{on } \partial\Omega_h, \end{cases}$$

then $u\leq 0$ in Ω_h, i.e. $F_{h,\tau}[\cdot]=F_h[\cdot]+\tau\cdot$ satisfies the Maximum Principle.

Proof. Suppose by contradiction that $\max_{\overline{\Omega}_h}\{u\}>0$. Let φ be as in the statement and set $L(\gamma)=\max_{\Omega_h}\{u-\gamma\varphi\}$ (note that the maximum is taken only with respect to the internal points). Then $L:[0,\infty)\to\mathbb{R}$ is continuous and decreasing, $L(0)>0$, and $L(\gamma)\to-\infty$ for $\gamma\to+\infty$. Hence, there exists $\gamma'>0$ such that $L(\gamma')=0$. Moreover, since $u-\gamma'\varphi\le 0$ on $\partial\Omega$, we also have $\max_{\overline{\Omega}_h}\{u-\gamma'\varphi\}=0$. Let $0<\gamma<\gamma'$ be such that

$$\frac{\gamma}{\gamma'}\lambda>\tau \tag{2.8}$$

and set $\psi=\gamma\varphi$. Then $F_h[\psi]+\lambda\psi\le 0$ and $M=\max_{\overline{\Omega}_h}\{u-\psi\}=(u-\psi)(\bar{x})>0$ for some $\bar{x}\in\Omega_h$. Hence, $\psi(\bar{x})+M=u(\bar{x})$ and $\psi(x)+M\ge u(x)$. Since F_h is of positive type, it follows that

$$\begin{aligned}F_h(x,\psi(\bar{x}),[\psi]_{\bar{x}})&\ge F_h((x,\psi(\bar{x})+M,[\psi+M]_{\bar{x}})=F_h(x,u(\bar{x}),[\psi+M]_{\bar{x}})\\&\ge F_h(x,u(\bar{x}),[u]_{\bar{x}}).\end{aligned}$$

Then

$$\tau u(\bar{x})\ge -F_h[u](\bar{x})\ge -F_h[\psi](\bar{x})\ge\lambda\psi(\bar{x})=\lambda\gamma\varphi(\bar{x})\ge\lambda\frac{\gamma}{\gamma'}u(\bar{x}),$$

and therefore contradicts Equation (2.8). □

The following result gives a comparison principle for Equation (2.4).

PROPOSITION 2.3. *Assume that F_h is of positive type and it satisfies either condition* (2.6) *or condition* (2.7). *Let u and v be a subsolution and respectively a supersolution of Equation* (2.4) *such that $u\le v$ on $\partial\Omega_h$. Then $u\le v$ in $\overline{\Omega}_h$.*

Proof. Suppose by contradiction that $M:=\max_{\overline{\Omega}_h}\{u-v\}>0$ and let $\bar{x}\in\Omega_h$ be such that $u(\bar{x})-v(\bar{x})=M$. Hence $v+M\ge u$ in Ω_h and it is not restrictive to assume that $\max_{y\in[\cdot]_{\bar{x}}}(v+M-u)>0$. It follows that

$$\begin{aligned}f(\bar{x})\le F_h(\bar{x},u(\bar{x}),[u]_{\bar{x}})=F_h(\bar{x},v(\bar{x})+M,[u]_{\bar{x}})&<F_h(\bar{x},v(\bar{x})+M,[v+M]_{\bar{x}})\\&\le F_h(\bar{x},v(\bar{x}),[v]_{\bar{x}})\le f(\bar{x})\end{aligned}$$

and therefore is a contradiction. A similar proof can be carried out under assumption (2.7).

□

3. Approximation of the principal eigenvalue

In this section we consider a specific class of finite difference schemes introduced in [18]. These schemes satisfy certain pointwise estimates which are the discrete analogues of those valid for a general class of fully-nonlinear uniformly-elliptic equations.

We assume that for all $x\in\mathbb{Z}^n_h$, the stencil $[\cdot]_x$ of the scheme is given by $x+hY$ where $Y=\{y_1,\dots,y_k\}\subset\mathbb{Z}^n$ is a finite set containing all the vectors of the canonical basis of $\mathbb{R}^n$. Then we consider a discrete operator F_h in the system (2.3) given by a finite difference scheme written in the form

$$F_h[u]=\mathcal{F}(x,u,\delta_h u,\delta_h^2 u), \tag{3.1}$$

where $\mathcal{F}:\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^Y\times\mathbb{R}^Y\to\mathbb{R}$ and for $y\in Y$, $u\in\mathcal{C}_h$

$$\begin{aligned}\delta^{\pm}_{h,y}u(x)&=\pm\frac{u(x\pm hy)-u(x)}{h|y|},\\ \delta_{h,y}u(x)&=\frac{1}{2}\{\delta^+_{h,y}u(x)+\delta^-_{h,y}u(x)\}=\frac{u(x+hy)-u(x-hy)}{2h|y|},\end{aligned}$$

$$\delta^2_{h,y}u(x)=\delta^+_{h,y}\delta^-_{h,y}u(x)=\frac{u(x+hy)+u(x-hy)-2u(x)}{h^2|y|^2},$$
$$\delta_h u=\{\delta_{h,y}u : y\in Y\},\quad \delta^2_h u=\{\delta^2_{h,y}u : y\in Y\}.$$

Set $\tilde\Gamma:=\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^Y\times\mathbb{R}^Y$ and denote by (x,z,q,s) the generic points in $\tilde\Gamma$. The operator F_h given by Equation (3.1) is of positive type if

$$\frac{\partial\mathcal{F}}{\partial s_y}-\frac{|hy|}{2}\left|\frac{\partial\mathcal{F}}{\partial q_y}\right|\ge 0\quad \forall y\in Y, \tag{3.2}$$

$$\frac{\partial\mathcal{F}}{\partial z}\le 0, \tag{3.3}$$

and positively homogeneous if

$$\mathcal{F}(x,tz,tq,ts)=t\mathcal{F}(x,z,q,s)\qquad \forall t\ge 0.$$

Moreover, if F in problem (1.5) satisfies the assumptions (2.1), then it is always possible to find a scheme of type (3.1) which is consistent with F and which, besides Equations (3.2)–(3.3), satisfies for all $y\in Y$ the bounds

$$\frac{\partial\mathcal{F}}{\partial s_y}-\frac{|hy|}{2}\left|\frac{\partial\mathcal{F}}{\partial q_y}\right|\ge\alpha_0,\qquad \frac{\partial\mathcal{F}}{\partial s_y}\le a_0,\qquad \left|\frac{\partial\mathcal{F}}{\partial q_y}\right|\le b_0, \tag{3.4}$$

where α_0, a_0, b_0 are constants depending on a, A, μ_0, μ_1 in the assumptions (2.1) (see [17, 18]). Note that in particular Equation (3.4) implies Equation (2.6).
We recall some important properties of the previous scheme (for the proof we refer to [18])

Proposition 3.1. *Assume Equations* (3.2)–(3.4) *and let f, g be two given mesh functions. Then for every $h>0$ sufficiently small there exists a unique solution $u_h:\Omega_h\to\mathbb{R}$ to the Dirichlet problem*

$$\begin{cases} F_h(x,u(x),[u]_x)=f & x\in\Omega_h,\\ u=g & x\in\partial\Omega_h.\end{cases} \tag{3.5}$$

Proposition 3.2. *Assume Equations* (3.2)–(3.4) *and let u_h be a subsolution of the problem* (3.5)*. Then*

$$\max_{\Omega_h}u_h\le\max_{\partial\Omega_h}g+\frac{C}{\alpha_0}\left\{\sum_{x\in\Omega_h}h^n|f(x)|^n\right\}^{\frac{1}{n}}, \tag{3.6}$$

where the constant C is independent of h. Moreover if u_h is a solution of the problem (3.5)*, then for any $x,y\in\Omega_h$*

$$|u_h(x)-u_h(y)|\le C\frac{|x-y|^\delta}{R}\left(\max_{B^h_R}u_h+\frac{R}{\alpha_0}\left\{\sum_{x\in\Omega_h}h^n|f(x)|^n\right\}^{\frac{1}{n}}\right), \tag{3.7}$$

where $R=\min\{\mathrm{dist}(x,\partial\Omega_h),\mathrm{dist}(x,\partial\Omega_h)\}$, $B^h_R=B(0,R)\cap\Omega_h$, and δ and C are positive constants independent of h.

We give an example of a scheme of the form (3.1). Consider the Hamilton-Jacobi-Bellman operator

$$F(x,u,Du(x),D^2u(x))=\sup_{\alpha\in A}\inf_{\beta\in B} L^{\alpha\beta}u(x),$$

where

$$L^{\alpha\beta}u(x)=a_{ij}^{\alpha\beta}(x)D_{ij}u+b_i^{\alpha\beta}(x)D_iu(x)+c^{\alpha\beta}(x)u(x). \tag{3.8}$$

It is always possible to rewrite the operator $L^{\alpha\beta}$ in Equation (3.8) in the following form (see [18]):

$$\overline{L}^{\alpha\beta}u(x)=\bar{a}_k^{\alpha\beta}(x)D^2_{y_k}u+\bar{b}_k^{\alpha\beta}(x)D_{y_k}u(x)+\bar{c}^{\alpha\beta}(x)u(x),$$

where $D_{y_k}u=\langle Du,y_k\rangle$ and $Y=\{y_1,\dots,y_k\}\subset\mathbb{Z}^n$ is a finite set containing all the vectors of the canonical basis in $\mathbb{R}^n$. Moreover, the coefficients $\bar{a}_k^{\alpha\beta}$, $\bar{b}_k^{\alpha\beta}$, and $\bar{c}^{\alpha\beta}$ satisfy the same properties of $a_{ij}^{\alpha\beta}$, $b_{ij}^{\alpha\beta}$, and $c^{\alpha\beta}$. Then we consider

$$F_h[u](x):=\sup_{\alpha\in A}\inf_{\beta\in B} L_h^{\alpha\beta}u(x), \tag{3.9}$$

where

$$L_h^{\alpha\beta}u(x)=\bar{a}_k^{\alpha\beta}(x)\delta^2_{h,y_k}u(x)+\bar{b}_k^{\alpha\beta}(x)\delta_{h,y_k}u(x)+\bar{c}^{\alpha\beta}(x)u(x). \tag{3.10}$$

For $x\in\mathbb{R}$ with $Y=\{1\}$, the previous scheme reads as

$$\sup_{\alpha\in A}\inf_{\beta\in B}\left\{a^{\alpha\beta}(x)\frac{u(x+h)+u(x-h)-2u(x)}{h^2}+b^{\alpha\beta}(x)\frac{u(x+h)-u(x-h)}{2h}+c^{\alpha\beta}(x)u(x)\right\}=0.$$

3.1. The linear case. In this part, we assume that the operator F in the problem (1.5) is linear, i.e. $F[u]=Lu$ with

$$Lu=a_{ij}(x)D_{ij}u+b_i(x)D_iu(x)+c(x)u(x),$$

and we consider a scheme defined as in Equations (3.9)–(3.10), obviously without the dependence on α, β.

Proposition 3.3. *Under the assumption* (3.4)*, the eigenvalue problem* (2.3) *has a simple eigenvalue* $\lambda_{1,h}\in\mathbb{R}$ *which corresponds to a positive eigenfunction. The other eigenvalues correspond to sign changing eigenfunctions.*

Proof. Choose $\xi>0$ large enough so that $c(x)-\xi<0$ and set

$$L_{h,\xi}(x,t,[u]_x)=L_h(x,t,[u]_x)-\xi t.$$

Let K be the positive cone of the nonnegative grid functions in $\mathcal{C}_h$. For a given grid function f, by Proposition 3.1 and Proposition 2.3 there exists a unique solution $u\in\mathcal{C}_h$ to

$$\begin{cases} L_{h,\xi}(y,u(y),[u]_y)+f=0 & \text{in } \Omega_h,\\ u=0 & \text{on } \partial\Omega_h.\end{cases}$$

Since $\mathcal{C}_h$ is a finite dimensional space, it follows that $T:\mathcal{C}_h \to \mathcal{C}_h$ defined by $Tf=u$ is a compact linear operator. Moreover, if $f \geq 0$, then by Proposition 2.1 $u \geq 0$ and if $f \in K \setminus \{0\}$, $u = Tf > 0$.
Therefore, by the Krein–Rutman theorem [19], $r(T)$, the spectral radius of T, is a simple real eigenvalue $r(T) > 0$ with a positive eigenfunction u such that $Tu = r(T)u$. Hence, for $\lambda_{1,h} = r(T)^{-1} - \xi$, $w_1 = Tu$ satisfies

$$\begin{cases} L_h(x, w_1(x), [w_1]_x) + \lambda_{1,h} w_1 = 0 & \text{in } \Omega_h, \\ w_1 = 0 & \text{on } \partial\Omega_h. \end{cases}$$

□

The following characterization of $\lambda_{1,h}$ is a simple consequence of Proposition 2.2.

Proposition 3.4. *We have*

$$\lambda_{1,h} = \sup\{\lambda : \exists\ \varphi > 0 \ s.t.\ L_h[\varphi] + \lambda\varphi \leq 0 \ in\ \Omega\} \tag{3.11}$$

or, equivalently,

$$\lambda_{1,h} = -\inf_{\varphi>0} \sup_{x\in\Omega_h} \left\{ \frac{L_h[\varphi](x)}{\varphi(x)} \right\}. \tag{3.12}$$

Proof. Denote by $\bar\lambda$ the right-hand side of Equation (3.11). Clearly $\lambda_{1,h} \leq \bar\lambda$. If $\lambda_{1,h} < \bar\lambda$ then there exist $\mu \in (\lambda_{1,h}, \bar\lambda)$ and $\varphi > 0$ such that $L_h[\varphi] + \mu\varphi \leq 0$. A contradiction follows immediately by Proposition 2.2 since the eigenfunction corresponding to $\lambda_{1,h}$ is positive. Hence, we have Equation (3.11).
Let $\varphi > 0$ such that $L_h[\varphi](x) + \lambda\varphi(x) \leq 0$ for $x \in \Omega_h$. Hence,

$$\lambda \leq \inf_{\Omega_h} \left\{ -\frac{L_h[\varphi]}{\varphi} \right\} = -\sup_{\Omega_h} \left\{ \frac{L_h[\varphi]}{\varphi} \right\}.$$

Consequently,

$$\lambda_{1,h} = \sup_{\varphi>0} \left(-\sup_{x\in\Omega_h} \left\{ \frac{L_h[\varphi](x)}{\varphi(x)} \right\} \right) = -\inf_{\varphi>0} \sup_{x\in\Omega_h} \left\{ \frac{L_h[\varphi](x)}{\varphi(x)} \right\}.$$

□

We give next an upper bound for $\lambda_{1,h}$ (compare with the corresponding estimate for λ_1 in [6], Lemma 1.1).

Lemma 3.1. *Let $n=1$ and assume that $B_R = \{|x| < R\}$ lies in Ω with $R \leq 1$. Then*

$$\lambda_{1,h}(\Omega_h) \leq \frac{C}{R^2}.$$

Proof. Given the linear operator

$$Lu = a(x)u'' + b(x)u'(x) + c(x)u(x),$$

let γ_0, Γ_0, b be positive constants such that $\gamma_0 \leq a(x) \leq \Gamma_0$ and $|b(x)|, |c(x)| \leq b$ in Ω. Let $r = R/2$ and assume for simplicity that $r = Nh$ for some $N \in \mathbb{N}$. Set $B_r = \{|x| < r\}$ and consider the grid function

$$\sigma_i = (r^2 - |ih|^2)^2 \qquad i = -N+1, \dots N-1.$$

Then for $i=-N+1,\dots,N-1$ we have

$$\frac{\sigma_{i+1}-\sigma_{i-1}}{2h}=-4hi(r^2-|ih|^2)+4h^3i$$
$$\frac{\sigma_{i+1}+\sigma_{i-1}-2\sigma_i}{h^2}=-4(r^2-|ih|^2)+2h^2(2i^2+1).$$

Denote by a_i, b_i, c_i the coefficients of the linear operator computed at the point $x=ih$. Since $h^2/(r^2-|ih|^2)\le 1$, it follows that

$$\begin{aligned}-\frac{L_h[\sigma](ih)}{4\sigma_i}&\le\frac{a_i}{(r^2-|ih|^2)}-\frac{a_i|hi|^2}{(r^2-|ih|^2)^2}+|b_i|\frac{2r}{(r^2-|ih|^2)}+\frac{c_i}{4}\\&\le\frac{\Gamma_0+br}{(r^2-|ih|^2)}-\frac{\gamma_0|hi|^2}{(r^2-|ih|^2)^2}+\frac{b}{4}.\end{aligned}\tag{3.13}$$

If

$$|ih|^2(\gamma_0+\Gamma_0+br)>r^2(\Gamma_0+br),$$

then the second term in Equation (3.13) dominates the first one and therefore

$$-\frac{L_h[\sigma](ih)}{4\sigma_i}\le\frac{b}{4}.\tag{3.14}$$

In the remaining part of B_r,

$$-\frac{L_h[\sigma](ih)}{4\sigma_i}\le\frac{\Gamma_0+br}{(r^2-|ih|^2)}+\frac{b}{4}\le\frac{b}{4}+\frac{1}{\gamma_0 r^2}(\Gamma_0+br)(\gamma_0+\Gamma_0+br).\tag{3.15}$$

By Equations (3.14) and (3.15), we get

$$\sup_{B_r}\left(-\frac{L_h[\sigma](ih)}{\sigma_i}\right)\le\frac{C}{R^2}\quad\text{for } i=-N+1,\dots,N-1.$$

To conclude the proof, we show that if for some positive function φ and $\lambda\in\mathbb{R}$, $L_h[\varphi]+\lambda\varphi\le 0$, then $\lambda\le\sup_{B_r}(-L_h[\sigma]/\sigma)$. For this purpose, assume that $\lambda>\sup_{B_r}(-L_h[\sigma]/\sigma):=\tau$; then $L_h[\sigma]+\tau\sigma\ge 0$ in B_r and $\sigma=0$ on ∂B_r, while $L_h[\varphi]+\tau\varphi\le 0$. Hence by Proposition 2.2, it follows $\sigma\le 0$ in B_r, a contradiction, and therefore $\lambda\le\tau$. □

3.2. The nonlinear case. We consider now a general discrete operator F_h given by Equation (3.1), and we study the corresponding eigenvalue problem (2.3). By analogy to formula (3.11), we define

$$\lambda_{1,h}=\sup\{\lambda:\exists\ \varphi>0\ \text{ such that } F_h[\varphi]+\lambda\varphi\le 0\}.\tag{3.16}$$

We prove for each h the existence of a pair $(\lambda_{1,h},w_{1,h})$ satisfying Equation (2.3) with $w_{1,h}>0$ in Ω_h.

PROPOSITION 3.5. *Assume that F_h satisfies Equation* (3.4)*, $f\le 0$, and $\lambda<\lambda_{1,h}$. Then there exists a nonnegative solution to*

$$\begin{cases}F_h(x,u(x),[u]_x)+\lambda u(x)=f(x) & x\in\Omega_h,\\ u(x)=0 & x\in\partial\Omega_h.\end{cases}\tag{3.17}$$

Proof. We can assume $\lambda\geq 0$, since for $\lambda<0$, $F_h[u]+\lambda u$ satisfies Equation (2.7) and therefore by propositions 3.1 and 2.3, there exists a unique solution to problem (3.17). Let us define by induction a sequence u_n by setting $u_1\equiv 0$. For $n\geq 1$ we consider the following equation:

$$\begin{cases} F_h(x,u_{n+1}(x),[u_{n+1}]_x)=f(x)-\lambda u_n, & x\in\Omega_h,\\ u_{n+1}(x)=0 & x\in\partial\Omega_h. \end{cases} \tag{3.18}$$

For any $n\in N$ there exists a non-negative solution u_{n+1} to problem (3.18). For $n=1$, existence follows by Proposition 3.1. Moreover, since $u_1\equiv 0$ is a subsolution to problem (3.18), by Proposition 2.3 we get $u_2\geq 0$. The existence of a non-negative solution at the $(n+1)$-step is proved in a similar way; moreover, the solution is non-negative since $f-\lambda u_n\leq 0$.

We claim now that, for any $n\geq 1$, $u_n\leq u_{n+1}$. For $n=1$, the claim is trivially true since $u_2\geq 0$. Assume then by induction that $u_n\geq u_{n-1}$. Since $f(x)-\lambda u_n\leq f(x)-\lambda u_{n-1}$, it follows that u_n is a subsolution of problem (3.18). By Proposition 2.3, we get that $u_n\leq u_{n+1}$.

Let us show now that the sequence u_n is bounded. Assume by contradiction that it is false and set $\overline{u}_n=u_n/|u_n|_\infty$. Then, by positive homogeneity, $\overline{u}_n$ is a solution of

$$F_h(x,\overline{u}_{n+1}(x),[\overline{u}_{n+1}]_x)=\frac{f(x)}{|u_{n+1}|_\infty}-\lambda\frac{u_n}{|u_{n+1}|_\infty}, \qquad x\in\Omega_h.$$

Since the sequence $\overline{u}_n$ is bounded, then up to a subsequence it converges to a function $\overline{u}$, while $u_n/|u_{n+1}|_\infty$ converges to $k\overline{u}$ where $k=\lim_{n\to\infty}|u_n|_\infty/|u_{n+1}|_\infty\leq 1$. Hence $\overline{u}\geq 0$, $|\overline{u}|_\infty=1$, $\overline{u}=0$ on $\partial\Omega_h$ and

$$F_h(x,\overline{u}(x),[\overline{u}]_x)+k\lambda\overline{u}=0, \qquad x\in\Omega_h.$$

Since $0\leq k\lambda\leq\lambda$ and using the fact that for $\lambda<\lambda_{1,h}$ there exists by definition $\varphi>0$ such that $F_h[\varphi]+\lambda\varphi\geq 0$ in Ω_h, we get a contradiction to Proposition 2.2. Hence, the sequence u_n is bounded and in addition monotone, it converges pointwise to a function u which solves problem (3.17). □

The next result shows that $\lambda_{1,h}$ is indeed an eigenvalue for the approximated operator F_h.

THEOREM 3.1. *Assume that F_h satisfies Equation* (3.4). *Then there exists $w_{1,h}>0$ in Ω_h satisfying*

$$\begin{cases} F_h(x,w_{1,h}(x),[w_{1,h}]_x)+\lambda_{1,h}w_{1,h}(x)=0 & x\in\Omega_h,\\ w_{1,h}=0 & x\in\partial\Omega_h. \end{cases} \tag{3.19}$$

Moreover, the characterization (3.12) *is still valid for the nonlinear operator F_h.*

Proof. Let λ_n be an increasing sequence converging to $\lambda_{1,h}$. By Proposition 3.5, there exists a positive solution u_n of

$$\begin{cases} F_h(x,u_n(x),[u_n]_x)+\lambda_n u_n=-1, & x\in\Omega_h,\\ u_n(x)=0 & x\in\partial\Omega_h. \end{cases}$$

We claim that u_n is not bounded. Assume by contradiction that u_n is bounded so that, up to a subsequence, u_n converges to a function $u>0$ which solves

$$\begin{cases} F_h(x,u(x),[u]_x)+\lambda_{1,h}u=-1, & x\in\Omega_h,\\ u(x)=0 & x\in\partial\Omega_h. \end{cases}$$

Then, for $\varepsilon>0$ small enough, u satisfies

$$F_h(x,u(x),[u]_x)+(\lambda_{1,h}+\varepsilon)u=-1+\varepsilon u\leq 0$$

which gives a contradiction to the definition (3.16). Hence $|u_n|_\infty\to\infty$.
Define now $w_n=u_n/|u_n|_\infty$ that solves

$$\begin{cases} F_h(x,w_n(x),[w_n]_x)+\lambda_n w_n=-\frac{1}{|u_n|_\infty}, & x\in\Omega_h,\\ w_n(x)=0 & x\in\partial\Omega_h. \end{cases}$$

Then, up to a subsequence, w_n converges to a bounded function $w_{1,h}$ which has norm 1 and which satisfies Equation (3.19), so that $w_{1,h}>0$. It is immediate that Equation (3.12) is still valid for F_h. □

Remark 3.1. There is a huge literature about the approximation of viscosity solutions of first- and second-order PDEs. In this framework, a well-established technique to prove the convergence of a numerical scheme is the Barles–Souganidis method [3]. Besides some natural properties of the scheme (stability, consistency, monotonicity), a key ingredient for this technique is a *strong comparison result* for the continuous problem, which allows us to show that a subsolution is always less than or equal to a supersolution. The comparison principle implies in particular that there is at most one viscosity solution of the problem. But it is immediate that problem (1.5) cannot satisfy a comparison principle since $w\equiv 0$ and the principal eigenfunction w_1 are two distinct solutions of the problem; hence, the convergence proof cannot rely on the Barles–Souganidis method and it needs a different argument.

We now discuss the convergence of the discrete principal eigenvalue $\lambda_{1,h}$ to the continuous one defined by (1.3). We recall the definition of weak limits in viscosity sense (see [3])

$$\limsup_{h\to 0}{}^{*} u_h(x):=\lim_{h\to 0^+}\sup\{u_\delta(y):|x-y|\leq h,\delta\leq h\},$$
$$\liminf_{h\to 0}{}_{*} u_h(x):=\lim_{h\to 0^+}\inf\{u_\delta(y):|x-y|\leq h,\delta\leq h\}.$$

Theorem 3.2. *Assume Equations* (2.1)–(2.2) *and* (3.2)–(3.4) *and that F_h is consistent with F. Let $(\lambda_{1,h},w_{1,h})$ be the sequence of the discrete eigenvalues and of the corresponding eigenfunctions, solutions of problem* (2.3). *Then $\lambda_{1,h}\to\lambda_1$ and $w_{1,h}\to w_1$ uniformly in $\overline{\Omega}$ as $h\to 0$, where λ_1 and w_1 are respectively the principal eigenvalue and a corresponding eigenfunction associated to F.*

Proof. By the positive homogeneity of the scheme, it is not restrictive to assume that $\max_{\Omega_h}\{w_{1,h}\}=1$; hence, the sequence $w_{1,h}$ is bounded. We first prove that

$$\liminf_{h\to 0}\lambda_{1,h}\geq\lambda_1. \tag{3.20}$$

Assume by contradiction that $\liminf_{h\to 0}\lambda_{1,h}=\tau$ for some some $\tau<\lambda_1$. Consider a subsequence, still denoted by $\lambda_{1,h}$, such that $\lim_{h\to 0}\lambda_{1,h}=\tau$. Set $\overline{w}=\limsup_{h\to 0}{}^{*}w_{1,h}$. By standard stability results in viscosity solution theory (see [3]), $\overline{w}$ satisfies in the viscosity sense

$$F[\overline{w}]+\tau\overline{w}\geq 0 \qquad \text{in } \Omega_h, \tag{3.21}$$

and

$$\max_{\Omega}\overline{w}=1. \tag{3.22}$$

Let $\eta>0$ be such that, for h sufficiently small, $\lambda_{1,h}\leq\tau+\eta$. Hence,

$$F_h[w_{1,h}]=-\lambda_{1,h}w_{1,h}\geq-\tau-\eta, \qquad x\in\Omega_h.$$

Set $f=-\tau-\eta$, $g\equiv 0$ and let u_h be the corresponding solution of problem (3.5), while u is the solution of

$$\begin{cases} F(x,u(x),Du,D^2u)=-\tau-\eta & x\in\Omega, \\ u=0 & x\in\partial\Omega. \end{cases}$$

Then by Proposition 2.3 and the consistency of the scheme for h sufficiently small

$$0\leq w_{1,h}\leq u_h\leq u+o(1) \qquad \text{in } \overline{\Omega}_h \tag{3.23}$$

and, therefore,

$$\overline{w}=0 \quad \text{on } \partial\Omega. \tag{3.24}$$

By Equations (3.21), (3.22), and (3.24) we get a contradiction to the maximum principle for the operator F (see [8, 11]) and therefore Equation (3.20) follows.
We now prove that

$$\limsup_{h\to 0}\lambda_{1,h}\leq\lambda_1. \tag{3.25}$$

Assume by contradiction that there exists $\eta>0$ such that

$$\bar{\lambda}:=\limsup_{h\to 0}\lambda_{1,h}\geq\lambda_1+\eta.$$

We consider a subsequence, still denoted by $\lambda_{1,h}$, such that $\lim_{h\to 0}\lambda_{1,h}=\bar{\lambda}$, and we set $\underline{w}=\liminf_{h\to 0}{}_{*}w_{1,h}$. By standard stability results, $\underline{w}$ satisfies $0\leq\underline{w}\leq 1$ and

$$\begin{cases} F[\underline{w}]+(\lambda_1+\eta)\underline{w}\leq 0 & \text{in } \Omega, \\ \underline{w}=0 & \text{on } \partial\Omega, \end{cases} \tag{3.26}$$

in viscosity sense. Let $x_h\in\Omega_h$ be a sequence such that $x_h\to x_0\in\overline{\Omega}$ and $w_{1,h}(x_h)=1$ for all $h>0$. By Equation (3.23), $x_0\in\Omega$. We claim that

$$\underline{w}(x_0)>0. \tag{3.27}$$

Assume by contradiction that $\underline{w}(x_0)=0$; hence, there exists a sequence $y_h \to x_0$ such that $\lim_{h\to 0} w_{1,h}(y_h)=0$. By Equation (3.7) with $u_h = w_{1,h}$ and $f=-\lambda_{1,h}w_{1,h}$, we get

$$|w_{1,h}(x_h)-w_{1,h}(y_h)| \le C\frac{|x_h-y_h|^\delta}{R}\left(\max_{B_R} w_{1,h}+\frac{R}{\alpha_0}\left\{\sum_{x\in\Omega_h} h^n|\lambda_{1,h}w_{1,h}|^n\right\}^{\frac{1}{n}}\right)$$
$$\le C\frac{|x_h-y_h|^\delta}{R}\left(1+\frac{R}{\alpha_0}|\lambda_{1,h}|\right).$$

Since $\lim_{h\to 0}(w_{1,h}(x_h)-w_{1,h}(y_h))=1$, we get a contradiction for h sufficiently small and therefore Equation (3.27) follows.

We are in a position to apply the maximum principle for the continuous problem (see [8]), and so we obtain that $\underline{w}>0$. But Equation (3.26) and the positivity of $\underline{w}$ give a contradiction to the definition of λ_1. By Equations (3.20) and (3.25), we get $\lim_{h\to 0}\lambda_{1,h}=\lambda_1$.

By Equation (3.7) and a local boundary estimate for $w_{1,h}$ (see [17, Thm. 5.1] and [18, Thm. 3]), we get the equicontinuity of the family $\{w_{1,h}\}$ and, therefore, the uniform convergence, up to a subsequence, of $w_{1,h}$ to w_1 with $\|w_1\|_\infty = 1$. The simplicity of the eigenfunction associated to the principal eigenvalue λ_1 gives the uniform convergence of all the sequence $w_{1,h}$ to w_1. □

4. An algorithm for computing the principal eigenvalue

In this section, we discuss an algorithm for the computation of the principal eigenvalue based on the inf-sup formula (3.12). In fact, we show that this formula results in a finite dimensional nonlinear optimization problem.

4.1. Discretization in one dimension. We first present the scheme in one dimension. Note that, since the eigenfunction corresponding to the principal eigenvalue vanishes on the boundary of Ω_h and it is strictly positive inside, then the minimization in Equation (3.12) can be restricted to the internal points. By the formula (3.1) and the homogeneity of $\mathcal{F}$, we have

$$\frac{F_h[u](x_i)}{u(x_i)}=\mathcal{F}\left(x_i,1,\frac{u(x_i+h)-u(x_i-h)}{2hu(x_i)},\frac{u(x_i+h)+u(x_i+h)}{h^2u(x_i)}-\frac{2}{h^2}\right).$$

We identify the function $u(x)$ with the values U_i, $i=0,\dots,N_h+1$, at the points of the grid (with $U_0=U_{N_h+1}=0$). Assume that $\mathcal{F}(x,z,q,s)$ is linear or more generally convex in (q,s). Then the functions $G:\mathbb{R}^{N_h}\to\mathbb{R}^{N_h}$, defined by

$$G_i(x,U_1,\dots,U_{N_h})=\mathcal{F}\left(x_i,1,\frac{U_{i+1}-U_{i-1}}{2hU_i},\frac{U_{i+1}+U_{i-1}}{h^2U_i}-\frac{2}{h^2}\right).$$

for $i=1,\dots,N_h$, is either linear or respectively convex in U_{i+1}, U_{i-1}. Moreover, since $U_i>0$, G is also convex in U_i. Taking the maximum of the functions G_i over the internal nodes of the grid gives a convex function $\mathcal{G}:\mathbb{R}^{N_h}\to\mathbb{R}$ defined by

$$\mathcal{G}(U_1,\dots,U_{N_h})=\max_{i=1,\dots,N_h} G_i(x_i,U_1,\dots,U_{N_h}) \tag{4.1}$$

Hence the computation of $\lambda_{1,h}$ is equivalent to the minimization of the convex function $\mathcal{G}$ of N_h variables: this problem can be solved by means of standard algorithms in convex optimization. Note also that the minimum is unique and the map is sparse, in the sense that the value of $\mathcal{G}$ at U_i depends only on the values at U_{i-1} and U_{i+1}.

In general, if $\mathcal{F}(x,z,q,s)$ is not convex, the computation of the principal eigenvalue is equivalent to the solution of a min-max problem in $\mathbb{R}^{N_h}$.
To solve min-max problem we use the routine `fminmax` available in the Optimization Toolbox of MATLAB. This routine is implemented on a laptop and therefore the number of variables is modest. A better implementation of the minimization procedure which takes advantage of the sparse structure of the problem would allow to solve larger problems.

Example 1. To validate the algorithm we begin by studying the eigenvalue problem:

$$\begin{cases} w''+\lambda w=0 & x\in(0,1), \\ w(x)=0 & x=0,1. \end{cases}$$

In this case the eigenvalue and the corresponding eigenfunction are given by

$$\lambda_1=\pi^2, \qquad w_1(x)=\sin(\pi x).$$

Note that since the eigenfunctions are defined up to multiplicative constant, we normalize the value by taking $\|w_1\|_\infty=\|w_{1,h}\|_\infty=1$ (the constraint for $w_{1,h}$ is included in the routine `fminmax`). Given a discretization step h and the corresponding grid points $x_i=ih$, $i=0,\dots,N_h+1$, the minimization problem (4.1) is

$$\lambda_{1,h}=-\min_{U\in\mathbb{R}^{N_h}}\left[\max_{i=1,\dots,N_h}\frac{U_{i+1}+U_{i-1}-2U_i}{h^2U_i}\right]$$

(with $U_0=U_{N_h+1}=0$). In Table 4.1, we compare the exact solution with the approximate one obtained by the scheme (4.1). We report the approximation error for λ_1 and w_1 (in L^∞-norm and L^2-norm) and the order of convergence for λ_1. We can observe an order of convergence close to 2 for λ_1 and therefore equivalent to one obtained by discretization of the Rayleigh quotient via finite elements (see [10]).

h	$Err(\lambda_1)$	$Order(\lambda_1)$	$Err_\infty(w_1)$	$Err_2(w_1)$
$1.00\cdot10^{-1}$	$8.0908\cdot10^{-2}$		$3.3662\cdot10^{-11}$	$5.7732\cdot10^{-11}$
$5.00\cdot10^{-2}$	$2.0277\cdot10^{-2}$	1.9964	$1.4786\cdot10^{-10}$	$3.8119\cdot10^{-10}$
$2.50\cdot10^{-2}$	$5.0723\cdot10^{-3}$	1.9991	$6.6613\cdot10^{-16}$	$1.8731\cdot10^{-15}$
$1.25\cdot10^{-2}$	$1.2683\cdot10^{-3}$	1.9998	$1.5543\cdot10^{-15}$	$6.2524\cdot10^{-15}$
$6.25\cdot10^{-3}$	$3.1708\cdot10^{-4}$	1.9999	$1.2212\cdot10^{-15}$	$7.1576\cdot10^{-15}$

TABLE 4.1. *Space step (first column), eigenvalue error (second column), convergence order (third column), eigenfunction error in L^∞ (fourth column), eigenfunction error in L^2 (last column).*

Example 2. In this example, we consider the eigenvalue problem for a linear equation with a discontinuous coefficient

$$\begin{cases} a(x)w''+\lambda w=0 & x\in(0,\pi), \\ w(x)=0 & x\in\{0,\pi\}, \end{cases} \tag{4.2}$$

where

$$a(x)=\begin{cases} 1 \text{ for } & x\in[0,\frac{\pi}{2k}), \\ 2 \text{ for } & x\in[\frac{\pi}{2k},\pi], \end{cases}$$

and $k := \frac{2+\sqrt{2}}{2\sqrt{2}} > 1$.

Proposition 4.1. *The principal eigenvalue λ_1 associated to problem (4.2) is given by $k^2 = \frac{3+2\sqrt{2}}{4}$.*

Proof. Let

$$w(x) = \begin{cases} \sin(kx) & \text{for } x \in [0, \frac{\pi}{2k}), \\ b\sin(\frac{kx}{\sqrt{2}} + c) & \text{for } x \in [\frac{\pi}{2k}, \pi]. \end{cases}$$

We choose b and c such that $w(0) = w(\pi) = 0$ and w is continuous in $\frac{\pi}{2k}$. Imposing these conditions, we get

$$\frac{k\pi}{\sqrt{2}} + c = \pi, \quad \text{and} \quad b\sin(\frac{\pi}{2\sqrt{2}} + c) = 1,$$

i.e.

$$c = \pi(1 - \frac{k}{\sqrt{2}}) = \pi(\frac{2-\sqrt{2}}{4}).$$

Furthermore, using that $\frac{\pi}{2\sqrt{2}} + c = \frac{\pi}{2}$, we get

$$\lim_{x \to \frac{\pi}{2k}^-} w'(x) = k\cos(\frac{\pi}{2}) = 0, \quad \lim_{x \to \frac{\pi}{2k}^+} w'(x) = bk\cos(\frac{\pi}{2\sqrt{2}} + c) = 0;$$

hence, $w \in C^1([0,\pi])$.
On the other hand, since w is not C^2 in $\frac{\pi}{2k}$, we show that it satisfies problem (4.2) in the sense of viscosity solutions. For any $(p,q) \in J^{2,+}w(\frac{\pi}{2k})$, we get $p = 0$ and $q \geq -\frac{k^2}{2}$. This implies that, for both $a = 1$ and $a = 2$;

$$aq \geq -k^2 w(\frac{\pi}{2k}),$$

so w is a subsolution. For any $(p,q) \in J^{2,-}w(\frac{\pi}{2k})$, we get $p = 0$ and $q \leq -k^2$. This implies that, for both $a = 1$ and $a = 2$;

$$aq \leq -k^2 w(\frac{\pi}{2k}),$$

so w is a supersolution. □

In Table 4.2, we compare the exact solution with the approximate one obtained by means of the scheme

$$\lambda_{1,h} = -\min_{U \in \mathbb{R}^{N_h}} \left[\max_{i=1,\ldots,N_h} a(ih) \frac{U_{i+1} + U_{i-1} - 2U_i}{h^2 U_i} \right]$$

(with $U_0 = U_{N^h+1} = 0$). The rates are not very good, but the problem is out of our setting since F is discontinuous and the error is very sensible to the chosen grid. In Figure 4.1, we report the graph of the exact and approximate eigenfunctions for $h = 0.1$.

Example 3. The Fucîk spectrum of Δ is the set of pairs $(\mu, \alpha\mu) \in \mathbb{R}^2$ for which the equation

$$-\Delta u = \mu u^+ - \alpha\mu u^-$$

h	$Err(\lambda_1)$	$Order(\lambda_1)$	$Err_\infty(w_1)$	$Err_2(w_1)$
0.1571	0.1197		0.0213	0.0563
0.0785	0.0476	1.3303	0.0090	0.0383
0.0393	0.0347	0.4576	0.0065	0.0391
0.0196	0.0157	1.1417	0.0030	0.0264
0.0098	0.0061	1.3596	0.0012	0.0149

TABLE 4.2. *Space step (first column), eigenvalue error (second column), eigenfunction error in* L^∞ *(fourth column), eigenfunction error in* L^2 *(last column).*

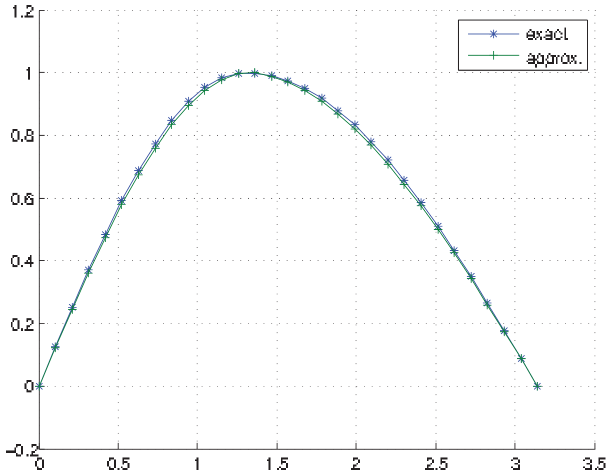

FIG. 4.1. *Exact and approximate eigenfunctions for* $h = 10^{-1}$.

has a non-zero solution, where $u^+(x) = \max\{u(x), 0\}$ and $u^-(x) = \max\{-u(x), 0\}$. For fixed $\alpha > 0$ the previous problem is equivalent to

$$\min\{\Delta u, \frac{1}{\alpha}\Delta u\} + \mu u = 0 \quad \text{if } \alpha \geq 1,$$
$$\max\{\Delta u, \frac{1}{\alpha}\Delta u\} + \mu u = 0 \quad \text{if } \alpha \leq 1.$$

For details, see [11]. Hence, the Fucîk spectrum can be seen as the spectrum of a nonlinear operators involving the maximum or minimum of two linear operators. To find the corresponding principal eigenvalue, we apply the scheme (4.1). In Table 4.3, we report the corresponding approximation error for λ_1 in the case $\alpha = 1/2$ and $\Omega = [0, \pi]$ (by the convexity of the solution, the eigenvalue for the continuous problem coincides with the one of the second derivative operator in $[0, \pi]$ i.e. $\lambda_1 = 1$).

Example 4. Consider the eigenvalue problem for the p-Laplace operator

$$\operatorname{div}(|Dw_1|^{p-2} Dw_1) + \lambda_1 |w_1|^{p-2} w_1 = 0.$$

This example does not fit exactly in the framework of this paper since the operator is

h	$Err(\lambda_1)$	$Order(\lambda_1)$
$1.00\cdot 10^{-1}$	0.0809	
$5.00\cdot 10^{-2}$	0.0203	1.9964
$2.50\cdot 10^{-2}$	0.0051	1.9991
$1.25\cdot 10^{-2}$	0.0013	1.9998
$6.25\cdot 10^{-3}$	0.0003	2.0000

TABLE 4.3. *Space step (first column), eigenvalue error (second column), convergence order (third column) for the Fucîk spectrum with* $\alpha = 1/2$.

not uniformly elliptic. However, the following formula

$$\lambda_{p,h} := -\inf_{\varphi>0}\sup_{y\in\Omega_h}\left\{\frac{F_{h,p}[\varphi](y)}{\varphi(y)^{p-1}}\right\},$$

where $F_{h,p}$ is a finite-difference approximations of F_p, produces a good approximation of the principal eigenvalue of the p-Laplace operator in the interval (a,b) whose exact value is given by

$$\sqrt[p]{\lambda_p} = \frac{2\pi\sqrt[p]{p-1}}{(b-a)p\sin(\frac{\pi}{p})}.$$

In Table 4.4, we report the approximation error and the corresponding order of convergence for the principal eigenvalue of the p-Laplace operator for $p=4$ (in this case $\lambda_4 \approx 73.0568$).

h	$Err(\lambda_4)$	$Order(\lambda_4)$
$1.00\cdot 10^{-1}$	2.6770	
$5.00\cdot 10^{-2}$	0.6210	2.1079
$2.50\cdot 10^{-2}$	0.1457	2.0912
$1.25\cdot 10^{-2}$	0.0347	2.0724
$6.25\cdot 10^{-3}$	0.0083	2.0581

TABLE 4.4. *Space step(first column), eigenvalue error (second column), convergence order (third column) for the p-Laplace operator with* $p=4$

It is also known (see [16]) that, if Ω is a ball, the eigenfunction w_p corresponding to the eigenvalue λ_p converges for $p\to\infty$ to $d(x,\partial\Omega)$. In Figure 4.2, we draw approximations of w_p computed by the scheme for various values of p and we observe the convergence of these functions to $d(x,\{0,1\})$ for p increasing, as expected by the theory.

4.2. Discretization in higher dimension. We now consider the eigenvalue problem in $\mathbb{R}^N$. Arguing as in the 1-dimensional case we write

$$\frac{F_h[u](x)}{u(x)} = \mathcal{F}\left(x,1,\left\{\frac{u(x+hy)-u(x-hy)}{2h|y|u(x)}\right\}_{y\in Y},\left\{\frac{u(x+hy)+u(x-hy)}{h^2|y|^2u(x)}-\frac{2}{h^2}\right\}_{y\in Y}\right) \tag{4.3}$$

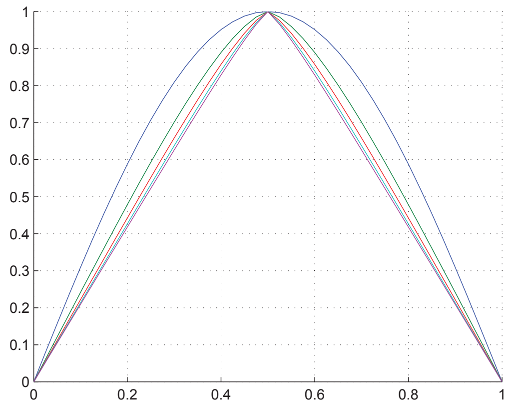

FIG. 4.2. *Approximate eigenfunction $u_{p,h}$ for $p=2,4,6,8,10$ and $h=10^{-3}$.*

for $i=1,\dots,N_h$ and F_h defined as in Equation (3.1), Y the stencil and N_h the cardinality of Ω_h. Hence, if the function $\mathcal{F}(x,z,\{q_y\}_{y\in Y},\{s_y\}_{y\in Y})$ is linear or more generally convex in the variables q_y and s_y, $y\in Y$, then the computation of the principal eigenvalue $\lambda_{1,h}$ is equivalent to the minimization with respect to the vector $U\in\mathbb{R}^{N_h}$ of the convex function $\mathcal{G}:\mathbb{R}^{N_h}\to\mathbb{R}$ obtained by taking the maximum with respect to $x\in\Omega_h$ in Equation (4.3). Therefore, this problem can be solved by means of some standard algorithms in convex optimization.

Example 5. Consider the problem

$$\begin{cases} \Delta w+\lambda w=0 & x\in(0,1)^2, \\ w(x)=0 & x\in\partial((0,1)^2). \end{cases}$$

The eigenvalue and the corresponding eigenfunction are given by

$$\lambda_1=2\pi^2, \qquad w(x_1,x_2)=\sin(\pi x_1)\sin(\pi x_2)$$

(the eigenfunctions are normalized by taking $\|w\|_\infty=\|w_h\|_\infty=1$). We use a standard five-point formula for the discretization of the Laplacian. In Table 4.5, we compare the exact solution with the approximate one obtained by the scheme (4.1). We report the approximation error for λ_1 and w_1 (in L^∞-norm and L^2-norm) and the order of convergence for λ_1. We can observe an order of convergence close to 2 for λ_1 and, therefore, equivalent to one obtained by discretization of the Rayleigh quotient via finite elements (see [10]).

Example 6. We consider the eigenvalue problem for the Ornstein–Uhlenbeck operator

$$\Delta w-x\cdot Dw+\lambda w=0, \qquad x\in(-1,1)^2$$

with homogeneous boundary conditions. The eigenvalue and the corresponding eigenfunction are given by

$$\lambda_1=4, \qquad w(x_1,x_2)=(1-x_1^2)(1-x_2^2),$$

h	$Err(\lambda_1)$	$Order(\lambda_1)$	$Err_\infty(w)$	$Err_2(w)$
$2.00\cdot 10^{-1}$	0.4469		0.0801	0.2256
$1.00\cdot 10^{-1}$	0.1338	1.7397	0.0203	0.1137
$5.00\cdot 10^{-2}$	0.0368	1.8629	0.0056	0.0590
$2.50\cdot 10^{-2}$	0.0097	1.9297	0.0015	0.0301

TABLE 4.5. *Space step (first column), eigenvalue error (second column), convergence order (third column), eigenfunction error in L^∞ (fourth column), eigenfunction error in L^2 (last column).*

with the eigenfunctions normalized by taking $\|w\|_\infty = \|w_{1,h}\|_\infty = 1$. The Laplacian is discretized by a five-point formula. In Table 4.6, we report the approximation error for λ_1 and the corresponding order of convergence.

h	$Err(\lambda_1)$	$Order(\lambda_1)$
$4.00\cdot 10^{-1}$	0.1524	
$2.00\cdot 10^{-1}$	0.0392	1.9592
$1.00\cdot 10^{-1}$	0.0103	1.9250
$5.00\cdot 10^{-2}$	0.0027	1.9580

TABLE 4.6. *Space step (first column), eigenvalue error (second column), convergence order (third column)*

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