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ENTROPY-DISSIPATING SEMI-DISCRETE RUNGE–KUTTA SCHEMES FOR NONLINEAR DIFFUSION EQUATIONS*

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Abstract. Semi-discrete Runge–Kutta schemes for nonlinear diffusion equations of parabolic type are analyzed. Conditions are determined under which the schemes dissipate the discrete entropy locally. The dissipation property is a consequence of the concavity of the difference of the entropies at two consecutive time steps. The concavity property is shown to be related to the Bakry–Emery approach and the geodesic convexity of the entropy. The abstract conditions are verified for quasilinear parabolic equations (including the porous-medium equation), a linear diffusion system, and the fourth-order quantum diffusion equation. Numerical experiments for various Runge–Kutta finite-difference discretizations of the one-dimensional porous-medium equation show that the entropy-dissipation property is in fact global.

Key words. Entropy-dissipative numerical schemes, Runge–Kutta schemes, entropy method, geodesic convexity, porous-medium equation, Derrida–Lebowitz–Speer–Spohn equation.

AMS subject classifications. 65J08, 65L06, 65M12, 65M20.

1. Introduction

Evolution equations often contain some structural information reflecting inherent physical properties such as positivity of solutions, conservation laws, and entropy dissipation. Numerical schemes should be designed in such a way that these structural features are preserved on the discrete level in order to obtain accurate and stable algorithms. In the last decades, concepts of structure-preserving schemes, geometric integration, and compatible discretization have been developed [7], but much less is known about the preservation of entropy dissipation and large-time asymptotics.

Entropy-stable schemes were derived by Tadmor already in the 1980s [23] in the context of conservation laws, thus without (physical) diffusion. Later, entropy-dissipative schemes were developed for (finite-volume) discretizations of diffusion equations in [2, 10, 11]. In [5], a finite-volume scheme which preserves the gradient-flow structure and hence the entropy structure is proposed. All these schemes are based on the implicit Euler method and are of first order (in time) only.

Higher-order schemes which diminish the total variation were developed for hyperbolic conservation laws, and they are often based on flux or slope limiters [22]. More general approaches are known under the name of strong stability preserving schemes ensuring stability in the same norm as the forward Euler scheme. They are used, for instance, for method-of-lines approximations of partial differential equations. For Runge–Kutta discretizations with this property, we refer to [12, 14].

Further numerical approaches of higher-order entropy-dissipating schemes include the second-order predictor-corrector approximation of [21] and the higher-order semi-implicit Runge–Kutta (DIRK) method of [3], together with a spatial fourth-order central finite-difference discretization. In [4, 19], multistep time approximations were employed,

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but they can be at most of second order, and they dissipate only one entropy (and not all functionals dissipated by the continuous equation). In this paper, we remove these restrictions by investigating time-discrete Runge–Kutta schemes of order $p\geq 1$ for general diffusion equations.

We stress the fact that we are interested in the analysis of entropy-dissipating schemes by "translating" properties for the continuous equation to the semi-discrete level, i.e. we study the stability of the schemes. However, we will not investigate convergence, stiffness, or computational issues here (see e.g. [3]).

More precisely, we consider time discretizations of the abstract Cauchy problem

$$\partial_t u(t)+A[u(t)]=0, \quad t>0, \quad u(0)=u^0, \tag{1.1}$$

where $A:D(A)\to X'$ is a (differential) operator defined on $D(A)\subset X$ and X is a Banach space with dual X'. In this paper, we restrict ourselves to diffusion operators $A[u]$ defined on some Sobolev space with solutions $u:\Omega\times(0,\infty)\to\mathbb{R}^n$, which may be vector-valued. A typical example is $A[u]=\operatorname{div}(a(u)\nabla u)$ defined on $X=L^2(\Omega)$ with domain $D(A)=H^2(\Omega)$, where $a:\mathbb{R}\to\mathbb{R}$ is a smooth function (see Section 3). Equation (1.1) often possesses a Lyapunov functional $H[u]=\int_\Omega h(u)dx$ (here called *entropy*), where $h:\mathbb{R}^n\to\mathbb{R}$, such that

$$\frac{dH}{dt}[u]=\int_\Omega h'(u)\partial_t u dx=-\int_\Omega h'(u)A[u]dx\leq 0,$$

at least when the *entropy production* $\int_\Omega h'(u)A[u]dx$ is nonnegative. Here, h' is the derivative of h and $h'(u)A[u]$ is interpreted as the inner product of $h'(u)$ and $A[u]$ in $\mathbb{R}^n$. Furthermore, if h is convex, the convex Sobolev inequality $\int_\Omega h'(u)A[u]dx\geq\kappa H[u]$ for some $\kappa>0$ may hold [6], which implies that $dH/dt\leq-\kappa H$ and hence implies exponential convergence of $H[u]$ to zero with rate κ. The aim is to design a higher-order time-discrete scheme which preserves this entropy-dissipation property.

To this end, we propose the following semi-discrete Runge–Kutta approximation of problem (1.1). Given $u^{k-1}\in X$, define

$$u^k=u^{k-1}+\tau\sum_{i=1}^s b_i K_i, \quad K_i=-A\left[u^{k-1}+\tau\sum_{j=1}^s a_{ij}K_j\right], \quad i=1,\dots,s, \tag{1.2}$$

where t^k are the time steps, $\tau=t^k-t^{k-1}>0$ is the uniform time step size, u^k approximates $u(t^k)$, and $s\geq 1$ denotes the number of Runge–Kutta stages. Since the Cauchy problem is autonomous, the knots $c_1,\dots,c_s$ are not needed here. In concrete examples (see below), u^k are functions from Ω to $\mathbb{R}^n$. If $a_{ij}=0$ for $j\geq i$, the Runge–Kutta scheme is explicit; otherwise, it is implicit and a nonlinear system of size s has to be solved to compute K_i. We assume that scheme (1.2) is solvable for $u^k:\Omega\to\mathbb{R}^n$.

Given $h:\mathbb{R}^n\to\mathbb{R}$, we wish to determine conditions under which the functional

$$H[u^k]=\int_\Omega h(u^k(x))dx \tag{1.3}$$

is dissipated by the numerical scheme (1.2),

$$H[u^k]+\tau\int_\Omega A[u^k]h'(u^k)dx\leq H[u^{k-1}], \quad k\in\mathbb{N}. \tag{1.4}$$

In many examples (see below), $\int_\Omega A[u^k]h'(u^k)dx \geq 0$ and, thus, the function $k \mapsto H[u^k]$ is decreasing. Such a property is the first step in proving the preservation of the large-time asymptotics of the numerical scheme (see Remark 1.1).

Our main results, stated on an informal level, are as follows:

(i) We determine an abstract condition under which the discrete entropy-dissipation inequality (1.4) holds for sufficiently small $\tau > 0$. This condition is made explicit for special choices of A and h, yielding entropy-dissipative implicit or explicit Runge–Kutta schemes of any order.

(ii) Numerical experiments for the porous-medium equation indicate that τ may be chosen independent of the time step k, thus yielding discrete entropy dissipation for all discrete times.

(iii) We show that, for Runge–Kutta schemes of order $p \geq 2$, the abstract condition in (i) is exactly the criterion of Liero and Mielke [20] to conclude geodesic 0-convexity of the entropy. In particular, it is related to the Bakry–Emery condition [1].

Let us describe the main results in more detail. We recall that the Runge–Kutta scheme (1.2) is consistent if $\sum_{j=1}^s a_{ij} = c_i$ and $\sum_{i=1}^s b_i = 1$. Furthermore, if $\sum_{i=1}^s b_i c_i = \frac{1}{2}$, it is at least of order two [13, Chap. II]. We introduce the number

$$C_{\mathrm{RK}} = 2\sum_{i=1}^s b_i(1-c_i), \tag{1.5}$$

which takes only three values:

$$\begin{aligned} C_{\mathrm{RK}} &= 0 \quad \text{for the implicit Euler scheme,} \\ C_{\mathrm{RK}} &= 1 \quad \text{for any Runge–Kutta scheme of order } p \geq 2, \text{ or} \\ C_{\mathrm{RK}} &= 2 \quad \text{for the explicit Euler scheme.} \end{aligned}$$

The *first main result* is an abstract entropy-dissipation property of scheme (1.2) for entropies of type (1.3).

THEOREM 1.1 (Entropy-dissipation structure I). *Let $h \in C^2(\mathbb{R}^n)$, let $A : D(A) \to X'$ be Fréchet differentiable with Fréchet derivative $DA[u] : X \to X'$ at $u \in D(A)$, and let (u^k) be the Runge–Kutta solution to scheme* (1.2). *Suppose that*

$$I_0^k := \int_\Omega \big(C_{\mathrm{RK}} h'(u^k) DA[u^k](A[u^k]) + h''(u^k)(A[u^k])^2\big) dx > 0. \tag{1.6}$$

Then there exists $\tau^k > 0$ such that, for all $0 < \tau \leq \tau^k$,

$$H[u^k] + \tau \int_\Omega A[u^k] h'(u^k) dx \leq H[u^{k-1}]. \tag{1.7}$$

Compared to strong stability preserving Runge–Kutta schemes [12, 14], we obtain not only a time-discrete dissipation property but also an estimate for $A[u^k]h'(u^k)$, which usually provides gradient bounds. Another difference is that we study semi-discrete problems, while the works [12, 14] are rather concerned with ordinary differential equations derived from method-of-lines approximations.

We assume that the solutions to scheme (1.2) are sufficiently regular such that the integral (1.6) can be defined. In the vector-valued case, $h''(u^k)$ is the Hessian matrix and we interpret $h''(u^k)(A[u^k])^2$ as the product $A[u^k]^\top h''(u^k) A[u^k]$. For Runge–Kutta schemes of order $p \geq 2$ (for which $C_{\mathrm{RK}} = 1$), the integral (1.6) corresponds exactly to the second-order time derivative of $H[u(t)]$ for solutions $u(t)$ to the *continuous* equation (1.1). Observe that the entropy-dissipation estimate (1.7) is only *local*, since the time step restriction depends on the time step k. For implicit Euler schemes (and convex entropies h), it is known that τ^k can be chosen independent of k. For general Runge–Kutta methods, we cannot prove rigorously that τ^k stays bounded from below as $k\to\infty$. However, our numerical experiments in Section 7 indicate that inequality (1.7) holds for sufficiently small $\tau>0$ uniformly in k.

REMARK 1.1 (Exponential decay of the discrete entropy). If the convex Sobolev inequality $\int_\Omega A[u^k]h'(u^k)dx\ge\kappa H[u^k]$ holds for some constant $\kappa>0$ and if there exists $\tau^*>0$ such that $\tau^k\ge\tau^*>0$ for all $k\in\mathbb{N}$, we infer from the estimate (1.7) that for $\tau:=\tau^*$,

$$H[u^k]\le(1+\kappa\tau)^{-k}H[u^0]=\exp(-\eta\kappa t^k)H[u^0],\quad\text{where }\eta=\frac{\log(1+\kappa\tau)}{\kappa\tau}<1,$$

which implies exponential decay of the discrete entropy with rate $\eta\kappa$. This rate converges to the continuous rate κ as $\tau\to0$, and therefore it is asymptotically sharp.

Theorem 1.1 can be generalized to a larger class of entropies, namely to so-called *first-order entropies*

$$F[u^k]=\int_\Omega|\nabla f(u^k)|^2dx, \tag{1.8}$$

where, for simplicity, we consider only the scalar case with $f:\mathbb{R}\to\mathbb{R}$. An important example is the Fisher information with $f(u)=\sqrt{u}$.

THEOREM 1.2 (Entropy-dissipating structure II). *Let $f\in C^2(\mathbb{R})$, let $A:D(A)\to X'$ be Fréchet differentiable, and let (u^k) be the Runge–Kutta solution to the scheme* (1.2). *Assume that the boundary condition $\nabla f(u^k)\cdot\nu=0$ on $\partial\Omega$ is satisfied. Furthermore, suppose that*

$$\begin{aligned}I_1^k:=\int_\Omega\Big(&|\nabla(f'(u^k)A[u^k]|^2-C_{\mathrm{RK}}\Delta f(u^k)f'(u^k)DA[u^k](A[u^k])\\&-\Delta f(u^k)f''(u^k)(A[u^k])^2\Big)dx>0.\end{aligned} \tag{1.9}$$

Then there exists $\tau^k>0$ such that, for all $0<\tau\le\tau^k$,

$$F[u^k]+\tau\int_\Omega A[u^k]f'(u^k)dx\le F[u^{k-1}].$$

The key idea of the proof of Theorem 1.1 (and similarly for Theorem 1.2) is a concavity property of the difference of the entropies at two consecutive time steps with respect to the time step size τ. To explain this idea, let $u:=u^k$ be fixed and introduce $v(\tau):=u^{k-1}$. Clearly, $v(0)=u$. A formal Taylor expansion of $G(\tau):=H[u]-H[v(\tau)]$ yields

$$H[u^k]-H[u^{k-1}]=G(\tau)=G(0)+\tau G'(0)+\frac{\tau^2}{2}G''(\xi^k),$$

where $0<\xi^k<\tau$. A computation, made explicit in Section 2, shows that $G'(0)=\int_\Omega A[u^k]h'(u^k)dx$ and $G''(0)=-I_0^k$. Now, if $G''(0)<0$, there exists $\tau^k>0$ such that

$G''(\tau)\le 0$ for $\tau\in[0,\tau^k]$ and in particular $G''(\xi^k)\le 0$. Consequently, $G(\tau)\le\tau G'(0)$, which equals Equation (1.4). The definition of $v(\tau)$ assumes implicitly that the scheme (1.2) is *backward* solvable. We prove in Proposition 2.1 below that this property holds if the operator A is a smooth self-mapping on X.

REMARK 1.2 (Discussion of τ^k). Since (u^k) is expected to converge to the stationary solution, $\lim_{k\to\infty} I_0^k=0$. Thus, in principle, for larger values of k, we expect that τ^k becomes smaller and smaller, thus restricting the choice of time step sizes τ. However, practically, the situation is better. For instance, for the implicit Euler scheme, if h is convex, we obtain

$$H[u^k]-H[u^{k-1}]\le\int_\Omega h'(u^k)(u^k-u^{k-1})dx=-\tau\int_\Omega h'(u^k)A[u^k]dx$$

for *any* value of $\tau>0$. Moreover, for other (higher-order) Runge–Kutta schemes, the numerical experiments in Section 7 indicate that there exists $\tau^*>0$ such that $G''(\tau)\le 0$ holds for all $\tau\in[0,\tau^*]$ uniformly in $k\in\mathbb{N}$. In this situation, the inequality (1.7) holds for all $0<\tau\le\tau^*$, and thus our estimate is global. In fact, the function G'' is numerically even nonincreasing in some interval $[0,\tau^*]$, but we are not able to prove this analytically.

The *second main result* is the specification of the abstract conditions (1.6) and (1.9) for a number of examples: a quasilinear diffusion equation, porous-medium or fast-diffusion equations, a linear diffusion system, and the fourth-order Derrida–Lebowitz–Speer–Spohn equation (see sections 3–6 for details). For instance, for the porous-medium equation

$$\partial_t u=\Delta(u^\beta)\text{ in }\Omega,\ t>0,\quad \nabla u^\beta\cdot\nu=0\text{ on }\partial\Omega,\quad u(0)=u^0,$$

we show that the Runge–Kutta scheme scheme satisfies

$$H[u^k]+\tau\beta\int_\Omega(u^k)^{\alpha+\beta-2}|\nabla u^k|^2dx\le H[u^{k-1}],\quad\text{where } H[u]=\frac{1}{\alpha(\alpha+1)}\int_\Omega u^{\alpha+1}dx$$

for $0<\tau\le\tau^k$ and all (α,β) belonging to some region in $[0,\infty)^2$ (see Figure 4.1 below). For $\alpha=0$, we write $H[u]=\int_\Omega u(\log u-1)dx$. In one space dimension and for Runge–Kutta schemes of order $p\ge 2$, this region becomes $-2<\alpha-\beta<1$, which is the same condition as for the continuous equation (except the boundary values). Furthermore, the first-order entropy (1.8) is dissipated for Runge–Kutta schemes of order $p\ge 2$, in one space dimension,

$$F[u^k]+\tau C_{\alpha,\beta}\int_\Omega(u^k)^{\alpha+\beta-2}(u^k)_{xx}^2dx\le F[u^{k-1}],\quad\text{where } F[u]=\int_\Omega(u^{\alpha/2})_x^2dx$$

for $0<\tau\le\tau^k$ and all (α,β) belonging to the region shown in Figure 4.2 below, and $C_{\alpha,\beta}>0$ is some constant. This region is smaller than the region of admissible values (α,β) for the continuous entropy. The borders of that region are indicated in the figure by dashed lines.

The proof of the above results, and namely of $G''(0)<0$, is based on systematic integration by parts [16]. The idea of the method is to formulate integration by parts as manipulations with polynomials and to conclude the inequality $G''(0)<0$ from a polynomial decision problem. This problem can be solved directly or by using computer algebra software.

Our *third main result* is the relation to geodesic 0-convexity of the entropy and the Bakry–Emery approach when $C_{\mathrm{RK}}=1$ (Runge–Kutta scheme of order $p\geq 2$). Liero and Mielke formulate in [20] the abstract Cauchy problem (1.1) as the gradient flow

$$\partial_t u = -K[u]DH[u], \quad t>0, \quad u(0)=u^0,$$

where the Onsager operator $K[u]$ describes the sum of diffusion and reaction terms. For instance, if $A[u]=\operatorname{div}(a(u)\nabla u)$, we can write $A[u]=\operatorname{div}(a(u)h''(u)^{-1}\nabla h'(u))$, and thus, identifying $h'(u)$ and $DH[u]$, we have $K[u]\xi=\operatorname{div}(a(u)h''(u)^{-1}\nabla\xi)$. It is shown in [20] that the entropy H is geodesic λ-convex if the inequality

$$M(u,\xi) := \langle \xi, DA[u]K[u]\xi\rangle - \frac{1}{2}\langle \xi, DK[u]A[u]\xi\rangle \geq \lambda\langle \xi, K[u]\xi\rangle \tag{1.10}$$

holds for all suitable u and ξ. We will prove in Section 2 that

$$G''(0) = 2M(u^k, h'(u^k)).$$

Hence, if $G''(0)\leq 0$ then Equation (1.10) with $\lambda=0$ is satisfied for $u=u^k$ and $\xi=h'(u^k)$, yielding geodesic 0-convexity for the semi-discrete entropy. Moreover, if $G''(0)\leq -\lambda G'(0)$ then we obtain geodesic λ-convexity. Since $G'(0)=-dH[u]/dt$ and $G''(0)=-d^2H[u]/dt^2$ in the continuous setting, the inequality $G''(0)\leq -\lambda G'(0)$ can be written as

$$\frac{d^2H}{dt^2}[u] \geq -\lambda \frac{dH}{dt}[u],$$

which corresponds to a variant of the Bakry–Emery condition [1], yielding exponential convergence of $H[u]$ (if $\tau^k\geq\tau^*>0$ for all k). Thus, our results constitute a first step towards a *discrete Bakry–Emery approach.*

The paper is organized as follows. The abstract method, i.e. the proof of backward solvability and of theorems 1.1 and 1.2, is presented in Section 2. The method is applied in the subsequent sections to a scalar diffusion equation (Section 3), the porous-medium equation (Section 4), a linear diffusion system (Section 5), and the fourth-order Derrida–Lebowitz–Speer–Spohn equation (Section 6). Finally, Section 7 is devoted to some numerical experiments showing that G'' is negative in some interval $[0,\tau^*]$.

2. The abstract method

In this section, we show that the Runge–Kutta scheme is backward solvable if A is a self-mapping and we prove theorems 1.1 and 1.2.

PROPOSITION 2.1 (Backward solvability). *Let $(\tau,u^k)\in[0,\infty)\times X$, where X is some Banach space, and let $A\in C^2(X,X)$ be a self-mapping. Then there exist $\tau_0>0$, a neighborhood $V\subset X$ of u^k, and a function $v\in C^2([0,\tau_0);X)$ such that (1.2) holds for $u^{k-1}:=v(\tau)$. Moreover,*

$$v(0)=0, \quad v'(0)=A[u], \quad \text{and} \quad v''(0)=C_{\mathrm{RK}}DA[u](A[u]). \tag{2.1}$$

The self-mapping assumption is strong for differential operators A but it is somehow natural in the context of Runge–Kutta methods and valid for smooth solutions.

Proof. The idea of the proof is to apply the implicit function theorem in Banach spaces (see [8, Corollary 15.1]). To this end, we set $u:=u^k$ and define the mapping $J=(J_0,\dots,J_s):\mathbb{R}\times X^{s+1}\to X^{s+1}$ by

$$J_0(\tau,y) = v - u + \tau\sum_{i=1}^{s} b_i k_i, \quad \text{where} y=(k_1,\dots,k_s,v),$$

$$J_i(\tau,y)=k_i+A\Big[v+\tau\sum_{j=1}^s a_{ij}k_j\Big],\quad i=1,\dots,s.$$

The Fréchet derivative of J in the direction of (τ_h,y_h), where $y_h=(k_{h1},\dots,k_{hs},v_h)$, reads as

$$\begin{aligned}
DJ_0(\tau,y)(\tau_h,y_h)&=v_h+\tau_h\sum_{i=1}^s b_ik_i+\tau\sum_{i=1}^s b_ik_{hi},\\
DJ_i(\tau,y)(\tau_h,y_h)&=k_{hi}+DA\Big[v+\tau\sum_{j=1}^s a_{ij}k_j\Big]\Big(v_h+\tau_h\sum_{j=1}^s a_{ij}k_j+\tau\sum_{j=1}^s a_{ij}k_{hj}\Big),
\end{aligned}$$

where $i=1,\dots,s$. Let $\tau_0=0$ and $y_0=(-A[u],\dots,-A[u],u)$. Then $J(\tau_0,y_0)=0$ and

$$DJ_0(\tau_0,y_0)(0,y_h)=v_h,\quad DJ_i(\tau_0,y_0)(0,y_h)=k_{hi}+DA[u](v_h),\quad i=1,\dots,s.$$

The mapping $y_h\mapsto DJ(\tau_0,y_0)(0,y_h)$ is clearly an isomorphism from X^{s+1} onto X^{s+1}. By the implicit function theorem, there exist an interval $U\subset[0,\tau_0)$, a neighborhood $V\subset X^{s+1}$ of y_0, and a function $(k,v)\in C^2([0,\tau_0);V)$ such that $(k,v)(0)=(-A[u],\dots,-A[u],u)$ and $J(\tau,k(\tau),v(\tau))=0$ for all $\tau\in[0,\tau_0)$.

Implicit differentiation of $J(\tau,k(\tau),v(\tau))=0$ yields

$$\begin{aligned}
0&=v'(\tau)+\sum_{i=1}^s b_ik_i(\tau)+\tau\sum_{i=1}^s b_ik_i'(\tau),\\
0&=k_i'(\tau)+DA\Big[v+\tau\sum_{j=1}^s a_{ij}k_j(\tau)\Big]\Big(v'(\tau)+\sum_{j=1}^s a_{ij}k_j(\tau)+\tau\sum_{j=1}^s a_{ij}k_j'(\tau)\Big),
\end{aligned}$$

where $i=1,\dots,s$ and $\tau\in[0,\tau_0)$. Using $\sum_{i=1}^s b_i=1$ and $\sum_{j=1}^s a_{ij}=c_i$, we infer that

$$\begin{aligned}
v'(0)&=-\sum_{i=1}^s b_ik_i(0)=\sum_{i=1}^s b_iA[u]=A[u],\\
k_i'(0)&=-DA[u]\Big(A[u]-\sum_{j=1}^s a_{ij}A[u]\Big)=-(1-c_i)DA[u](A[u]). \qquad (2.2)
\end{aligned}$$

Differentiating $J_0(\tau,k(\tau),v(\tau))=0$ twice leads to

$$0=v''(\tau)+2\sum_{i=1}^s b_ik_i'(\tau)+\tau\sum_{i=1}^s b_ik_i''(\tau).$$

Because of Equation (2.2), this reads at $\tau=0$ as

$$v''(0)=-2\sum_{i=1}^s b_ik_i'(0)=2\sum_{i=1}^s b_i(1-c_i)DA[u](A[u])=C_{\rm RK}DA[u](A[u]).$$

This finishes the proof. ∎

We prove now theorems 1.1 and 1.2.

Proof. **(Proof of Theorem 1.1).** We set $u:=u^k$. By Proposition 2.1, there exists a backward solution $v\in C^2([0,\tau_0))$ such that $v(0)=u$, $v'(0)=A[u]$, and

$v''(0) = C_{\mathrm{RK}} DA[u](A[u])$. Furthermore, the function $G(\tau) = \int_\Omega (h(u) - h(v(\tau)))dx$ satisfies $G(0) = 0$,

$$G'(0) = -\int_\Omega h'(v(0))v'(0)dx = -\int_\Omega h'(u)A[u]dx,$$
$$G''(0) = -\int_\Omega \big(h'(v(0))v''(0) + h''(v(0))v'(0)^2\big)dx$$
$$= -\int_\Omega \big(C_{\mathrm{RK}} h'(u)DA[u](A[u]) + h''(u)(A[u])^2\big)dx = -I_0^k < 0,$$

using the assumption. By continuity, there exists $0 < \tau^k < \tau_0$ such that $G''(\xi) \le 0$ for $0 \le \xi \le \tau^k$. Then the Taylor expansion $G(\tau) = G(0) + G'(0)\tau + \frac{1}{2}G''(\xi)\tau^2 \le G'(0)\tau$ concludes the proof. ∎

Proof. **(Proof of Theorem 1.2.)** Following the lines of the previous proof, it is sufficient to compute $G'(0)$ and $G''(0)$, where now $G(\tau) = \int_\Omega (|\nabla f(u)|^2 - |\nabla f(v(\tau))|^2)dx$. Using integration by parts and the boundary condition $\nabla f(v) \cdot \nu = 0$ on $\partial\Omega$, we compute

$$G'(0) = -\int_\Omega \nabla f(v(0)) \cdot \nabla\big(f'(v(0))v'(0)\big)dx = \int_\Omega \Delta f(u) f'(v(\tau))A[u]dx,$$

since $v(0) = u$ and $v'(0) = A[u]$. Furthermore, again integrating by parts,

$$G''(\tau) = -\int_\Omega \Big(\big|\nabla\big(f'(v(\tau))v'(\tau)\big)\big|^2 + \nabla f(v(\tau)) \cdot \nabla\big(f''(v(\tau))(v'(\tau))^2\big)$$
$$+ \nabla f(v(\tau)) \cdot \nabla\big(f'(v(\tau))v''(\tau)\big)\Big)dx$$
$$= -\int_\Omega \Big(\big|\nabla\big(f'(v(\tau))v'(\tau)\big)\big|^2 - \Delta f(v(\tau)) f''(v(\tau))(v'(\tau))^2$$
$$- \Delta f(v(\tau)) f'(v(\tau))v''(\tau)\Big)dx.$$

Since $v''(0) = C_{\mathrm{RK}} DA[u](A[u])$, this reduces at $\tau = 0$ to

$$G''(0) = -\int_\Omega \Big(|\nabla(f'(u)A[u])|^2 - \Delta f(u) f''(u)(A[u])^2 - C_{\mathrm{RK}} \Delta f(u) f'(u) DA[u](A[u])\Big)dx.$$

This expression equals $-I_1^k$, and the result follows. ∎

Finally, we show that $G''(0)$ for entropies (1.3) is related to the geodesic convexity condition of [20].

Lemma 2.1. *Let $A[u] = K(u)DH[u]$ for some symmetric operator $K: D(A) \to X$ and Fréchet derivative $DH[u]$, let G be defined as in the proof of Theorem 1.1 for a solution u^k to the Runge–Kutta scheme* (1.2) *of order $p \ge 2$, and let $M(u, \xi)$ be given by Equation* (1.10). *Then*

$$G''(0) = -2M(u^k, DH[u^k]).$$

Proof. The proof is just a (formal) calculation. Recall that for Runge–Kutta schemes of order $p \ge 2$, we have $C_{\mathrm{RK}} = 1$. Set $u := u^k$ and identify $DH[u]$ with $\xi = h'(u)$. Inserting the expression $DA[u](v) = DK[u](v)h'(u) + K[u]h''(u)v$ into the definition of $G''(0)$, we find that

$$-G''(0) = \langle \xi, DA[u](A[u])\rangle + \langle A[u], h''(u)A[u]\rangle$$

$$
\begin{aligned}
&= \langle \xi, DK[u](A[u])\xi + K[u]h''(u)A[u]\rangle + \langle A[u], h''(u)A[u]\rangle \\
&= \langle \xi, DK[u](K[u]\xi)\xi\rangle + \langle \xi, K[u]h''(u)K[u]\xi\rangle + \langle K[u]\xi, h''(u)K[u]\xi\rangle \\
&= \langle \xi, DK[u](K[u]\xi)\xi\rangle + 2\langle \xi, K[u]h''(u)K[u]\xi\rangle,
\end{aligned}
$$

since $K[u]$ is assumed to be symmetric. Rearranging the terms, we obtain

$$
\begin{aligned}
-G''(0) &= 2\langle \xi, DK[u](K[u]\xi)\xi\rangle + 2\langle \xi, K[u]h''(u)K[u]\xi\rangle - \langle \xi, DK[u](K[u]\xi)\rangle \\
&= 2\langle \xi, DA[u](K[u]\xi)\xi\rangle - \langle \xi, DK[u](A[u])\rangle = 2M(u,\xi),
\end{aligned}
$$

which proves the claim. ∎

3. Scalar diffusion equation

In this section, we analyze time-discrete Runge–Kutta schemes of the diffusion equation

$$
\partial_t u = \operatorname{div}(a(u)\nabla u), \quad t > 0, \quad u(0) = u^0, \tag{3.1}
$$

with periodic or homogeneous Neumann boundary conditions. This equation, also including a drift term, was analyzed in [20] in the context of geodesic convexity. Our results are similar to those in [20], but we consider the time-discrete and not the continuous equation and we employ systematic integration by parts [16].

Setting $\mu(u) = a(u)/h''(u)$, we can write the diffusion equation as a formal gradient flow

$$
\partial_t u = -A[u] := \operatorname{div}(\mu(u)\nabla h'(u)), \quad t > 0.
$$

We prove that the Runge–Kutta scheme (1.2) dissipates all convex entropies subject to some conditions on the functions μ and h.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^d$ be convex with smooth boundary. Let (u^k) be a sequence of (smooth) solutions to the Runge–Kutta scheme* (1.2) *of the diffusion equation* (3.1)*. Let $k \in \mathbb{N}$ be fixed and u^k be not equal to the constant steady state of Equation* (3.1)*. We suppose that, for all admissible u, it holds that $a(u) \geq 0$, $h''(u) \geq 0$,*

$$
b(u) := \frac{2}{3}(C_{\mathrm{RK}} + 1)\int_{u_0}^{u} \mu(v)\mu'(v)h''(v)dv \geq 0, \tag{3.2}
$$

$$
\frac{d-1}{d} b(u) \leq (C_{\mathrm{RK}} + 1)h''(u)\mu(u)^2, \tag{3.3}
$$

$$
(C_{\mathrm{RK}} + 2)\mu(u)\mu''(u) + (C_{\mathrm{RK}} - 1)\mu'(u)^2 < 0. \tag{3.4}
$$

Then there exists $\tau^k > 0$ such that for all $0 < \tau < \tau^k$,

$$
H[u^k] + \tau \int_\Omega h''(u^k)a(u^k)|\nabla u^k|^2 dx \leq H[u^{k-1}].
$$

Conditions (3.2)–(3.3) correspond to Equation (4.12) in [20]. Condition (3.4) is satisfied for concave functions μ, except for the explicit Euler scheme ($C_{\mathrm{RK}} = 2$), for which we need additionally $4\mu\mu'' + (\mu')^2 < 0$. For the implicit Euler scheme, we may allow even for nonconcave mobilities μ, e.g. $\mu(u) = u^\gamma$ for $1 < \gamma < 2$.

Proof. According to Theorem 1.1, we only need to show that $I_0^k = -G''(0) > 0$. To simplify, we set $u := u^k$. First, we observe that the boundary condition $\nabla u \cdot \nu = 0$

Abbreviation	definition	Abbreviation	definition
ξ	$h'(u)$		
ξ_L	$\Delta\xi$	ξ_G	$\lvert\nabla\xi\rvert$
ξ_H	$\lvert\nabla^2\xi\rvert$	ξ_{GHG}	$\nabla\xi^\top\nabla^2\xi\nabla\xi$
ξ_S	$(d-1)^{-1}\xi_G^{-2}(\xi_{GHG}-\xi_L\xi_G^2/d)$	ξ_R^2	$\xi_H^2-\xi_L^2/d-d(d-1)\xi_S^2$

TABLE 3.1. *Overview of the abbreviations for the proof of Theorem 3.1.*

on Ω implies that $0=\partial_t\nabla u\cdot\nu=\nabla\partial_t u\cdot\nu=-\nabla A[u]\cdot\nu$ on $\partial\Omega$. Using $DA[u](A[u])=\operatorname{div}(a'(u)A[u]\nabla u+a(u)\nabla A[u])=\Delta(a(u)A[u])$, the abbreviation $\xi=h'(u)$, and integration by parts, we compute

$$\begin{aligned}G''(0)&=-\int_\Omega\Big(C_{\mathrm{RK}}h'(u)\Delta(a(u)A[u])+h''(u)\big(\operatorname{div}(\mu(u)\nabla h'(u))\big)^2\Big)dx\\&=\int_\Omega\Big(C_{\mathrm{RK}}\nabla h'(u)\cdot\nabla(a(u)A[u])-h''(u)\big(\mu'(u)\nabla u\cdot\nabla h'(u)+\mu(u)\Delta h'(u)\big)^2\Big)dx\\&=-\int_\Omega\left(C_{\mathrm{RK}}\Delta\xi a(u)A[u]+h''(u)\left(\frac{\mu'(u)}{h''(u)}|\nabla\xi|^2+\mu(u)\Delta\xi\right)^2\right)dx.\end{aligned}$$

The boundary integrals vanish since $\nabla u\cdot\nu=\nabla A[u]\cdot\nu=0$ on $\partial\Omega$. Replacing $A[u]$ by $\operatorname{div}(\mu(u)\nabla\xi)=\mu(u)\Delta\xi+\mu'(u)|\nabla\xi|^2/h''(u)$ and expanding the square, we arrive at

$$\begin{aligned}G''(0)&=-\int_\Omega\bigg(\big(C_{\mathrm{RK}}a(u)\mu(u)+h''(u)\mu(u)^2\big)(\Delta\xi)^2\\&\qquad+\left(C_{\mathrm{RK}}a(u)\frac{\mu'(u)}{h''(u)}+2\mu(u)\mu'(u)\right)\Delta\xi|\nabla\xi|^2+\frac{\mu'(u)^2}{h''(u)}|\nabla\xi|^4\bigg)dx\\&=-\int_\Omega\big((C_{\mathrm{RK}}+1)h''(u)\mu(u)^2\xi_L^2+(C_{\mathrm{RK}}+2)\mu(u)\mu'(u)\xi_L\xi_G^2\\&\qquad+\mu'(u)^2h''(u)^{-1}\xi_G^4\big)dx,\end{aligned}\tag{3.5}$$

where we have employed the identity $a(u)=\mu(u)h''(u)$ and the abbreviations $\xi_G=|\nabla\xi|$ and $\xi_L=\Delta\xi$; see Table 3.1 for an overview of the various abbreviations.

We apply now the method of systematic integration by parts [16]. The idea is to identify useful integration-by-parts formulas and to add them to $G''(0)$ without changing the sign of $G''(0)$. The first formula is given by

$$\int_\Omega\operatorname{div}\big(\Gamma_1(u)(\nabla^2\xi-\Delta\xi\mathbb{I})\cdot\nabla\xi\big)dx=\int_{\partial\Omega}\Gamma_1(u)\nabla\xi^\top(\nabla^2\xi-\Delta\xi\mathbb{I})\nu ds,\tag{3.6}$$

where $\Gamma_1(u)\le 0$ is an arbitrary (smooth) scalar function which still needs to be chosen and $\mathbb{I}$ is the unit matrix in $\mathbb{R}^{d\times d}$. Computing the divergence and using the property $\nabla u=\nabla\xi/h''(u)$, the left-hand side can be expanded as

$$\begin{aligned}&\int_\Omega\Big(\Gamma_1'(u)\nabla u^\top(\nabla^2\xi-\Delta\xi\mathbb{I})\nabla\xi+\Gamma_1(u)(\nabla^2\xi-\Delta\xi\mathbb{I}):\nabla^2\xi\Big)dx\\&=\int_\Omega\left(\frac{\Gamma_1'(u)}{h''(u)}\nabla\xi^\top\nabla^2\xi\nabla\xi-\frac{\Gamma_1'(u)}{h''(u)}\Delta\xi|\nabla\xi|^2+\Gamma_1(u)|\nabla^2\xi|^2-\Gamma_1(u)(\Delta\xi)^2\right)dx\end{aligned}$$

$$=\int_\Omega\left(\frac{\Gamma_1(u)}{h''(u)}\xi_{GHG}-\frac{\Gamma_1'(u)}{h''(u)}\xi_L\xi_G^2+\Gamma_1(u)\xi_H^2-\Gamma_1(u)\xi_L^2\right)dx,$$

where we have set $\xi_{GHG}=\nabla\xi^\top\nabla^2\xi\nabla\xi$ and $\xi_H=|\nabla^2\xi|$. The boundary integral in Equation (3.6) becomes

$$\int_{\partial\Omega}\Gamma_1(u)\left(\frac{1}{2}\nabla(|\nabla\xi|^2)-\Delta\xi\nabla\xi\right)\cdot\nu ds=\frac{1}{2}\int_{\partial\Omega}\Gamma_1(u)\nabla(|\nabla\xi|^2)\cdot\nu ds\geq 0,$$

since $\Gamma_1(u)\leq 0$, $\nabla\xi\cdot\nu=0$ on $\partial\Omega$, and it holds that $\nabla(|\nabla\xi|^2)\cdot\nu\leq 0$ on $\partial\Omega$ for all smooth functions satisfying $\nabla\xi\cdot\nu=0$ on $\partial\Omega$ [20, Prop. 4.2]. Here, we need the convexity of Ω. Thus, the first integration-by-parts formula becomes

$$J_1:=\int_\Omega\left(\frac{\Gamma_1'(u)}{h''(u)}\xi_{GHG}-\frac{\Gamma_1'(u)}{h''(u)}\xi_L\xi_G^2+\Gamma_1(u)\xi_H^2-\Gamma_1(u)\xi_L^2\right)dx\geq 0. \tag{3.7}$$

The second formula reads as

$$\begin{aligned}0&=\int_\Omega \operatorname{div}\left(\Gamma_2(u)|\nabla\xi|^2\nabla\xi\right)dx\\&=\int_\Omega\left(\frac{\Gamma_2'(u)}{h''(u)}\xi_G^4+2\Gamma_2(u)\xi_{GHG}+\Gamma_2(u)\xi_L\xi_G^2\right)dx=:J_2,\end{aligned} \tag{3.8}$$

where Γ_2 is an arbitrary scalar function. The goal is to find functions $\Gamma_1(u)\leq 0$ and $\Gamma_2(u)$ such that $G''(0)\leq G''(0)+J_1+J_2<0$.

According to [17], the computations simplify if we introduce the variables ξ_R and ξ_S satisfying

$$(d-1)\xi_G^2\xi_S=\xi_{GHG}-\frac{1}{d}\xi_L\xi_G^2,\quad \xi_H^2=\frac{1}{d}\xi_L^2+d(d-1)\xi_S^2+\xi_R^2.$$

The existence of ξ_R follows from the inequality

$$\xi_H^2=|\nabla^2\xi|^2\geq\frac{1}{d}(\Delta\xi)^2+\frac{d}{d-1}\left(\frac{\nabla\xi^\top\nabla^2\xi\nabla\xi}{\nabla\xi^2}-\frac{\Delta\xi}{d}\right)^2=\frac{1}{d}\xi_L^2+d(d-1)\xi_S^2,$$

which is proven in [17, Lemma 2.1]. Then

$$G''(0)\leq G''(0)+J_1+J_2=-\int_\Omega\left(a_1\xi_L^2+a_2\xi_L\xi_G^2+a_3\xi_G^4+a_4\xi_S\xi_G^2+a_5\xi_R^2+a_6\xi_S^2\right)dx, \tag{3.9}$$

where

$$\begin{aligned}a_1&=(C_{\mathrm{RK}}+1)h''(u)\mu(u)^2+\left(1-\frac{1}{d}\right)\Gamma_1(u),\\a_2&=(C_{\mathrm{RK}}+2)\mu(u)\mu'(u)+\left(1-\frac{1}{d}\right)\frac{\Gamma_1'(u)}{h''(u)}-\left(\frac{2}{d}+1\right)\Gamma_2(u),\\a_3&=\frac{\mu'(u)^2-\Gamma_2'(u)}{h''(u)},\quad a_4=-(d-1)\left(\frac{\Gamma_1'(u)}{h''(u)}+2\Gamma_2(u)\right),\\a_5&=-\Gamma_1(u),\quad a_6=-d(d-1)\Gamma_1(u).\end{aligned} \tag{3.10}$$

The aim now is to determine conditions on $a_1,\ldots,a_6$ such that the polynomial $P(\xi)=a_1\xi_L^2+a_2\xi_L\xi_G^2+a_3\xi_G^4+a_4\xi_S\xi_G^2+a_5\xi_R^2+a_6\xi_S^2$ is nonnegative, as this implies that

$G''(0)\le 0$. In the general case, this leads to nonlinear ordinary differential equations for Γ_1 and Γ_2 which cannot be easily solved. A possible solution is to require that the coefficients of the mixed terms vanish, i.e. $a_2=a_4=0$, and that the remaining coefficients are nonnegative. With the case $d=1$ being simpler than the general case (since J_1 is not necessary), we assume that $d>1$. Then $a_4=0$ implies that $\Gamma_1'(u)/h''(u)=-2\Gamma_2(u)$. Replacing $\Gamma_1'(u)/h''(u)$ by $-2\Gamma_2(u)$ in $a_2=0$ gives

$$\Gamma_2(u)=\frac{C_{\rm RK}+2}{3}\mu(u)\mu'(u).$$

On the other hand, replacing $\Gamma_2(u)$ by $-\Gamma_1'(u)/(2h''(u))$ in $a_2=0$, we find that

$$\Gamma_1'(u)=-\frac{2}{3}(C_{\rm RK}+2)\mu(u)\mu'(u)h''(u)$$

or, after integration,

$$\Gamma_1(u)=-\frac{2}{3}(C_{\rm RK}+2)\int_{u_0}^{u}\mu(v)\mu'(v)h''(v)dv.$$

These functions have to satisfy the conditions

$$\begin{aligned} &a_1\ge 0 \quad \text{or} \quad \frac{d-1}{d}\Gamma_1(u)\ge -(C_{\rm RK}+1)h''(u)\mu(u)^2,\\ &a_3\ge 0 \quad \text{or} \quad (C_{\rm RK}+2)\mu(u)\mu''(u)+(C_{\rm RK}-1)\mu'(u)^2\le 0,\\ &a_5\ge 0 \quad \text{or} \quad \Gamma_1(u)\le 0 \quad \text{for all } u. \end{aligned}$$

Note that $a_1\ge 0$ and $a_5\ge 0$ correspond to Equations (3.3) and (3.2), respectively. This shows that $P(\xi)\ge 0$ for all $\xi\in\mathbb{R}^4$ and $G''(0)\le 0$.

If $G''(0)=0$, the nonnegative polynomial P, which depends on $x\in\Omega$ via ξ, has to vanish. In particular, $a_3\xi_G^4=a_3|\nabla u|^4=0$ in Ω. As $a_3>0$ by assumption, $u(x)=$const. for $x\in\Omega$. This contradicts the hypothesis that u is not a steady state. Consequently, $G''(0)<0$, and we finish the proof by setting $b(u)=-\Gamma_1(u)$. ∎

4. Porous-medium equation

The results of the previous section can be applied in principle to the Runge–Kutta scheme for the porous-medium or fast-diffusion equation

$$\partial_t u=\Delta(u^\beta) \quad \text{in } \Omega,\ t>0, \quad \nabla u^\beta\cdot\nu=0 \quad \text{on } \partial\Omega, \quad u(0)=u^0, \tag{4.1}$$

where $\beta>0$. It can be seen that conditions (3.2)–(3.4) are not optimal for particular entropies. This is not surprising since we have neglected the mixed terms in the polynomial in Equation (3.9) (i.e. $a_2=a_4=0$), which is not optimal. In this section, we make a different approach by making an ansatz for the functions Γ_1 and Γ_2, considering both zeroth-order and first-order entropies.

4.1. Zeroth-order entropies.

We prove the following result.

THEOREM 4.1. *Let $\Omega\subset\mathbb{R}^d$ be convex with smooth boundary. Let (u^k) be a sequence of (smooth) solutions to the Runge–Kutta scheme* (1.2) *for Equation* (4.1). *Let the entropy be given by $H[u]=\alpha^{-1}(\alpha+1)^{-1}\int_\Omega u^{\alpha+1}dx$ with $\alpha>0$, let $k\in\mathbb{N}$, and let u^k be not the constant steady state of Equation* (4.1). *There exists a nonempty region $R_0(d)\subset(0,\infty)^2$ and $\tau^k>0$ such that, for all $(\alpha,\beta)\in R_0(d)$ and $0<\tau\le\tau^k$,*

$$H[u^k]+\tau\beta\int_\Omega (u^k)^{\alpha+\beta-2}|\nabla u^k|^2dx\le H[u^{k-1}], \quad k\in\mathbb{N}.$$

In one space dimension, we have the following:

implicit Euler:	$R_0(1)=(0,\infty)^2$,
Runge–Kutta of order $p\geq 2$:	$R_0(1)=\{(\alpha,\beta)\in(0,\infty)^2:-2<\alpha-\beta<1\}$,
explicit Euler:	$R_0(1)=\{(\alpha,\beta)\in(0,\infty)^2:-1<\alpha-\beta<1\}$.

For the implicit Euler scheme, the theorem shows that any positive values for (α,β) is admissible, which corresponds to the continuous situation. For the Runge–Kutta case with $C_{\mathrm{RK}}=1$, our condition is more restrictive. As expected, the explicit Euler scheme requires the most restrictive condition. The set $R_0(d)$ is illustrated in Figure 4.1 for $d=2$ and $d=10$.

Proof. Since $k\in\mathbb{N}$ is fixed, we set $u:=u^k$. We choose the functions

$$\Gamma_1(u)=c_1\beta^2u^{2\beta-\alpha-1},\quad \Gamma_2(u)=c_2\beta^2u^{2\beta-2\alpha-1}.$$

It holds that $h''(u)=u^{\alpha-1}$ and $\mu(u)=\beta u^{\beta-\alpha}$. Then the coefficients in Equation (3.10) are as follows:

$$\begin{aligned}
a_1&=\beta^2\big((C_{\mathrm{RK}}+1)+(1-\tfrac{1}{d})c_1\big)u^{2\beta-\alpha-1},\\
a_2&=\beta^2\big((C_{\mathrm{RK}}+2)(\beta-\alpha)+(1-\tfrac{1}{d})(2\beta-\alpha-1)c_1-(\tfrac{2}{d}+1)c_2\big)u^{2\beta-2\alpha-1},\\
a_3&=\beta^2\big((\beta-\alpha)^2-(2\beta-2\alpha-1)c_2\big)u^{2\beta-3\alpha-2},\\
a_4&=-\beta^2(d-1)\big((2\beta-\alpha-1)c_1+2c_2\big)u^{2\beta-2\alpha-1},\\
a_5&=-\beta^2c_1u^{2\beta-\alpha-1},\quad a_6=-\beta^2d(d-1)c_1u^{2\beta-\alpha-1}.
\end{aligned}$$

Introducing the variables $\eta_j=\xi_j/u^\alpha$ for $j\in\{G,L,R,S\}$, we can write Equation (3.9) as

$$\begin{aligned}
&G''(0)\leq G''(0)+J_1+J_2=-\beta^2\int_\Omega u^{2\beta+\alpha-1}Q(\eta)dx,\\
&\text{where } Q(\eta)=b_1\eta_L^2+b_2\eta_L\eta_G^2+b_3\eta_G^4+b_4\eta_S\eta_G^2+b_5\eta_R^2+b_6\eta_S^2
\end{aligned}$$

with coefficients

$$\begin{aligned}
b_1&=(C_{\mathrm{RK}}+1)+(1-\tfrac{1}{d})c_1,\\
b_2&=(C_{\mathrm{RK}}+2)(\beta-\alpha)+(1-\tfrac{1}{d})(2\beta-\alpha-1)c_1-(\tfrac{2}{d}+1)c_2,\\
b_3&=(\beta-\alpha)^2-(2\beta-2\alpha-1)c_2,\\
b_4&=-(d-1)\big((2\beta-\alpha-1)c_1+2c_2\big),\\
b_5&=-c_1,\quad b_6=-d(d-1)c_1.
\end{aligned}$$

We need to determine all (α,β) such that there exist $c_1\leq 0$, $c_2\in\mathbb{R}$ such that $Q(\eta)\geq 0$ for all $\eta=(\eta_G,\eta_L,\eta_R,\eta_S)$. Without loss of generality, we exclude the cases $b_1=b_2=0$ and $b_4=b_6=0$ since they lead to parameters (α,β) included in the region calculated below. Thus, let $b_1>0$ and $b_6>0$. These inequalities give the bound $-(C_{\mathrm{RK}}+1)/(1-1/d)<c_1<0$. Thus, we may introduce the parameter $\lambda\in(0,1)$ by setting $c_1=-\lambda(C_{\mathrm{RK}}+1)/(1-1/d)$. The polynomial $Q(\eta)$ can be rewritten as

$$Q(\eta)=b_1\left(\eta_L+\frac{b_2}{2b_1}\eta_G^2\right)^2+b_6\left(\eta_S+\frac{b_4}{2b_6}\eta_G^2\right)^2+b_5\eta_R^2+\eta_G^4\left(b_3-\frac{b_2^2}{4b_1}-\frac{b_4^2}{4b_6}\right)$$

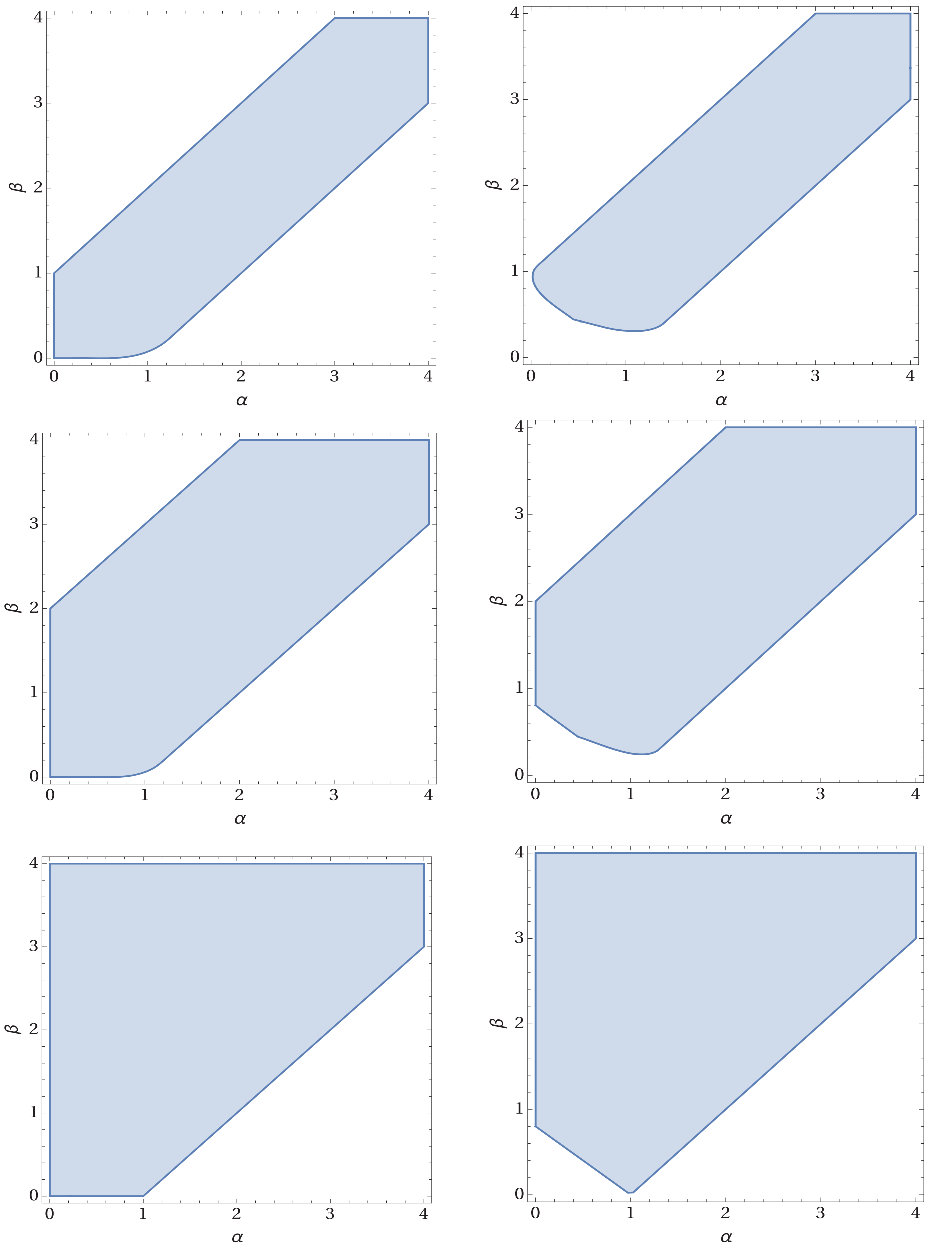

FIG. 4.1. *Set $R_0(d)$ of all (α,β) for which the zeroth-order entropy is dissipating. Left column: $d=2$; right column: $d=10$. Top row: explicit Euler scheme with $C_{\mathrm{RK}}=2$; middle row: implicit Euler scheme with $C_{\mathrm{RK}}=1$; bottom row: Runge–Kutta scheme of order $p\geq 2$ with $C_{\mathrm{RK}}=0$.*

$$\geq \eta_G^4\left(b_3-\frac{b_4^2}{4b_6}-\frac{b_2^2}{4b_1}\right)=:\frac{\eta_G^4(C_{\mathrm{RK}}+1)}{4b_1b_6}R(c_2;\lambda,\alpha,\beta),$$

where $R(c_2;\lambda,\alpha,\beta)$ is a quadratic polynomial in c_2 with the nonpositive leading term $-d^2(4-3\lambda)+4(2-3\lambda)d-4$. The polynomial $R(c_2;\lambda,\alpha,\beta)$ is nonnegative for some c_2 if and only if its discriminant $4d^2\lambda(1-\lambda)S(\lambda;\alpha,\beta)$ is nonnegative. Here, $S(\lambda;\alpha,\beta)$ is a quadratic polynomial in λ. In order to derive the conditions on (α,β) such that $S(\lambda;\alpha,\beta)\geq 0$ for some $\lambda\in(0,1)$, we employ the computer-algebra system `Mathematica`. The result of the command

```
Resolve[Exists[LAMBDA, S[LAMBDA] >= 0 && LAMBDA > 0
&& LAMBDA < 1], Reals]
```

gives all $(\alpha,\beta)\in\mathbb{R}^2$ such that there exist $c_1\leq 0$, $c_2\in\mathbb{R}$ such that $Q(\eta)\geq 0$. The interior of this region equals the set $R_0(d)$, defined in the statement of the theorem. This shows that $G''(0)\leq 0$ for all $(\alpha,\beta)\in R_0(d)$.

If $G''(0)=0$, the nonnegative polynomial Q has to vanish. In particular, $b_1\eta_L^2=0$. If $\eta_L=0$ in Ω, the boundary conditions imply that u is constant, which contradicts our assumption that u is not the steady state. Thus $b_1=0$. Similarly, $b_2=b_3=b_4=0$. This gives a system of four inhomogeneous linear equations for (c_1,c_2) which is unsolvable. Consequently, $G''(0)<0$.

The set $R_0(d)$ is nonempty since e.g. $(1,1)\in R_0(d)$. Indeed, choosing $c_1=-1$ and $c_2=0$, we find that $Q(\eta)=(C_{\mathrm{RK}}+\frac{1}{d})\eta_L^2+\eta_R^2+d(d-1)\eta_S^2\geq 0$.

In one space dimension, the situation simplifies since the Laplacian coincides with the Hessian, and thus the integration-by-parts formula (3.7) is not needed. Then (see Equation (3.8))

$$G''(0)=G''(0)+J_1=-\beta^2\int_\Omega u^{2\beta+\alpha-1}\big(a_1\xi_L^2+a_2\xi_L\xi_G^2+a_3\xi_G^4\big)dx,$$

where

$$a_1=C_{\mathrm{RK}}+1,\quad a_2=(C_{\mathrm{RK}}+2)(\beta-\alpha)-3c_2,\quad a_3=(\beta-\alpha)^2-(2\beta-2\alpha-1)c_2.$$

The polynomial $P(\xi)=\xi_G^4(a_1y^2+a_2y+a_3)$ with $y=\xi_L/\xi_G^2$ is nonnegative if and only if $a_1\geq 0$ and $4a_1a_3-a_2^2\geq 0$, which is equivalent to

$$-9c_2^2+2\big((C_{\mathrm{RK}}-2)(\alpha-\beta)+2(C_{\mathrm{RK}}+1)\big)c_2-C_{\mathrm{RK}}^2(\alpha-\beta)^2\geq 0. \tag{4.2}$$

This inequality has a solution $c_2\in\mathbb{R}$ if and only if the quadratic polynomial has real roots, i.e. if its discriminant is nonnegative,

$$\begin{aligned}0&\leq\big((C_{\mathrm{RK}}-2)(\alpha-\beta)+2(C_{\mathrm{RK}}+1)\big)^2-9C_{\mathrm{RK}}^2(\alpha-\beta)^2\\&=4(C_{\mathrm{RK}}+1)\big(-(2C_{\mathrm{RK}}-1)(\alpha-\beta)^2+(C_{\mathrm{RK}}-2)(\alpha-\beta)+(C_{\mathrm{RK}}+1)\big).\end{aligned}$$

The polynomial $-(2C_{\mathrm{RK}}-1)z^2+(C_{\mathrm{RK}}-2)z+(C_{\mathrm{RK}}+1)$ with $z=\alpha-\beta$ is always nonnegative if $C_{\mathrm{RK}}=0$ (implicit Euler). For $C_{\mathrm{RK}}=1$ and $C_{\mathrm{RK}}=2$, this property holds if and only if $-(C_{\mathrm{RK}}+1)/(2C_{\mathrm{RK}}-1)\leq\alpha-\beta\leq 1$. This concludes the proof. ☐

4.2. First-order entropies. We consider the one-dimensional case and first-order entropies with $f(u)=u^{\alpha/2}$, $\alpha>0$.

THEOREM 4.2. *Let $\Omega\subset\mathbb{R}$ be a bounded interval. Let (u^k) be a sequence of (smooth) solutions to the Runge–Kutta scheme* (1.2) *of order $p\geq 2$ for Equation* (4.1) *in one space dimension. Let the entropy be given by $F[u]=\int_\Omega(u^{\alpha/2})_x^2dx$ with $\alpha>0$, let $k\in\mathbb{N}$ be fixed, and let u^k be not the constant steady state of Equation* (4.1). *There exists a nonempty region $R_1\in[0,\infty)^2$ and $\tau^k>0$ such that, for all $(\alpha,\beta)\in R_1$, there is a constant $C_{\alpha,\beta}>0$ such that, for all $0<\tau\leq\tau^k$,*

$$F[u^k]+\tau C_{\alpha,\beta}\int_\Omega(u^k)^{\alpha+\beta-3}(u_{xx}^k)^2dx\leq F[u^{k-1}],\quad k\in\mathbb{N}.$$

Figure 4.2 illustrates the set R_1. The set of admissible value (α,β) for the continuous equation is given by $\{-2\leq\alpha-2\beta<1\}$ (the borders of this set are depicted in the figure by the dashed lines).

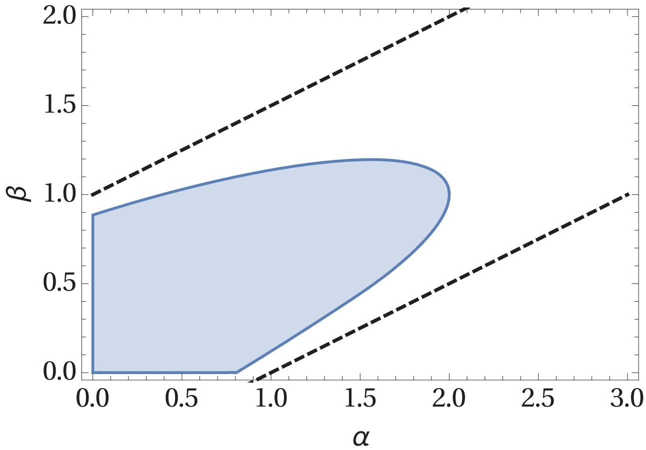

Fig. 4.2. *Set of all (α,β) for which the discrete first-order entropy for solutions to the one-dimensional porous-medium equation is dissipating. The continuous first-order entropy is dissipated for $-2\leq\alpha-2\beta<1$. The borders of this set is indicated in the figure by dashed lines.*

Proof. First, we compute $G'(0)$ according to Theorem 1.2

$$G'(0)=-\alpha\int_\Omega u^{\alpha/2-1}(u^{\alpha/2})_{xx}(u^\beta)_{xx}dx.$$

We show that $G'(0)$ is nonpositive in a certain range of values (α,β). We formulate $G'(0)$ as

$$G'(0)=-\frac{\alpha^2\beta}{4}\int_\Omega u^{\alpha+\beta-1}\big((\alpha-2)(\beta-1)\xi_1^4+(\alpha+2\beta-4)\xi_1^2\xi_2+2\xi_2^2\big)dx,$$

where $\xi_1=u_x/u$, $\xi_2=u_{xx}/u$. We employ the integration-by-parts formula

$$0=\int_\Omega(u^{\alpha+\beta-4}u_x^3)_xdx=\int_\Omega u^{\alpha+\beta-1}\big((\alpha+\beta-4)\xi_1^4+3\xi_1^2\xi_2\big)dx=:J.$$

Therefore,

$$G'(0)=G'(0)-\frac{\alpha^2\beta}{4}cJ=-\frac{\alpha^2\beta}{4}\int_\Omega u^{\alpha+\beta-1}P(\xi)dx,$$

where

$$P(\xi)=\big((\alpha-2)(\beta-1)+(\alpha+\beta-4)c\big)\xi_1^4+\big(\alpha+2\beta-4+3c\big)\xi_1^2\xi_2+2\xi_2^2.$$

This polynomial is nonnegative if and only if

$$8\big((\alpha-2)(\beta-1)+(\alpha+\beta-4)c\big)-(\alpha+2\beta-4+3c)^2\geq0,$$

which is equivalent to

$$g(c) := -9c^2 + 2(\alpha - 2\beta - 4)c - (\alpha - 2\beta)^2 \geq 0.$$

The maximizing value $c^* = (\alpha - 2\beta - 4)/9$, obtained from $g'(c) = 0$, yields

$$g(c^*) = -\frac{8}{9}(\alpha - 2\beta - 1)(\alpha - 2\beta + 2) \geq 0$$

and consequently $G'(0) \leq 0$ if $-2 \leq \alpha - 2\beta \leq 1$. This condition is the same as in [6, Theorem 13] for the continuous equation.

Next, we turn to the proof of $G''(0) < 0$. The proof of Theorem 1.2 shows that

$$G''(0) = -\frac{\alpha}{2}\int_\Omega \bigg(\frac{\alpha}{2}\big(u^{\alpha/2-1}(u^\beta)_{xx}\big)_x^2 - \Big(\frac{\alpha}{2} - 1\Big)u^{\alpha/2-2}(u^{\alpha/2})_{xx}(u^\beta)_{xx}^2 \\ - \beta C_{\mathrm{RK}} u^{\alpha/2-1}(u^{\alpha/2})_{xx}\big(u^{\beta-1}(u^\beta)_{xx}\big)_{xx}\bigg)dx.$$

We integrate by parts in the last term and use $(\beta u^{\beta-1}(u^\beta)_{xx})_x = 0$ on $\partial\Omega$

$$G''(0) = -\frac{1}{8}\alpha^2\beta^2\int_\Omega u^{\alpha+2\beta-2} \\ \times \big(a_1\xi_1^6 + a_2\xi_1^4\xi_2 + a_3\xi_1^3\xi_3 + a_4\xi_1^2\xi_2^2 + a_5\xi_1\xi_2\xi_3 + a_6\xi_2^3 + a_7\xi_3^2\big)dx,$$

where $\xi_1 = u_x/u$, $\xi_2 = u_{xx}/u$, $\xi_3 = u_{xxx}/u$, and

$$\begin{aligned}
a_1 &= (\beta - 1)\big(2C_{\mathrm{RK}}\alpha^2\beta - 3C_{\mathrm{RK}}\alpha^2 + 2\alpha\beta^2 - 2(5C_{\mathrm{RK}} + 3)\alpha\beta + (15C_{\mathrm{RK}} + 4)\alpha \\
&\quad + 2\beta^3 - 14\beta^2 + 4(3C_{\mathrm{RK}} + 7)\beta - 2(9C_{\mathrm{RK}} + 8)\big), \\
a_2 &= (\beta - 1)\big(4C_{\mathrm{RK}}\alpha^2 + (8C_{\mathrm{RK}} + 7)\alpha\beta - (32C_{\mathrm{RK}} + 9)\alpha + 12\beta^2 - 2(8C_{\mathrm{RK}} + 25)\beta \\
&\quad + 6(8C_{\mathrm{RK}} + 7)\big), \\
a_3 &= C_{\mathrm{RK}}\alpha^2 + 2\alpha\beta - (5C_{\mathrm{RK}} + 2)\alpha + 4(C_{\mathrm{RK}} + 1)\beta^2 - 2(5C_{\mathrm{RK}} + 8)\beta + 12(C_{\mathrm{RK}} + 1), \\
a_4 &= 2(\beta - 1)\big(2(4C_{\mathrm{RK}} + 1)\alpha + 9\beta - (16C_{\mathrm{RK}} + 13)\big), \\
a_5 &= 2(2C_{\mathrm{RK}} + 1)\alpha + 4(2C_{\mathrm{RK}} + 3)\beta - 16(C_{\mathrm{RK}} + 1), \\
a_6 &= 2 - \alpha, \quad a_7 = 2(C_{\mathrm{RK}} + 1).
\end{aligned}$$

We employ three integration-by-parts formulas

$$\begin{aligned}
0 &= \int_\Omega \big(u^{\alpha+2\beta-5}u_{xx}^2u_x\big)_x dx = \int_\Omega u^{\alpha+2\beta-2}\big((\alpha + 2\beta - 5)\xi_1^2\xi_2^2 + 2\xi_1\xi_2\xi_3 + \xi_2^3\big)dx =: J_1, \\
0 &= \int_\Omega \big(u^{\alpha+2\beta-6}u_{xx}u_x^3\big)_x dx = \int_\Omega u^{\alpha+2\beta-2}\big((\alpha + 2\beta - 6)\xi_1^4\xi_2 + \xi_1^3\xi_3 + 3\xi_1^2\xi_2^2\big)dx =: J_2, \\
0 &= \int_\Omega \big(u^{\alpha+2\beta-7}u_x^5\big)_x dx = \int_\Omega u^{\alpha+2\beta-2}\big((\alpha + 2\beta - 7)\xi_1^6 + 5\xi_1^4\xi_2\big)dx =: J_3.
\end{aligned}$$

Then

$$G''(0) = G''(0) - \frac{1}{8}\alpha^2\beta^2(c_1J_1 + c_2J_2 + c_3J_3) = -\frac{1}{8}\alpha^2\beta^2\int_\Omega u^{\alpha+2\beta-2}P(\xi)dx,$$

where

$$P(\xi) = b_1\xi_1^6 + b_2\xi_1^4\xi_2 + b_3\xi_1^3\xi_3 + b_4\xi_1^2\xi_2^2 + b_5\xi_1\xi_2\xi_3 + b_6\xi_2^3 + b_7\xi_3^2$$

and the coefficients are given by

$$
\begin{aligned}
b_1 &= a_1 + (\alpha+2\beta-7)c_3, & b_2 &= a_2 + (\alpha+2\beta-6)c_2 + 5c_3,\\
b_3 &= a_3 + c_2, & b_4 &= a_4 + (\alpha+2\beta-5)c_1 + 3c_2,\\
b_5 &= a_5 + 2c_1, & b_6 &= a_6 + c_1,\\
b_7 &= a_7.
\end{aligned}
$$

Choosing $c_1 = -a_6$, we eliminate the cubic term ξ_2^3. Furthermore, setting, $x = \xi_2/\xi_1^2$ and $y = \xi_3/\xi_1^3$, we can write the polynomial P as a quadratic polynomial in (x,y)

$$Q(x,y) = \xi_1^6 P(\xi) = b_1 + b_2 x + b_3 y + b_4 x^2 + b_5 xy + b_7 y^2.$$

The following lemma is a consequence of the proof of Lemma 2.2 in [18].

LEMMA 4.1. *The polynomial $p(x,y) = A + Bx + Cy + Dx^2 + Exy + Fy^2$ with $F>0$ is nonnegative for all $(x,y)\in\mathbb{R}^2$ if and only if*

(i) $4DF - E^2 > 0$ *and* $A(4DF-E^2) - B^2F - C^2D + BCE \geq 0$, *or*

(ii) $4DF - E^2 = 0$ *and* $2BF - CE = 0$ *and* $4AF - C^2 \geq 0$.

Note that in case $4DF - E^2 = 0$ and $E \neq 0$, we may replace $2BF - CE = 0$ by the condition $2BEF = CE^2 = 4CDF$ or (since $F>0$) $BE = 2CD$.

The first inequality in case (i),

$$
\begin{aligned}
0 < 4b_4b_7 - b_5^2 = &-(C_{\mathrm{RK}}+1)(2C_{\mathrm{RK}}+1)\alpha^2 + (2C_{\mathrm{RK}}+2)(4C_{\mathrm{RK}}-3)\alpha\beta + (9C_{\mathrm{RK}}+9)\alpha\\
&- 2C_{\mathrm{RK}}(4C_{\mathrm{RK}}+3)\beta^2 + (8C_{\mathrm{RK}}+12)\beta + (3C_{\mathrm{RK}}+3)c_2 - (12C_{\mathrm{RK}}+14),
\end{aligned}
$$

is linear in c_2 and provides a lower bound for c_2

$$
\begin{aligned}
c_2 > \frac{1}{3(C_{\mathrm{RK}}+1)}\Big(&(C_{\mathrm{RK}}+1)(2C_{\mathrm{RK}}+1)\alpha^2 - (2C_{\mathrm{RK}}+2)(4C_{\mathrm{RK}}-3)\alpha\beta - (9C_{\mathrm{RK}}+9)\alpha\\
&+ 2C_{\mathrm{RK}}(4C_{\mathrm{RK}}+3)\beta^2 - (8C_{\mathrm{RK}}+12)\beta + (12C_{\mathrm{RK}}+14)\Big) =: c_2^*.
\end{aligned}
$$

The second inequality in case (i) becomes

$$0 \leq b_1(4b_4b_7 - b_5^2) - b_2^2 b_7 - b_3^2 b_4 + b_2b_3b_5 = -50(C_{\mathrm{RK}}+1)c_3^2 + p_1(\alpha,\beta,c_2)c_3 + p_2(\alpha,\beta,c_2),$$

where p_1 and p_2 are some polynomials in α, β, and c_2. This quadratic expression in c_3 is nonnegative if and only if its discriminant is nonnegative,

$$
\begin{aligned}
0 &\leq -200(C_{\mathrm{RK}}+1)p_2(\alpha,\beta,c_2) - p_1(\alpha,\beta,c_2)^2\\
&= -8\big(4b_4b_7 - b_5^2\big)\big(25c_2^2 + p_3(\alpha,\beta)c_2 + p_4(\alpha,\beta)\big),
\end{aligned}
$$

where $p_3(\alpha,\beta)$ and $p_4(\alpha,\beta)$ are some polynomials in α and β. The factor $4b_4b_7 - b_5^2$ is positive, so we have to ensure that $R_{\alpha,\beta}(c_2) = 25c_2^2 + p_3(\alpha,\beta)c_2 + p_4(\alpha,\beta) \leq 0$ for some $c_2 > c_2^*$. Therefore, we must ensure that the rightmost root of $R_{\alpha,\beta}(c_2)$ is greater than or equal to the lower bound for c_2, i.e., $-p_3(\alpha,\beta) + \sqrt{p_3^2(\alpha,\beta) - 100p_4(\alpha,\beta)} \geq 50c_2^*$. For $C_{\mathrm{RK}} = 1$, the values (α,β) for which there exists $c_2 > c_2^*$ such that $R_{\alpha,\beta}(c_2) \leq 0$ is depicted in Figure 4.2. In case (ii), we may immediately calculate c_2 and c_3, but this results in a region which is already contained in the first one. This shows that $G''(0) \leq 0$.

If $G''(0) = 0$, the polynomial Q vanishes. Thus, either $u_x/u = \xi_1 = 0$ or $P(\xi) = 0$ in Ω. The first case is impossible since u is not constant in Ω. As $b_7 = a_7 = 2(C_{\mathrm{RK}}+1) > 0$, the second case $P(\xi) = 0$ implies that $\xi_3 = 0$. Hence, u is a quadratic polynomial. In view of the boundary conditions, u must be constant, but this contradicts our assumption. Hence, $G''(0) < 0$. ∎

5. Linear diffusion system

We consider the following linear diffusion system:

$$\partial_t u_1 - \rho_1 \Delta u_1 = \mu(u_2 - u_1) \quad \partial_t u_2 - \rho_2 \Delta u_2 = \mu(u_1 - u_2), \tag{5.1}$$

with initial and homogeneous Neumann boundary conditions, $\rho_1, \rho_2, \mu > 0$, and the entropy

$$H[u] = \int_\Omega h(u)dx = \int_\Omega \sum_{i=1}^2 u_i(\log u_i - 1)dx, \tag{5.2}$$

where $u = (u_1, u_2)$. If the initial data is nonnegative, the maximum principle shows that the solutions to Equation (5.1) are nonnegative, too.

THEOREM 5.1. *Let (u^k) be a sequence of (smooth) nonnegative solutions to the Runge–Kutta scheme* (1.2) *for Equation* (5.1) *with $C_{\mathrm{RK}} = 1$ and $\rho := \rho_1 = \rho_2$. Let the entropy H be given by Equation* (5.2). *Let $k \in \mathbb{N}$ be fixed and let u^k be not the steady state of the scheme* (1.2). *Then there exists $\tau^k > 0$ such that, for all $0 < \tau < \tau^k$,*

$$H[u^k] + \tau \int_\Omega \left(\rho \sum_{i=1}^2 \frac{|\nabla u_i^k|^2}{u_i^k} + \mu(\log u_1^k - \log u_2^k)(u_1^k - u_2^k) \right) dx \le H[u^{k-1}].$$

Note that we need equal diffusivities $\rho_1 = \rho_2$ and higher-order schemes ($C_{\mathrm{RK}} = 1$). These conditions are in accordance with [20], where the continuous equation was studied. In order to highlight the step where these conditions are needed, the following proof is slightly more general than actually needed.

Proof. We fix $k \in \mathbb{N}$ and set $u := u^k$. Let $A[u] = (A_1[u], A_2[u]) = (\rho_1 \Delta u_1 + \mu(u_2 - u_1), \rho_2 \Delta u_2 + \mu(u_1 - u_2))$. Since A is linear, $DA[u](h) = A[h]$. Thus,

$$G''(0) = -\int_\Omega \big(C_{\mathrm{RK}} h'(u)^\top A[A[u]] + A[u]^\top h''(u) A[u] \big) dx = -G_1 - G_2.$$

In the following, we set $\partial_i h = \partial h / \partial u_i$ for $i = 1, 2$. We integrate by parts twice, using the boundary conditions $\nabla u_i \cdot \nu = 0$ and $\nabla A_i[u] \cdot \nu = 0$ on $\partial\Omega$, and collect the terms

$$\begin{aligned}
G_1 &= C_{\mathrm{RK}} \int_\Omega \Big(\partial_1 h(u) \big(\rho_1 \Delta A_1[u] + \mu(A_2[u] - A_1[u]) \big) \\
&\quad + \partial_2 h(u) \big(\rho_2 \Delta A_2[u] + \mu(A_1[u] - A_2[u]) \big) \Big) dx \\
&= C_{\mathrm{RK}} \int_\Omega \Big(\rho_1 \Delta \partial_1 h(u) A_1[u] + \rho_2 \Delta \partial_2 h(u) A_2[u] \\
&\quad + \mu(\partial_1 h(u) - \partial_2 h(u))(A_2[u] - A_1[u]) \Big) dx \\
&= C_{\mathrm{RK}} \int_\Omega \Big(\rho_1 \big(\partial_1^2 h(u) \Delta u_1 + \partial_1^3 h(u) |\nabla u_1|^2 \big) \big(\rho_1 \Delta u_1 + \mu(u_2 - u_1) \big) \\
&\quad + \rho_2 \big(\partial_2^2 h(u) \Delta u_2 + \partial_2^3 h(u) |\nabla u_2|^2 \big) \big(\rho_2 \Delta u_2 + \mu(u_1 - u_2) \big) \\
&\quad + \mu(\partial_2 h(u) - \partial_1 h(u)) \big(\rho_1 \Delta u_1 - \rho_2 \Delta u_2 + 2\mu(u_2 - u_1) \big) \Big) dx \\
&= C_{\mathrm{RK}} \int_\Omega \Big(\rho_1^2 \partial_1^2 h(u) (\Delta u_1)^2 + \rho_2^2 \partial_2^2 h(u) (\Delta u_2)^2 + \rho_1^2 \partial_1^3 h(u) \Delta u_1 |\nabla u_1|^2
\end{aligned}$$

$$
\begin{aligned}
&+\rho_2^2\partial_2^3 h(u)\Delta u_2|\nabla u_2|^2+\rho_1\mu\big(\partial_1^2 h(u)(u_2-u_1)+\partial_2 h(u)-\partial_1 h(u)\big)\Delta u_1\\
&+\rho_2\mu\big(\partial_2^2 h(u)(u_1-u_2)+\partial_1 h(u)-\partial_2 h(u)\big)\Delta u_2+\rho_1\mu\partial_1^3 h(u)(u_2-u_1)|\nabla u_1|^2\\
&+\rho_2\mu\partial_2^3 h(u)(u_1-u_2)|\nabla u_2|^2+2\mu^2(\partial_2 h(u)-\partial_1 h(u))(u_2-u_1)\Big)dx.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
G_2&=\int_\Omega\Big(\partial_1^2 h(u)\big(\rho_1\Delta u_1+\mu(u_2-u_1)\big)^2+\partial_2^2 h(u)\big(\rho_2\Delta u_2+\mu(u_1-u_2)\big)^2\Big)dx\\
&=\int_\Omega\Big(\rho_1^2\partial_1^2 h(u)(\Delta u_1)^2+\rho_2^2\partial_2^2 h(u)(\Delta u_2)^2+2\rho_1\mu\partial_1^2 h(u)(u_2-u_1)\Delta u_1\\
&\quad+2\rho_2\mu\partial_2^2 h(u)(u_1-u_2)\Delta u_2+\mu^2(\partial_1^2 h(u)+\partial_2^2 h(u))(u_1-u_2)^2\Big)dx.
\end{aligned}
$$

Adding G_1 and G_2, we arrive at

$$
\begin{aligned}
G''(0)&=-\sum_{i=1}^2\int_\Omega\Big(\rho_i^2(C_{\rm RK}+1)\partial_i^2 h(u)(\Delta u_i)^2+\rho_i^2 C_{\rm RK}\partial_i^3 h(u)\Delta u_i|\nabla u_i|^2\Big)dx\\
&\quad-\int_\Omega\Big(\rho_1\mu\big((C_{\rm RK}+2)\partial_1^2 h(u)(u_2-u_1)+C_{\rm RK}(\partial_2 h(u)-\partial_1 h(u))\big)\Delta u_1\\
&\quad+\rho_2\mu\big((C_{\rm RK}+2)\partial_2^2 h(u)(u_1-u_2)+C_{\rm RK}(\partial_1 h(u)-\partial_2 h(u))\big)\Delta u_2\\
&\quad+\rho_1\mu C_{\rm RK}\partial_1^3 h(u)(u_2-u_1)|\nabla u_1|^2+\rho_2\mu C_{\rm RK}\partial_2^3 h(u)(u_1-u_2)|\nabla u_2|^2\Big)dx\\
&\quad-\int_\Omega\mu^2\Big(2(\partial_1 h(u)-\partial_2 h(u))+(\partial_1^2 h(u)+\partial_2^2 h(u))(u_1-u_2)\Big)(u_1-u_2)dx\\
&=-I_2-I_1-I_0.
\end{aligned}
$$

The idea of [20] is to show that each integral I_i, involving only derivatives of order i, is nonnegative. In contrast to [20], we employ systematic integration by parts, which allows for a simpler and more general proof in our context. For the term I_2, we use the following integration-by-parts formula:

$$
0=\int_\Omega\operatorname{div}\big(u_i^{-2}|\nabla u_i|^3\big)dx=\int_\Omega\big(-2u_i^{-3}|\nabla u_i|^4+3u_i^{-2}\Delta u_i|\nabla u_i|^2\big)dx=:J_i.
$$

Then, for $\varepsilon>0$,

$$
\begin{aligned}
&I_2-c\sum_{i=1}^2\rho_i^2 J_i-\varepsilon\sum_{i=1}^2 u_i^{-3}|\nabla u_i|^4dx\\
&=\sum_{i=1}^2\rho_i^2\int_\Omega\Big((C_{\rm RK}+1)u_i^{-1}(\Delta u_i)^2-(3c+C_{\rm RK})u_i^{-2}\Delta u_i|\nabla u_i|^2+(2c-\varepsilon)u_i^{-3}|\nabla u_i|^4\Big)dx.
\end{aligned}
$$

The integrand defines a quadratic polynomial in the variables Δu_i and $|\nabla u_i|^2$ and is nonnegative if its discriminant satisfies $4(2c-\varepsilon)(C_{\rm RK}+1)-(3c+C_{\rm RK})^2\ge 0$. It turns out that this inequality holds true for $C_{\rm RK}\in\{0,1\}$ if we choose $c=2/3$ and $\varepsilon>0$ sufficiently small. When $C_{\rm RK}=2$, we can show only that $I_2\ge 0$, which is not sufficient to prove that $G''(0)<0$ (see below). We conclude that

$$
I_2\ge\varepsilon\sum_{i=1}^2\int_\Omega u_i^{-3}|\nabla u_i|^4dx. \tag{5.3}
$$

Integrating by parts in I_1 in order to obtain only first-order derivatives, we find after some rearrangements that

$$I_1 = \mu \int_\Omega \big(a_1 |\nabla \log u_1|^2 + a_2 \nabla \log u_1 \cdot \nabla \log u_2 + a_3 |\nabla \log u_2|^2 \big) dx,$$

where

$$a_1 = 2\rho_1 (C_{\rm RK} u_1 + u_2), \quad a_3 = 2\rho_2 (C_{\rm RK} u_2 + u_1),$$
$$a_2 = -(C_{\rm RK}(\rho_1+\rho_2) + 2\rho_2) u_1 - (C_{\rm RK}(\rho_1+\rho_2) + 2\rho_1) u_2.$$

The integrand is nonnegative if and only if $4a_1a_3 - a_2^2 \geq 0$ for all (u_1,u_2). We compute

$$C_{\rm RK} = 0: \quad 4a_1a_3 - a_2^2 = -4(\rho_1 u_2 - \rho_2 u_1)^2,$$
$$C_{\rm RK} = 1: \quad 4a_1a_3 - a_2^2 = (\rho_1 - \rho_2)\big(\rho_1(u_1^2 + 6u_1u_2 + 9u_2^2) - \rho_2(9u_1^2 + 6u_1u_2 + u_2^2)\big),$$
$$C_{\rm RK} = 2: \quad 4a_1a_3 - a_2^2 = -4\big(\rho_1(u_1 + 2u_2) - \rho_2(2u_1 + u_2)\big).$$

Thus, $4a_1a_3 - a_2^2 \geq 0$ is possible only if $\rho_1 = \rho_2$ and $C_{\rm RK} = 1$.

Finally, we see immediately that the remaining term

$$I_0 = \mu^2 \int_\Omega \left(2(\log u_1 - \log u_2)(u_1 - u_2) + \left(\frac{1}{u_1} + \frac{1}{u_2} \right)(u_1 - u_2)^2 \right) dx$$

is nonnegative. This shows that $G''(0) \leq 0$. If $G''(0) = 0$, we infer from (5.3) that $u_i = $ const., but this contradicts our hypothesis that u_i is not a steady state. $\square$

6. The Derrida–Lebowitz–Speer–Spohn equation

Consider the one-dimensional fourth-order equation

$$\partial_t u = -(u(\log u)_{xx})_{xx} \quad \text{in } \Omega,\ t > o, \quad u(0) = u^0 \tag{6.1}$$

with periodic boundary conditions. This equation appears as a scaling limit of the so-called (time-discrete) Toom model, which describes interface fluctuations in a two-dimensional spin system [9]. The variable u is the limit of a random variable related to the deviation of the spin interface from a straight line. The multi-dimensional version of Equation (6.1) models the electron density u in a quantum semiconductor, and the equation is the zero-temperature, zero-field approximation of the quantum drift-diffusion model [15]. For existence results for Equation (6.1), we refer to [17] and references therein.

To simplify our calculations, we analyze only the logarithmic entropy $H[u] = \int_\Omega u(\log u - 1)dx$. It is possible to verify condition (1.6) also for entropies of the form $\int_\Omega u^\alpha dx$, but it turns out that only sufficiently small $\alpha > 0$ are admissible (about $0 < \alpha < 0.15\ldots$) and the computations are very tedious. Therefore, we restrict ourselves to the case $\alpha = 0$.

THEOREM 6.1. *Let (u^k) be a sequence of (smooth) solutions to the Runge–Kutta scheme* (1.2) *with $C_{\rm RK} = 1$ for Equation* (6.1). *Let the entropy be given by $H[u] = \int_\Omega u(\log u - 1)dx$, let $k \in \mathbb{N}$ be fixed, and let u^k be not a steady state. Then there exists $\tau^k > 0$ such that, for all $0 < \tau < \tau^k$,*

$$H[u^k] + \tau q \int_\Omega u(\log u)_x^8 dx + \tau \int_\Omega u(\log u)_{xx}^2 dx \leq H[u^{k-1}], \quad q \approx 0.0045.$$

Proof. First, we observe that $G'(0) = -\int_\Omega (u(\log u)_{xx})_{xx} \log u dx = -\int_\Omega u(\log u)_{xx}^2 dx$. With $A[u] = (u(\log u)_{xx})_{xx}$ and $DA[u](h) = \big(h_{xx} - 2(\log u)_x h_x + (\log u)_x^2 h\big)_{xx}$, we can write $G''(0) = -I_0^k$ according to Equation (1.6) as

$$\begin{aligned} G''(0) &= -\int_\Omega \Big(\log u \big(A[u]_{xx} - 2(\log u)_x A[u]_x + (\log u)_x^2 A[u]\big)_{xx} + \frac{1}{u} A[u]^2 \Big) dx \\ &= -\int_\Omega \Big((\log u)_{xx} \big(A[u]_{xx} - 2(\log u)_x A[u]_x + (\log u)_x^2 A[u]\big) + \frac{1}{u} A[u]^2 \Big) dx \\ &= -\int_\Omega \Big(\big(v_{xxxx} + 2(v_x v_{xx})_x + v_x^2 v_{xx}\big) A[u] + \frac{1}{u} A[u]^2 \Big) dx, \end{aligned}$$

where we have integrated by parts several times and have set $v = \log u$. Then $A[u] = u(v_x^2 v_{xx} + 2v_x v_{xxx} + v_{xx}^2 + v_{xxxx})$ and, with the abbreviations $\xi_1 = v_x, \ldots, \xi_4 = v_{xxxx}$,

$$\begin{aligned} G''(0) = -\int_\Omega u \Big(& 2\xi_1^4\xi_2^2 + 8\xi_1^3\xi_2\xi_3 + 5\xi_1^2\xi_2^3 + 4\xi_1^2\xi_2\xi_4 + 8\xi_1^2\xi_3^2 + 10\xi_1\xi_2^2\xi_3 \\ & + 8\xi_1\xi_3\xi_4 + 3\xi_2^4 + 5\xi_2^2\xi_4 + 2\xi_4^2 \Big) dx. \end{aligned}$$

We employ the following integration-by-parts formulas:

$$\begin{aligned} 0 &= \int_\Omega (uv_x^7)_x dx = \int_\Omega u(\xi_1^8 + 7\xi_1^6\xi_2) dx =: J_1, \\ 0 &= \int_\Omega (uv_{xx}v_x^5)_x dx = \int_\Omega u(\xi_1^6\xi_2 + \xi_1^5\xi_3 + 5\xi_1^4\xi_2^2) dx =: J_2, \\ 0 &= \int_\Omega (uv_{xxx}v_x^4)_x dx = \int_\Omega u(\xi_1^5\xi_3 + \xi_1^4\xi_4 + 4\xi_1^3\xi_2\xi_3) dx =: J_3, \\ 0 &= \int_\Omega (uv_{xx}^2 v_x^3)_x dx = \int_\Omega u(\xi_1^4\xi_2^2 + 2\xi_1^3\xi_2\xi_3 + 3\xi_1^2\xi_2^3) dx =: J_4, \\ 0 &= \int_\Omega (uv_{xx}v_{xxx}v_x^2)_x dx = \int_\Omega u(\xi_1^3\xi_2\xi_3 + \xi_1^2\xi_2\xi_4 + \xi_1^2\xi_3^2 + 2\xi_1\xi_2^2\xi_3) dx =: J_5, \\ 0 &= \int_\Omega (uv_{xxx}^2 v_x)_x dx = \int_\Omega u(\xi_1^2\xi_3^2 + 2\xi_1\xi_3\xi_4 + \xi_2\xi_3^2) dx =: J_6, \\ 0 &= \int_\Omega (uv_{xx}^3 v_x)_x dx = \int_\Omega u(\xi_1^2\xi_2^3 + 3\xi_1\xi_2^2\xi_3 + \xi_2^4) dx =: J_7, \\ 0 &= \int_\Omega (uv_{xxx}v_{xx}^2)_x dx = \int_\Omega u(\xi_1\xi_2^2\xi_3 + 2\xi_2\xi_3^2 + \xi_2^2\xi_4) dx =: J_8. \end{aligned}$$

Then

$$\begin{aligned} G''(0) = G''(0) - 4\sum_{i=1}^{8} c_i J_i = -\int_\Omega u \Big(& a_1\xi_1^8 + a_2\xi_1^6\xi_2 + a_3\xi_1^5\xi_3 + a_4\xi_1^4\xi_2^2 + a_5\xi_1^4\xi_4 \\ & + a_6\xi_1^3\xi_2\xi_3 + a_7\xi_1^2\xi_2^3 + a_8\xi_1^2\xi_2\xi_4 + a_9\xi_1^2\xi_3^2 + a_{10}\xi_1\xi_2^2\xi_3 + a_{11}\xi_1\xi_3\xi_4 + a_{12}\xi_2^4 \\ & + a_{13}\xi_2^2\xi_4 + a_{14}\xi_2\xi_3^2 + a_{15}\xi_4^2 \Big) dx, \end{aligned}$$

where

$$\begin{aligned}
&a_1 = 4c_1, && a_2 = 28c_1 + 4c_2, && a_3 = 4c_2 + 4c_3,\\
&a_4 = 2 + 20c_2 + 4c_4, && a_5 = 4c_3, && a_6 = 8 + 16c_3 + 8c_4 + 4c_5,\\
&a_7 = 5 + 12c_4 + 4c_7, && a_8 = 4 + 4c_5, && a_9 = 8 + 4c_5 + 4c_6,\\
&a_{10} = 10 + 8c_5 + 12c_7 + 4c_8, && a_{11} = 8 + 8c_6, && a_{12} = 3 + 4c_7,\\
&a_{13} = 5 + 4c_8, && a_{14} = 4c_6 + 8c_8, && a_{15} = 2.
\end{aligned}$$

Next, we eliminate all terms involving ξ_4 by formulating the following square:

$$\begin{aligned}
G''(0) = -\int_\Omega u\bigg[& a_{15}\left(\xi_4 + \frac{a_5}{2a_{15}}\xi_1^4 + \frac{a_8}{2a_{15}}\xi_1^2\xi_2 + \frac{a_{11}}{2a_{15}}\xi_1\xi_3 + \frac{a_{13}}{2a_{15}}\xi_2^2\right)^2\\
&+ \left(a_1 - \frac{a_5^2}{4a_{15}}\right)\xi_1^8 + \left(a_2 - \frac{a_5a_8}{2a_{15}}\right)\xi_1^6\xi_2 + \left(a_3 - \frac{a_5a_{11}}{2a_{15}}\right)\xi_1^5\xi_3\\
&+ \left(a_4 - \frac{a_8^2}{4a_{15}} - \frac{a_5a_{13}}{2a_{15}}\right)\xi_1^4\xi_2^2 + \left(a_6 - \frac{a_8a_{11}}{2a_{15}}\right)\xi_1^3\xi_2\xi_3 + \left(a_7 - \frac{a_8a_{13}}{2a_{15}}\right)\xi_1^2\xi_2^3\\
&+ \left(a_9 - \frac{a_{11}^2}{4a_{15}}\right)\xi_1^2\xi_3^2 + \left(a_{10} - \frac{a_{11}a_{13}}{2a_{15}}\right)\xi_1\xi_2^2\xi_3 + \left(a_{12} - \frac{a_{13}^2}{4a_{15}}\right)\xi_2^4 + a_{14}\xi_2\xi_3^2\bigg]dx.
\end{aligned}$$

We eliminate all terms involving ξ_3 and set the corresponding coefficients to zero. From $a_{14} = 0$, we conclude that $c_6 = -2c_8$. Furthermore,

$$\begin{aligned}
a_9 - \frac{a_{11}^2}{4a_{15}} &= 0 &&\text{gives} && c_5 = 8c_8^2 - 6c_8,\\
a_{10} - \frac{a_{11}a_{13}}{2a_{15}} &= 0 &&\text{gives} && c_7 = -\frac{20}{3}c_8^2 + \frac{8}{3}c_8,\\
a_6 - \frac{a_8a_{11}}{2a_{15}} &= 0 &&\text{gives} && c_4 = -2c_3 - 16c_8^3 + 16c_8^2 - 5c_8,\\
a_3 - \frac{a_5a_{11}}{2a_{15}} &= 0 &&\text{gives} && c_2 = c_3 - 4c_3c_8.
\end{aligned}$$

By these choices, we obtain

$$b_{12} := a_{12} - \frac{a_{11}^2}{4a_{15}} = -\frac{86}{3}c_8^2 + \frac{17}{3}c_8 - \frac{1}{8}.$$

This quadratic polynomial in c_8 admits its maximal value at $c_8^* = 17/172$ with value $b_{12} = 20/129$. The integral can now be written as

$$G''(0) \le -\int_\Omega u\big(b_1\xi_1^8 + b_2\xi_1^6\xi_2 + b_4\xi_1^4\xi_2^2 + b_7\xi_1^2\xi_2^3 + b_{12}\xi_2^4\big)dx,$$

where

$$\begin{aligned}
b_1 &= a_1 - \frac{a_5^2}{4a_{15}} = 4c_1 - 2c_3^2,\\
b_2 &= a_2 - \frac{a_5a_8}{2a_{15}} = 28c_1 - 32c_3c_8^2 + 8c_3c_8,\\
b_4 &= a_4 - \frac{a_8^2}{4a_{15}} - \frac{a_5a_{13}}{2a_{15}} = 7c_3 - 84c_3c_8 - 128c_8^4 + 128c_8^3 - 40c_8^2 + 4c_8,
\end{aligned}$$

$$b_7 = a_7 - \frac{a_8 a_{13}}{2a_{15}} = -24c_3 - 244c_8^3 + \frac{448}{3}c_8^2 - \frac{70}{3}c_8.$$

If $b_4 = 2b_2b_{12}/b_7 + b_7^2/(4b_{12})$, we can write the integal as the sum of two squares, noting that b_{12} is positive,

$$G''(0) \le -\int_\Omega u\left(b_{12}\left(\xi_2^2 + \frac{b_7}{2b_{12}}\xi_1^2\xi_2 + \frac{b_2}{b_7}\xi_1^4\right)^2 + \left(b_1 - \frac{b_2^2 b_{12}}{b_7^2}\right)\xi_1^8\right)dx.$$

The expression $b_4b_7 - 2b_2b_{12} - b_7^3/(4b_{12}) = 0$ defines a polynomial in (c_1, c_3) which is linear in c_1. Solving it for c_1 gives

$$c_1 = \frac{449307}{175}c_3^3 + \frac{741681}{2150}c_3^2 + \frac{35780649411}{2393160700}c_3 + \frac{34135130165539}{163091166664200}.$$

It remains to show that $p(c_3) := b_1 - b_2^2 b_{12}/b_7^2$, which is a polynomial of fourth order in c_3, is positive. Choosing $c_3^* = -0.029$, we find that $p(c_3^*) \approx 0.0045 > 0$. This shows that

$$G''(0) \le -q(c_3^*)\int_\Omega u\xi_1^8 dx = -q(c_3^*)\int_\Omega u(\log u)_x^8 dx \le 0.$$

Finally, if $G''(0) = 0$, we infer that u is constant which is excluded. Therefore, $G''(0) < 0$, which ends the proof. ∎

7. Numerical examples

The aim of this section is to explore the numerical behavior of the second-order derivative of the function $G(\tau)$, defined in the introduction, for the porous-medium equation (4.1) in one space dimension. The equation is discretized by standard finite differences, and we employ periodic boundary conditions. The discrete solution u_i^k approximates the solution $u(x_i, t^k)$ to Equation (4.1) with $x_i = i\triangle x$, $t^k = k\tau$, and $\triangle x$, τ are the space and time step sizes, respectively. We choose the Barenblatt profile

$$u^0(x) = t_0^{-1/(\beta+1)} \max\left(0, C - \frac{\beta-1}{2\beta(\beta+1)}\frac{(x-1/2)^2}{t_0^{2/(\beta+1)}}\right)^{1/(\beta-1)}, \quad 0 \le x \le 1, \tag{7.1}$$

where

$$t_0 = 0.01, \quad C = \frac{\beta-1}{2\beta(\beta+1)}\frac{(x_R - 1/2)^2}{t_0^{2/(\beta+1)}}, \quad x_R = \frac{1}{4},$$

as the initial datum. Its support is contained in $[\frac{1}{2} - x_R, \frac{1}{2} + x_R]$; see Figure 7.1 (left). We choose the exponent $\beta = 2$. The semi-logarithmic plot of the discrete entropy $H_d[u^k] = \sum_{i=0}^N (u_i^k)^\alpha \triangle x$ with $\alpha = 5$ versus time is illustrated in Figure 7.1 (right), using the implicit Euler scheme with parameters $\tau = 10^{-4}$ and the number of grid points $N = 1/\triangle x = 64$. The decay is exponential for "large" times. The nonlinear discrete system is solved by Newton's method with the tolerance $\texttt{tol} = 10^{-15}$. We have highlighted four time steps t_i at which we will compute numerically the function $G(\tau)$ for the following Runge–Kutta schemes:

explicit Euler scheme: $u^k - u^{k-1} = -\tau A[u^{k-1}]$,

implicit Euler scheme: $u^k - u^{k-1} = -\tau A[u^k]$,

second-order trapezoidal rule: $u^k - u^{k-1} = -\frac{\tau}{2}(A[u^k] + A[u^{k-1}])$,

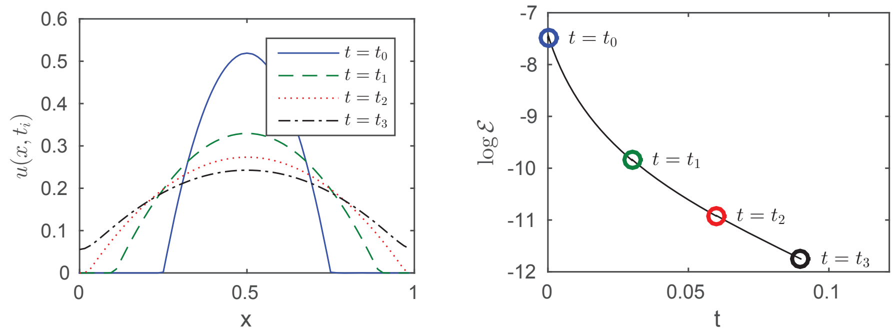

FIG. 7.1. *Left: Evolution of the intial datum (7.1) for $\beta=2$ at various time steps t_i, $i=0,1,2,3$; right: Semi-logarithmic plot of the discrete entropy $H_d[u^k]$ versus time.*

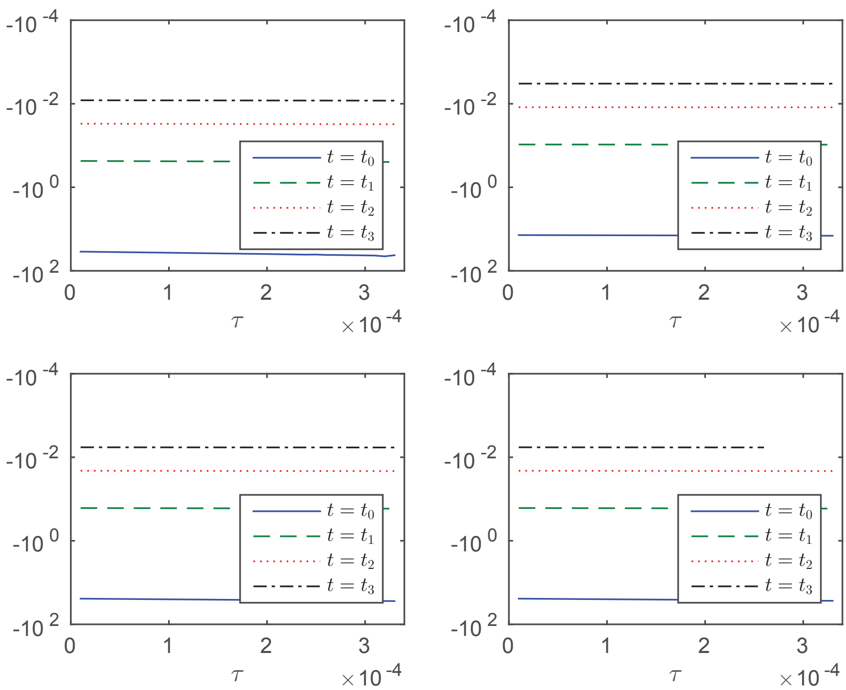

FIG. 7.2. *Numerical evaluation of the discrete version of $G''(\tau)$ for various Runge–Kutta schemes at the time steps t_i. Top left: explicit Euler scheme; top right: implicit Euler scheme; bottom left: implicit trapezoidal rule; bottom right: Simpson's rule.*

third-order Simpson rule: $$u^k - u^{k-1} = -\frac{\tau}{6}\big(A[u^k] + 4A[(u^k+u^{k-1})/2] + A[u^{k-1}]\big).$$

We set as before $u := u^k$, $v(\tau) := u^{k-1}$ and compute $G(\tau) = H_d[u] - H_d[v(\tau)]$ and the discrete second-order derivative $\partial^2 G$ of G (using central differences). The result is presented in Figure 7.2. As expected, the discrete derivative $\partial^2 G$ is negative on a (small) interval for all times t_i, $i=1,2,3$. We observe that $\partial^2 G$ is even slightly decreasing, but we expect that it becomes positive for sufficiently large values of τ. Clearly, the values

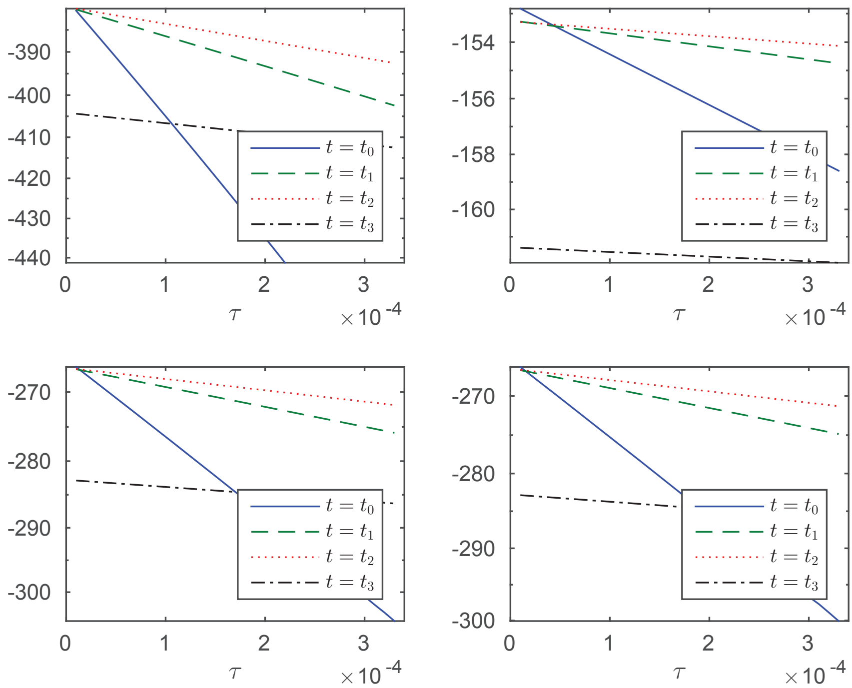

FIG. 7.3. *Numerical evaluation of the discrete version of $Q(\tau)$, defined in Equation* (7.2)*, for various Runge–Kutta schemes at the time steps t_i. Top left: explicit Euler scheme; top right: implicit Euler scheme; bottom left: implicit trapezoidal rule; bottom right: Simpson's rule.*

for $\partial^2 G$ tend to zero as we approach the steady state (see Remark 1.2). This experiment indicates that τ^k from Theorem 1.1 is bounded from below by $\tau^* = 3\cdot 10^{-4}$, for instance.

In order to understand the behavior of $G(\tau)$ in a better way, it is convenient to study the discrete version of the quotient

$$Q(\tau) := \frac{G''(\tau)}{\|u^{\alpha+2\beta-2}u_x^4\|_{L^1}}. \tag{7.2}$$

Indeed, the analysis in Section 4 gives an estimate of the type $G''(0) \le -C\int_\Omega u^{2\beta+\alpha-5}u_x^4 dx$ for some constant $C>0$. Thus, we expect that for sufficiently small $\tau > 0$, $Q(\tau)$ is bounded from above by some negative constant. This expectation is confirmed in Figure 7.3. In the examples, $Q(\tau)$ is a decreasing function of τ and $Q(0)$ is decreasing with increasing time.

All these results indicate that the threshold parameter τ^k in Theorem 1.1 can be chosen independently of the time step k.

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