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ASYMPTOTIC STABILITY AND SEMI-CLASSICAL LIMIT FOR BIPOLAR QUANTUM HYDRODYNAMIC MODEL*

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Abstract. In this paper, the initial-boundary value problem of a 1-D bipolar quantum semiconductor hydrodynamic model is investigated under a non-linear boundary condition which means the quantum effect vanishes on the boundary. First of all, the existence and uniqueness of the corresponding stationary solution are established. Then the exponentially asymptotic stability of the stationary solution and the semi-classical limits are further studied. The adopted approach is the elementary energy method but with some new developments.

Key words. Bipolar quantum hydrodynamic model, stationary solution, energy estimates, asymptotic stability, semi-classical limit.

AMS subject classifications. 35A01, 35B35, 35B40, 35B45, 35M33, 82D37.

1. Introduction

We consider the following bipolar isothermal quantum hydrodynamic (QHD for abbreviation) model for semiconductors

$$\begin{cases} n_{it}+j_{ix}=0, \\ j_{it}+\left(j_i^2 n_i^{-1}+K_i n_i\right)_x-\varepsilon^2 n_i\left[\left(\sqrt{n_i}\right)_{xx}/\sqrt{n_i}\right]_x=(-1)^{i-1}n_i\phi_x-j_i, \\ \phi_{xx}=n_1-n_2-D(x), \quad i=1,2, \quad \forall (t,x)\in(0,+\infty)\times\Omega, \end{cases} \tag{1.1}$$

where $\Omega:=(0,1)$ is a bounded interval occupied by the semiconductor device, and the quantum effects contribute to the dispersion terms based on the Bohm potential. The unknown functions $n_i(t,x)$ and $j_i(t,x)$ stand for the charge density, current distribution for electrons ($i=1$) and holes ($i=2$), respectively, and ϕ is the electrostatic potential. $P_i(n_i)=K_i n_i$ ($i=1,2$) are the pressure functions corresponding to n_i. The positive constants ε, K_1, and K_2 are the scaled Planck constant, temperature constant of electrons and temperature constant of holes, respectively. The given function $D(x)$ means the nonconstant doping profile, the density of impurities in semiconductor devices.

The system (1.1) is derived from the bipolar quantum Boltzmann equations through the momentum method developed in [3,4,18]. Mathematically, in the sense of quantum corrections, it takes the form of the compressible fluids coupled with self-consistent Poisson equation, which leads to a hyperbolic-elliptic system with higher order dispersion terms.

In the present paper, we are interested in the initial-boundary value problem (IBVP for abbreviation) of the system (1.1). The initial data is given by

$$(n_i,j_i)(0,x)=(n_{i0},j_{i0})(x), \quad x\in[0,1] \tag{1.2}$$

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and the physically motivated boundary data are prescribed as

$$n_i(t,0)=n_{il}>0, \qquad n_i(t,1)=n_{ir}>0, \tag{1.3a}$$
$$(\sqrt{n_i})_{xx}(t,0)=(\sqrt{n_i})_{xx}(t,1)=0, \tag{1.3b}$$
$$\phi(t,0)=0, \qquad \phi(t,1)=\phi_r>0, \tag{1.3c}$$

where n_{il}, n_{ir}, and ϕ_r are positive constants. The condition (1.3a) is the physical contact boundary, and the nonlinear boundary condition (1.3b) represents the vanishing quantum effect on the boundary, which, derived in [4,21], is also physically reasonable. The boundary (1.3c) stands for the applied bias voltage. For the compatibility, we further assume

$$n_{i0}(0)=n_{il}, \quad n_{i0}(1)=n_{ir}, \quad j_{i0x}(0)=j_{i0x}(1)=(\sqrt{n_{i0}})_{xx}(0)=(\sqrt{n_{i0}})_{xx}(1)=0. \tag{1.4}$$

Through out this paper, the flow is considered as subsonic, namely,

$$\begin{aligned}\text{velocity of the flow} := (u_1,u_2) &= \left(\frac{j_1}{n_1},\frac{j_2}{n_2}\right)\\ &< \left(\sqrt{p_1'(n_1)},\sqrt{p_2'(n_2)}\right)=\left(\sqrt{K_1},\sqrt{K_2}\right) =: \text{sound speed}.\end{aligned}$$

This is equivalent to

$$\inf_{x\in\Omega} S_i[n_i,j_i]>0, \text{ with } S_i[n_i,j_i]:=K_i-j_i^2n_i^{-2}, \tag{1.5a}$$
$$\inf_{x\in\Omega} n_i>0, \tag{1.5b}$$

where condition (1.5a) is called the subsonic condition, and condition (1.5b) refers to the positivity of the carrier density. Apparently, if we want to construct the solution in the physical region where the conditions (1.5) hold, then the initial data (1.2) must satisfy the same conditions

$$\inf_{x\in\Omega} S_i[n_{i0},j_{i0}]>0, \qquad \inf_{x\in\Omega} n_{i0}>0. \tag{1.6}$$

The QHD stationary problem of the IBVP (1.1)–(1.3) reads

$$\begin{cases}\tilde{j}_{ix}=0, & (1.7\text{a})\\ S_i[\tilde{n}_i,\tilde{j}_i]\tilde{n}_{ix}-\varepsilon^2\tilde{n}_i\left[\left(\sqrt{\tilde{n}_i}\right)_{xx}\Big/\sqrt{\tilde{n}_i}\right]_x=(-1)^{i-1}\tilde{n}_i\tilde{\phi}_x-\tilde{j}_i, & (1.7\text{b})\\ \tilde{\phi}_{xx}=\tilde{n}_1-\tilde{n}_2-D(x), \quad i=1,2, \quad \forall x\in\Omega, & (1.7\text{c})\end{cases}$$

with the boundary data

$$\tilde{n}_i(0)=n_{il}>0, \qquad \tilde{n}_i(1)=n_{ir}>0, \tag{1.8a}$$
$$\left(\sqrt{\tilde{n}_i}\right)_{xx}(0)=\left(\sqrt{\tilde{n}_i}\right)_{xx}(1)=0, \tag{1.8b}$$
$$\tilde{\phi}(0)=0, \qquad \tilde{\phi}(1)=\phi_r>0. \tag{1.8c}$$

Formally, consider the quantum effect vanishing in Equation (1.1), we could expect to reduce the IBVP (1.1)–(1.3) to the following IBVP of the bipolar HD model

$$\begin{cases}n_{it}^0+j_{ix}^0=0, & (1.9\text{a})\\ j_{it}^0+\left[\left(j_i^0\right)^2\left(n_i^0\right)^{-1}+K_in_i^0\right]_x=(-1)^{i-1}n_i^0\phi_x^0-j_i^0, & (1.9\text{b})\\ \phi_{xx}^0=n_1^0-n_2^0-D(x), \quad i=1,2, \quad \forall(t,x)\in(0,+\infty)\times\Omega, & (1.9\text{c})\end{cases}$$

with the initial and boundary data

$$(n_i^0, j_i^0)(0,x) = (n_{i0}, j_{i0})(x), \tag{1.10}$$

and

$$n_i^0(t,0) = n_{il} > 0, \qquad n_i^0(t,1) = n_{ir} > 0, \tag{1.11a}$$

$$\phi^0(t,0) = 0, \qquad \phi^0(t,1) = \phi_r > 0. \tag{1.11b}$$

The corresponding stationary problem of the bipolar HD model reads

$$\begin{cases} \tilde{j}_{ix}^0 = 0, & (1.12a) \\ S_i[\tilde{n}_i^0, \tilde{j}_i^0]\tilde{n}_{ix}^0 = (-1)^{i-1}\tilde{n}_i^0 \tilde{\phi}_x^0 - \tilde{j}_i^0, & (1.12b) \\ \tilde{\phi}_{xx}^0 = \tilde{n}_1^0 - \tilde{n}_2^0 - D(x), \quad i = 1,2, \quad \forall x \in \Omega, & (1.12c) \end{cases}$$

with the same boundary conditions (1.11).

Over the past two decades, the research on the hydrodynamic model for semiconductors makes an attractive progress. One of the hot spots is to study the device filled within quantum material, because the quantum function makes the semiconductor device working more effective but makes the device more expensive. For the unipolar hydrodynamic system of semiconductor devices with quantum effect (called, unipolar QHD), the relevant studies are prolific and intensive. Among them, Jüngel [11] first considered a unipolar stationary QHD model for potential flows in multi-dimensional bounded domain. The existence of solutions was proved under the assumption that the electric energy was small compared to the thermal energy, where Dirichlet boundary conditions were addressed. This result was then generalized and developed by Gyi and Jüngel [5], Hao, Jia, and Li [6], and Jüngel and Li [12]. Furthermore, the convergence of the original time-dependent QHD solutions to their corresponding stationary solutions (we also call them as *stationary waves*) were intensively studied by Jüngel and Li [13], and Huang, Li, and Matsumura [8], respectively. The semi-classic limits of QHD model to HD model was technically showed by Nishibata and Suzuki in [20], and the relaxation time limit of both stationary and transient unipolar QHD model over the whole space $\mathbb{R}^3$ was further derived by Jüngel, Li, and Matsumura [14].

Regarding the unipolar QHD model for irrotational fluid in spatial periodic domain, the global existence of the nD solutions and the exponential convergence to their equilibria were artfully proved by Li and Marcati in [15]. Remarkably, the weak solutions with large initial data for the quantum hydrodynamic system in multiple dimensions were further obtained by Antonelli and Marcati in [1,2]. For the bipolar case of QHD models, the studies are very limited and challenging due to much more complexity of the systems themselves. The first frame work on the existence and semi-classical limit of the isothermal solutions of the bipolar stationary QHD model in the multi-dimensional bounded domain was given by Unterreiter [22] by the variational approach.

Later, Liang and Zhang [16] generalized the result in [11] to the bipolar case and also obtained the relaxation time limit and dispersive limit on the bipolar and unipolar equations respectively. G. Zhang and K. Zhang [23] established the existence of a unique thermal equilibrium solution of the bipolar multidimensional QHD model over the whole space and obtained the relevant semi-classical limit and a combined Planck–Debye length limit. Furthermore, Li, G. Zhang, and K. Zhang [17] investigated the large-time behavior of solutions to the initial value problem of the QHD model in $\mathbb{R}^3$ and obtained the algebraic time-decay rate, and further showed in [24] the global in time semi-classical and relaxation limits of the same problem. For other interesting research, we refer to [3,4] and in particular, the survey paper [7].

The main goal of the present paper is to investigate the asymptotic behavior of the bipolar QHD model (1.1) with the physical initial-boundary conditions (1.2) and (1.3). We first prove the existence and uniqueness of both the time-dependent QHD solutions $(n_i,j_i,\phi)(t,x)$ of the system (1.1)–(1.3) and the stationary QHD solutions $(\tilde{n}_i,\tilde{j}_i,\tilde{\phi})(x)$ of (1.7)–(1.8). Then we prove the time-asymptotic convergence of the time-dependent QHD solutions $(n_i,j_i,\phi)(t,x)$ of the system (1.1)–(1.3) to their corresponding stationary QHD solutions $(\tilde{n}_i,\tilde{j}_i,\tilde{\phi})(x)$ of the system (1.7)–(1.8) as $t\to\infty$. Furthermore, we prove the semi-classical limit of the time-dependent QHD solutions $(n_i^{\varepsilon},j_i^{\varepsilon},\phi^{\varepsilon})(t,x)$ of the sysetm (1.1)–(1.3) to the corresponding time-dependent HD solutions $(n_i^0,j_i^0,\phi^0)(t,x)$ of the system (1.9)–(1.11) as $\varepsilon\to 0$, namely, the vanishing effect of quantum in the device. The idea of the proof is inspired by [20] for the unipolar QHD case, but with some new techniques and developments. Now we outline it as follows, particularly, the technical points.

First, the existence of the stationary waves for the QHD model (see Theorem 1.1 below) can be proved using a new approach developed in our previous work [9], which is based on the regular perturbation, linearization, and Banach fixed point argument. The quantum effect makes us to have to handle a strongly elliptic system with a singular parameter ε. The main difficulty is that we must refine the elliptic estimate which is essentially important for the long time behavior and the semi-classical limit analysis. However, we have to fix the boundary information of the nonconstant doping profile in order to obtain the desired sharp elliptic estimate. Secondly, a standard argument (the same to unipolar problem [13,20]) gives the local existence of our bipolar problem. In order to obtain the global existence and the exponential decay rate, we must establish the uniform a priori estimate. However, the quantum effect and the bipolar coupled effect make this task more difficult. Through a detailed analysis, we find that the quantum effect not only boosts the regularity of the estimate but also makes the bipolar effect terms more complex.

Fortunately, we also find that all the bipolar effect terms can be decomposed into a well-controlled structure: nonnegative terms plus cubic nonlinearity. These observations together with the Poincaré inequality then help us to get the desired decay estimate. Finally, comparing with the unipolar problem [20], we choose some new multipliers and weights to overcome the influence of the bipolar effect during establishing the semi-classical limit of the stationary solution.

Before stating our main results, we firstly list the notations and settings used in this paper,

- $\mathcal{B}^l(\overline{\Omega})$: The space of l-times bounded differentiable functions on $\overline{\Omega}$ with the norm $|\cdot|_l := \sum_{m=0}^{l}\sup_{x\in\overline{\Omega}}|\partial_x^m\cdot|$ (integer $l\geq 0$). The stationary solution will be found in this class of function spaces.
- $H^l(\Omega)$: The usual L^2-Sobolev space over Ω of integer order l with the norm $\|\cdot\|_l$ ($l\geq 0$). In particular, $\|\cdot\|_0=\|\cdot\|$.
- $C^l([0,T];H^m(\Omega))$: The space of l-times continuously differentiable functions on time interval $[0,T]$ with values in $H^m(\Omega)$. The non-stationary solution will be constructed in this class of function spaces.

The solution spaces used in QHD problem:

$$\bar{\mathfrak{X}}_m^l([0,T]) := \bigcap_{k=0}^{[m/2]} C^k([0,T];H^{l+m-2k}(\Omega)), \qquad \bar{\mathfrak{X}}_m := \bar{\mathfrak{X}}_m^0,$$

$$\mathfrak{Y}([0,T]) := C^2([0,T]; H^2(\Omega)),$$

and in HD problem:

$$\mathfrak{X}_m^l([0,T]) := \bigcap_{k=0}^{m} C^k([0,T]; H^{l+m-k}(\Omega)), \qquad \mathfrak{X}_m := \mathfrak{X}_m^0.$$

- Assume the doping function to be $D \in \mathcal{H} := \left\{ D \in H^2(\Omega) \;\middle|\; D(0) = \bar{d},\ D(1) = \tilde{d} \right\}$, where $\bar{d} := n_{1l} - n_{2l}$ and $\tilde{d} := n_{1r} - n_{2r}$.
- The strength parameter of the given data is defined as

$$\delta := \sum_{i=1}^{2} |n_{il} - n_{ir}| + |\phi_r| + \|D - \bar{d}\|_2,$$

and the assumption $\delta \ll 1$ will play an important role in what follows.
- C denotes the generic positive constant and N, γ_k, C_k, C_{kl}, and C_{kr} $(k = 1,2,\ldots)$ stand for the specific positive constants. It is worth mentioning that all these constants only depend on the state constants n_{1l}, n_{2l}, K_1, and K_2 throughout the paper. This fact allows us to establish the semi-classical limits.

In addition, we also need to introduce the existence, uniqueness and long-time stability of the stationary solution to the corresponding bipolar HD model, which were proved in the previous work [9, 19]. Seeing that they will be used in the present paper, it is better for us to review them briefly.

We first state the known results on the existence of both the time-dependent HD solutions (1.9)–(1.11) and the stationary HD solutions (1.12) and (1.11).

LEMMA 1.1 (see [9, 19]). *Assume that $D \in \mathcal{H}$, for arbitrary constants $n_{il}, K_i > 0$, there exist two constants $\delta_1, C > 0$ such that if $\delta \le \delta_1$, then there is a unique solution $\left(\tilde{n}_1^0, \tilde{j}_1^0, \tilde{n}_2^0, \tilde{j}_2^0, \tilde{\phi}^0\right) \in \left[\mathcal{B}^2(\overline{\Omega})\right]^5$ of the stationary problem* (1.12) *satisfying the condition* (1.5) *and the estimates*

$$0 < \frac{1}{4} n_{il} \le \tilde{n}_i^0(x) \le 4 n_{il}, \quad \forall x \in \overline{\Omega}, \quad i = 1,2, \tag{1.13a}$$

$$\sum_{i=1}^{2} \left(|\tilde{n}_i^0 - n_{il}|_2 + |\tilde{j}_i^0| \right) + |\tilde{\phi}^0|_2 \le C\delta. \tag{1.13b}$$

LEMMA 1.2 (see [9, 19]). *Suppose that $D \in \mathcal{H}$, the initial data $n_{i0}, j_{i0} \in H^2(\Omega)$ satisfy the condition* (1.6) *and are compatible with the boundary data* (1.11). *For arbitrary constants $n_{il}, K_i > 0$, there exist three constants $\delta_2, \gamma_1, C > 0$ such that if $\delta + \sum_{i=1}^{2} \|(n_{i0} - \tilde{n}_i^0, j_{i0} - \tilde{j}_i^0)\|_2 \le \delta_2$, then there is a unique global solution $\left(n_1^0, j_1^0, n_2^0, j_2^0, \phi^0\right) \in \left[(\mathfrak{X}_2)^4 \times \mathfrak{Y}\right]([0,+\infty))$ of the IBVP* (1.9)–(1.11) *satisfying the condition* (1.5), *the additional regularity $\phi^0 - \tilde{\phi}^0 \in \mathfrak{X}_2^2([0,+\infty))$, and the decay estimate*

$$\sum_{i=1}^{2} \|(n_i^0 - \tilde{n}_i^0, j_i^0 - \tilde{j}_i^0)(t)\|_2 + \|(\phi^0 - \tilde{\phi}^0)(t)\|_4$$
$$\le C \sum_{i=1}^{2} \|(n_{i0} - \tilde{n}_i^0, j_{i0} - \tilde{j}_i^0)\|_2 \, e^{-\gamma_1 t}, \quad \forall t \in [0,+\infty). \tag{1.14}$$

Now we can state the main results in the present paper as follows.

Theorem 1.1 (Existence and uniqueness of stationary wave). *Assume that $D \in \mathcal{H}$, for arbitrary constants $n_{il}, K_i > 0$, there exist two constants $\delta_3, C > 0$ such that if $\delta \le \delta_3$, then for all $0 < \varepsilon \le 1$ there is a unique solution $\big(\tilde{n}_1^\varepsilon, \tilde{j}_1^\varepsilon, \tilde{n}_2^\varepsilon, \tilde{j}_2^\varepsilon, \tilde{\phi}^\varepsilon\big) \in \big[(\mathcal{B}^4)^4 \times \mathcal{B}^3\big](\overline{\Omega})$ of the BVP* (1.7)–(1.8) *satisfying the condition* (1.5) *and the estimates*

$$0 < \frac{1}{4} n_{il} \le \tilde{n}_i^\varepsilon(x) \le 4 n_{il}, \quad \forall x \in \overline{\Omega}, \quad i = 1,2, \tag{1.15a}$$

$$\sum_{i=1}^{2} \Big(\varepsilon^2 \|\partial_x^4 \tilde{n}_i^\varepsilon\| + \varepsilon \|\partial_x^3 \tilde{n}_i^\varepsilon\| + \|\tilde{n}_i^\varepsilon - n_{il}\|_2 + |\tilde{j}_i^\varepsilon| \Big) + \|\tilde{\phi}^\varepsilon\|_3 \le C\delta. \tag{1.15b}$$

Theorem 1.2 (Stability of stationary wave). *Suppose that $D \in \mathcal{H}$, the initial data $(n_{i0}, j_{i0}) \in (H^4 \times H^3)(\Omega)$ satisfy the condition* (1.6) *and the compatible condition* (1.4) *with boundary data* (1.3). *For arbitrary constants $n_{il}, K_i > 0$, there exist three constants $\delta_4, \gamma_2, C > 0$ such that if $\delta + \sum_{i=1}^2 \big(\|(n_{i0} - \tilde{n}_i^\varepsilon, j_{i0} - \tilde{j}_i^\varepsilon)\|_2 + \|(\varepsilon \partial_x^3 (n_{i0} - \tilde{n}_i^\varepsilon), \varepsilon \partial_x^3 (j_{i0} - \tilde{j}_i^\varepsilon), \varepsilon^2 \partial_x^4 (n_{i0} - \tilde{n}_i^\varepsilon))\| \big) \le \delta_4$, then for all $0 < \varepsilon \le 1$ there is a unique global solution $\big(n_1^\varepsilon, j_1^\varepsilon, n_2^\varepsilon, j_2^\varepsilon, \phi^\varepsilon\big) \in \big[(\bar{\mathfrak{X}}_4 \times \bar{\mathfrak{X}}_3)^2 \times \mathfrak{Y}\big]([0,+\infty))$ of the IBVP* (1.1)–(1.3) *satisfying the condition* (1.5), *the additional regularity $\phi^\varepsilon - \tilde{\phi}^\varepsilon \in \bar{\mathfrak{X}}_4^2([0,+\infty))$ and the decay estimate*

$$\begin{aligned}
&\sum_{i=1}^{2} \Big(\|(n_i^\varepsilon - \tilde{n}_i^\varepsilon, j_i^\varepsilon - \tilde{j}_i^\varepsilon)(t)\|_2 \\
&\qquad + \|(\varepsilon \partial_x^3 (n_i^\varepsilon - \tilde{n}_i^\varepsilon), \varepsilon \partial_x^3 (j_i^\varepsilon - \tilde{j}_i^\varepsilon), \varepsilon^2 \partial_x^4 (n_i^\varepsilon - \tilde{n}_i^\varepsilon))(t)\| \Big) + \|(\phi^\varepsilon - \tilde{\phi}^\varepsilon)(t)\|_4 \\
&\le C e^{-\gamma_2 t} \sum_{i=1}^{2} \Big(\|(n_{i0} - \tilde{n}_i^\varepsilon, j_{i0} - \tilde{j}_i^\varepsilon)\|_2 \\
&\qquad + \|(\varepsilon \partial_x^3 (n_{i0} - \tilde{n}_i^\varepsilon), \varepsilon \partial_x^3 (j_{i0} - \tilde{j}_i^\varepsilon), \varepsilon^2 \partial_x^4 (n_{i0} - \tilde{n}_i^\varepsilon))\| \Big),
\end{aligned} \tag{1.16}$$

for $t \in [0, +\infty)$.

Theorem 1.3 (Semi-classical limit of QHD stationary wave to HD stationary wave). *Under the conditions of Lemma 1.1 and Theorem 1.1 , let $(\tilde{n}_i^\varepsilon, \tilde{j}_i^\varepsilon)(x)$ be the stationary solution of the QHD model* (1.7) *and* (1.8), *and $(\tilde{n}_i^0, \tilde{j}_i^0)(x)$ be the stationary solution for the HD model* (1.12) *and* (1.11), *then for arbitrary constants $n_{il}, K_i > 0$, there exist two constants $\delta_5, C > 0$ such that if $\delta \le \delta_5$, such that, for all $0 < \varepsilon \le 1$, the following convergence estimates hold:*

$$\sum_{i=1}^{2} \Big(\|\tilde{n}_i^\varepsilon - \tilde{n}_i^0\|_1 + |\tilde{j}_i^\varepsilon - \tilde{j}_i^0| \Big) + \|\tilde{\phi}^\varepsilon - \tilde{\phi}^0\|_3 \le C\varepsilon, \tag{1.17a}$$

and there exists a subsequence $\{\varepsilon_k > 0\}$ of $\varepsilon > 0$ such that the following semi-classic limits hold:

$$\sum_{i=1}^{2} \Big(\|\partial_x^2 (\tilde{n}_i^{\varepsilon_k} - \tilde{n}_i^0)\| + \varepsilon_k \|\partial_x^3 \tilde{n}_i^{\varepsilon_k}\| + \varepsilon_k^2 \|\partial_x^4 \tilde{n}_i^{\varepsilon_k}\| \Big) + \|\partial_x^4 (\tilde{\phi}^{\varepsilon_k} - \tilde{\phi}^0)\| \to 0, \ \text{as } \varepsilon_k \to 0^+. \tag{1.17b}$$

THEOREM 1.4 (Semi-classical limit of QHD to HD for time-dependent solutions). *Under the conditions of Lemma 1.2 and Theorem 1.2, let $(n_i^\varepsilon, j_i^\varepsilon)(t,x)$ be the time-dependent solution of the QHD model (1.1) and (1.3), and $(n_i^0, j_i^0)(t,x)$ be the time-dependent solution for the HD model (1.9) and (1.11), then for arbitrary constants $n_{il}, K_i > 0$, there exist four constants $\delta_6, \gamma_3, \gamma_4, C > 0$ such that if*

$$\begin{aligned}
&\varepsilon_k + \delta + \sum_{i=1}^{2} \Big(\|(n_{i0} - \tilde{n}_i^{\varepsilon_k}, j_{i0} - \tilde{j}_i^{\varepsilon_k})\|_2 \\
&\quad + \|\big(\varepsilon_k \partial_x^3 (n_{i0} - \tilde{n}_i^{\varepsilon_k}), \varepsilon_k \partial_x^3 (j_{i0} - \tilde{j}_i^{\varepsilon_k}), \varepsilon_k^2 \partial_x^4 (n_{i0} - \tilde{n}_i^{\varepsilon_k})\big)\| \Big) \\
&\le \delta_6,
\end{aligned} \tag{1.18}$$

then the following convergence estimates hold

$$\sum_{i=1}^{2} \|(n_i^{\varepsilon_k} - n_i^0, j_i^{\varepsilon_k} - j_i^0)(t)\|_1 + \|(\phi^{\varepsilon_k} - \phi^0)(t)\|_3 \le C e^{\gamma_3 t} \varepsilon_k^{1/2}, \quad \forall t \in [0, +\infty), \tag{1.19}$$

and

$$\sup_{t \in [0,+\infty)} \Big(\sum_{i=1}^{2} \|(n_i^{\varepsilon_k} - n_i^0, j_i^{\varepsilon_k} - j_i^0)(t)\|_1 + \|(\phi^{\varepsilon_k} - \phi^0)(t)\|_3 \Big) \le C \varepsilon_k^{\gamma_4}, \tag{1.20}$$

where $\varepsilon_k \to 0^+$ is the subsequence given in Theorem 1.3 and $0 < \gamma_4 \le 1/4$.

The paper is organized as follows. In Section 2, we prove the existence and uniqueness of a stationary solution. Subsection 3.1 devotes to the reformulation of the original evolution problem and the local existence theorem. In Subsections 3.2–3.4, we show the asymptotic stability of the stationary solution. In Section 4, we establish the semi-classical limits for the stationary solution and the global solution in Subsection 4.1 and Subsection 4.2, respectively.

2. Existence and uniqueness of a stationary solution

In this section, we prove the existence and uniqueness of a stationary solution of the BVP (1.7)–(1.8) (i.e. Theorem 1.1). For simplicity, the solution $\big(\tilde{n}_1^\varepsilon, \tilde{j}_1^\varepsilon, \tilde{n}_2^\varepsilon, \tilde{j}_2^\varepsilon, \tilde{\phi}^\varepsilon\big)$ will be denoted by the notations $\big(\tilde{n}_1, \tilde{j}_1, \tilde{n}_2, \tilde{j}_2, \tilde{\phi}\big)$ in what follows.

Based on the idea of rationalization, it is convenient to introduce the new variables

$$\tilde{w}_i := \sqrt{\tilde{n}_i}, \quad i = 1, 2. \tag{2.1}$$

Then the conditions (1.5) transform to

$$\inf_{x \in \Omega} S_i[\tilde{w}_i^2, \tilde{j}_i] > 0, \qquad \inf_{x \in \Omega} \tilde{w}_i > 0, \tag{2.2}$$

and the above BVP (1.7)–(1.8) can be written as

$$\begin{cases}
\tilde{j}_{ix} = 0, & (2.3a) \\
2 S_i[\tilde{w}_i^2, \tilde{j}_i] \tilde{w}_i \tilde{w}_{ix} - \varepsilon^2 \tilde{w}_i^2 (\tilde{w}_{ixx}/\tilde{w}_i)_x = (-1)^{i-1} \tilde{w}_i^2 \tilde{\phi}_x - \tilde{j}_i, & (2.3b) \\
\tilde{\phi}_{xx} = \tilde{w}_1^2 - \tilde{w}_2^2 - D(x), \quad i = 1, 2, \quad \forall x \in \Omega, & (2.3c)
\end{cases}$$

with the boundary data

$$\tilde{w}_i(0) = w_{il} := \sqrt{n_{il}} > 0, \qquad \tilde{w}_i(1) = w_{ir} := \sqrt{n_{ir}} > 0, \tag{2.4a}$$

$$\tilde{w}_{ixx}(0) = \tilde{w}_{ixx}(1) = 0, \tag{2.4b}$$

$$\tilde{\phi}(0) = 0, \qquad \tilde{\phi}(1) = \phi_r > 0. \tag{2.4c}$$

In order to treat the third-order dispersion terms, we perform the following procedure

$$\int_0^x (2.3\text{b})/\tilde{w}_i^2 \, dy, \tag{2.5}$$

and by using the boundary conditions (2.4b), we have

$$\varepsilon^2 \frac{\tilde{w}_{ixx}}{\tilde{w}_i} = F_i(\tilde{w}_i^2, \tilde{j}_i) - F_i(n_{il}, \tilde{j}_i) + (-1)^i \tilde{\phi} + \tilde{j}_i \int_0^x \tilde{w}_i^{-2}(y) \, dy, \tag{2.6a}$$

where

$$F_i(a,b) := \frac{b^2}{2a^2} + K_i \ln a. \tag{2.6b}$$

Furthermore, let $x = 1$ in Equation (2.6a) and by using the boundary conditions (2.4) again, we obtain an important relationship in semiconductor equations, namely, the current-voltage characterization,

$$(-1)^{i-1}\phi_r = F_i(n_{ir}, \tilde{j}_i) - F_i(n_{il}, \tilde{j}_i) + \tilde{j}_i \int_0^1 \tilde{w}_i^{-2} \, dx, \quad i = 1,2. \tag{2.7}$$

Based on the conditions (2.2) and the assumption $\delta \ll 1$, we can uniquely and explicitly solve the stationary current densities from Equation (2.7) as follows

$$\tilde{j}_i = J_i[\tilde{w}_i^2] := 2B_{ib} \left(\int_0^1 \tilde{w}_i^{-2} \, dx + \sqrt{ \left(\int_0^1 \tilde{w}_i^{-2} \, dx \right)^2 + 2B_{ib} \left(n_{ir}^{-2} - n_{il}^{-2} \right) } \right)^{-1}, \tag{2.8a}$$

where

$$B_{ib} := (-1)^{i-1}\phi_r - K_i(\ln n_{ir} - \ln n_{il}). \tag{2.8b}$$

It is easy to see that $\tilde{j}_i \gtreqless 0$ if and only if $B_{ib} \gtreqless 0$. Multiply Equation (2.6a) by $\tilde{w}_i$, we can obtain a new boundary value problem

$$\begin{cases} \varepsilon^2 \tilde{w}_{ixx} = \tilde{w}_i \Big(F_i(\tilde{w}_i^2, \tilde{j}_i) - F_i(n_{il}, \tilde{j}_i) + (-1)^i \tilde{\phi} + \tilde{j}_i \int_0^x \tilde{w}_i^{-2}(y) \, dy \Big), & (2.9\text{a}) \\ \tilde{\phi}_{xx} = \tilde{w}_1^2 - \tilde{w}_2^2 - D(x), \quad i = 1,2, \quad \forall x \in \Omega, & (2.9\text{b}) \end{cases}$$

with the boundary data

$$\tilde{w}_i(0) = w_{il} > 0, \qquad \tilde{w}_i(1) = w_{ir} > 0, \tag{2.9c}$$

$$\tilde{\phi}(0) = 0, \qquad \tilde{\phi}(1) = \phi_r > 0, \tag{2.9d}$$

where $\tilde{j}_i = J_i[\tilde{w}_i^2]$ is given by Equation (2.8a). In addition, for the classical solution satisfying the conditions (2.2), we have the following equivalent relationships

$$\text{BVP (1.7)–(1.8)} \Longleftrightarrow \text{BVP (2.3)–(2.4)} \Longleftrightarrow \text{BVP (2.9) with (2.8)},$$

thanks to the current-voltage characterization (2.7).

In the rest of this section, we focus on the BVP (2.9) and summarize the existence theorem in the following lemma.

LEMMA 2.1. *Assume that $D\in\mathcal{H}$, for arbitrary constants $n_{il},K_i>0$, there exist two constants $\delta_3,C>0$ such that if $\delta\le\delta_3$, then for all $\varepsilon>0$ there is a unique solution $\big(\tilde{w}_1,\tilde{j}_1,\tilde{w}_2,\tilde{j}_2,\tilde{\phi}\big)\in\big[(\mathcal{B}^4)^4\times\mathcal{B}^3\big](\overline{\Omega})$ of the BVP (2.9) satisfying the condition (2.2) and the estimates*

$$0<\frac{1}{2}w_{il}\le\tilde{w}_i(x)\le 2w_{il},\quad \forall x\in\overline{\Omega},\quad i=1,2, \tag{2.10a}$$

$$\sum_{i=1}^{2}\Big(\varepsilon^2\|\partial_x^4\tilde{w}_i\|+\varepsilon\|\partial_x^3\tilde{w}_i\|+\|\tilde{w}_i-w_{il}\|_2+|\tilde{j}_i|\Big)+\|\tilde{\phi}\|_3\le C\delta. \tag{2.10b}$$

Proof.

Step I. *Reformulation of the problem.* Our proof starts with the observation that there is a unique constant subsonic solution $\big(w_{1l},0,w_{2l},0,0\big)$ of the BVP (2.9) provided $\delta=0$. Therefore, we can naturally regard the subsonic solution of the BVP (2.9) with $\delta>0$ as a perturbation around the above constant state $\big(w_{1l},0,w_{2l},0,0\big)$, because we have assumed that the doping profile D is a perturbation of the constant $\bar{d}$.

In order to implement the above idea of the regular perturbation, we introduce the perturbation variables

$$\begin{aligned} \tilde{w}_i^{\delta}(x):=\tilde{w}_i(x)-w_{il},\qquad \tilde{j}_i^{\delta}:=\tilde{j}_i-0=\tilde{j}_i,\\ \tilde{\phi}^{\delta}(x):=\tilde{\phi}(x)-0=\tilde{\phi}(x). \end{aligned} \tag{2.11}$$

Substituting Equation (2.11) into the BVP (2.9) and linearizing the resultant equations around the above constant state. In matrix notation, we obtain the equivalent boundary value problem

$$\begin{cases} -(A_\varepsilon U_x)_x+BU=F(U)+G(x), & x\in\Omega, \qquad (2.12a)\\ U|_{\partial\Omega}=H(x), & \qquad (2.12b)\end{cases}$$

satisfied by the perturbations, where the vector-valued unknown is

$$U:=\Big(\tilde{w}_1^{\delta},\tilde{w}_2^{\delta},\tilde{\phi}\Big)^T, \tag{2.13}$$

the coefficient matrices are

$$A_\varepsilon:=\begin{pmatrix}\varepsilon^2 & 0 & 0\\ 0 & \varepsilon^2 & 0\\ 0 & 0 & 1/2\end{pmatrix},\quad B:=\begin{pmatrix}2K_1 & 0 & -w_{1l}\\ 0 & 2K_2 & w_{2l}\\ w_{1l} & -w_{2l} & 0\end{pmatrix}, \tag{2.14}$$

the quadratic nonlinearity and the well-estimated nonlocal terms are

$$F(U):=\Big(f_1(U),f_2(U),f_3(U)\Big)^T, \tag{2.15}$$

$$\begin{aligned} f_i(U):=&-\tilde{w}_i\left(\tilde{j}_i^2/(2\tilde{w}_i^4)-\tilde{j}_i^2/(2n_{il}^2)+\tilde{j}_i\int_0^x\tilde{w}_i^{-2}(y)\,dy\right)\\ &-2K_i\tilde{w}_i\Big(\ln\tilde{w}_i-\ln w_{il}-\tilde{w}_i^{\delta}/w_{il}\Big)-2K_i(\tilde{w}_i^{\delta})^2/w_{il}+(-1)^{i-1}\tilde{w}_i^{\delta}\tilde{\phi}, \end{aligned} \tag{2.16a}$$

$$f_3(U) := -(\tilde{w}_1^\delta)^2/2 + (\tilde{w}_2^\delta)^2/2, \tag{2.16b}$$

where $\tilde{j}_i = J_i[(\tilde{w}_i^\delta + w_{il})^2]$. The inhomogeneous term

$$G(x) := \Big(0, 0, (D(x) - \bar{d})/2\Big)^T, \tag{2.17}$$

and the $H^4(\Omega)$-extension of the boundary data

$$H(x) := \Big(h_1(x), h_2(x), h_3(x)\Big)^T, \tag{2.18a}$$

$$h_i(x) := (w_{ir} - w_{il})x, \quad h_3(x) := \phi_r x. \tag{2.18b}$$

For all $\varepsilon > 0$, we note that

$$\lambda_\varepsilon := \min\big\{\varepsilon^2, 1/2\big\} > 0. \tag{2.19}$$

It means that the linear differential operator $L_\varepsilon U := -(A_\varepsilon U_x)_x + BU$ is a parameter-dependent strongly elliptic operator of second order.

Step II. *The sharp elliptic estimate.* Because we discuss both the existence of the stationary solution and its semi-classical limit, we have to sharpen the elliptic estimate of the strong solution to the linear Dirichlet problem

$$\begin{cases} -(A_\varepsilon U_x)_x + BU = M(x), & x \in \Omega, \quad (2.20a) \\ U|_{\partial\Omega} = H(x). & \quad (2.20b) \end{cases}$$

The proof strongly depends on the assumption that the inhomogeneous term must satisfy

$$M \in \mathcal{M} := \Big\{ M \in H^2(\Omega) \;\Big|\; M(0) = BH(0) = 0, \quad M(1) = BH(1) \Big\}. \tag{2.21}$$

It follows from Equation (2.21) immediately that the strong solution (if it exists) satisfies

$$U \in H^4(\Omega), \quad U_{xx}(0) = U_{xx}(1) = 0. \tag{2.22}$$

In fact, according to the Fredholm alternative (uniqueness implies existence) and the L^2-regularity theory of the Dirichlet problem of the linear strongly elliptic system, we can conclude that the above strong solution U to the linear problem (2.20) indeed uniquely exists. To this end, we need only consider the corresponding homogeneous problem

$$\begin{cases} -(A_\varepsilon U_x)_x + BU = 0, & x \in \Omega, \quad (2.23a) \\ U|_{\partial\Omega} = 0. & \quad (2.23b) \end{cases}$$

Applying $\int_0^1 (2.23\text{a}) \cdot U\,dx$ and integrating by parts yields $U(x) \equiv 0$.

Assume that the condition

$$M \in \mathcal{M}, \quad H \in H^4(\Omega), \text{ and } \partial_x^k H \equiv 0, \; k = 2, 3, 4, \tag{2.24}$$

holds, we will establish the desired elliptic estimate step by step. For simplicity of notations, we write

$$U := \Big(u^{(1)}, u^{(2)}, u^{(3)}\Big)^T, \tag{2.25a}$$

$$U_H := U - H = \left(u_H^{(1)}, u_H^{(2)}, u_H^{(3)}\right)^T, \tag{2.25b}$$

in the present step only. First, substituting U_H into the linear BVP (2.20), we have

$$\begin{cases} -A_\varepsilon (U_H)_{xx} + BU_H = M(x) - BH(x), \quad x \in \Omega, & (2.26a) \\ U_H(0) = U_H(1) = 0, & (2.26b) \\ (U_H)_{xx}(0) = (U_H)_{xx}(1) = 0. & (2.26c) \end{cases}$$

Here we have adopted the assumption (2.24) and utilized Equation (2.22). Secondly, performing the following procedure

$$\left(\sum_{l=0}^{2} \int_0^1 \partial_x^l (2.26a) \cdot \partial_x^l U_H \, dx \right)^{1/2}, \tag{2.27}$$

we can obtain the estimate

$$\sum_{i=1}^{2} \left(\varepsilon \|\partial_x^3 u_H^{(i)}\| + \|u_H^{(i)}\|_2 \right) + \|u_H^{(3)}\|_3 \leq C\Big(\|M\|_2 + \|H\|_1 \Big), \quad \forall \varepsilon > 0, \tag{2.28}$$

by using the integration by parts and Yong's inequality, where the generic positive constant C is independent of the parameter ε. Thirdly, we solve the quantity $A_\varepsilon \partial_x^4 U_H$ from the system ∂_x^2(2.26a) and obtain

$$A_\varepsilon \partial_x^4 U_H = B \partial_x^2 U_H - \partial_x^2 M(x). \tag{2.29}$$

Taking the L^2-norm on the both sides of Equation (2.29) directly, we have

$$\sum_{i=1}^{2} \varepsilon^2 \|\partial_x^4 u_H^{(i)}\| + \|\partial_x^4 u_H^{(3)}\| \leq C\Big(\|M\|_2 + \|H\|_1 \Big), \quad \forall \varepsilon > 0, \tag{2.30}$$

again, the generic positive constant C is independent of the parameter ε. Finally, we summarize the above unique solvability result and the estimates (2.28) and (2.30) together and can obtain the following important fact:

(Sharp elliptic estimate.) If the assumption (2.24) *holds, then there is a unique strong solution* $U = \left(u^{(1)}, u^{(2)}, u^{(3)}\right)^T \in H^4(\Omega)$ *of the linear BVP* (2.20) *satisfying the sharp elliptic estimate*

$$\sum_{i=1}^{2} \left(\varepsilon^2 \|\partial_x^4 u^{(i)}\| + \varepsilon \|\partial_x^3 u^{(i)}\| + \|u^{(i)}\|_2 \right) + \|u^{(3)}\|_4 \leq C\Big(\|M\|_2 + \|H\|_1 \Big), \quad \forall \varepsilon > 0, \tag{2.31}$$

where the generic positive constant C is independent of the parameter ε.

Step III. *Banach fixed point argument.* On account of the above fact (2.31) in step II and the observation of the nonlinearity (2.16), we introduce a metric space

$$\mathbb{U}[N] := \Big\{ U \in H^2(\Omega) \;\Big|\; \|U\|_2 \leq N\delta, \quad U|_{\partial\Omega} = H \Big\} \tag{2.32}$$

equipped with the metric associated with the norm $\|\cdot\|_2$. Here the positive constant N will be determined later. In fact, it follows from the trace theorem that $\mathbb{U}[N]$ is a closed subspace of $H^2(\Omega)$ for any $N > 0$ and $\delta \geq 0$. Thus $\mathbb{U}[N]$ is a complete metric space.

Next, for all $V=\left(\tilde{v}_1^\delta,\tilde{v}_2^\delta,\tilde{\varphi}\right)^T\in\mathbb{U}[N]$, let $\tilde{k}_i:=J_i[(\tilde{v}_i^\delta+w_{il})^2]$, we have $F(V)\in H^2(\Omega)$ by Equations (2.15) and (2.16). Moreover, let $M:=F(V)+G$, one can easily see that $M\in\mathcal{M}$ if $D\in\mathcal{H}$. Then we can define a solution operator $S:\mathbb{U}[N]\to H^4(\Omega)$, $V\mapsto U=:SV$ by solving the linear BVP (2.20).

Now we tend to determine the positive constant N to ensure that the operator S is a contraction mapping on $\mathbb{U}[N]$ if $\delta\ll 1$. To this end, we separately show that S is onto and contractive below.

S is onto. Apparently, the boundary data (2.18a) and $M=F(V)+G$, $\forall V\in\mathbb{U}[N]$ satisfy the assumption (2.24), then we get the elliptic estimate by using Equation (2.31) directly

$$
\begin{aligned}
\|SV\|_2=\|U\|_2&\le C\Big(\|F(V)+G\|_2+\|H\|_1\Big)\\
&\le C\Big(\|F(V)\|_2+\underbrace{\|G\|_2+\|H\|_1}_{O(1)\delta}\Big)\\
&\le C\Big(\|F(V)\|_2+\delta\Big).
\end{aligned}
\tag{2.33}
$$

Next, we need to estimate $\|F(V)\|_2$. Before doing so, we have to estimate $\tilde{k}_i:=J_i[(\tilde{v}_i^\delta+w_{il})^2]$. By using Equation (2.8a) and the a priori assumption $N\delta\ll 1$, we have

$$
|\tilde{k}_i|\le C\delta,\quad V\in\mathbb{U}[N]. \tag{2.34}
$$

Because of the estimate (2.34), we can easily control the nonlocal terms in $F(V)$. On the other hand, the quadratic nonlinear terms in $F(V)$ can also be estimated as well by Sobolev embedding theorem. Through the same methods in [9], we get the estimate

$$
\|F(V)\|_2\le C\Big(\big(N^2+N\big)\delta+1\Big)\delta. \tag{2.35}
$$

Substituting Equation (2.35) into Equation (2.33), we have

$$
\|SV\|_2\le\Big(C_1\big(N^2+N\big)\delta+C_2\Big)\delta. \tag{2.36}
$$

Define

$$
N:=2C_2>0. \tag{2.37}
$$

If

$$
\delta\le C_2\big/\big(4C_1C_2^2+2C_1C_2\big), \tag{2.38}
$$

then

$$
\|SV\|_2\le 2C_2\delta=N\delta. \tag{2.39}
$$

S is contractive. For $\forall V_1,V_2\in\mathbb{U}[N]$, we need to estimate the difference $U:=SV_1-SV_2$. To this end, we define $M:=F(V_1)-F(V_2)$, by the definition of the solution operator S, we know that the difference U satisfies the following BVP

$$
\begin{cases}
-(A_\varepsilon U_x)_x+BU=M,\quad x\in\Omega, & (2.40\text{a})\\
U|_{\partial\Omega}=0. & (2.40\text{b})
\end{cases}
$$

Therefore, applying the sharp elliptic estimate (2.31) to the difference $U=SV_1-SV_2$, we have

$$\|SV_1-SV_2\|_2\leq C\|F(V_1)-F(V_2)\|_2. \tag{2.41}$$

Our next goal is to estimate $\|F(V_1)-F(V_2)\|_2$. In fact, form the mean value theorem, Equation (2.8a), and the a priori assumption $N\delta\ll 1$, we get

$$|\tilde{k}_{i1}-\tilde{k}_{i2}|\leq C\delta\|\tilde{v}_{i1}^{\delta}-\tilde{v}_{i2}^{\delta}\|\leq C\delta\|V_1-V_2\|, \quad i=1,2. \tag{2.42}$$

Based on Equation (2.42), by using the same method in [9], we have

$$\|F(V_1)-F(V_2)\|_2\leq C\delta\|V_1-V_2\|_2. \tag{2.43}$$

Here N has been defined in Equation (2.37). Substituting Equation (2.43) into Equation (2.41), we obtain

$$\|SV_1-SV_2\|_2\leq C_3\delta\|V_1-V_2\|_2, \quad \forall V_1,V_2\in\mathbb{U}[N]. \tag{2.44}$$

If

$$\delta\leq 1\big/\big(2C_3\big)>0, \tag{2.45}$$

then S is contractive.

Note that we have actually proved that there are positive constants δ_3 and N such that if $\delta\leq\delta_3$, then $S:\mathbb{U}[N]\to\mathbb{U}[N]$ is a contraction mapping. According to the Banach fixed point theorem, we get a unique fixed point $U=\big(\tilde{w}_1^{\delta},\tilde{w}_2^{\delta},\tilde{\phi}\big)^T\in\mathbb{U}[N]\cap H^4(\Omega)$ of the solution operator S. It is obvious that this fixed point U is our desired solution of the perturbation problem (2.12). Moreover, it also satisfies the sharp elliptic estimate

$$\sum_{i=1}^{2}\Big(\varepsilon^2\|\partial_x^4\tilde{w}_i^{\delta}\|+\varepsilon\|\partial_x^3\tilde{w}_i^{\delta}\|+\|\tilde{w}_i^{\delta}\|_2\Big)+\|\tilde{\phi}\|_4\leq N\delta, \quad \forall\varepsilon>0, \tag{2.46}$$

which follows from Equations (2.31) and (2.33)–(2.39). Analysis similar to that in Equation (2.34) shows that $\tilde{j}_i=J_i[(\tilde{w}_i^{\delta}+w_{il})^2]$ satisfies the estimate $|\tilde{j}_i|\leq C\delta$. Hence the inequality (2.10b) is proved. On the other hand, the inequality (2.10a) follows from Equation (2.46) and Sobolev's embedding theorem.

Step IV. *More regularity.* We conclude from Equations (2.17) and (2.16a), the first two equations of the elliptic system (2.12a) and Sobolev embedding theorem that $f_i(U)\in H^4(\Omega)$, hence that the above strong solution has the additional regularity $U=\big(\tilde{w}_1^{\delta},\tilde{w}_2^{\delta},\tilde{\phi}\big)^T\in\big(H^6\times H^6\times H^4\big)(\Omega)$, and finally that $U\in\big(\mathcal{B}^5\times\mathcal{B}^5\times\mathcal{B}^3\big)(\overline{\Omega})$ which implies the desired regularity in Lemma 2.1. This completes the proof. □

REMARK 2.1. Once Lemma 2.1 is proven, Theorem 1.1 immediately follows by using the transformation $\tilde{n}_i=\tilde{w}_i^2$. From the proof, we can also see that we have to fix the boundary information of the doping profile $D(x)$ in order to obtain the sharper elliptic estimate independent of the parameter ε.

3. Asymptotic stability of the stationary solution

To simplify notations, we let (n_1,j_1,n_2,j_2,ϕ) stand for the solution $\big(n_1^{\varepsilon},j_1^{\varepsilon},n_2^{\varepsilon},j_2^{\varepsilon},\phi^{\varepsilon}\big)$ of the IBVP (1.1)–(1.3) in this section.

3.1. Reformulation and local existence. To consider the IBVP (1.1)–(1.3), it is convenient to rewrite the problem in terms of $(w_1, j_1, w_2, j_2, \phi)$, where

$$w_i := \sqrt{n_i}, \quad i = 1, 2. \tag{3.1}$$

Then the conditions (1.5) become

$$\inf_{x\in\Omega} S_i[w_i^2, j_i] > 0, \qquad \inf_{x\in\Omega} w_i > 0, \tag{3.2}$$

and the IBVP (1.1)–(1.3) can be written as

$$\begin{cases} 2w_i w_{it} + j_{ix} = 0, & (3.3\text{a}) \\ j_{it} + 2S_i[w_i^2, j_i] w_i w_{ix} + 2j_i w_i^{-2} j_{ix} - \varepsilon^2 w_i^2 (w_{ixx}/w_i)_x = (-1)^{i-1} w_i^2 \phi_x - j_i, & (3.3\text{b}) \\ \phi_{xx} = w_1^2 - w_2^2 - D(x), \quad i = 1, 2, \quad \forall (t,x) \in (0, +\infty) \times \Omega, & (3.3\text{c}) \end{cases}$$

with the initial data

$$(w_i, j_i)(0, x) = (w_{i0}, j_{i0})(x) := (\sqrt{n_{i0}}, j_{i0})(x), \tag{3.4}$$

and the boundary data

$$\begin{aligned} w_i(t,0) = w_{il} > 0, \qquad w_i(t,1) = w_{ir} > 0, & \qquad (3.5\text{a}) \\ w_{ixx}(t,0) = w_{ixx}(t,1) = 0, & \qquad (3.5\text{b}) \\ \phi(t,0) = 0, \qquad \phi(t,1) = \phi_r > 0. & \qquad (3.5\text{c}) \end{aligned}$$

The compatibility (1.4) between the initial data and boundary data transforms to

$$w_{i0}(0) = w_{il}, \quad w_{i0}(1) = w_{ir}, \quad j_{i0x}(0) = j_{i0x}(1) = w_{i0xx}(0) = w_{i0xx}(1) = 0. \tag{3.6}$$

Let the stationary solution $(\tilde{w}_1, \tilde{j}_1, \tilde{w}_2, \tilde{j}_2, \tilde{\phi})(x)$ be given by Lemma 2.1. It satisfies the BVP (2.3)–(2.4). We denote the perturbation by

$$\begin{aligned} \psi_i(t,x) := w_i(t,x) - \tilde{w}_i(x), \qquad \eta_i(t,x) := j_i(t,x) - \tilde{j}_i, \\ \sigma(t,x) := \phi(t,x) - \tilde{\phi}(x). \end{aligned} \tag{3.7}$$

From

$$\begin{aligned} (3.3\text{a}) - -(2.3\text{a}), \quad (3.3\text{b})/w_i^2 - (2.3\text{b})/\tilde{w}_i^2, \quad (3.3\text{c}) - -(2.3\text{c}), & \qquad (3.8\text{a}) \\ (3.5) - -(2.4), & \qquad (3.8\text{b}) \end{aligned}$$

we thus deduce that

$$\begin{cases} 2\big(\psi_i + \tilde{w}_i\big)\psi_{it} + \eta_{ix} = 0, & (3.9\text{a}) \\ \Big[\big(\eta_i + \tilde{j}_i\big)/\big(\psi_i + \tilde{w}_i\big)^2\Big]_t + \dfrac{1}{2}\Big\{\Big[\big(\eta_i + \tilde{j}_i\big)/\big(\psi_i + \tilde{w}_i\big)^2\Big]^2 - \big(\tilde{j}_i/\tilde{w}_i^2\big)^2\Big\}_x \\ \quad + K_i\Big[\ln\big(\psi_i + \tilde{w}_i\big)^2 - \ln \tilde{w}_i^2\Big]_x - \varepsilon^2\Big[\big(\psi_i + \tilde{w}_i\big)_{xx}/\big(\psi_i + \tilde{w}_i\big) - \tilde{w}_{ixx}/\tilde{w}_i\Big]_x \\ \quad + (-1)^i \sigma_x + \big(\eta_i + \tilde{j}_i\big)/\big(\psi_i + \tilde{w}_i\big)^2 - \tilde{j}_i/\tilde{w}_i^2 = 0, & (3.9\text{b}) \\ \sigma_{xx} = \big(\psi_1 + 2\tilde{w}_1\big)\psi_1 - \big(\psi_2 + 2\tilde{w}_2\big)\psi_2, \quad i = 1, 2, & (3.9\text{c}) \end{cases}$$

with the initial data

$$(\psi_i,\eta_i)(0,x)=(\psi_{i0},\eta_{i0})(x):=(w_{i0}-\tilde{w}_i,j_{i0}-\tilde{j}_i)(x), \tag{3.10}$$

and the boundary data

$$\psi_i(t,0)=\psi_i(t,1)=0,\quad \psi_{ixx}(t,0)=\psi_{ixx}(t,1)=0,\quad \sigma(t,0)=\sigma(t,1)=0. \tag{3.11}$$

The task is now to find the local-in-time solution of the IBVP (3.9)–(3.11) in the subsonic region. In fact, the local existence result of unipolar problem has been proved by the iteration method and compactness argument in [20] for nonlinear boundary condition (and in [13] for linear boundary condition). The methods employed in [13,20] can be applied to our bipolar problem directly. The proof is straightforward, and we have

LEMMA 3.1 (Local existence). *Suppose that the initial data $(\psi_{i0},\eta_{i0})\in(H^4\times H^3)(\Omega)$ and $(\psi_{i0}+\tilde{w}_i,\eta_{i0}+\tilde{j}_i)$ satisfy the conditions* (3.2) *and* (3.6). *Then there exists a finite time $T_*>0$ such that the IBVP* (3.9)–(3.11) *has a unique local solution $(\psi_1,\eta_1,\psi_2,\eta_2,\sigma)\in\left[(\bar{\mathfrak{X}}_4\times\bar{\mathfrak{X}}_3)^2\times\bar{\mathfrak{X}}_4^2\right]([0,T_*])$ and $(\psi_i+\tilde{w}_i,\eta_i+\tilde{j}_i)$ also satisfy the condition* (3.2).

Owing to Lemma 3.1, it suffices to use the standard continuation argument together with an uniform a priori estimate in order to show the existence of the global solution. For this purpose, it is convenient to use notations

$$n_\varepsilon(t):=\sum_{i=1}^{2}\Big(\|(\psi_i,\eta_i)(t)\|_2+\|(\varepsilon\partial_x^3\psi_i,\varepsilon\partial_x^3\eta_i,\varepsilon^2\partial_x^4\psi_i)(t)\|\Big),\quad \forall t\in[0,T], \tag{3.12a}$$

$$N_\varepsilon(T):=\sup_{t\in[0,T]}n_\varepsilon(t). \tag{3.12b}$$

We conclude from Sobolev's embedding theorem and Equation (3.12) that for all $0<\varepsilon\le 1$ we have

$$\sum_{i=1}^{2}\Big(|(\psi_i,\eta_i)(t)|_1+|(\varepsilon\psi_{ixx},\varepsilon\eta_{ixx},\varepsilon^2\partial_x^3\psi_i)(t)|_0\Big)\le CN_\varepsilon(T),\quad \forall t\in[0,T]. \tag{3.13}$$

What left now is to establish the uniform a priori estimate as follows

LEMMA 3.2 (Uniform a priori estimate). *Let $(\psi_1,\eta_1,\psi_2,\eta_2,\sigma)\in\left[(\bar{\mathfrak{X}}_4\times\bar{\mathfrak{X}}_3)^2\times\bar{\mathfrak{X}}_4^2\right]([0,T])$ be a local solution on a finite time interval $[0,T]$ of the IBVP* (3.9)–(3.11). *For arbitrary constants $n_{il},K_i>0$, there exist three constants $\delta_4,\gamma_2,C>0$ such that if $N_\varepsilon(T)+\delta\le\delta_4$, then for all $0<\varepsilon\le 1$ it holds that*

$$n_\varepsilon(t)\le Cn_\varepsilon(0)\,e^{-\gamma_2 t},\quad t\in[0,T]. \tag{3.14}$$

REMARK 3.1. Once Lemma 3.2 is proven, Theorem 1.2 immediately holds true by using the transformations $\tilde{n}_i=\tilde{w}_i^2$ and $n_i=w_i^2$.

Actually, we can prove Lemma 3.2 via a series of estimates in Subsection 3.2, Subsection 3.3 and Subsection 3.4. However, we also need the auxiliary estimate of the stationary solution

$$\sum_{i=1}^{2}\Big(|\tilde{w}_i-w_{il}|_1+|(\varepsilon\tilde{w}_{ixx},\varepsilon^2\partial_x^3\tilde{w}_i)|_0\Big)\le C\delta,\quad \forall\ 0<\varepsilon\le 1, \tag{3.15}$$

which follows from Sobolev embedding theorem and Equation (2.10).

3.2. Basic estimate. In this subsection, we derive the basic energy estimate. To this end, we employ an energy form $\mathcal{E}$ defined by

$$\mathcal{E}(t,x) := \frac{1}{2}(\sigma_x)^2 + \sum_{i=1}^{2}\left(\frac{1}{2}\eta_i^2 w_i^{-2} + \Psi_i\left(w_i^2, \tilde{w}_i^2\right) + \varepsilon^2(\psi_{ix})^2\right), \tag{3.16}$$

where

$$\Psi_i\left(w_i^2, \tilde{w}_i^2\right) := K_i \int_{\tilde{w}_i^2}^{w_i^2} \left(\ln\xi - \ln\tilde{w}_i^2\right) d\xi, \tag{3.17}$$

and we can easily prove that $\Psi_i\left(w_i^2, \tilde{w}_i^2\right)$ is equivalent to ψ_i^2 provided $N_\varepsilon(T) + \delta \ll 1$.

LEMMA 3.3. *Under the same hypotheses of Lemma 3.2, we have*

$$\|\partial_t^l \sigma(t)\|_2 \le C\left(\|(\partial_t^l\psi_1, \partial_t^l\psi_2)(t)\| + \frac{l(l-1)}{2} N_\varepsilon(T)\|(\psi_{1t}, \psi_{2t})(t)\|\right), \ l = 0,1,2, \tag{3.18}$$

$$\|\sigma_{xt}(t)\| \le C\|(\eta_1, \eta_2)(t)\|, \tag{3.19}$$

$$\sum_{i=1}^{2} |(\psi_{it}, \eta_{it})(t)|_0 \le C N_\varepsilon(T), \quad t \in [0,T], \quad 0 < \varepsilon \le 1. \tag{3.20}$$

Proof. A slight change in the proof of Lemma 3.5 in [9] actually shows the estimates (3.18), (3.19), and (3.20). □

Now the basic estimate is as follows.

LEMMA 3.4 (Basic estimate). *Under the same hypotheses of Lemma 3.2, for all $t \in [0,T]$ and $0 < \varepsilon \le 1$, we get*

$$\frac{d}{dt}\int_0^1 \mathcal{E}(t,x)\, dx + \int_0^1 \sum_{i=1}^{2} \eta_i^2 \tilde{w}_i^{-2}\, dx = \int_0^1 R_2\, dx, \tag{3.21a}$$

and there exist the positive constants C, C_{1l}, C_{1r} such that

$$\left|\int_0^1 R_2\, dx\right| \le C(N_\varepsilon(T) + \delta)\sum_{i=1}^{2} \|(\psi_i, \eta_i)(t)\|_1^2, \tag{3.21b}$$

$$C_{1l}\sum_{i=1}^{2} \|(\psi_i, \eta_i, \varepsilon\psi_{ix})(t)\|^2 \le \int_0^1 \mathcal{E}(t,x)\, dx \le C_{1r}\sum_{i=1}^{2} \|(\psi_i, \eta_i, \varepsilon\psi_{ix})(t)\|^2, \tag{3.21c}$$

provided $N_\varepsilon(T) + \delta \ll 1$.

Proof. From

$$\sum_{i=1}^{2} (3.9\text{b})\eta_i, \tag{3.22}$$

we deduce that

$$\overbrace{\sum_{i=1}^{2}\left(j_iw_i^{-2}\right)_t\eta_i}^{\mathrm{IV}_1}+\overbrace{\sum_{i=1}^{2}\frac{1}{2}\left(j_i^2w_i^{-4}-\tilde{j}_i^2\tilde{w}_i^{-4}\right)_x\eta_i}^{\mathrm{IV}_2}+\overbrace{\sum_{i=1}^{2}K_i\left(\ln w_i^2-\ln\tilde{w}_i^2\right)_x\eta_i}^{\mathrm{IV}_3}$$
$$\underbrace{-\sum_{i=1}^{2}\varepsilon^2\left(w_{ixx}w_i^{-1}-\tilde{w}_{ixx}\tilde{w}_i^{-1}\right)_x\eta_i}_{\mathrm{IV}_4:\text{ quantum effect}}+\underbrace{\sum_{i=1}^{2}(-1)^i\sigma_x\eta_i}_{\mathrm{IV}_5:\text{ bipolar effect}}+\underbrace{\sum_{i=1}^{2}\left(j_iw_i^{-2}-\tilde{j}_i\tilde{w}_i^{-2}\right)\eta_i}_{\mathrm{IV}_6}=0. \tag{3.23}$$

By Leibniz's formula and Equation (3.9a), we thus get

$$\mathrm{IV}_1=\left(\sum_{i=1}^{2}\frac{1}{2}\eta_i^2w_i^{-2}\right)_t-\sum_{i=1}^{2}\left(\frac{1}{2}\eta_iw_i^{-4}-j_iw_i^{-4}\right)\eta_i\eta_{ix}, \tag{3.24}$$

$$\mathrm{IV}_2\text{ keeps intact,} \tag{3.25}$$

$$\mathrm{IV}_3=\left(\sum_{i=1}^{2}\Psi_i\left(w_i^2,\tilde{w}_i^2\right)\right)_t+\left(\sum_{i=1}^{2}K_i\left(\ln w_i^2-\ln\tilde{w}_i^2\right)\eta_i\right)_x, \tag{3.26}$$

$$\mathrm{IV}_6=\sum_{i=1}^{2}\eta_i^2\tilde{w}_i^{-2}-\sum_{i=1}^{2}j_i\left(\tilde{w}_i+w_i\right)\left(\tilde{w}_iw_i\right)^{-2}\psi_i\eta_i. \tag{3.27}$$

Since we focus on both the quantum effect and the bipolar effect, we give the detailed computations of I_4 and I_5 by Leibniz's formula, Equation (3.9a), and Equation (3.9c) as follows:

$$\begin{aligned}
\mathrm{IV}_4&=-\sum_{i=1}^{2}\varepsilon^2\Big\{\overbrace{\left[\left(w_{ixx}w_i^{-1}-\tilde{w}_{ixx}\tilde{w}_i^{-1}\right)\eta_i\right]_x}^{\mathrm{IV}_{4,1}}-\left(w_{ixx}w_i^{-1}-\tilde{w}_{ixx}\tilde{w}_i^{-1}\right)\eta_{ix}\Big\}\\
&=-\sum_{i=1}^{2}\varepsilon^2\Big\{\mathrm{IV}_{4,1}+\left(w_{ixx}w_i^{-1}-\tilde{w}_{ixx}\tilde{w}_i^{-1}\right)(w_i^2)_t\Big\}\\
&=-\sum_{i=1}^{2}\varepsilon^2\Big\{\mathrm{IV}_{4,1}+(\psi_{ixx}\tilde{w}_i-\tilde{w}_{ixx}\psi_i)(\tilde{w}_iw_i)^{-1}(w_i^2)_t\Big\}\\
&=-\sum_{i=1}^{2}\varepsilon^2\Big\{\mathrm{IV}_{4,1}+2\psi_{ixx}\psi_{it}+\underbrace{\tilde{w}_{ixx}(\tilde{w}_iw_i)^{-1}\psi_i\eta_{ix}}_{\mathrm{IV}_4^{(2)}}\Big\}\\
&=-\sum_{i=1}^{2}\varepsilon^2\Big\{\mathrm{IV}_{4,1}+(2\psi_{ix}\psi_{it})_x-[(\psi_{ix})^2]_t+\mathrm{IV}_{4,2}\Big\}\\
&=\Big[\sum_{i=1}^{2}\varepsilon^2(\psi_{ix})^2\Big]_t-\sum_{i=1}^{2}\varepsilon^2\tilde{w}_{ixx}(\tilde{w}_iw_i)^{-1}\psi_i\eta_{ix}\\
&\qquad\qquad-\Big\{\sum_{i=1}^{2}\varepsilon^2\Big[(w_{ixx}w_i^{-1}-\tilde{w}_{ixx}\tilde{w}_i^{-1})\eta_i+2\psi_{ix}\psi_{it}\Big]\Big\}_x,
\end{aligned} \tag{3.28}$$

$$\mathrm{IV}_5=\sum_{i=1}^{2}(-1)^i\Big[(\sigma\eta_i)_x-\sigma\eta_{ix}\Big]=\overbrace{-\Big[\sigma(\eta_1-\eta_2)\Big]_x}^{\mathrm{IV}_{5,1}}+\sigma(\eta_{1x}-\eta_{2x})$$

$$
\begin{aligned}
&= \mathrm{IV}_{5,1} - \sigma(w_1^2 - w_2^2)_t = \mathrm{IV}_{5,1} - \sigma\sigma_{xxt} = \mathrm{IV}_{5,1} - \Big[(\sigma\sigma_{xt})_x - \sigma_x\sigma_{xt}\Big] \\
&= -\Big[\sigma(\eta_1 - \eta_2) + \sigma\sigma_{xt}\Big]_x + \left[\frac{1}{2}(\sigma_x)^2\right]_t . \qquad (3.29)
\end{aligned}
$$

Substituting Equations (3.24)–(3.29) into Equation (3.23), we obtain

$$
\mathcal{E}_t + \sum_{i=1}^{2} \eta_i^2 \tilde{w}_i^{-2} = R_{1x} + R_2, \qquad (3.30a)
$$

where

$$
\begin{aligned}
R_1 := \sigma(\eta_1 - \eta_2) + \sigma\sigma_{xt} - \sum_{i=1}^{2} \Big\{ & K_i \left(\ln w_i^2 - \ln \tilde{w}_i^2\right)\eta_i \\
& - \varepsilon^2 \Big[(w_{ixx} w_i^{-1} - \tilde{w}_{ixx}\tilde{w}_i^{-1})\eta_i + 2\psi_{ix}\psi_{it} \Big] \Big\}, \qquad (3.30b)
\end{aligned}
$$

$$
\begin{aligned}
R_2 := \sum_{i=1}^{2} \Bigg[& \left(\frac{1}{2}\eta_i w_i^{-4} - j_i w_i^{-4} \right) \eta_i \eta_{ix} - \frac{1}{2}\left(j_i^2 w_i^{-4} - \tilde{j}_i^2 \tilde{w}_i^{-4} \right)_x \eta_i \\
& + j_i (\tilde{w}_i + w_i)(\tilde{w}_i w_i)^{-2} \psi_i \eta_i + \varepsilon^2 \tilde{w}_{ixx} (\tilde{w}_i w_i)^{-1} \psi_i \eta_{ix} \Bigg]. \qquad (3.30c)
\end{aligned}
$$

Applying the boundary condtions (2.4), (3.5), and (3.11), we can assert that $\int_0^1 R_{1x}\,dx = 0$. For this reason, integrating Equation (3.30a) over Ω yields Equation (3.21a). Combining the inequalities (2.10), (3.15), (3.13), and the Cauchy–Schwarz inequality with Equation(3.30c), we obtain the estimate

$$
|R_2(t,x)| \le C(N_\varepsilon(T) + \delta) \sum_{i=1}^{2} |(\psi_i, \psi_{ix}, \eta_i, \eta_{ix})(t,x)|^2, \quad 0 < \varepsilon \le 1, \qquad (3.31)
$$

if $N_\varepsilon(T) + \delta \ll 1$. Note that we have actually proved the estimate (3.21b) by integrating Equation (3.31) over Ω. Based on the elliptic estimate (3.18) with $l = 0$ and the equivalent relationship below (3.17), a trivial verification gives the equivalence estimate (3.21c). □

3.3. Higher order estimates. This subsection is devoted to the derivation of the higher order estimates. Before stating the main results to be proved, we indicate that the computations of the higher order estimates in several steps are formal since the regularity of the local solution is insufficient. However, we can rigorously justify these formal computations by using the mollifier with respect to t. Because the argument is standard, we omit the detailed verification.

In order to use the homogeneous boundary condition (3.11), only the operation ∂_t^k is legitimate. Therefore, it is convenient to introduce notations

$$
A_{-1}^2(t) := \sum_{i=1}^{2} \|(\psi_i, \eta_i)(t)\|^2, \qquad (3.32a)
$$

$$
A_k^2(t) := A_{-1}^2(t) + \sum_{l=0}^{k} \sum_{i=1}^{2} \|(\partial_t^l \psi_{it}, \partial_t^l \psi_{ix}, \varepsilon \partial_t^l \psi_{ixx})(t)\|^2, \quad k = 0, 1. \qquad (3.32b)
$$

Next, we derive the working equations which are used to obtain the higher order estimates. From

$$-\partial_t^k\Big[\partial_x(3.3\text{b})/w_i-\partial_x(2.3\text{b})/\tilde{w}_i\Big],\quad i=1,2,\quad k=0,1, \tag{3.33}$$

together with Equations (3.3a) and (2.3a), we deduce that

$$\begin{aligned}&2\partial_t^k\psi_{itt}-2K_i\partial_t^k\psi_{ixx}+\varepsilon^2\partial_x^4\partial_t^k\psi_i+2\partial_t^k\psi_{it}\\ &=2j_iw_i^{-3}\partial_t^k\eta_{ixx}-2j_i^2w_i^{-4}\partial_t^k\psi_{ixx}+\varepsilon^2\big[(k+1)\psi_{ixx}+2\tilde{w}_{ixx}\big]w_i^{-1}\partial_t^k\psi_{ixx}\\ &\quad\underbrace{+(-1)^i\partial_t^k(\sigma_{xx}w_i)}_{\text{bipolar effect}}+\partial_t^kF_i+G_{ik},\quad i=1,2,\quad k=0,1,\end{aligned} \tag{3.34a}$$

where

$$\begin{aligned}F_i:=&\,2(\eta_{ix})^2w_i^{-3}-2(\psi_{it})^2w_i^{-1}+2K_i(w_{ix})^2(w_i^{-1}-\tilde{w}_i^{-1})\\ &+2K_i(w_{ix}+\tilde{w}_{ix})\tilde{w}_i^{-1}\psi_{ix}-8j_i\eta_{ix}w_i^{-4}w_{ix}+6(w_{ix}+\tilde{w}_{ix})j_i^2w_i^{-5}\psi_{ix}\\ &+6(\tilde{w}_{ix})^2(j_i+\tilde{j}_i)w_i^{-5}\eta_i+6(\tilde{w}_{ix})^2\tilde{j}_i^2(w_i^{-5}-\tilde{w}_i^{-5})\\ &+\varepsilon^2(\tilde{w}_{ixx})^2(w_i^{-1}-\tilde{w}_i^{-1})+(-1)^i(\tilde{\phi}_{xx}\psi_i+2w_{ix}\sigma_x+2\tilde{\phi}_x\psi_{ix})\\ &-2\tilde{w}_{ixx}(j_i+\tilde{j}_i)w_i^{-4}\eta_i-2\tilde{w}_{ixx}\tilde{j}_i^2(w_i^{-4}-\tilde{w}_i^{-4}),\end{aligned} \tag{3.34b}$$

and

$$G_{i0}:=0,\quad G_{i1}:=2(j_iw_i^{-3})_t\eta_{ixx}-2(j_i^2w_i^{-4})_t\psi_{ixx}-\varepsilon^2(w_{ixx}+\tilde{w}_{ixx})w_i^{-2}\psi_{it}\psi_{ixx}. \tag{3.34c}$$

To deal with $\partial_t^kF_i$ and G_{ik}, we need to use the inequalities (2.10), (3.13), (3.15), (3.18), (3.19), and the mean value theorem. Through a tedious but straightforward computation, we get

$$\|\partial_t^kF_i\|\le C(N_\varepsilon(T)+\delta)\|\partial_t^k(\psi_1,\psi_2,\psi_{ix},\psi_{it},\eta_i,\eta_{ix})\|, \tag{3.35a}$$
$$\|G_{i1}\|\le C(N_\varepsilon(T)+\delta)\|(\psi_{ixx},\eta_{ixx})\|,\quad 0<\varepsilon\le 1. \tag{3.35b}$$

It is worth mentioning that the quantum effect will make the bipolar effect in the working Equations (3.34a) more difficult to handle.

Moreover, based on the previous estimates (2.10), (3.13), (3.15), and (3.20) and Equations (3.9a), (3.9b), and $(3.34\text{a})|_{k=0}$, the similar proof in unipolar problem [20] works for our bipolar problem, we thus obtain

$$\partial_t^k\eta_{ixx}=-2w_i\partial_t^k\psi_{ixt}+H_{ik},\quad i=1,2,\quad k=0,1, \tag{3.36a}$$
$$\text{where } H_{i0}:=-2w_{ix}\psi_{it},\quad H_{i1}:=-4\psi_{it}\psi_{ixt}-2w_{ix}\psi_{itt}, \tag{3.36b}$$
$$\|H_{ik}\|\le CA_k(t), \tag{3.36c}$$
$$\|\eta_{it}(t)\|_1\le CA_1(t), \tag{3.36d}$$

and

$$C_{2l}A_1(t)\le n_\varepsilon(t)\le C_{2r}A_1(t),\quad \forall 0<\varepsilon\le 1, \tag{3.37}$$

provided $N_\varepsilon(T)+\delta\ll 1$.

Lemma 3.5 (Higher order estimates). *Under the same hypotheses of Lemma 3.2, for all $t\in[0,T]$ and $0<\varepsilon\le 1$, we have*

$$\frac{d}{dt}\mathrm{III}^k(t)+\mathrm{VII}^k(t)=\mathrm{XII}^k(t),\quad k=0,1, \tag{3.38a}$$

where $\mathrm{III}^k(t)$, $\mathrm{VII}^k(t)$, *and* $\mathrm{XII}^k(t)$ *are defined in Equations* (3.54b), (3.54c), *and* (3.54d), *respectively. Moreover, there exists a positive constant* C *such that*

$$\left|\mathrm{III}^k(t)\right| \le CA_k^2(t), \quad \left|\mathrm{XII}^k(t)\right| \le C(N_\varepsilon(T)+\delta)A_k^2(t), \quad k=0,1, \tag{3.38b}$$

provided $N_\varepsilon(T)+\delta \ll 1$.

Proof. Actually, based on the homogeneous boundary conditions

$$(\psi_i,\partial_t^k\psi_{it},\partial_t^k\psi_{ixx})(t,0)=(\psi_i,\partial_t^k\psi_{it},\partial_t^k\psi_{ixx})(t,1)=0, \quad i=1,2, \quad k=0,1, \tag{3.39}$$

Lemma 3.5 can be proved by using the procedure

$$\sum_{i=1}^2\int_0^1 (3.34\mathrm{a})(\partial_t^k\psi_i+2\partial_t^k\psi_{it})\,dx, \quad k=0,1. \tag{3.40}$$

However, due to the complexity, the proof will be divided into three steps.

Step I. From

$$\sum_{i=1}^2\int_0^1 (3.34\mathrm{a})\partial_t^k\psi_i\,dx, \quad k=0,1, \tag{3.41}$$

we have

$$\begin{aligned}
&\underbrace{\sum_{i=1}^2\int_0^1 2\partial_t^k\psi_{itt}\partial_t^k\psi_i\,dx}_{\mathrm{I}_1^k}\underbrace{-\sum_{i=1}^2\int_0^1 2K_i\partial_t^k\psi_{ixx}\partial_t^k\psi_i\,dx}_{\mathrm{I}_2^k}\\
&\underbrace{+\sum_{i=1}^2\int_0^1 \varepsilon^2\partial_x^4\partial_t^k\psi_i\partial_t^k\psi_i\,dx}_{\mathrm{I}_3^k}\underbrace{+\sum_{i=1}^2\int_0^1 2\partial_t^k\psi_{it}\partial_t^k\psi_i\,dx}_{\mathrm{I}_4^k}\\
=&\underbrace{\sum_{i=1}^2\int_0^1 2j_iw_i^{-3}\partial_t^k\eta_{ixx}\partial_t^k\psi_i\,dx}_{\mathrm{I}_5^k}\underbrace{-\sum_{i=1}^2\int_0^1 2j_i^2w_i^{-4}\partial_t^k\psi_{ixx}\partial_t^k\psi_i\,dx}_{\mathrm{I}_6^k}\\
&\underbrace{+\sum_{i=1}^2\int_0^1 \varepsilon^2\left[(k+1)\psi_{ixx}+2\tilde{w}_{ixx}\right]w_i^{-1}\partial_t^k\psi_{ixx}\partial_t^k\psi_i\,dx}_{\mathrm{I}_7^k}\\
&\underbrace{+\sum_{i=1}^2\int_0^1 (-1)^i\partial_t^k(\sigma_{xx}w_i)\partial_t^k\psi_i\,dx}_{\mathrm{I}_8^k:\ \text{bipolar effect}}\underbrace{+\sum_{i=1}^2\int_0^1 (\partial_t^kF_i+G_{ik})\partial_t^k\psi_i\,dx}_{\mathrm{I}_9^k},
\end{aligned}\tag{3.42}$$

By integration by parts, we can easily obtain

$$\mathrm{I}_1^k=\frac{d}{dt}\int_0^1\sum_{i=1}^2 2\partial_t^k\psi_{it}\partial_t^k\psi_i\,dx-\int_0^1\sum_{i=1}^2 2(\partial_t^k\psi_{it})^2\,dx, \tag{3.43a}$$

$$\mathrm{I}_2^k = \int_0^1 \sum_{i=1}^2 2K_i(\partial_t^k \psi_{ix})^2\,dx, \tag{3.43b}$$

$$\mathrm{I}_3^k = \int_0^1 \sum_{i=1}^2 (\varepsilon \partial_t^k \psi_{ixx})^2\,dx, \tag{3.43c}$$

$$\mathrm{I}_4^k = \frac{d}{dt}\int_0^1 \sum_{i=1}^2 (\partial_t^k \psi_i)^2\,dx, \tag{3.43d}$$

$$\mathrm{I}_5^k = -\int_0^1 \sum_{i=1}^2 2(j_i w_i^{-3}\partial_t^k \psi_i)_x \partial_t^k \eta_{ix}\,dx, \tag{3.43e}$$

$$\mathrm{I}_6^k = \int_0^1 \sum_{i=1}^2 2(j_i^2 w_i^{-4}\partial_t^k \psi_i)_x \partial_t^k \psi_{ix}\,dx. \tag{3.43f}$$

Let I_7^k and I_9^k keep intact. Since the main difficulties arise from the bipolar effect, we need to pay more attention to I_8^k,

$$\begin{aligned}\text{bipolar effect: } \mathrm{I}_8^k &= \int_0^1 \sum_{i=1}^2 (-1)^i \partial_t^k(\sigma_{xx} w_i)\partial_t^k \psi_i\,dx \\ &= -\int_0^1 \left[\partial_t^k(\sigma_{xx}w_1)\partial_t^k\psi_1 - \partial_t^k(\sigma_{xx}w_2)\partial_t^k\psi_2\right]dx, \quad k=0,1.\end{aligned} \tag{3.44a}$$

From Equation (3.9c), we get

$$\sigma_{xx} = (w_1+\tilde{w}_1)\psi_1 - (w_2+\tilde{w}_2)\psi_2, \tag{3.44b}$$

$$\sigma_{xxt} = 2w_1\psi_{1t} - 2w_2\psi_{2t}. \tag{3.44c}$$

Our next goal is to deal with I_8^0 and I_8^1 in Equation (3.44a), respectively.

$$\begin{aligned}\mathrm{I}_8^0 &= -\int_0^1 \big(\sigma_{xx}w_1\psi_1 - \sigma_{xx}w_2\psi_2\big)\,dx = -\int_0^1 \sigma_{xx}\big(w_1\psi_1 - w_2\psi_2\big)\,dx \\ &= -\int_0^1 \big[(w_1+\tilde{w}_1)\psi_1 - (w_2+\tilde{w}_2)\psi_2\big]\big(w_1\psi_1 - w_2\psi_2\big)\,dx \\ &= -\int_0^1 \big[(w_1\psi_1 - w_2\psi_2) + (\tilde{w}_1\psi_1 - \tilde{w}_2\psi_2)\big]\big(w_1\psi_1 - w_2\psi_2\big)\,dx \\ &= -\int_0^1 \big[\underbrace{(w_1\psi_1 - w_2\psi_2)^2}_{\mathrm{I}_{8,1}^0} + (\tilde{w}_1\psi_1 - \tilde{w}_2\psi_2)(w_1\psi_1 - w_2\psi_2)\big]\,dx \\ &= -\int_0^1 \Big\{\mathrm{I}_{8,1}^0 + (\tilde{w}_1\psi_1 - \tilde{w}_2\psi_2)\big[(\psi_1+\tilde{w}_1)\psi_1 - (\psi_2+\tilde{w}_2)\psi_2\big]\Big\}\,dx \\ &= -\int_0^1 \Big\{\mathrm{I}_{8,1}^0 + (\tilde{w}_1\psi_1 - \tilde{w}_2\psi_2)\big[(\tilde{w}_1\psi_1 - \tilde{w}_2\psi_2) + (\psi_1^2 - \psi_2^2)\big]\Big\}\,dx \\ &= -\int_0^1 \Big[\underbrace{(w_1\psi_1 - w_2\psi_2)^2 + (\tilde{w}_1\psi_1 - \tilde{w}_2\psi_2)^2}_{\textit{nonnegative}}\ \underbrace{+(\tilde{w}_1\psi_1 - \tilde{w}_2\psi_2)(\psi_1^2 - \psi_2^2)}_{\mathrm{I}_{8,2}^0\text{: cubic nonlinearity}}\Big]\,dx,\end{aligned} \tag{3.44d}$$

$$
\begin{aligned}
\mathrm{I}_8^1 =& -\int_0^1 \left[(\sigma_{xx}w_1)_t\psi_{1t} - (\sigma_{xx}w_2)_t\psi_{2t}\right]dx \\
=& -\int_0^1 \left[(\sigma_{xxt}w_1 + \sigma_{xx}\psi_{1t})\psi_{1t} - (\sigma_{xxt}w_2 + \sigma_{xx}\psi_{2t})\psi_{2t}\right]dx \\
=& -\int_0^1 \left\{\sigma_{xxt}(w_1\psi_{1t} - w_2\psi_{2t}) + \sigma_{xx}\left[(\psi_{1t})^2 - (\psi_{2t})^2\right]\right\}dx \\
=& -\int_0^1 \Big\{\underbrace{2(w_1\psi_{1t} - w_2\psi_{2t})^2}_{\text{nonnegative}} \underbrace{+\left[(w_1+\tilde{w}_1)\psi_1 - (w_2+\tilde{w}_2)\psi_2\right]\left[(\psi_{1t})^2 - (\psi_{2t})^2\right]}_{\mathrm{I}_{8,1}^1\text{: cubic nonlinearity}}\Big\}dx.
\end{aligned}
\tag{3.44e}
$$

We indicate that the bipolar effect in Equation (3.44a) will only influence the final dissipation rate $\tilde{F}(t)$ in the next subsection.

Substituting Equations (3.43) and (3.44) into Equation (3.42), we have

$$
\frac{d}{dt}\mathrm{I}^k(t) + \mathrm{V}^k(t) = \mathrm{X}^k(t), \quad k = 0,1, \tag{3.45a}
$$

where

$$
\mathrm{I}^k(t) := \int_0^1 \sum_{i=1}^2 \left[2\partial_t^k\psi_{it}\partial_t^k\psi_i + (\partial_t^k\psi_i)^2\right]dx, \tag{3.45b}
$$

$$
\mathrm{V}^k(t) := \int_0^1 \sum_{i=1}^2 \left[2K_i(\partial_t^k\psi_{ix})^2 + (\varepsilon\partial_t^k\psi_{ixx})^2 - 2(\partial_t^k\psi_{it})^2\right]dx - \mathrm{I}_8^k(t), \tag{3.45c}
$$

$$
\begin{aligned}
\mathrm{X}^k(t) :=& -\int_0^1 \sum_{i=1}^2 2(j_i w_i^{-3}\partial_t^k\psi_i)_x \partial_t^k\eta_{ix}\,dx \\
&+ \int_0^1 \sum_{i=1}^2 2(j_i^2 w_i^{-4}\partial_t^k\psi_i)_x \partial_t^k\psi_{ix}\,dx + \mathrm{I}_7^k(t) + \mathrm{I}_9^k(t),
\end{aligned}
\tag{3.45d}
$$

and by using the estimates (2.10), (3.13), (3.15), (3.20) and (3.35) together with the Cauchy–Schwarz inequality and Equation (3.36), for all $0 < \varepsilon \le 1$, we obtain

$$
\left|\int_0^1 \mathrm{I}_{8,2}^0(t,x)\,dx\right| \le CN_\varepsilon(T)\|(\psi_1,\psi_2)(t)\|^2, \tag{3.46a}
$$

$$
\left|\int_0^1 \mathrm{I}_{8,1}^1(t,x)\,dx\right| \le CN_\varepsilon(T)\|(\psi_{1t},\psi_{2t})(t)\|^2, \tag{3.46b}
$$

$$
\left|\mathrm{I}^k(t)\right| \le CA_k^2(t), \quad \left|\mathrm{X}^k(t)\right| \le C(N_\varepsilon(T)+\delta)A_k^2(t), \quad k = 0,1, \tag{3.46c}
$$

provided $N_\varepsilon(T) + \delta \ll 1$.

Step II. From

$$
\sum_{i=1}^2 \int_0^1 (3.34\text{a})\partial_t^k\psi_{it}\,dx, \quad k = 0,1, \tag{3.47}
$$

we obtain

$$
\begin{aligned}
&\underbrace{\sum_{i=1}^{2}\int_0^1 2\partial_t^k\psi_{itt}\partial_t^k\psi_{it}\,dx}_{\mathrm{II}_1^k}\ \underbrace{-\sum_{i=1}^{2}\int_0^1 2K_i\partial_t^k\psi_{ixx}\partial_t^k\psi_{it}\,dx}_{\mathrm{II}_2^k}\\
&\quad\underbrace{+\sum_{i=1}^{2}\int_0^1 \varepsilon^2\partial_x^4\partial_t^k\psi_i\partial_t^k\psi_{it}\,dx}_{\mathrm{II}_3^k}\ \underbrace{+\sum_{i=1}^{2}\int_0^1 2(\partial_t^k\psi_{it})^2\,dx}_{\mathrm{II}_4^k}\\
=&\underbrace{\sum_{i=1}^{2}\int_0^1 2j_iw_i^{-3}\partial_t^k\eta_{ixx}\partial_t^k\psi_{it}\,dx}_{\mathrm{II}_5^k}\ \underbrace{-\sum_{i=1}^{2}\int_0^1 2j_i^2w_i^{-4}\partial_t^k\psi_{ixx}\partial_t^k\psi_{it}\,dx}_{\mathrm{II}_6^k}\\
&\quad\underbrace{+\sum_{i=1}^{2}\int_0^1 \varepsilon^2\left[(k+1)\psi_{ixx}+2\tilde{w}_{ixx}\right]w_i^{-1}\partial_t^k\psi_{ixx}\partial_t^k\psi_{it}\,dx}_{\mathrm{II}_7^k}\\
&\quad\underbrace{+\sum_{i=1}^{2}\int_0^1 (-1)^i\partial_t^k(\sigma_{xx}w_i)\partial_t^k\psi_{it}\,dx}_{\mathrm{II}_8^k:\ \text{bipolar effect}}\underbrace{+\sum_{i=1}^{2}\int_0^1 (\partial_t^kF_i+G_{ik})\partial_t^k\psi_{it}\,dx}_{\mathrm{II}_9^k},
\end{aligned}
\tag{3.48}
$$

By integration by parts and Equation (3.36a), we can easily obtain

$$\mathrm{II}_1^k=\frac{d}{dt}\int_0^1\sum_{i=1}^{2}(\partial_t^k\psi_{it})^2\,dx, \tag{3.49a}$$

$$\mathrm{II}_2^k=\frac{d}{dt}\int_0^1\sum_{i=1}^{2}K_i(\partial_t^k\psi_{ix})^2\,dx, \tag{3.49b}$$

$$\mathrm{II}_3^k=\frac{d}{dt}\int_0^1\sum_{i=1}^{2}\frac{1}{2}(\varepsilon\partial_t^k\psi_{ixx})^2\,dx, \tag{3.49c}$$

$$\mathrm{II}_4^k=\int_0^1\sum_{i=1}^{2}2(\partial_t^k\psi_{it})^2\,dx, \tag{3.49d}$$

$$\mathrm{II}_5^k=\int_0^1\sum_{i=1}^{2}2(j_iw_i^{-2})_x(\partial_t^k\psi_{it})^2\,dx+\int_0^1\sum_{i=1}^{2}2j_iw_i^{-3}H_{ik}\partial_t^k\psi_{it}\,dx, \tag{3.49e}$$

$$
\begin{aligned}
\mathrm{II}_6^k=&\frac{d}{dt}\int_0^1\sum_{i=1}^{2}j_i^2w_i^{-4}(\partial_t^k\psi_{ix})^2\,dx\\
&+\int_0^1\sum_{i=1}^{2}2(j_i^2w_i^{-4})_x\partial_t^k\psi_{it}\partial_t^k\psi_{ix}\,dx-\int_0^1\sum_{i=1}^{2}(j_i^2w_i^{-4})_t(\partial_t^k\psi_{ix})^2\,dx.
\end{aligned}
\tag{3.49f}
$$

Let II_7^k and II_9^k keep intact. Since the main difficulties arise from the bipolar effect, we need to pay more attention to II_8^k,

$$\text{bipolar effect: } \mathrm{II}_8^k = \int_0^1 \sum_{i=1}^2 (-1)^i \partial_t^k(\sigma_{xx} w_i)\partial_t^k \psi_{it}\, dx, \quad k=0,1. \tag{3.50a}$$

Our next goal is to deal with II_8^0 and II_8^1 in Equation (3.50a), respectively.

$$\begin{aligned}
\mathrm{II}_8^0 =& \int_0^1 \sum_{i=1}^2 (-1)^i (\sigma_{xx} w_i)\psi_{it}\, dx \\
=& \int_0^1 \sum_{i=1}^2 (-1)^i \big[(\sigma_{xx} w_i \psi_i)_t - (\sigma_{xx} w_i)_t \psi_i\big]\, dx \\
=& \underbrace{\frac{d}{dt}\int_0^1 \sum_{i=1}^2 (-1)^i \sigma_{xx} w_i \psi_i\, dx}_{\mathrm{II}_{8,1}^0} - \int_0^1 \sum_{i=1}^2 (-1)^i (\sigma_{xx} w_i)_t \psi_i\, dx \\
=& \mathrm{II}_{8,1}^0 - \int_0^1 \sum_{i=1}^2 (-1)^i (\sigma_{xxt} w_i + \sigma_{xx}\psi_{it})\psi_i\, dx \\
=& \mathrm{II}_{8,1}^0 - \int_0^1 \sum_{i=1}^2 (-1)^i \sigma_{xxt} w_i \psi_i\, dx \underbrace{- \int_0^1 \sum_{i=1}^2 (-1)^i \sigma_{xx} \psi_{it} \psi_i\, dx}_{\mathrm{II}_{8,2}^0} \\
=& \mathrm{II}_{8,1}^0 + \int_0^1 \sigma_{xxt}(w_1\psi_1 - w_2\psi_2)\, dx + \mathrm{II}_{8,2}^0 \\
=& \mathrm{II}_{8,1}^0 + \int_0^1 2(w_1\psi_{1t} - w_2\psi_{2t})(w_1\psi_1 - w_2\psi_2)\, dx + \mathrm{II}_{8,2}^0 \\
=& \mathrm{II}_{8,1}^0 + \int_0^1 2\big[(w_1\psi_1 - w_2\psi_2)_t - (\psi_{1t}\psi_1 - \psi_{2t}\psi_2)\big](w_1\psi_1 - w_2\psi_2)\, dx + \mathrm{II}_{8,2}^0 \\
=& \mathrm{II}_{8,1}^0 + \frac{d}{dt}\int_0^1 (w_1\psi_1 - w_2\psi_2)^2\, dx - \int_0^1 2(w_1\psi_1 - w_2\psi_2)(\psi_{1t}\psi_1 - \psi_{2t}\psi_2)\, dx + \mathrm{II}_{8,2}^0 \\
=& \underbrace{\frac{d}{dt}\int_0^1 \sum_{i=1}^2 (-1)^i \sigma_{xx} w_i \psi_i\, dx}_{=\mathrm{I}_8^0 \text{ by (3.44d)}} \\
&+ \frac{d}{dt}\int_0^1 (w_1\psi_1 - w_2\psi_2)^2\, dx - \int_0^1 2(w_1\psi_1 - w_2\psi_2)(\psi_{1t}\psi_1 - \psi_{2t}\psi_2)\, dx \\
&- \int_0^1 \sum_{i=1}^2 (-1)^i \sigma_{xx} \psi_{it} \psi_i\, dx \\
=& -\frac{d}{dt}\int_0^1 \Big[(\tilde{w}_1\psi_1 - \tilde{w}_2\psi_2)^2 + (\tilde{w}_1\psi_1 - \tilde{w}_2\psi_2)(\psi_1^2 - \psi_2^2)\Big]\, dx \\
&- \int_0^1 \big[2(w_1\psi_1 - w_2\psi_2) - \sigma_{xx}\big](\psi_{1t}\psi_1 - \psi_{2t}\psi_2)\, dx
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{d}{dt}\int_0^1 \Big[(\tilde{w}_1\psi_1 - \tilde{w}_2\psi_2)^2 + (\tilde{w}_1\psi_1 - \tilde{w}_2\psi_2)(\psi_1^2 - \psi_2^2)\Big]\,dx \\
&\quad - \int_0^1 (\psi_1^2 - \psi_2^2)(\psi_{1t}\psi_1 - \psi_{2t}\psi_2)\,dx \\
&= -\frac{d}{dt}\int_0^1 \Bigg[\underbrace{\frac{1}{4}(\psi_1^2 - \psi_2^2)^2 + (\tilde{w}_1\psi_1 - \tilde{w}_2\psi_2)^2}_{\text{nonnegative}} \underbrace{+(\tilde{w}_1\psi_1 - \tilde{w}_2\psi_2)(\psi_1^2 - \psi_2^2)}_{=\mathrm{I}_{8,2}^0\text{: cubic nonlinearity; (3.46a)}}\Bigg]\,dx, \qquad (3.50b)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{II}_8^1 &= \int_0^1 \sum_{i=1}^2 (-1)^i (\sigma_{xx} w_i)_t \psi_{itt}\,dx \\
&= \int_0^1 \sum_{i=1}^2 (-1)^i (\sigma_{xxt} w_i + \sigma_{xx}\psi_{it})\psi_{itt}\,dx \\
&= \underbrace{\int_0^1 \sum_{i=1}^2 (-1)^i \sigma_{xxt} w_i \psi_{itt}\,dx}_{\mathrm{II}_{8,1}^1} + \int_0^1 \sum_{i=1}^2 (-1)^i \sigma_{xx}\psi_{it}\psi_{itt}\,dx \\
&= \mathrm{II}_{8,1}^1 - \int_0^1 \sigma_{xx}(\psi_{1t}\psi_{1tt} - \psi_{2t}\psi_{2tt})\,dx \\
&= \mathrm{II}_{8,1}^1 - \underbrace{\int_0^1 \big[(w_1 + \tilde{w}_1)\psi_1 - (w_2 + \tilde{w}_2)\psi_2\big](\psi_{1t}\psi_{1tt} - \psi_{2t}\psi_{2tt})\,dx}_{\mathrm{II}_{8,2}^1\text{: cubic nonlinearity}} \\
&= \int_0^1 \sum_{i=1}^2 (-1)^i \sigma_{xxt} w_i \psi_{itt}\,dx - \mathrm{II}_{8,2}^1 \\
&= -\int_0^1 \sigma_{xxt}(w_1\psi_{1tt} - w_2\psi_{2tt})\,dx - \mathrm{II}_{8,2}^1 \\
&= -\int_0^1 \sigma_{xxt}(w_1\psi_{1t} - w_2\psi_{2t})_t\,dx + \int_0^1 \sigma_{xxt}\big[(\psi_{1t})^2 - (\psi_{2t})^2\big]\,dx - \mathrm{II}_{8,2}^1 \\
&= -\frac{d}{dt}\int_0^1 (w_1\psi_{1t} - w_2\psi_{2t})^2\,dx + \underbrace{\int_0^1 2(w_1\psi_{1t} - w_2\psi_{2t})\big[(\psi_{1t})^2 - (\psi_{2t})^2\big]\,dx}_{\mathrm{II}_{8,3}^1\text{: cubic nonlinearity}} - \mathrm{II}_{8,2}^1 \\
&= -\frac{d}{dt}\int_0^1 \underbrace{(w_1\psi_{1t} - w_2\psi_{2t})^2}_{\text{nonnegative}}\,dx \underbrace{+\mathrm{II}_{8,3}^1 - \mathrm{II}_{8,2}^1}_{\text{cubic nonlinearity}}. \qquad (3.50c)
\end{aligned}
$$

From Equations (3.50b) and (3.50c), we can also see that the bipolar effect in Equation (3.50a) will influence not only the final dissipation rate $\tilde{F}(t)$ but also the final energy $\tilde{E}(t)$ in the next subsection. However, due to the structure found in Equations (3.44d), (3.44e), (3.50b), and (3.50c), the bipolar effects I_8^k in Equation (3.44a) and II_8^k in Equation (3.50a) can be well controlled.

Substituting Equations (3.49) and (3.50) into Equation (3.48), we get

$$
\frac{d}{dt}\mathrm{II}^k(t) + \mathrm{VI}^k(t) = \mathrm{XI}^k(t), \quad k = 0,1, \qquad (3.51a)
$$

where

$$\mathrm{II}_{10}^k(t) := \int_0^1 \sum_{i=1}^2 \left[(\partial_t^k \psi_{it})^2 + \overbrace{(K_i - j_i^2 w_i^{-4})}^{=S_i[w_i^2, j_i]} (\partial_t^k \psi_{ix})^2 + \frac{1}{2}(\varepsilon \partial_t^k \psi_{ixx})^2 \right] dx, \tag{3.51b}$$

$$\mathrm{II}^0(t) := \mathrm{II}_{10}^0(t) \underbrace{+ \int_0^1 \left[\frac{1}{4}(\psi_1^2 - \psi_2^2)^2 + (\tilde{w}_1 \psi_1 - \tilde{w}_2 \psi_2)^2 + (\tilde{w}_1 \psi_1 - \tilde{w}_2 \psi_2)(\psi_1^2 - \psi_2^2) \right] dx}_{\mathrm{II}_b^0(t):\ \text{bipolar effect}}, \tag{3.51c}$$

$$\mathrm{II}^1(t) := \mathrm{II}_{10}^1(t) \overbrace{+ \int_0^1 (w_1 \psi_{1t} - w_2 \psi_{2t})^2 \, dx}^{\mathrm{II}_b^1(t):\ \text{bipolar effect}}, \tag{3.51d}$$

$$\mathrm{VI}^k(t) := \int_0^1 \sum_{i=1}^2 2(\partial_t^k \psi_{it})^2 \, dx, \tag{3.51e}$$

$$\begin{aligned} \mathrm{XI}^k(t) := & \int_0^1 \sum_{i=1}^2 2(j_i w_i^{-2})_x (\partial_t^k \psi_{it})^2 \, dx + \int_0^1 \sum_{i=1}^2 2 j_i w_i^{-3} H_{ik} \partial_t^k \psi_{it} \, dx \\ & + \int_0^1 \sum_{i=1}^2 2(j_i^2 w_i^{-4})_x \partial_t^k \psi_{it} \partial_t^k \psi_{ix} \, dx - \int_0^1 \sum_{i=1}^2 (j_i^2 w_i^{-4})_t (\partial_t^k \psi_{ix})^2 \, dx \\ & + \mathrm{II}_7^k + \mathrm{II}_9^k + k\left(\mathrm{II}_{8,3}^1 - \mathrm{II}_{8,2}^1 \right), \end{aligned} \tag{3.51f}$$

and by using the estimates (2.10), (3.13), (3.15), (3.20), (3.35), and (3.46a) together with the Cauchy–Schwarz inequality and Equation (3.36), for all $0 < \varepsilon \le 1$, we have

$$\left| \mathrm{II}^k(t) \right| \le C A_k^2(t), \quad \left| \mathrm{XI}^k(t) \right| \le C(N_\varepsilon(T) + \delta) A_k^2(t), \quad k = 0, 1, \tag{3.52}$$

provided $N_\varepsilon(T) + \delta \ll 1$.

Step III. From

$$(3.45\text{a}) + 2(3.51\text{a}), \tag{3.53}$$

namely Equation (3.40), we obtain

$$\frac{d}{dt} \mathrm{III}^k(t) + \mathrm{VII}^k(t) = \mathrm{XII}^k(t), \quad k = 0, 1, \tag{3.54a}$$

where

$$\begin{aligned} \mathrm{III}^k(t) := & \mathrm{I}^k(t) + 2\mathrm{II}^k(t) \\ = & \mathrm{I}^k(t) + 2\left(\mathrm{II}_{10}^k(t) + \mathrm{II}_b^k(t) \right) \\ = & \left(\mathrm{I}^k(t) + 2\mathrm{II}_{10}^k(t) \right) + 2\mathrm{II}_b^k(t) \\ = & \int_0^1 \sum_{i=1}^2 \Big[\underbrace{(\partial_t^k \psi_i + \partial_t^k \psi_{it})^2}_{\text{nonnegative}} + (\partial_t^k \psi_{it})^2 + \underbrace{2 S_i[w_i^2, j_i]}_{\ge K_i > 0} (\partial_t^k \psi_{ix})^2 + (\varepsilon \partial_t^k \psi_{ixx})^2 \Big] dx \end{aligned}$$

$$+2\mathrm{II}_b^k(t), \tag{3.54b}$$

$$\begin{aligned}\mathrm{VII}^k(t):=&\mathrm{V}^k(t)+2\mathrm{VI}^k(t)\\=&\int_0^1\sum_{i=1}^2\Big[2K_i(\partial_t^k\psi_{ix})^2+(\varepsilon\partial_t^k\psi_{ixx})^2+2(\partial_t^k\psi_{it})^2\Big]\,dx-\mathrm{I}_8^k(t),\end{aligned} \tag{3.54c}$$

$$\mathrm{XII}^k(t):=\mathrm{X}^k(t)+2\mathrm{XI}^k(t). \tag{3.54d}$$

We note that Equation (3.54a) follows from Equation (3.38a). According to the estimates (3.46) and (3.52), obviously, the estimate (3.38b) is followed. This completes the proof. ∎

3.4. Decay estimate. In this subsection, we can combine the basic estimate with the higher order estimates to prove Lemma 3.2 as follows.

Proof. From

$$(3.21\mathrm{a})+\sum_{k=0}^{1}(3.38\mathrm{a}), \tag{3.55}$$

we get the final energy equality

$$\frac{d}{dt}\tilde{E}(t)+\tilde{F}(t)=0, \tag{3.56a}$$

where

$$\tilde{E}(t):=\int_0^1\mathcal{E}(t,x)\,dx+\mathrm{III}^0(t)+\mathrm{III}^1(t), \tag{3.56b}$$

$$\tilde{F}(t):=\int_0^1\sum_{i=1}^2\eta_i^2\tilde{w}_i^{-2}\,dx+\mathrm{VII}^0(t)+\mathrm{VII}^1(t)-\int_0^1R_2\,dx-\mathrm{XII}^0(t)-\mathrm{XII}^1(t). \tag{3.56c}$$

Now we claim an important fact which reveals the dissipation mechanism in our bipolar problem:

> *Equivalence.* There are four positive constants $C_{3l},C_{3r},C_{4l},C_{4r}$ such that for all $0<\varepsilon\le 1$ if $N_\varepsilon(T)+\delta\ll 1$, then we have the equivalent relationships
>
> $$C_{3l}A_1^2(t)\le\tilde{E}(t)\le C_{3r}A_1^2(t), \tag{3.57a}$$
>
> $$C_{4l}A_1^2(t)\le\tilde{F}(t)\le C_{4r}A_1^2(t),\quad\forall t\in[0,T]. \tag{3.57b}$$

In fact, by $0<\varepsilon\le 1$ and the previous estimates (3.21b), (3.21c), (3.38b), a standard argument gives the upper bound estimates in Equation (3.57) under the a priori assumption $N_\varepsilon(T)+\delta\ll 1$.

The key point is how to establish the lower bound estimates in Equation (3.57). To this end, we first recall an optimal Poincaré inequality

$$\|f\|^2\le\frac{1}{4}\|f_x\|^2,\quad\forall f\in H_0^1(\Omega). \tag{3.58}$$

It will be very useful in estimating the lower bound of the final dissipation rate $\tilde{F}(t)$. Precisely, with the help of the Poincaré inequality (3.58), we can create the zero order dissipation rate $\|(\psi_1,\psi_2)(t)\|^2$ from the higher order estimate $\mathrm{VII}^0(t)$, which is lost in the basic estimate by bipolar effect.

Based on the structure analysis of the bipolar effect terms I_8^k in Equation (3.44) and II_8^k in Equation (3.50), namely,

$$\text{bipolar effect terms} = \text{nonnegative terms} + \text{cubic nonlinearities}, \tag{3.59}$$

we can abandon some certain nonnegative terms in $\tilde{E}(t)$ and $\tilde{F}(t)$ in order to get the desired lower bound.

$$\begin{aligned}
\tilde{E}(t) &= \int_0^1 \mathcal{E}(t,x)\,dx + \mathrm{III}^0(t) + \mathrm{III}^1(t)\\
&\geq C_{1l}\sum_{i=1}^{2}\|(\psi_i,\eta_i,\underbrace{\varepsilon\psi_{ix}}_{\geq 0})(t)\|^2\\
&\quad + \sum_{k=0}^{1}\int_0^1\sum_{i=1}^{2}\Big[\underbrace{(\partial_t^k\psi_i+\partial_t^k\psi_{it})^2}_{\geq 0} + (\partial_t^k\psi_{it})^2 + 2S_i[w_i^2,j_i](\partial_t^k\psi_{ix})^2 + (\varepsilon\partial_t^k\psi_{ixx})^2\Big]\,dx\\
&\quad + \sum_{k=0}^{1} 2\mathrm{II}_b^k(t)\\
&\geq C_{1l}\sum_{i=1}^{2}\|(\psi_i,\eta_i)(t)\|^2 + \sum_{k=0}^{1}\int_0^1\sum_{i=1}^{2}\Big[(\partial_t^k\psi_{it})^2 + K_i(\partial_t^k\psi_{ix})^2 + (\varepsilon\partial_t^k\psi_{ixx})^2\Big]\,dx\\
&\quad + \int_0^1\Bigg[\underbrace{\frac{1}{4}(\psi_1^2-\psi_2^2)^2 + (\tilde{w}_1\psi_1-\tilde{w}_2\psi_2)^2 + (\tilde{w}_1\psi_1-\tilde{w}_2\psi_2)(\psi_1^2-\psi_2^2)}_{\geq 0}\Bigg]\,dx\\
&\quad + \int_0^1 \underbrace{(w_1\psi_{1t}-w_2\psi_{2t})^2}_{\geq 0}\,dx\\
&\geq C_{1l}\sum_{i=1}^{2}\|(\psi_i,\eta_i)(t)\|^2 + \sum_{k=0}^{1}\int_0^1\sum_{i=1}^{2}\Big[(\partial_t^k\psi_{it})^2 + K_i(\partial_t^k\psi_{ix})^2 + (\varepsilon\partial_t^k\psi_{ixx})^2\Big]\,dx\\
&\quad - CN_\varepsilon(T)\sum_{i=1}^{2}\|\psi_i(t)\|^2\\
&\geq C_{3l}A_1^2(t),
\end{aligned}\tag{3.60a}$$

and

$$\begin{aligned}
\tilde{F}(t) &= \int_0^1\sum_{i=1}^{2}\eta_i^2\tilde{w}_i^{-2}\,dx + \sum_{k=0}^{1}\mathrm{VII}^k(t) - \int_0^1 R_2\,dx - \sum_{k=0}^{1}\mathrm{XII}^k(t)\\
&\geq C\sum_{i=1}^{2}\|\eta_i(t)\|^2\\
&\quad + \sum_{k=0}^{1}\Bigg\{\int_0^1\sum_{i=1}^{2}\Big[2K_i(\partial_t^k\psi_{ix})^2 + (\varepsilon\partial_t^k\psi_{ixx})^2 + 2(\partial_t^k\psi_{it})^2\Big]\,dx - \mathrm{I}_8^k(t)\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&-\left|\int_0^1 R_2\,dx\right|-\sum_{k=0}^{1}\left|\mathrm{XII}^k(t)\right|\\
\geq& C\sum_{i=1}^{2}\|\eta_i(t)\|^2+\sum_{i=1}^{2}\sum_{k=0}^{1}\Big[2K_i\|\partial_t^k\psi_{ix}(t)\|^2+\|\varepsilon\partial_t^k\psi_{ixx}(t)\|^2+2\|\partial_t^k\psi_{it}(t)\|^2\Big]\\
&-\sum_{k=0}^{1}\mathrm{I}_8^k(t)-C(N_\varepsilon(T)+\delta)\sum_{i=1}^{2}\|(\psi_i,\eta_i)(t)\|_1^2-C(N_\varepsilon(T)+\delta)\sum_{k=0}^{1}A_k^2(t)\\
\geq&\sum_{i=1}^{2}\big(K_i\underbrace{\|\psi_{ix}(t)\|^2}_{\mathrm{by}(3.58)}+K_i\|\psi_{ix}(t)\|^2\big)\\
&+\sum_{i=1}^{2}\Big[C\|\eta_i(t)\|^2+2K_i\|\partial_t^1\psi_{ix}(t)\|^2+\sum_{k=0}^{1}\big(\|\varepsilon\partial_t^k\psi_{ixx}(t)\|^2+2\|\partial_t^k\psi_{it}(t)\|^2\big)\Big]\\
&-C(N_\varepsilon(T)+\delta)\sum_{i=1}^{2}\|(\psi_i,\eta_i)(t)\|_1^2-C(N_\varepsilon(T)+\delta)\sum_{k=0}^{1}A_k^2(t)\\
&+\int_0^1\Big[\underbrace{(w_1\psi_1-w_2\psi_2)^2+(\tilde w_1\psi_1-\tilde w_2\psi_2)^2}_{\geq 0}+(\tilde w_1\psi_1-\tilde w_2\psi_2)(\psi_1^2-\psi_2^2)\Big]\,dx\\
&+\int_0^1\Big\{\underbrace{2(w_1\psi_{1t}-w_2\psi_{2t})^2}_{\geq 0}+\big[(w_1+\tilde w_1)\psi_1-(w_2+\tilde w_2)\psi_2\big]\big[(\psi_{1t})^2-(\psi_{2t})^2\big]\Big\}\,dx\\
\geq&\sum_{i=1}^{2}\big(4K_i\|\psi_i(t)\|^2+K_i\|\psi_{ix}(t)\|^2+C\|\eta_i(t)\|^2+2K_i\|\partial_t^1\psi_{ix}(t)\|^2\big)\\
&+\sum_{i=1}^{2}\sum_{k=0}^{1}\big(\|\varepsilon\partial_t^k\psi_{ixx}(t)\|^2+2\|\partial_t^k\psi_{it}(t)\|^2\big)\\
&-C(N_\varepsilon(T)+\delta)\sum_{i=1}^{2}\|(\psi_i,\eta_i)(t)\|_1^2-C(N_\varepsilon(T)+\delta)\sum_{k=0}^{1}A_k^2(t)\\
&-CN_\varepsilon(T)\sum_{i=1}^{2}\|\psi_i(t)\|^2-CN_\varepsilon(T)\sum_{i=1}^{2}\|\psi_{it}(t)\|^2\\
\geq& C_{4l}A_1^2(t).
\end{aligned}
\tag{3.60b}
$$

From Equation (3.57), we know that $\tilde E(t)$ and $\tilde F(t)$ are also equivalent to each other. Therefore, there exists a positive constant C_5 such that

$$C_5\tilde E(t)\leq \tilde F(t),\quad \forall t\in[0,T]. \tag{3.61}$$

Substituting Equation (3.61) into Equation (3.56a), we obtain the ordinary differential inequality

$$\frac{d}{dt}\tilde E(t)+C_5\tilde E(t)\leq 0,\quad \forall t\in[0,T]. \tag{3.62}$$

Gronwall's inequality argument yields that

$$\tilde E(t)\leq \tilde E(0)e^{-C_5 t},\quad \forall t\in[0,T]. \tag{3.63}$$

By Equations (3.37), (3.57a), and (3.63), there is a positive constant C such that

$$n_\varepsilon^2(t) \le C n_\varepsilon^2(0) e^{-C_5 t}, \quad \forall t \in [0,T]. \tag{3.64}$$

Let $\gamma_2 := C_5/2$, then we complete the proof. ∎

4. Semi-classical limits

In this section, we discuss the semi-classical limit from the QHD model to the HD model. First, we show the semi-classical limit for the stationary solution in Subsection 4.1. And then, we study the semi-classical limit for the global solution in Subsection 4.2.

4.1. Stationary solution case. We let $(\tilde{n}_1^\varepsilon, \tilde{j}_1^\varepsilon, \tilde{n}_2^\varepsilon, \tilde{j}_2^\varepsilon, \tilde{\phi}^\varepsilon)$ stand for the solution of the QHD boundary value problem (1.7)–(1.8), and continue to write $(\tilde{n}_1^0, \tilde{j}_1^0, \tilde{n}_2^0, \tilde{j}_2^0, \tilde{\phi}^0)$ for the solution of the HD boundary value problem (1.12).

It is worth mentioning that a similar result for unipolar problem has already been proved under the exponential transformation in [20]. However, the multiplier $(\ln \tilde{n}_i^\varepsilon - \ln \tilde{n}_i^0)_x$ used for unipolar problem are not applicable to our bipolar problem.

Through a careful observation on the bipolar structure of our problem, we can successfully overcome the main difficulty caused by the bipolar effect by choosing some new multipliers and estimating the error variables in terms of their original form. Now, we can prove Theorem 1.3 as follows.

Proof. First, we introduce the error variables between the QHD model and the HD model as follows

$$\tilde{\mathcal{N}}_i^\varepsilon := \tilde{n}_i^\varepsilon - \tilde{n}_i^0, \quad \tilde{\mathcal{J}}_i^\varepsilon := \tilde{j}_i^\varepsilon - \tilde{j}_i^0, \quad \tilde{\Phi}^\varepsilon = \tilde{\phi}^\varepsilon - \tilde{\phi}^0, \quad i = 1,2. \tag{4.1}$$

From

$$(1.12\text{b})/\tilde{n}_i^0, \quad (1.7\text{b})/\tilde{n}_i^\varepsilon, \quad (1.7\text{b})/\tilde{n}_i^\varepsilon - -(1.12\text{b})/\tilde{n}_i^0, \quad (1.7\text{c}) - -(1.12\text{c}), \quad i = 1,2, \tag{4.2}$$

we deduce that

$$\tilde{S}_i^0 (\tilde{n}_i^0)^{-1} \tilde{n}_{ix}^0 + (-1)^i \tilde{\phi}_x^0 = -\tilde{j}_i^0 (\tilde{n}_i^0)^{-1}, \tag{4.3a}$$

$$\tilde{S}_i^\varepsilon (\tilde{n}_i^\varepsilon)^{-1} \tilde{n}_{ix}^\varepsilon - \varepsilon^2 \left[\left(\sqrt{\tilde{n}_i^\varepsilon}\right)_{xx} \Big/ \sqrt{\tilde{n}_i^\varepsilon} \right]_x + (-1)^i \tilde{\phi}_x^\varepsilon = -\tilde{j}_i^\varepsilon (\tilde{n}_i^\varepsilon)^{-1}, \tag{4.3b}$$

$$\begin{aligned} &\tilde{S}_i^\varepsilon (\tilde{n}_i^\varepsilon)^{-1} \tilde{n}_{ix}^\varepsilon - \tilde{S}_i^0 (\tilde{n}_i^0)^{-1} \tilde{n}_{ix}^0 + (-1)^i \tilde{\Phi}_x^\varepsilon - \varepsilon^2 \left[\left(\sqrt{\tilde{n}_i^\varepsilon}\right)_{xx} \Big/ \sqrt{\tilde{n}_i^\varepsilon} \right]_x \\ &= -\left[\tilde{j}_i^\varepsilon (\tilde{n}_i^\varepsilon)^{-1} - \tilde{j}_i^0 (\tilde{n}_i^0)^{-1} \right], \end{aligned} \tag{4.3c}$$

$$\tilde{\Phi}_{xx}^\varepsilon = \tilde{\mathcal{N}}_1^\varepsilon - \tilde{\mathcal{N}}_2^\varepsilon, \quad i = 1,2, \quad \forall x \in \Omega, \tag{4.3d}$$

where $\tilde{S}_i^\varepsilon := S_i[\tilde{n}_i^\varepsilon, \tilde{j}_i^\varepsilon]$ and $\tilde{S}_i^0 := S_i[\tilde{n}_i^0, \tilde{j}_i^0]$.

By

$$\sum_{i=1}^2 \int_0^1 (4.3\text{c}) \tilde{\mathcal{N}}_{ix}^\varepsilon \, dx, \tag{4.4}$$

we get

$$\overbrace{\sum_{i=1}^{2}\int_0^1\Big[\tilde{S}_i^\varepsilon(\tilde{n}_i^\varepsilon)^{-1}\tilde{n}_{ix}^\varepsilon-\tilde{S}_i^0(\tilde{n}_i^0)^{-1}\tilde{n}_{ix}^0\Big]\tilde{\mathcal{N}}_{ix}^\varepsilon\,dx}^{\Theta_1}\;\overbrace{+\sum_{i=1}^{2}\int_0^1(-1)^i\tilde{\Phi}_x^\varepsilon\tilde{\mathcal{N}}_{ix}^\varepsilon\,dx}^{\Theta_2\text{: bipolar effect}}$$
$$=\underbrace{\varepsilon^2\sum_{i=1}^{2}\int_0^1\Big[\Big(\sqrt{\tilde{n}_i^\varepsilon}\Big)_{xx}\Big/\sqrt{\tilde{n}_i^\varepsilon}\Big]_x\tilde{\mathcal{N}}_{ix}^\varepsilon\,dx}_{\Theta_3}\;\underbrace{-\sum_{i=1}^{2}\int_0^1\big[\tilde{j}_i^\varepsilon(\tilde{n}_i^\varepsilon)^{-1}-\tilde{j}_i^0(\tilde{n}_i^0)^{-1}\big]\tilde{\mathcal{N}}_{ix}^\varepsilon\,dx}_{\Theta_4}. \quad (4.5)$$

Secondly, from [9], we have $\tilde{j}_i^0=J_i[\tilde{n}_i^0]$ where $J_i[\cdot]$ is given by Equation (2.8a). Thus, a standard argument gives

$$|\mathcal{J}_i^{\varepsilon_k}|=|J_i[\tilde{n}_i^\varepsilon]-J_i[\tilde{n}_i^0]|\le C\delta\|\tilde{\mathcal{N}}_i^\varepsilon\|. \quad (4.6)$$

Furthermore, we know that

$$\tilde{\mathcal{N}}_i^\varepsilon\in H_0^1(\Omega), \quad (4.7)$$

follows from the boundary conditions (1.8a) and (1.11a). Thus, the Poincaré inequality (3.58) can also be used to estimate $\tilde{\mathcal{N}}_i^\varepsilon$. Namely, by using Equations (1.13) and (1.15), Hölder's inequality, and integration by parts, for all $0<\varepsilon\le 1$, we get

$$\begin{aligned}\Theta_1=&\sum_{i=1}^{2}\int_0^1\Big\{\big[\tilde{S}_i^\varepsilon(\tilde{n}_i^\varepsilon)^{-1}-\tilde{S}_i^0(\tilde{n}_i^0)^{-1}\big]\tilde{n}_{ix}^\varepsilon+\tilde{S}_i^0(\tilde{n}_i^0)^{-1}\tilde{\mathcal{N}}_{ix}^\varepsilon\Big\}\tilde{\mathcal{N}}_{ix}^\varepsilon\,dx\\
\ge& C_6\sum_{i=1}^{2}\|\tilde{\mathcal{N}}_{ix}^\varepsilon\|^2+\sum_{i=1}^{2}\int_0^1\big[\tilde{S}_i^\varepsilon(\tilde{n}_i^\varepsilon)^{-1}-\tilde{S}_i^0(\tilde{n}_i^0)^{-1}\big]\tilde{n}_{ix}^\varepsilon\tilde{\mathcal{N}}_{ix}^\varepsilon\,dx\\
\ge& C_6\sum_{i=1}^{2}\|\tilde{\mathcal{N}}_{ix}^\varepsilon\|^2-C\delta\sum_{i=1}^{2}(\|\tilde{\mathcal{N}}_i^\varepsilon\|+|\tilde{\mathcal{J}}_i^\varepsilon|)\|\tilde{\mathcal{N}}_{ix}^\varepsilon\|\\
\ge& C_6\sum_{i=1}^{2}\|\tilde{\mathcal{N}}_{ix}^\varepsilon\|^2-C\delta\sum_{i=1}^{2}\|\tilde{\mathcal{N}}_{ix}^\varepsilon\|^2\\
\ge& C_6\sum_{i=1}^{2}\|\tilde{\mathcal{N}}_{ix}^\varepsilon\|^2, \qquad (4.8a)\\
\Theta_2=&\sum_{i=1}^{2}\int_0^1(-1)^{i-1}\tilde{\Phi}_{xx}^\varepsilon\tilde{\mathcal{N}}_i^\varepsilon\,dx=\int_0^1\tilde{\Phi}_{xx}^\varepsilon\sum_{i=1}^{2}(-1)^{i-1}\tilde{\mathcal{N}}_i^\varepsilon\,dx\\
=&\int_0^1\tilde{\Phi}_{xx}^\varepsilon(\tilde{\mathcal{N}}_1^\varepsilon-\tilde{\mathcal{N}}_2^\varepsilon)\,dx=\int_0^1(\tilde{\Phi}_{xx}^\varepsilon)^2\,dx\\
\ge& 0, \qquad (4.8b)\\
\Theta_3=&-\varepsilon^2\sum_{i=1}^{2}\int_0^1\Big[\Big(\sqrt{\tilde{n}_i^\varepsilon}\Big)_{xx}\Big/\sqrt{\tilde{n}_i^\varepsilon}\Big]\tilde{\mathcal{N}}_{ixx}^\varepsilon\,dx\\
=&-\varepsilon^2\sum_{i=1}^{2}\int_0^1\big(\tilde{w}_{ixx}\tilde{w}_i^{-1}\big)\tilde{\mathcal{N}}_{ixx}^\varepsilon\,dx\le\varepsilon^2\sum_{i=1}^{2}\|\tilde{w}_i^{-1}\|_\infty\|\tilde{w}_{ixx}\|\|\tilde{\mathcal{N}}_{ixx}^\varepsilon\|\\
\le&\varepsilon^2\sum_{i=1}^{2}\|\tilde{w}_i^{-1}\|_\infty\|\tilde{w}_{ixx}\|(\|\tilde{n}_{ixx}^\varepsilon\|+\|\tilde{n}_{ixx}^0\|)\end{aligned}$$

$$\leq C\varepsilon^2, \tag{4.8c}$$

we continue in the above fashion obtaining

$$\Theta_4 \leq C\delta \sum_{i=1}^{2} \|\tilde{\mathcal{N}}_{ix}^{\varepsilon}\|^2. \tag{4.8d}$$

Substituting the estimates (4.8) into Equation (4.5), we obtain

$$\sum_{i=1}^{2} \|\tilde{\mathcal{N}}_{ix}^{\varepsilon}\|^2 \leq C\varepsilon^2. \tag{4.9}$$

Combining Equation (4.9) with Equation (4.7), we get

$$\sum_{i=1}^{2} \|\tilde{\mathcal{N}}_{i}^{\varepsilon}\|_1 \leq C\varepsilon, \quad \forall 0<\varepsilon\leq 1. \tag{4.10}$$

From Equations (4.10) and (4.6), and the elliptic estimate $\|\tilde{\Phi}^{\varepsilon}\|_3 \leq C\sum_{i=1}^{2}\|\tilde{\mathcal{N}}_i^{\varepsilon}\|_1$, we can easily see that the convergence estimate (1.17a) holds true.

We are now in a position to show the convergence result (1.17b). From Equation (1.15b) and $\delta<1$, we have

$$\|\tilde{n}_{ixx}^{\varepsilon}\| \leq C\delta \leq C, \quad \forall 0<\varepsilon\leq 1, \quad i=1,2. \tag{4.11}$$

Combining Equation (4.11) with Equation (4.10), we conclude that there exists a subsequence $\{0<\varepsilon_k\leq 1\}$ of $\{0<\varepsilon\leq 1\}$ such that

$$\tilde{n}_{ixx}^{\varepsilon_k} \rightharpoonup \tilde{n}_{ixx}^{0} \quad \text{in } L^2 \text{ weakly as } \varepsilon_k\to 0^+, \quad i=1,2. \tag{4.12}$$

In order to improve the weak convergence (4.12) into the strong convergence, we have to use a standard functional analysis argument in a certain weighted L^2 space. Of course, we also need to use Equations (4.3a) and (4.3b) to establish some necessary estimates and limit results. Precisely, by using

$$\int_0^1 2\partial_x(4.3\text{b}) \frac{(\sqrt{\tilde{n}_i^{\varepsilon}})_{xx}}{\sqrt{\tilde{n}_i^{\varepsilon}}}\, dx, \quad i=1,2, \tag{4.13}$$

together with the homogeneous boundary condition (1.8b), we get

$$\int_0^1 \alpha_i^{\varepsilon}(\tilde{n}_{ixx}^{\varepsilon})^2\, dx + \underbrace{2\varepsilon^2 \int_0^1 \left\{\left[\left(\sqrt{\tilde{n}_i^{\varepsilon}}\right)_{xx} \Big/ \sqrt{\tilde{n}_i^{\varepsilon}}\right]_x\right\}^2 dx}_{\geq 0} = Q_i^{\varepsilon}, \tag{4.14a}$$

where

$$\alpha_i^{\varepsilon} := \tilde{S}_i^{\varepsilon}(\tilde{n}_i^{\varepsilon})^{-2}, \tag{4.14b}$$

$$Q_i^{\varepsilon} = Q_i[\tilde{n}_i^{\varepsilon},\tilde{j}_i^{\varepsilon},\tilde{\phi}^{\varepsilon}] := \int_0^1 \left\{ \frac{1}{2}\tilde{S}_i^{\varepsilon}(\tilde{n}_i^{\varepsilon})^{-3}(\tilde{n}_{ix}^{\varepsilon})^2\tilde{n}_{ixx}^{\varepsilon} - 2\left[\tilde{S}_{ix}^{\varepsilon}(\tilde{n}_i^{\varepsilon})^{-1}\tilde{n}_{ix}^{\varepsilon} - \tilde{S}_i^{\varepsilon}(\tilde{n}_i^{\varepsilon})^{-2}(\tilde{n}_{ix}^{\varepsilon})^2\right.\right.$$

$$+(-1)^i\tilde{\phi}^\varepsilon_{xx}+\left(\tilde{j}^\varepsilon_i(\tilde{n}^\varepsilon_i)^{-1}\right)_x\Big]\frac{(\sqrt{\tilde{n}^\varepsilon_i})_{xx}}{\sqrt{\tilde{n}^\varepsilon_i}}\Bigg\}\,dx,\ i=1,2. \quad (4.14c)$$

Let

$$\alpha^0_i:=\alpha^\varepsilon_i|_{\varepsilon=0}=\tilde{S}^0_i(\tilde{n}^0_i)^{-2}, \quad (4.15a)$$

$$Q^0_i:=Q^\varepsilon_i|_{\varepsilon=0}. \quad (4.15b)$$

By

$$\int_0^1 2\partial_x(4.3\text{a})\frac{(\sqrt{\tilde{n}^0_i})_{xx}}{\sqrt{\tilde{n}^0_i}}\,dx, \quad i=1,2, \quad (4.16)$$

we can easily check that

$$\int_0^1 \alpha^0_i(\tilde{n}^0_{ixx})^2\,dx=Q^0_i. \quad (4.17)$$

In addition, for the subsequence $\{\varepsilon_k\}$ in Equation (4.12), we claim that the limit of the number sequence

$$\lim_{\varepsilon_k\to 0^+}Q^{\varepsilon_k}_i=Q^0_i, \quad i=1,2, \quad (4.18)$$

holds true. In fact, from what has already been proved, a standard argument yields

$$\begin{aligned}\left|Q^{\varepsilon_k}_i-Q^0_i\right|\le& C\sum_{i=1}^2\|\tilde{\mathcal{N}}^{\varepsilon_k}_i\|_1+\left|\int_0^1\overbrace{\frac{1}{2}\tilde{S}^0_i(\tilde{n}^0_i)^{-3}(\tilde{n}^0_{ix})^2}^{\in L^2}\,(\tilde{n}^{\varepsilon_k}_{ixx}-\tilde{n}^0_{ixx})\,dx\right|\\ &+\left|\int_0^1\underbrace{A^0_i(\tilde{n}^0_i)^{-1}}_{\in L^2}\,(\tilde{n}^{\varepsilon_k}_{ixx}-\tilde{n}^0_{ixx})\,dx\right|\\ &\to 0, \quad \text{as } \varepsilon_k\to 0^+, \end{aligned}\quad (4.19a)$$

where

$$A^0_i:=\tilde{S}^0_{ix}(\tilde{n}^0_i)^{-1}\tilde{n}^0_{ix}-\tilde{S}^0_i(\tilde{n}^0_i)^{-2}(\tilde{n}^0_{ix})^2+(-1)^i\tilde{\phi}^0_{xx}+\left(\tilde{j}^0_i(\tilde{n}^0_i)^{-1}\right)_x, \quad i=1,2. \quad (4.19b)$$

Due to Equations (4.10), (1.13) and (1.15), a similar computation shows that

$$\lim_{\varepsilon\to 0^+}\int_0^1(\alpha^\varepsilon_i-\alpha^0_i)(\tilde{n}^\varepsilon_{ixx})^2\,dx=0, \quad i=1,2. \quad (4.20)$$

Actually, for all $0<\varepsilon\le 1$, we have

$$\begin{aligned}&\left|\int_0^1(\alpha^\varepsilon_i-\alpha^0_i)(\tilde{n}^\varepsilon_{ixx})^2\,dx\right|\\ =&\left|\int_0^1\left\{(\tilde{S}^\varepsilon_i-\tilde{S}^0_i)(\tilde{n}^\varepsilon_i)^{-2}+\tilde{S}^0_i\left[(\tilde{n}^\varepsilon_i)^{-2}-(\tilde{n}^0_i)^{-2}\right]\right\}(\tilde{n}^\varepsilon_{ixx})^2\,dx\right|\\ \le& C(|\tilde{\mathcal{J}}^\varepsilon_i|+|\tilde{\mathcal{N}}^\varepsilon_i|_0)\|\tilde{n}^\varepsilon_{ixx}\|^2\end{aligned}$$

$$
\begin{aligned}
&\le C\delta \|\tilde{\mathcal{N}}_i^\varepsilon\|_1 \\
&\le C\varepsilon \to 0^+.
\end{aligned} \tag{4.21}
$$

Based on Equations (4.14a), (4.18), (4.17), and (4.20), and the properties of the limit supremum of the number sequence, we have actually proved that

$$
\limsup_{\varepsilon_k\to 0^+} \int_0^1 \alpha_i^0 (\tilde{n}_{ixx}^{\varepsilon_k})^2\,dx = \limsup_{\varepsilon_k\to 0^+} \int_0^1 \alpha_i^{\varepsilon_k} (\tilde{n}_{ixx}^{\varepsilon_k})^2\,dx \le \int_0^1 \alpha_i^0 (\tilde{n}_{ixx}^0)^2\,dx, \quad i=1,2. \tag{4.22}
$$

From Lemma 1.1, we know that the function $\alpha_i^0 = \alpha_i^0(x)$ defined in Equation (4.15a) is strictly positive and continuous on the interval $\overline{\Omega} = [0,1]$. Then we choose α_i^0 as a weight and introduce a weighted L^2 space as follows

$$
L^2_{\alpha_i^0}(\Omega) := \left\{ f:\Omega\to\mathbb{R} \text{ is measurable} \;\middle|\; \int_0^1 \alpha_i^0 |f|^2\,dx < +\infty \right\}, \tag{4.23a}
$$

with the inner product

$$
(f,g)_{L^2_{\alpha_i^0}} := \int_0^1 \alpha_i^0 f g\,dx, \tag{4.23b}
$$

and the associated norm

$$
\|f\|_{L^2_{\alpha_i^0}} := \left(\int_0^1 \alpha_i^0 |f|^2\,dx \right)^{1/2}, \quad i=1,2. \tag{4.23c}
$$

We see at once that $L^2_{\alpha_i^0}(\Omega)$ is a Hilbert space.

Furthermore, Equation (4.22) implies that

$$
\limsup_{\varepsilon_k\to 0^+} \|\tilde{n}_{ixx}^{\varepsilon_k}\|_{L^2_{\alpha_i^0}} \le \|\tilde{n}_{ixx}^0\|_{L^2_{\alpha_i^0}}, \quad i=1,2, \tag{4.24a}
$$

and Equation (4.12) implies that

$$
\tilde{n}_{ixx}^{\varepsilon_k} \rightharpoonup \tilde{n}_{ixx}^0 \quad \text{in } L^2_{\alpha_i^0} \text{ weakly as } \varepsilon_k\to 0^+, \quad i=1,2. \tag{4.24b}
$$

We conclude from Equation (4.24) that

$$
\tilde{n}_{ixx}^{\varepsilon_k} \to \tilde{n}_{ixx}^0 \quad \text{in } L^2_{\alpha_i^0} \text{ strongly as } \varepsilon_k\to 0^+, \tag{4.25}
$$

hence that

$$
\lim_{\varepsilon_k\to 0^+} \|\tilde{\mathcal{N}}_{ixx}^{\varepsilon_k}\| = 0, \quad i=1,2. \tag{4.26}
$$

Next, from Equations (4.20), (4.25), and (4.14a), we have

$$
\varepsilon_k \left\| \left[\left(\sqrt{\tilde{n}_i^{\varepsilon_k}} \right)_{xx} \Big/ \sqrt{\tilde{n}_i^{\varepsilon_k}} \right]_x \right\| \to 0, \quad \text{as } \varepsilon_k\to 0^+, \quad i=1,2. \tag{4.27}
$$

This together with Equation (1.15), we deduce that

$$
\varepsilon_k \|\partial_x^3 \tilde{n}_i^{\varepsilon_k}\| = \varepsilon_k \left\| 2\tilde{n}_i^{\varepsilon_k} \left[\left(\sqrt{\tilde{n}_i^{\varepsilon_k}} \right)_{xx} \Big/ \sqrt{\tilde{n}_i^{\varepsilon_k}} \right]_x + 2(\tilde{n}_i^{\varepsilon_k})^{-1} \tilde{n}_{ix}^{\varepsilon_k} \tilde{n}_{ixx}^{\varepsilon_k} - (\tilde{n}_i^{\varepsilon_k})^{-2} (\tilde{n}_{ix}^{\varepsilon_k})^3 \right\|
$$

$$\leq C\left(\varepsilon_k\left\|\left[\left(\sqrt{\tilde{n}_i^{\varepsilon_k}}\right)_{xx}\Big/\sqrt{\tilde{n}_i^{\varepsilon_k}}\right]_x\right\|+\varepsilon_k\right)\to 0,\quad \text{as } \varepsilon_k\to 0^+,\quad i=1,2. \tag{4.28}$$

Finally, by $\|\partial_x(4.3\text{b})|_{\varepsilon_k}\|$, we get

$$\varepsilon_k^2\left\|\left[\left(\sqrt{\tilde{n}_i^{\varepsilon_k}}\right)_{xx}\Big/\sqrt{\tilde{n}_i^{\varepsilon_k}}\right]_{xx}\right\|=\hat{n}_i^{\varepsilon_k},\quad i=1,2, \tag{4.29a}$$

where

$$\hat{n}_i^{\varepsilon_k}=\hat{n}_i[\tilde{n}_i^{\varepsilon_k},\tilde{j}_i^{\varepsilon_k},\tilde{\phi}^{\varepsilon_k}]:=\left\|\left[\tilde{S}_i^{\varepsilon_k}(\tilde{n}_i^{\varepsilon_k})^{-1}\tilde{n}_{ix}^{\varepsilon_k}\right]_x+(-1)^i\tilde{\phi}_{xx}^{\varepsilon_k}+\left(\tilde{j}_i^{\varepsilon_k}(\tilde{n}_i^{\varepsilon_k})^{-1}\right)_x\right\|. \tag{4.29b}$$

Formally, let $\hat{n}_i^0:=\hat{n}_i^{\varepsilon_k}|_{\varepsilon_k=0}$. By $\|\partial_x(4.3\text{a})\|$, we rigorously obtain that $\hat{n}_i^0=0,\ i=1,2$. Similar analysis to that in Equation (4.18) shows that

$$\lim_{\varepsilon_k\to 0^+}\hat{n}_i^{\varepsilon_k}=\hat{n}_i^0=0,\quad i=1,2. \tag{4.30}$$

Combining Equation (4.29a) with Equation (4.30), we have

$$\lim_{\varepsilon_k\to 0^+}\varepsilon_k^2\left\|\left[\left(\sqrt{\tilde{n}_i^{\varepsilon_k}}\right)_{xx}\Big/\sqrt{\tilde{n}_i^{\varepsilon_k}}\right]_{xx}\right\|=0,\quad i=1,2. \tag{4.31}$$

We continue in the fashion of Equation (4.28) to obtain

$$\varepsilon_k^2\|\partial_x^4\tilde{n}_i^{\varepsilon_k}\|\leq C\left(\varepsilon_k^2\left\|\left[\left(\sqrt{\tilde{n}_i^{\varepsilon_k}}\right)_{xx}\Big/\sqrt{\tilde{n}_i^{\varepsilon_k}}\right]_{xx}\right\|+\varepsilon_k^2\|\partial_x^3\tilde{n}_i^{\varepsilon_k}\|+\varepsilon_k^2\right)\to 0,\quad \text{as } \varepsilon_k\to 0^+. \tag{4.32}$$

Combining Equations (4.26), (4.28), and (4.32), and using the elliptic estimate $\|\partial_x^4\tilde{\Phi}^\varepsilon\|\leq\sum_{i=1}^2\|\tilde{\mathcal{N}}_{ixx}^\varepsilon\|$, we thus prove that the convergence result (1.17b) holds true. □

4.2. Global solution case. We let $(n_1^\varepsilon,j_1^\varepsilon,n_2^\varepsilon,j_2^\varepsilon,\phi^\varepsilon)$ stand for the global solution of the QHD initial-boundary value problem (1.1)–(1.3), and continue to write $(n_1^0,j_1^0,n_2^0,j_2^0,\phi^0)$ for the global solution of the HD initial-boundary value problem (1.9)–(1.11).

In order to prove Theorem 1.4, we have to use Lemma 1.2, Theorem 1.2, and Theorem 1.3 simultaneously. Therefore, the semi-classical limit of the global solution is based on the subsequence $\{\varepsilon_k\}$ in Theorem 1.3. Below, we give the proof of Theorem 1.4.

Proof. We first choose the appropriate initial data to ensure that there are the global solution of QHD problem and that of the HD problem at the same time. To this end, we let

$$\delta_0:=\frac{1}{2}\min\{\delta_2,\ \delta_4\}>0, \tag{4.33}$$

where δ_2 is given in Lemma 1.2 and δ_4 in Theorem 1.2. By Theorem 1.3, for the above δ_0, there is a positive constant ε_0 such that if $\varepsilon_k\leq\varepsilon_0$, then

$$\sum_{i=1}^{2}\left(\|\tilde{n}_i^{\varepsilon_k}-\tilde{n}_i^0\|_1+|\tilde{j}_i^{\varepsilon_k}-\tilde{j}_i^0|\right)\leq\delta_0. \tag{4.34}$$

Now, we define

$$\delta_6 := \min\{\delta_5,\ \varepsilon_0,\ \delta_0\} > 0. \tag{4.35}$$

It is obvious that the condition (1.18) in Theorem 1.4 implies

$$\delta + \sum_{i=1}^{2} \|(n_{i0} - \tilde{n}_i^0, j_{i0} - \tilde{j}_i^0)\|_2 \le 2\delta_0 \le \delta_2. \tag{4.36}$$

Secondly, for $0 < \varepsilon_k \le \delta_6$, we introduce the error variables as follows

$$\mathcal{N}_i^{\varepsilon_k} := n_i^{\varepsilon_k} - n_i^0, \quad \mathcal{J}_i^{\varepsilon_k} := j_i^{\varepsilon_k} - j_i^0, \quad \Phi^{\varepsilon_k} := \phi^{\varepsilon_k} - \phi^0, \quad \forall (t,x) \in [0,+\infty) \times \Omega. \tag{4.37}$$

From

$$(1.1) - -(1.9), \quad (1.3a) - -(1.11a), \quad (1.3c) - -(1.11b), \tag{4.38}$$

we deduce that

$$\begin{cases} \mathcal{N}_{it}^{\varepsilon_k} + \mathcal{J}_{ix}^{\varepsilon_k} = 0, & (4.39a) \\ \mathcal{J}_{it}^{\varepsilon_k} + K_i \mathcal{N}_{ix}^{\varepsilon_k} + 2\left[j_i^{\varepsilon_k} j_{ix}^{\varepsilon_k} (n_i^{\varepsilon_k})^{-1} - j_i^0 j_{ix}^0 (n_i^0)^{-1} \right] & \\ \qquad - \left[(j_i^{\varepsilon_k})^2 (n_i^{\varepsilon_k})^{-2} n_{ix}^{\varepsilon_k} - (j_i^0)^2 (n_i^0)^{-2} n_{ix}^0 \right] & \\ \qquad + (-1)^i \left(\mathcal{N}_i^{\varepsilon_k} \phi_x^{\varepsilon_k} + n_i^0 \Phi_x^{\varepsilon_k} \right) + \mathcal{J}_i^{\varepsilon_k} = \varepsilon_k^2 n_i^{\varepsilon_k} \left[\left(\sqrt{n_i^{\varepsilon_k}} \right)_{xx} \Big/ \sqrt{n_i^{\varepsilon_k}} \right]_x, & (4.39b) \\ \Phi_{xx}^{\varepsilon_k} = \mathcal{N}_1^{\varepsilon_k} - \mathcal{N}_2^{\varepsilon_k}, \quad i = 1,2, \quad \forall (t,x) \in [0,+\infty) \times \Omega, & (4.39c) \end{cases}$$

with the homogeneous boundary conditions

$$\mathcal{N}_i^{\varepsilon_k}(t,0) = \mathcal{N}_i^{\varepsilon_k}(t,1) = \mathcal{N}_{it}^{\varepsilon_k}(t,0) = \mathcal{N}_{it}^{\varepsilon_k}(t,1) = \Phi^{\varepsilon_k}(t,0) = \Phi^{\varepsilon_k}(t,1) = 0. \tag{4.40}$$

From $-\partial_x$(4.39b) together with Equation (4.39a), we get

$$\begin{aligned} &\mathcal{N}_{itt}^{\varepsilon_k} - K_i \mathcal{N}_{ixx}^{\varepsilon_k} - 2\left[j_i^{\varepsilon_k} j_{ix}^{\varepsilon_k} (n_i^{\varepsilon_k})^{-1} - j_i^0 j_{ix}^0 (n_i^0)^{-1} \right]_x \\ &\qquad + \left[(j_i^{\varepsilon_k})^2 (n_i^{\varepsilon_k})^{-2} n_{ix}^{\varepsilon_k} - (j_i^0)^2 (n_i^0)^{-2} n_{ix}^0 \right]_x + (-1)^{i-1} \left(\mathcal{N}_i^{\varepsilon_k} \phi_x^{\varepsilon_k} + n_i^0 \Phi_x^{\varepsilon_k} \right)_x \\ &\qquad\qquad + \mathcal{N}_{it}^{\varepsilon_k} = -\varepsilon_k^2 \left\{ n_i^{\varepsilon_k} \left[\left(\sqrt{n_i^{\varepsilon_k}} \right)_{xx} \Big/ \sqrt{n_i^{\varepsilon_k}} \right]_x \right\}_x, \quad i = 1,2. \end{aligned} \tag{4.41}$$

Repeated application of Lemma 1.1 and Lemma 1.2 enables us to write

$$n_i^0(t,x),\ S_i^0 := S_i[n_i^0, j_i^0] \ge C_7 > 0, \quad i = 1,2, \quad \forall (t,x) \in [0,+\infty) \times \Omega, \tag{4.42a}$$

$$\sum_{i=1}^{2} \Big(\|(n_i^0 - n_{il}, j_i^0)(t)\|_2 + \|(n_{it}^0, j_{it}^0)(t)\|_1 \Big) + \|\phi^0(t)\|_2 \le C\delta_2 \le C, \quad \forall t \in [0,+\infty). \tag{4.42b}$$

Based on Theorem 1.1 and Theorem 1.2, we continue in the above fashion obtaining

$$n_i^{\varepsilon_k}(t,x),\ S_i^{\varepsilon_k} := S_i[n_i^{\varepsilon_k}, j_i^{\varepsilon_k}] \ge C_8 > 0, \quad i = 1,2, \quad \forall (t,x) \in [0,+\infty) \times \Omega, \tag{4.43a}$$

$$\sum_{i=1}^{2} \Big(\|(n_i^{\varepsilon_k} - n_{il}, j_i^{\varepsilon_k})(t)\|_2 + \|(n_{it}^{\varepsilon_k}, j_{it}^{\varepsilon_k})(t)\|_1$$

$$+\|(\varepsilon_k\partial_x^3 n_i^{\varepsilon_k},\varepsilon_k\partial_x^3 j_i^{\varepsilon_k},\varepsilon_k^2\partial_x^4 n_i^{\varepsilon_k})(t)\|\Big)+\|\phi^{\varepsilon_k}(t)\|_3\le C\delta_6\le C,\quad \forall t\in[0,+\infty),\tag{4.43b}$$

for $\forall 0<\varepsilon_k\le\delta_6$, where C_8,C,δ_6 are independent of ε_k.

The remaining part of the proof will be divided into three steps.

Step I. From

$$\sum_{i=1}^{2}\int_0^1 (4.39\text{b})\mathcal{J}_i^{\varepsilon_k}\,dx,\tag{4.44}$$

we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_0^1\sum_{i=1}^{2}\frac{1}{2}(\mathcal{J}_i^{\varepsilon_k})^2\,dx+\int_0^1\sum_{i=1}^{2}(\mathcal{J}_i^{\varepsilon_k})^2\,dx\\
=&-\sum_{i=1}^{2}\int_0^1 K_i\mathcal{N}_{ix}^{\varepsilon_k}\mathcal{J}_i^{\varepsilon_k}\,dx-\sum_{i=1}^{2}\int_0^1 2\left[j_i^{\varepsilon_k}j_{ix}^{\varepsilon_k}(n_i^{\varepsilon_k})^{-1}-j_i^0 j_{ix}^0(n_i^0)^{-1}\right]\mathcal{J}_i^{\varepsilon_k}\,dx\\
&+\sum_{i=1}^{2}\int_0^1\left[(j_i^{\varepsilon_k})^2(n_i^{\varepsilon_k})^{-2}n_{ix}^{\varepsilon_k}-(j_i^0)^2(n_i^0)^{-2}n_{ix}^0\right]\mathcal{J}_i^{\varepsilon_k}\,dx\\
&+\sum_{i=1}^{2}\int_0^1(-1)^{i-1}\left(\mathcal{N}_i^{\varepsilon_k}\phi_x^{\varepsilon_k}+n_i^0\Phi_x^{\varepsilon_k}\right)\mathcal{J}_i^{\varepsilon_k}\,dx\\
&+\varepsilon_k^2\sum_{i=1}^{2}\int_0^1 n_i^{\varepsilon_k}\left[\left(\sqrt{n_i^{\varepsilon_k}}\right)_{xx}\Big/\sqrt{n_i^{\varepsilon_k}}\right]_x\mathcal{J}_i^{\varepsilon_k}\,dx\\
=&\Gamma_1+\Gamma_2+\Gamma_3+\Gamma_4+\Gamma_5,
\end{aligned}\tag{4.45}$$

By using Equations (4.42) and (4.43), we estimate Γ_1–Γ_5 as follows

$$\Gamma_1\le C\sum_{i=1}^{2}\|(\mathcal{N}_{ix}^{\varepsilon_k},\mathcal{J}_i^{\varepsilon_k})(t)\|^2,\tag{4.46a}$$

$$\begin{aligned}
\Gamma_2\le& C\sum_{i=1}^{2}\int_0^1(|\mathcal{J}_i^{\varepsilon_k}|+|\mathcal{J}_{ix}^{\varepsilon_k}|+|\mathcal{N}_i^{\varepsilon_k}|)|\mathcal{J}_i^{\varepsilon_k}|\,dx\\
\le& C\sum_{i=1}^{2}\|(\mathcal{J}_i^{\varepsilon_k},\mathcal{J}_{ix}^{\varepsilon_k},\mathcal{N}_i^{\varepsilon_k})(t)\|^2,
\end{aligned}\tag{4.46b}$$

$$\begin{aligned}
\Gamma_3\le& C\sum_{i=1}^{2}\int_0^1(|\mathcal{J}_i^{\varepsilon_k}|+|\mathcal{N}_i^{\varepsilon_k}|+|\mathcal{N}_{ix}^{\varepsilon_k}|)|\mathcal{J}_i^{\varepsilon_k}|\,dx\\
\le& C\sum_{i=1}^{2}\|(\mathcal{J}_i^{\varepsilon_k},\mathcal{N}_i^{\varepsilon_k},\mathcal{N}_{ix}^{\varepsilon_k})(t)\|^2,
\end{aligned}\tag{4.46c}$$

$$\begin{aligned}
\Gamma_4\le& C\sum_{i=1}^{2}\int_0^1(|\mathcal{N}_i^{\varepsilon_k}|+|\Phi_x^{\varepsilon_k}|)|\mathcal{J}_i^{\varepsilon_k}|\,dx\le C\sum_{i=1}^{2}\|(\mathcal{N}_i^{\varepsilon_k},\Phi_x^{\varepsilon_k},\mathcal{J}_i^{\varepsilon_k})(t)\|^2\\
\le& C\sum_{i=1}^{2}\|(\mathcal{N}_i^{\varepsilon_k},\mathcal{J}_i^{\varepsilon_k})(t)\|^2,
\end{aligned}\tag{4.46d}$$

$$
\begin{aligned}
\Gamma_5 =&\varepsilon_k^2 \sum_{i=1}^{2} \int_0^1 \left[\left(\sqrt{n_i^{\varepsilon_k}} \right)_{xx} \Big/ \sqrt{n_i^{\varepsilon_k}} \right]_x n_i^{\varepsilon_k} \mathcal{J}_i^{\varepsilon_k} \, dx \\
=& -\varepsilon_k^2 \sum_{i=1}^{2} \int_0^1 \left[\left(\sqrt{n_i^{\varepsilon_k}} \right)_{xx} \Big/ \sqrt{n_i^{\varepsilon_k}} \right] (n_i^{\varepsilon_k} \mathcal{J}_i^{\varepsilon_k})_x \, dx \\
\leq& \varepsilon_k^2 \sum_{i=1}^{2} \left| 1 \Big/ \sqrt{n_i^{\varepsilon_k}} \right|_0 \left\| \left(\sqrt{n_i^{\varepsilon_k}} \right)_{xx} \right\| \Big(|n_{ix}^{\varepsilon_k}|_0 \|\mathcal{J}_i^{\varepsilon_k}\| + |n_i^{\varepsilon_k}|_0 \|\mathcal{J}_{ix}^{\varepsilon_k}\| \Big) \\
\leq& C\varepsilon_k^2 \sum_{i=1}^{2} \left\| \left(\sqrt{n_i^{\varepsilon_k}} \right)_{xx} \right\| \|(\mathcal{J}_i^{\varepsilon_k}, \mathcal{J}_{ix}^{\varepsilon_k})(t)\| \leq C\varepsilon_k^2.
\end{aligned} \tag{4.46e}
$$

Substituting Equation (4.46) into Equation (4.45), we get

$$
\frac{d}{dt} \sum_{i=1}^{2} \frac{1}{2} \|\mathcal{J}_i^{\varepsilon_k}(t)\|^2 + \sum_{i=1}^{2} \|\mathcal{J}_i^{\varepsilon_k}(t)\|^2 \leq C \Big(\sum_{i=1}^{2} \|(\mathcal{N}_i^{\varepsilon_k}, \mathcal{J}_i^{\varepsilon_k})(t)\|_1^2 + \varepsilon_k^2 \Big). \tag{4.47}
$$

Step II. From

$$
\sum_{i=1}^{2} \int_0^1 (4.41) \mathcal{N}_{it}^{\varepsilon_k} \, dx, \tag{4.48}
$$

repeated application of Equations (4.42) and (4.43) together with the above standard argument gives

$$
\frac{d}{dt} \int_0^1 \frac{1}{2} \sum_{i=1}^{2} \left[(\mathcal{N}_{it}^{\varepsilon_k})^2 + S_i^0 (\mathcal{N}_{ix}^{\varepsilon_k})^2 \right] dx + \sum_{i=1}^{2} \|\mathcal{N}_{it}^{\varepsilon_k}(t)\|^2 = B(t), \tag{4.49a}
$$

where

$$
\begin{aligned}
B(t) :=& \sum_{i=1}^{2} \int_0^1 2 \Big\{ \left[j_i^{\varepsilon_k} (n_i^{\varepsilon_k})^{-1} - j_i^0 (n_i^0)^{-1} \right] j_{ix}^{\varepsilon_k} \Big\}_x \mathcal{N}_{it}^{\varepsilon_k} \, dx \\
&- \sum_{i=1}^{2} \int_0^1 \left[j_i^0 (n_i^0)^{-1} \right]_x (\mathcal{N}_{it}^{\varepsilon_k})^2 \, dx - \sum_{i=1}^{2} \int_0^1 \frac{1}{2} \left[(j_i^0)^2 (n_i^0)^{-2} \right]_t (\mathcal{N}_{ix}^{\varepsilon_k})^2 \, dx \\
&- \sum_{i=1}^{2} \int_0^1 \Big\{ \left[(j_i^{\varepsilon_k})^2 (n_i^{\varepsilon_k})^{-2} - (j_i^0)^2 (n_i^0)^{-2} \right] n_{ix}^{\varepsilon_k} \Big\}_x \mathcal{N}_{it}^{\varepsilon_k} \, dx \\
&+ \sum_{i=1}^{2} \int_0^1 (-1)^i \left(\mathcal{N}_i^{\varepsilon_k} \phi_x^{\varepsilon_k} + n_i^0 \Phi_x^{\varepsilon_k} \right)_x \mathcal{N}_{it}^{\varepsilon_k} \, dx \\
&+ \varepsilon_k^2 \sum_{i=1}^{2} \int_0^1 n_i^{\varepsilon_k} \left[\left(\sqrt{n_i^{\varepsilon_k}} \right)_{xx} \Big/ \sqrt{n_i^{\varepsilon_k}} \right]_x \mathcal{N}_{ixt}^{\varepsilon_k} \, dx
\end{aligned} \tag{4.49b}
$$

$$
\leq C \Big(\sum_{i=1}^{2} \|(\mathcal{N}_i^{\varepsilon_k}, \mathcal{J}_i^{\varepsilon_k})(t)\|_1^2 + \varepsilon_k \Big). \tag{4.49c}
$$

Substituting Equation (4.49c) into Equation (4.49a), we have

$$
\frac{d}{dt} \int_0^1 \frac{1}{2} \sum_{i=1}^{2} \left[(\mathcal{N}_{it}^{\varepsilon_k})^2 + S_i^0 (\mathcal{N}_{ix}^{\varepsilon_k})^2 \right] dx + \sum_{i=1}^{2} \|\mathcal{N}_{it}^{\varepsilon_k}(t)\|^2 \leq C \Big(\sum_{i=1}^{2} \|(\mathcal{N}_i^{\varepsilon_k}, \mathcal{J}_i^{\varepsilon_k})(t)\|_1^2 + \varepsilon_k \Big). \tag{4.50}
$$

Step III. From

$$(4.47)+(4.50), \tag{4.51}$$

we have

$$\frac{d}{dt}E^{\varepsilon_k}(t)+\underbrace{\sum_{i=1}^{2}\|\mathcal{J}_i^{\varepsilon_k}(t)\|_1^2}_{\geq 0}\leq C\Big(\sum_{i=1}^{2}\|(\mathcal{N}_i^{\varepsilon_k},\mathcal{J}_i^{\varepsilon_k})(t)\|_1^2+\varepsilon_k\Big), \tag{4.52a}$$

where

$$E^{\varepsilon_k}(t):=\int_0^1\frac{1}{2}\sum_{i=1}^{2}\left[(\mathcal{J}_i^{\varepsilon_k})^2+(\mathcal{J}_{ix}^{\varepsilon_k})^2+S_i^0(\mathcal{N}_{ix}^{\varepsilon_k})^2\right]dx. \tag{4.52b}$$

Here we have used Equation (4.39a) and $0<\varepsilon_k<1$. By Equation (4.42a) and Equation (3.58), we can easily check the equivalence

$$C_{9l}\sum_{i=1}^{2}\|(\mathcal{N}_i^{\varepsilon_k},\mathcal{J}_i^{\varepsilon_k})(t)\|_1^2\leq E^{\varepsilon_k}(t)\leq C_{9r}\sum_{i=1}^{2}\|(\mathcal{N}_i^{\varepsilon_k},\mathcal{J}_i^{\varepsilon_k})(t)\|_1^2,\quad \forall t\in[0,+\infty). \tag{4.53}$$

Combining Equation (4.52a) with Equation (4.53), we obtain the ordinary differential inequality

$$\frac{d}{dt}E^{\varepsilon_k}(t)\leq C_{10}E^{\varepsilon_k}(t)+C\varepsilon_k,\quad \forall t\in[0,+\infty). \tag{4.54}$$

Applying Gronwall's inequality to Equation (4.54) and noting that $E^{\varepsilon_k}(0)=0$, we get

$$E^{\varepsilon_k}(t)\leq Ce^{C_{10}t}\varepsilon_k,\quad \forall t\in[0,+\infty). \tag{4.55}$$

Substituting Equation (4.53) into Equation (4.55), we have

$$\sum_{i=1}^{2}\|(\mathcal{N}_i^{\varepsilon_k},\mathcal{J}_i^{\varepsilon_k})(t)\|_1\leq Ce^{C_{10}t/2}\varepsilon_k^{1/2},\quad \forall t\in[0,+\infty). \tag{4.56}$$

Let $\gamma_3:=C_{10}/2>0$, by using the elliptic estimate $\|\Phi^{\varepsilon_k}(t)\|_3\leq\sum_{i=1}^{2}\|\mathcal{N}_i^{\varepsilon_k}(t)\|_1$ together with Equation (4.56), we see that Equation (1.19) is true.

Finally, we show the convergence estimate (1.20). For this purpose, fix $0<\varepsilon_k<1$, define

$$T_k=T(\varepsilon_k):=-\frac{\ln\varepsilon_k}{4\gamma_3}>0, \tag{4.57}$$

Substituting (4.57) into (4.56), we get

$$\sum_{i=1}^{2}\|(\mathcal{N}_i^{\varepsilon_k},\mathcal{J}_i^{\varepsilon_k})(t)\|_1\leq Ce^{\gamma_3 t}\varepsilon_k^{1/2}\leq Ce^{\gamma_3 T_k}\varepsilon_k^{1/2}=C\varepsilon_k^{1/4},\quad \forall t\in[0,T_k]. \tag{4.58}$$

For $\forall t\geq T_k$, using the estimates (1.14), (1.16), and (1.17a), we obtain

$$\sum_{i=1}^{2}\|(\mathcal{N}_i^{\varepsilon_k},\mathcal{J}_i^{\varepsilon_k})(t)\|_1\leq\sum_{i=1}^{2}\Big(\|(n_i^{\varepsilon_k}-\tilde{n}_i^{\varepsilon_k},j_i^{\varepsilon_k}-\tilde{j}_i^{\varepsilon_k})(t)\|_1$$

$$
\begin{aligned}
&\qquad\qquad +\|(\tilde{n}_i^{\varepsilon_k}-\tilde{n}_i^0,\tilde{j}_i^{\varepsilon_k}-\tilde{j}_i^0)(t)\|_1+\|(n_i^0-\tilde{n}_i^0,j_i^0-\tilde{j}_i^0)(t)\|_1\Big)\\
&\le C\delta_4 e^{-\gamma_2 t}+C\varepsilon_k+C\delta_2 e^{-\gamma_1 t}\\
&\le C\big(e^{-\gamma_1 T_k}+e^{-\gamma_2 T_k}+\varepsilon_k\big)\\
&= C\big(\varepsilon_k^{\gamma_1/(4\gamma_3)}+\varepsilon_k^{\gamma_2/(4\gamma_3)}+\varepsilon_k\big)\\
&\le C\varepsilon_k^{\gamma_4},
\end{aligned}
\tag{4.59}
$$

where

$$
\gamma_4 := \min\left\{\frac{\gamma_1}{4\gamma_3},\frac{\gamma_2}{4\gamma_3},\frac{1}{4}\right\}>0. \tag{4.60}
$$

Since $\gamma_4\le 1/4$ and $0<\varepsilon_k<1$, combining Equation (4.58) with Equation (4.59), we hence have

$$
\sum_{i=1}^{2}\|(\mathcal{N}_i^{\varepsilon_k},\mathcal{J}_i^{\varepsilon_k})(t)\|_1+\|\Phi^{\varepsilon_k}(t)\|_3\le C\varepsilon_k^{\gamma_4},\quad \forall t\in[0,+\infty). \tag{4.61}
$$

Here we have used the elliptic estimate $\|\Phi^{\varepsilon_k}(t)\|_3\le\sum_{i=1}^2\|\mathcal{N}_i^{\varepsilon_k}(t)\|_1$ again. Note that the right-hand side of Equation (4.61) is independent of time $t\in[0,+\infty)$, this immediately implies Equation (1.20). ∎

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