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ANALYSIS OF KINETIC AND MACROSCOPIC MODELS OF PURSUIT-EVASION DYNAMICS*

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Abstract. We analyse kinetic and macroscopic models intended to describe pursuit-evasion dynamics. We investigate well-posedness issues and the connection between the two model, based on asymptotic analysis. In particular, in dimension 2, we show that the macroscopic system has some regularizing effects: bounded solutions are produced, even when starting from integrable but possibly unbounded data. Our proof is based on De Giorgi's method.

Keywords. Collective behaviour, self-propelling particles, self-organization, kinetic models, hydrodynamic models, regularity of solutions, De Giorgi's method.

AMS subject classifications. 92D25, 92C17, 74A25, 76N10.

1. Introduction

In [13], a hierarchy of equations has been introduced in order to model some simple pursuit-evasion dynamics. Roughly speaking, these equations describe the interaction between prey and chasers, governed by the following simple rule: prey are repelled by the chasers while chasers are attracted by the presence of prey. The proposed models range from individual-based models, which have the form of systems of ODEs, to macroscopic equations, where the unknowns are the local concentrations of prey and chasers. Connections between these equations are formally drawn in [13], based on suitable rescaling and asymptotic arguments. Here, we wish to analyse in more detail some aspects of this hierarchy.

More precisely, we are mainly interested in the following system of PDEs:

$$\begin{aligned} &\partial_t \rho_c - \mathrm{div}_x(\rho_c \nabla_x \Phi_c) = \Delta_x \rho_c, \\ &\partial_t \rho_p - \mathrm{div}_x(\rho_p \nabla_x \Phi_p) = \Delta_x \rho_p, \end{aligned} \tag{1.1}$$

where the potentials are self-consistently defined by

$$\Delta_x \Phi_c = \rho_p, \qquad -\Delta_x \Phi_p = \alpha \rho_c \quad (\alpha > 0). \tag{1.2}$$

Here, the equations are considered on the whole space $\mathbb{R}^N$; the functions $(t,x) \mapsto \rho_c(t,x)$ and $(t,x) \mapsto \rho_p(t,x)$ stand for the concentration of chasers and prey, respectively. It means that, for any subdomain $\Omega \subset \mathbb{R}^N$, $\int_\Omega \rho_j(t,x)\,\mathrm{d}x$ gives the number of the individuals in the population labelled by j that can be found in the domain Ω at time t. The system (1.1)–(1.2) is complemented by initial conditions

$$\rho_p\Big|_{t=0} = \rho_{p,0}, \qquad \rho_c\Big|_{t=0} = \rho_{c,0}$$

which are thus naturally non-negative and integrable functions. The definition of the potentials Φ_c and Φ_p in Equation (1.2) is intended to describe the attractive effect of

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the prey on the chasers population and the repulsive effect of the chasers on the prey: the population $j\in\{c,p\}$ is driven according to the gradient of the potentials Φ_j, which is itself defined by the other population, through the Poisson equation (1.2). The sign determines whether a population has an attractive or a repulsive effect on the other. This should be thought of by analogy with the definition of repulsive electrostatic forces and attractive gravitational forces. For further details on the modelling issues and alternate definitions of the potentials, we refer the reader to [13]. We bear in mind that Equation (1.2) should be understood as a convolution relation involving the elementary solution of $-\Delta_x$. We shall also pay attention to the following kinetic version of pursuit-evasion dynamics:

$$\begin{aligned}
&\partial_t f_c+v\cdot\nabla_x f_c-\nabla_x\Phi_c\cdot\nabla_v f_c=L(f_c),\\
&\partial_t f_p+v\cdot\nabla_x f_p-\nabla_x\Phi_p\cdot\nabla_v f_p=L(f_p),\\
&\Delta_x\Phi_c=\int_{\mathbb{R}^N} f_p\,\mathrm{d}v,\qquad -\Delta_x\Phi_p=\alpha\int_{\mathbb{R}^N} f_c\,\mathrm{d}v,
\end{aligned}\tag{1.3}$$

where L stands for the Fokker–Planck operator

$$L(f)=\mathrm{div}_v(vf+\nabla_v f)=\mathrm{div}_v\Big(M\nabla_v\Big(\frac{f}{M}\Big)\Big),\qquad M(v)=\frac{1}{(2\pi)^{N/2}}e^{-v^2/2}.\tag{1.4}$$

It corresponds to a statistical description of the population: v is interpreted as the velocity variable, and $f_j(t,x,v)$ is the distribution in phase space of the population j. In other words, $\int_\Omega\int_{\mathscr{V}} f_j(t,x,v)\,\mathrm{d}v\,\mathrm{d}x$ gives the number of the individuals in the population j which are in the domain $\Omega\subset\mathbb{R}^N$, with a velocity $v\in\mathscr{V}\subset\mathbb{R}^N$, at time t. System (1.3) is written in dimensionless variables. It can be rescaled by introducing a parameter $0<\epsilon\ll 1$ which leads to

$$\begin{aligned}
&\partial_t f_c^\epsilon+\frac{1}{\epsilon}(v\cdot\nabla_x f_c^\epsilon-\nabla_x\Phi_c^\epsilon\cdot\nabla_v f_c^\epsilon)=\frac{1}{\epsilon^2}L(f_c^\epsilon),\\
&\partial_t f_p^\epsilon+\frac{1}{\epsilon}(v\cdot\nabla_x f_p^\epsilon-\nabla_x\Phi_p^\epsilon\cdot\nabla_v f_p^\epsilon)=\frac{1}{\epsilon^2}L(f_p^\epsilon),\\
&\Delta_x\Phi_c^\epsilon=\int_{\mathbb{R}^N} f_p^\epsilon\,\mathrm{d}v,\\
&-\Delta_x\Phi_p^\epsilon=\alpha\int_{\mathbb{R}^N} f_c^\epsilon\,\mathrm{d}v.
\end{aligned}\tag{1.5}$$

We refer the reader to [13, Section 2.3] for details on the scaling; see also [33] for a similar discussion in a different context. The regime can be roughly motivated as follows. The Fokker–Planck operator describes drag effects, which make the velocity of the individuals relax towards the gradient of the potential. We are assuming that the relaxation time associated to this friction force is small compared to the time scale of observation (in other words the strength of the friction is strong). In the meantime, the typical velocity of the individuals is supposed to be large compared to the observation units, while the strength of the coupling force is weak. The arguments developed in [13] indicate that the system (1.1)–(1.2) can be obtained from the system (1.5) in the regime $\epsilon\to 0$. In this work, we wish to address the following problems:

- well-posedness of the system (1.1)–(1.2) and qualitative properties of the solutions
- well-posedness of the system (1.3).

- derivation of the macroscopic model (1.1)–(1.2) from the kinetic equations rescaled as in the system (1.5).

At first sight, the system (1.1)–(1.2), shares the structure of the following Keller–Segel type system, introduced in [21, 22, 30]:

$$\begin{aligned} &\partial_t \rho - \mathrm{div}_x(\rho \nabla_x \Phi) = \Delta_x \rho, \\ &\Delta_x \Phi = \rho. \end{aligned} \tag{1.6}$$

As revealed in [15, 20], this system (1.6) is known to produce Dirac masses in finite time when the integral $\int_{\mathbb{R}^N} \rho(0,x)\,\mathrm{d}x$ exceeds a certain threshold. We refer the reader to the surveys [17, 18] for further information and references on the system (1.6). Many mechanisms have been discussed that can prevent the blow-up of the solutions in such PDE systems describing chemotactic phenomena. For instance, adding a logistic-like source term or cross-diffusion terms might have such a regularizing effect, as studied in [44] and [7, 16], respectively. Another option, relevant in several physical situations, consists in introducing non-linearities in the convection and/or diffusion coefficients. Depending on growth assumptions on the non-linearities, the modified system can be shown to admit bounded solutions [19, 41]. Closer to our purposes, coupling between several species might also have some regularizing effects that lead to bounded solutions. This is particularly the case for chemotaxis-haptotaxis models that describe the invasion of tissues by tumor cells; see [28, 32, 37, 38, 39, 40, 43] or, for systems modelling ants foraging, see [2]. Hence, for the system (1.1)–(1.2) it is natural to wonder whether or not solutions become singular in finite time. In fact, we shall show that the system (1.1)–(1.2) admits bounded solutions and, furthermore, that the system has a regularizing effect: we shall prove that integrable data, possibly unbounded, lead to bounded solutions, at least in dimension $N=2$ (and $N=1$). This is in contrast with the behaviour of the system (1.6). Our results in this direction are complementary to the recent work [42], where similar two-species models are analysed, with equations set on a bounded domain with Neumann boundary conditions (for both the densities and the potentials): [42] justifies the well-posedness of the system for continuous initial data. Here, we show that the solutions become instantaneously bounded for general, possibly unbounded, data. The analysis of the boundedness of solutions for such PDEs systems which involve some attractive self-consistent potential usually relies either on semigroup techniques or on suitable adaptations of Moser's iteration reasoning, a method inspired from [1]. Here, the proof we propose uses De Giorgi's approach, in the spirit of [2, 14, 32]. Concerning asymptotic issues, connections between Keller–Segel models of type (1.6) have been studied via hydrodynamical limits in [27] and the derivation of drift-diffusion systems like (1.6) from kinetic models has been investigated for instance in [9, 12, 26, 33].

The paper is organized as follows. In Section 2, we set up a few notations and give the precise statements of the main results. Section 3 is devoted to the analysis of the system (1.1)–(1.2). In Section 4, we turn to the investigation of the system (1.3) and of the asymptotic regime.

2. Main results

We start with the statements concerned with the existence and regularity theory for the macroscopic system (1.1)–(1.2). We refer the reader to [42, Thm. 1.1] for, among others, existence-uniqueness results for the system (1.1)–(1.2) in a bounded domain with Neumann conditions, when $N \leq 3$ and starting with continuous initial data. It is remarkable that in dimension $N=2$ the system produces bounded, and thus smooth, solutions, while the data can be unbounded. (A similar result holds in dimension $N=1$; see Section 3.5 below.)

THEOREM 2.1. *Let $(\rho_{p,0},\rho_{c,0})$ be a pair of functions in $L^1\cap L^\infty(\mathbb{R}^N)$. Furthermore, we assume that $|x|^2(\rho_{p,0}+\rho_{c,0})\in L^1(\mathbb{R}^N)$. Then, for any $T>0$, the system (1.1)–(1.2) with data $(\rho_{p,0},\rho_{c,0})$ admits a unique solution which is bounded on $[0,T]\times\mathbb{R}^N$ and lies in $C^\infty((0,T)\times\mathbb{R}^N)$.*

THEOREM 2.2. *Let $N=2$. We suppose that $\rho_{p,0},\rho_{c,0}$ belong to in $L^1\cap L^{1+\delta}(\mathbb{R}^2)$ for some $\delta>0$, with $|x|^2(\rho_{p,0}+\rho_{c,0})\in L^1(\mathbb{R}^2)$. Then, there exists a solution (ρ_p,ρ_c) in $C([0,\infty);L^1(\mathbb{R}^2)-\text{weak})$ of the system (1.1)–(1.2) with initial data $(\rho_{p,0},\rho_{c,0})$. Furthermore, for any $t_\star>0$, there exists a constant $M_\star>0$ such that $0\le\rho_p(t,x),\ \rho_c(t,x)\le M_\star$ holds for a.e. $(t,x)\in[t_\star,\infty)\times\mathbb{R}^2$, and the solution lies in $C^\infty([t_\star,\infty)\times\mathbb{R}^2)$.*

The system (1.3) is a two-species version of the Vlasov–Poisson–Fokker–Planck equations. We can use the methods introduced in [33] to justify the existence of solutions, as well as to investigate the behaviour of the solutions of the system (1.5) as $\epsilon\to0$. In order to state the results, let us introduce the norm

$$|||f|||_q:=\left(\iint f^q M^{1-q}\,\mathrm{d}v\,\mathrm{d}x\right)^{1/q}.$$

Given $0<T<\infty$, we also define the following functional space:

$$\mathcal{M}_{q,T}:=\left\{f_c,f_p:[0,T]\times\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R},\ \sup_{0\le t\le T}\left(|||f_c(t,\cdot)|||_q+|||f_p(t,\cdot)|||_q\right)<+\infty\right\}.$$

Given $0<m_c,m_p<\infty$, we shall denote by $\mathscr{C}_{q,T}$ the convex subset in $\mathcal{M}_{q,T}$ made of non-negative functions in $\mathcal{M}_{q,T}$ which satisfy

$$\iint\begin{pmatrix}f_c\\ f_p\end{pmatrix}\mathrm{d}v\,\mathrm{d}x=\begin{pmatrix}m_c\\ m_p\end{pmatrix}.$$

THEOREM 2.3. *Let $f_{c,0},f_{p,0}$ be a pair of non-negative functions such that*

$$\iint f_{c,0}\,\mathrm{d}v\,\mathrm{d}x=m_c,\qquad \iint f_{p,0}\,\mathrm{d}v\,\mathrm{d}x=m_p,$$
$$|||f_{c,0}|||_q+|||f_{p,0}|||_q<\infty,$$

for some $q>\max(N,2)$. We also assume that

$$\iint f_{j,0}\left(|\ln(f_{j,0})|+|x|+\frac{v^2}{2}\right)\mathrm{d}v\,\mathrm{d}x<\infty$$

for $j\in\{c,p\}$. Then, there exists $T>0$ such that the system (1.3) complemented with the initial data $(f_c,f_p)\big|_{t=0}=(f_{c,0},f_{p,0})$ has a solution which belongs to $\mathscr{C}_{q,T}$.

THEOREM 2.4. *Let $\left(f^\epsilon_{c,0},f^\epsilon_{p,0}\right)_{\epsilon>0}$ be a sequence of non-negative functions bounded in the $|||\cdot|||_q$-norm for some $q>\max(N,2)$, with $\int f^\epsilon_{c,0}\,\mathrm{d}v\,\mathrm{d}x=m_c$ and $\int f^\epsilon_{p,0}\,\mathrm{d}v\,\mathrm{d}x=m_p$. Furthermore, we assume that*

$$\sup_{\epsilon>0}\left(\iint f^\epsilon_{j,0}\left(|\ln(f^\epsilon_{j,0})|+|x|+\frac{v^2}{2}\right)\mathrm{d}v\,\mathrm{d}x\right)<\infty$$

holds for $j\in\{p,c\}$. Let $(f^\epsilon_c,f^\epsilon_p)$ be a solution in $\mathscr{C}_{q,T}$ of the system (1.5) complemented with the initial data $(f^\epsilon_{c,0},f^\epsilon_{p,0})$. Then, provided $0<T<\infty$ is small enough, up to a *subsequence (still labelled by ϵ), the macroscopic concentrations* $\rho_c^\epsilon = \int f_c^\epsilon\,\mathrm{d}v$ *and* $\rho_p^\epsilon = \int f_p^\epsilon\,\mathrm{d}v$ *converge strongly to* ρ_c *and* ρ_p*, respectively, in* $L^s(0,T;L^r(\mathbb{R}^N))$ *for any* $1 \le s < \infty$*,* $1 \le r < q$*, where* ρ_c *and* ρ_p *are solutions of the system* (1.1)–(1.2)*, with initial data defined by the weak limits of* $\int f_{c,0}^\epsilon\,\mathrm{d}v$ *and* $\int f_{p,0}^\epsilon\,\mathrm{d}v$*.*

The last two results are only local in time. This is due to the adopted functional framework, directly inspired by [33], and the restriction comes from non-linear estimates for the norm $|\!|\!|\cdot|\!|\!|_q$. Again, difficulties are related to the meaning and the stability of the product between the densities and the force field. It would be interesting to further investigate the existence theory of the kinetic model, for instance by using the techniques introduced in [5]. In order to obtain global statements for the asymptotic analysis, it could be worth trying to adapt the tricky renormalisation arguments designed for the scalar case in [9].

3. Analysis of the macroscopic model: boundedness of solutions

The main ingredient of the analysis consists in finding a priori estimates satisfied by the solutions of the system (1.1)–(1.2). Therefore, we start by assuming that we have at hand non-negative and mass-preserving solutions of the system (1.1)–(1.2), with enough regularity and fast decay at infinity to perform manipulations like permutation of derivatives and integrals, integration by parts, etc. We establish some uniform estimates on these solutions that will depend only on certain L^q-norms of the initial data. Then, we shall need to construct solutions that satisfy such estimates, possibly at the price of restricting the set of initial data. Then, using the uniformity of the obtained estimates the result can be extended to more general data.

3.1. A priori estimates. To start with, let us show that solutions associated with bounded data remain in L^∞. This property is already in contrast to the Keller–Segel system. In dimension $N=2$, we can prove the propagation of L^q estimates for any exponent $q \ge 1$. Let us summarize our findings concerning the propagation of L^∞ and L^q bounds as follows.

LEMMA 3.1. *If* $\rho_{p,0}$ *and* $\rho_{c,0}$ *belong to* $L^1 \cap L^\infty(\mathbb{R}^N)$*, then for any* $1 \le q \le \infty$ *we have*

$$\begin{aligned} &\|\rho_p(t,\cdot)\|_q \le \|\rho_{p,0}\|_q && \text{for any } t \ge 0,\\ &\|\rho_c(t,\cdot)\|_q \le e^{T\|\rho_{p,0}\|_\infty}\|\rho_{c,0}\|_q && \text{for any } 0 \le t \le T < \infty. \end{aligned}$$

Furthermore, for any $0 < T < \infty$ *and* $1 < q < \infty$*,* $\nabla_x \rho_p^{q/2}$ *and* $\nabla_x \rho_c^{q/2}$ *belong to* $L^2((0,\infty)\times\mathbb{R}^N)$*.*

LEMMA 3.2. *Let us assume* $N=2$*. If* $\rho_{p,0}$ *and* $\rho_{c,0}$ *belong to* $L^1 \cap L^q(\mathbb{R}^N)$ *for some* $q > 1$*, then*

$$\|\rho_p(t,\cdot)\|_q \le \|\rho_{p,0}\|_q$$

and there exists $C > 0$ *which depends only on* q*,* m_c*,* m_p*,* $\|\rho_{p,0}\|_q$*, and* $\|\rho_{c,0}\|_q$ *such that*

$$\|\rho_c(t,\cdot)\|_q \le C$$

holds for any $t \ge 0$*. Furthermore,* $\nabla_x \rho_p^{q/2}$ *and* $\nabla_x \rho_c^{q/2}$ *belong to* $L^2((0,\infty)\times\mathbb{R}^2)$*.*

Proof. **(Proof of Lemma 3.1.)** The proof is quite simple and relies on standard Stampacchia's reasoning. Let $G : \mathbb{R} \to (0,\infty)$ be a convex function. Multiply

$$\partial_t \rho_p - \mathrm{div}_x(\rho_p \nabla_x \Phi_p + \nabla_x \rho_p) = 0$$

by $G'(\rho_p)$ and integrate by parts. We obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int G(\rho_p)\,\mathrm{d}x+\int G''(\rho_p)|\nabla_x\rho_p|^2\,\mathrm{d}x=-\int G''(\rho_p)\nabla_x\rho_p\cdot\rho_p\nabla_x\Phi_p\,\mathrm{d}x. \tag{3.1}$$

Let Z be a primitive of $\rho\mapsto G''(\rho)\rho$. By using Equation (1.2), the right-hand side of Equation (3.1) becomes

$$-\int\nabla_x Z(\rho_p)\cdot\nabla_x\Phi_p\,\mathrm{d}x=\int Z(\rho_p)\Delta_x\Phi_p\,\mathrm{d}x=-\alpha\int\rho_c Z(\rho_p)\,\mathrm{d}x,$$

which is non-positive since G is convex. Therefore, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\int G(\rho_p)\,\mathrm{d}x+\int G''(\rho_p)|\nabla_x\rho_p|^2\,\mathrm{d}x\le 0. \tag{3.2}$$

We use this relation with $G(\rho):=\frac{1}{2}\big[\rho-\|\rho_{p,0}\|_\infty\big]_+^2$ to deduce the uniform estimate on ρ_p in $L^\infty(\mathbb{R}^N)$. More generally, with $G(\rho)=\rho^q$, (3.2) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\rho_p^q\,\mathrm{d}x+4\frac{q-1}{q}\int|\nabla_x\rho_p^{q/2}|^2\,\mathrm{d}x\le 0, \tag{3.3}$$

which gives the estimates on the different L^q-norms.

We turn to the estimates on the chaser density. We repeat the same argument on

$$\partial_t\rho_c-\mathrm{div}_x(\rho_p\nabla_x\Phi_c+\nabla_x\rho_c)=0$$

with the function $G(\rho):=\rho^q$, $q>1$. This yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\rho_c^q\,\mathrm{d}x+\frac{4(q-1)}{q}\int|\nabla_x\rho_c^{q/2}|^2\,\mathrm{d}x=(q-1)\int\rho_p\rho_c^q\,\mathrm{d}x, \tag{3.4}$$

where we have made use of Equation (1.2). The obtained estimate for ρ_p allows us to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\rho_c^q\,\mathrm{d}x+\frac{4(q-1)}{q}\int|\nabla_x\rho_c^{q/2}|^2\,\mathrm{d}x\le(q-1)\|\rho_{p,0}\|_\infty\int\rho_c^q\,\mathrm{d}x.$$

Grönwall's Lemma leads us to

$$\int\rho_c^q(t,x)\,\mathrm{d}x\le e^{qt\|\rho_{p,0}\|_\infty}\int\rho_{c,0}^q(x)\,\mathrm{d}x,$$

which recasts as $\|\rho_c(t,\cdot)\|_q\le e^{T\|\rho_{p,0}\|_\infty}\|\rho_{c,0}\|_q\le e^{T\|\rho_{p,0}\|_\infty}\|\rho_{c,0}\|_\infty^{1-1/q}m_c^{1/q}$ for any $1<q<\infty$. We let q go to ∞ to obtain the L^∞ estimate. ∎

Proof. **(Proof of Lemma 3.2.)** Of course, Equation (3.3) implies that $\rho_p\in L^\infty(0,\infty;L^q(\mathbb{R}^N))$ and $\nabla_x\rho_p^{q/2}\in L^2((0,\infty)\times\mathbb{R}^N)$ when $\rho_{p,0}$ belongs to $L^q(\mathbb{R}^N)$. What is remarkable is to improve the L^q estimate for ρ_c in Lemma 3.1 and to make it uniform with respect to time when $N=2$. The restriction on the space dimension arises when we estimate the right-hand side of Equation (3.4). To this end, we make use of the Gagliardo–Nirenberg–Sobolev inequality (see, e.g., [29, p. 125] or [6, Eq. (85) p. 195]), which holds in $\mathbb{R}^2$ for any $\alpha\ge 1$:

$$\int\xi^{\alpha+1}\,\mathrm{d}x\le C\int\xi\,\mathrm{d}x\int|\nabla_x(\xi^{\alpha/2})|^2\,\mathrm{d}x. \tag{3.5}$$

Then, by using Hölder's (with conjugate exponents $q+1$ and $(q+1)'=\frac{q+1}{q}$) and Young's inequalities, we get

$$(q-1)\int \rho_p\rho_c^q\,\mathrm{d}x \le q\left(\int \rho_c^{q+1}\,\mathrm{d}x\right)^{q/(q+1)}\left(\int \rho_p^{q+1}\,\mathrm{d}x\right)^{1/(q+1)}$$
$$\le q\delta^{1/q}\int \rho_c^{q+1}\,\mathrm{d}x+\frac{1}{\delta}\int \rho_p^{q+1}\,\mathrm{d}x,$$

for $\delta>0$ to be determined. With Equation (3.5), we are led to

$$(q-1)\int \rho_p\rho_c^q\,\mathrm{d}x \le Cq\delta^{1/q}\int \rho_c\,\mathrm{d}x\int |\nabla_x\rho_c^{q/2}|^2\,\mathrm{d}x+\frac{1}{\delta}\int \rho_p^{q+1}\,\mathrm{d}x.$$

By mass conservation, we have $\int \rho_c\,\mathrm{d}x=m_c$. We go back to Equation (3.4). Choosing $\delta>0$ small enough, we find two constants, C_1 and C_2, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \rho_c^q\,\mathrm{d}x+C_1\int |\nabla_x\rho_c^{q/2}|^2\,\mathrm{d}x\le C_2\int \rho_p^{q+1}\,\mathrm{d}x.$$

The constants depend only on the Gagliardo–Nirenberg–Sobolev, q, and m_c. For instance we can set $C_1=2\frac{q-1}{q}$ by choosing $\delta=\left(2\frac{q-1}{q^2Cm_c}\right)^q$; accordingly $C_2=\left(\frac{q^2Cm_c}{2(q-1)}\right)^q$. By using Equation (3.5) again, we are led to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \rho_c^q\,\mathrm{d}x+C_1\int |\nabla_x\rho_c^{q/2}|^2\,\mathrm{d}x\le C_2Cm_p\int |\nabla_x\rho_p^{q/2}|^2\,\mathrm{d}x,$$

where the bound in $L^1((0,\infty))$ on the right-hand side has already been discussed in Equation (3.3). $\square$

3.2. Boundedness implies regularity. As a consequence of the L^∞ estimate, we can establish the regularity of the solution.

Lemma 3.3. *Assume that the solution (ρ_p,ρ_c) of the system (1.1)–(1.2) lies in $L^\infty((t_\star,T)\times\mathbb{R}^N)$ for some $0\le t_\star<T\le\infty$. Then, ρ_p,ρ_c are actually C^∞ on $(t_\star,T)\times\mathbb{R}^N$.*

The proof uses the following elementary estimate on the velocity field, bearing in mind the definition of the potentials in Equation (1.2) by means of a convolution formula.

Lemma 3.4. *Let $\rho\in L^1\cap L^\infty(\mathbb{R}^N)$. Set*

$$\nabla_x\Phi(x)=\int \frac{x-y}{|x-y|^N}\rho(y)\,\mathrm{d}y.$$

There exists a constant $C_N>0$ such that

$$|\nabla_x\Phi(x)|\le C_N\|\rho\|_1^{1/N}\|\rho\|_\infty^{1-1/N}.$$

Proof. For a given $A>0$, we split

$$\nabla_x\Phi(x)=\int_{|x-y|\le A}\frac{x-y}{|x-y|^N}\rho(y)\,\mathrm{d}y+\int_{|x-y|>A}\frac{x-y}{|x-y|^N}\rho(y)\,\mathrm{d}y.$$

The first integral is dominated by

$$\|\rho\|_\infty\ |\mathbb{S}^{N-1}|\int_0^A \frac{1}{r^{N-1}}\ r^{N-1}\,\mathrm{d}r = \|\rho\|_\infty\ |\mathbb{S}^{N-1}|A$$

while the second is dominated by

$$\frac{1}{A^{N-1}}\int |\rho(y)|\,\mathrm{d}y.$$

Optimizing with respect to A yields $A=\big(\frac{N-1}{|\mathbb{S}^{N-1}|}\big)^{1/N}\|\rho\|_1^{1/N}\|\rho\|_\infty^{-1/N}$, which allows us to conclude. ∎

Proof. **(Proof of Lemma 3.3.)** Lemma 3.4 implies that $\nabla_x\Phi_p(t,\cdot)$ (resp. $\nabla_x\Phi_c(t,\cdot)$) is bounded a.e. when $\rho_c(t,\cdot)$ (resp. $\rho_p(t,\cdot)$) lies in $L^1\cap L^\infty(\mathbb{R}^N)$. Let $0\le t_\star < T\le\infty$. Going back to the convection-diffusion equations satisfied by the densities ρ_p,ρ_c, we can apply standard results from the theory of parabolic equations (see for instance [23, Thm. VII.6.1]) to assert that $\nabla_x\rho_p,\nabla_x\rho_c\in L^\infty((t_\star,T)\times\Omega)$ for any $\Omega\subset\mathbb{R}^N$ provided ρ_p and ρ_c lie in $L^\infty((t_\star,T)\times\mathbb{R}^N)$. Then, for $j\in\{p,c\}$ and any $k\in\{1,\dots,N\}$, the function $u_j=\partial_{x_k}\rho_j$ verifies

$$\partial_t u_j - \mathrm{div}_x(u_j\nabla_x\Phi_j) - \Delta_x u_j = \mathrm{div}_x(u_j\nabla_x\Psi_j),$$

where $\Psi_j=\partial_{x_k}\Phi_j$ is defined by the Poisson equation $\Delta_x\Psi_c=u_p$ or $-\Delta_x\Psi_p=\alpha u_c$. We deduce from standard results (see for instance [10, Thm. 3.9 & Prob. 8.4]), that $\nabla_x\Psi_j$ is a (locally) bounded function which in turn permits us to conclude that $\nabla_x u_j$ is bounded on any subdomain $(t_\star,T)\times\Omega$. Continuing this reasoning by induction as in [14, Proposition A.1] establishes that ρ_c and ρ_p are C^∞ functions. ∎

3.3. De Giorgi's analysis. We wish to relax the boundedness and integrability conditions on the initial data, showing that they are improved by the dynamics itself. The proof splits into two steps. Firstly, we pay attention to L^q estimates for finite q's; secondly we discuss the L^∞ bound by adapting the De Giorgi technique. We refer the reader to [2, 14, 32] for similar reasoning. The first step aims at establishing the following claim, where a restriction on the space dimension appears.

LEMMA 3.5. *Let us assume $N=2$. Let $1<q<\infty$. There exists a constant $\mathscr{M}$ which only depends on the initial mass m_c, m_p, and q, such that*

$$\int \rho_p^q(t,x)\,\mathrm{d}x + \int \rho_c^q(t,x)\,\mathrm{d}x \le \mathscr{M}\Big(1+\frac{1}{t^{q-1}}\Big)$$

holds for any $t\ge 0$.

Proof. We go back to the proof of Lemma 3.2. By using Equation (3.5), which gives rise to the restriction on the space dimension $N=2$, and mass conservation, we have actually obtained the following differential inequalities:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int \rho_p^q\,\mathrm{d}x + 4\frac{q-1}{q}\int |\nabla_x\rho_p^{q/2}|^2\,\mathrm{d}x \le 0,\\ &\frac{\mathrm{d}}{\mathrm{d}t}\int \rho_c^q\,\mathrm{d}x + C_1\int |\nabla_x\rho_c^{q/2}|^2\,\mathrm{d}x \le C_2\int \rho_p^{q+1}\,\mathrm{d}x.\end{aligned} \tag{3.6}$$

Owing to Equation (3.5) and mass conservation again, we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int \rho_p^q\,\mathrm{d}x + 4\frac{q-1}{q}\frac{1}{Cm_p}\int \rho_p^{q+1}\,\mathrm{d}x \le 0.\\ &\frac{\mathrm{d}}{\mathrm{d}t}\int \rho_c^q\,\mathrm{d}x + \frac{C_1}{Cm_c}\int \rho_c^{q+1}\,\mathrm{d}x \le C_2\int \rho_p^{q+1}\,\mathrm{d}x.\end{aligned}$$

Let us set

$$\mathscr{X}(t) = \int \rho_p^q\,\mathrm{d}x + A\int \rho_c^q\,\mathrm{d}x$$

for some $A>0$ to be determined. We get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{X} + \Big(4\frac{q-1}{q}\frac{1}{Cm_p} - AC_2\Big)\int \rho_p^{q+1}\,\mathrm{d}x + A\frac{C_1}{Cm_c}\int \rho_c^{q+1}\,\mathrm{d}x \le 0.$$

We choose $A>0$ small enough such that the constant the factor in front of $\int \rho_p^{q+1}\,\mathrm{d}x$ remains positive (for instance $A=2\frac{q-1}{qC_2Cm_p}$). Finally, we make use of the interpolation inequality

$$\int \xi^q\,\mathrm{d}x \le \Big(\int \xi\,\mathrm{d}x\Big)^{1/q}\Big(\int \xi^{q+1}\,\mathrm{d}x\Big)^{(q-1)/q}.$$

Together with the mass conservation property, it permits us to find two constants $a,b>0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{X} + a\Big(\int \rho_c^q\,\mathrm{d}x\Big)^{q/(q-1)} + b\Big(\int \rho_p^q\,\mathrm{d}x\Big)^{q/(q-1)} \le 0.$$

With the elementary inequality $(s+t)^{q/(q-1)} \le C_q(s^{q/(q-1)} + t^{q/(q-1)})$, we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{X} + \beta\mathscr{X}^{q/(q-1)} \le 0$$

holds for a certain constant $\beta>0$. By a comparison argument (see Appendix A) we deduce that

$$\mathscr{X}(t) \le \mathscr{M}(1+1/t^{q-1}),$$

where the constant $\mathscr{M}$ only depends on q and β. □

Lemma 3.5 already indicates that L^q-norms of the solutions become instantaneously finite, for any positive time, even if the L^q-norm of the data is infinite. We shall use this information to obtain that the L^∞-norm becomes finite too, by using the De Giorgi scheme, as in [2, 14, 32]. This is the second step of our approach. As it will be clear within the proof, the restriction on the space dimension comes from the use of Lemma 3.5.

LEMMA 3.6. *Let us assume $N=2$. Let $t_\star>0$. There exists a constant $M_\star$ which depends on $t_\star$ in such a way that it blows up as $t_\star\to 0$, such that*

$$|\rho_p(t,x)| \le M_\star, \qquad |\rho_c(t,x)| \le M_\star$$

holds for almost every $t\ge t_\star$, $x\in\mathbb{R}^N$.

Proof. We are working on a finite time time interval $0<t_\star<T<\infty$ which does not contain 0. Let $M>0$ to be determined. We define the following sequences:

$$M_k:=M(1-1/2^k), \qquad t_k:=t_\star(1-1/2^{k+1})=t_\star/2^{k+1}+(1-1/2^k)t_\star.$$

We denote

$$\rho_p^{(k)}=(\rho_p-M_k)\mathbf{1}_{\rho_p>M_k},$$

where $\mathbf{1}_\Omega$ stands for the characteristic function of the set Ω. We multiply Equation (1.1) by $\rho_p^{(k)}$ and integrate the result. It yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\rho_p^{(k)}|^2\,\mathrm{d}x+\int|\nabla_x\rho_p^{(k)}|^2\,\mathrm{d}x=-\alpha\int\rho_c|\rho_p^{(k)}|^2\,\mathrm{d}x. \tag{3.7}$$

We integrate Equation (3.7) over the interval $[s,t]$, with $t_{k-1}<s<t_k<t<T$. We get

$$\begin{aligned}&\frac{1}{2}\int|\rho_p^{(k)}|^2(t,x)\,\mathrm{d}x+\int_s^t\int|\nabla_x\rho_p^{(k)}(\tau,x)|^2\,\mathrm{d}x\,\mathrm{d}\tau+\alpha\int_s^t\int\rho_c|\rho_p^{(k)}|^2(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau\\ =&\frac{1}{2}\int|\rho_p^{(k)}(s,x)|^2\,\mathrm{d}x\geq\frac{1}{2}\int|\rho_p^{(k)}|^2(t,x)\,\mathrm{d}x+\int_{t_k}^t\int|\nabla_x\rho_p^{(k)}(\tau,x)|^2\,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned} \tag{3.8}$$

Let us define the sequence

$$\mathscr{V}_k=\sup_{t_k\leq t\leq T}\frac{1}{2}\int|\rho_p^{(k)}|^2(t,x)\,\mathrm{d}x+\int_{t_k}^T\int|\nabla_x\rho_p^{(k)}(\tau,x)|^2\,\mathrm{d}x\,\mathrm{d}\tau.$$

We average Equation (3.8) over $s\in[t_k,t_{k-1}]$ and obtain

$$\mathscr{V}_k\leq\frac{1}{2}\frac{1}{t_k-t_{k-1}}\int_{t_{k-1}}^{t_k}\int|\rho_p^{(k)}(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s.$$

However, for any $\beta\geq0$, we have

$$|\rho_p^{(k)}|^2\leq|\rho_p^{(k-1)}|^2\left(\frac{2^k}{M}\rho_p^{(k-1)}\right)^\beta,$$

which yields

$$\mathscr{V}_k\leq\frac{1}{2}\frac{2^{k+1}}{t_\star}\int_{t_{k-1}}^T\int|\rho_p^{(k)}(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s\leq\frac{1}{2}\frac{2^{k+1}}{t_\star}\frac{2^{\beta k}}{M^\beta}\int_{t_{k-1}}^T\int|\rho_p^{(k-1)}(s,x)|^{2+\beta}\,\mathrm{d}x\,\mathrm{d}s.$$

The choice of the exponent β relies on the Gagliardo–Nirenberg–Sobolev inequality

$$\int|\xi|^{2+\beta}\,\mathrm{d}x\leq C\int|\nabla_x\xi|^2\,\mathrm{d}x\left(\int|\xi|^2\,\mathrm{d}x\right)^{\beta/2},$$

which holds for $2+\beta=2\frac{N+2}{N}=2+4/N$. Bearing in mind that $N=2$, we arrive at

$$\begin{aligned}\mathscr{V}_k&\leq\frac{C}{2}\frac{2^{k+1}}{t_\star}\frac{2^{2k}}{M^2}\int_{t_{k-1}}^T\left\{\int|\nabla_x\rho_p^{(k-1)}|^2\,\mathrm{d}x\times\int|\rho_p^{(k-1)}|^2\,\mathrm{d}x\right\}\mathrm{d}s\\&\leq\frac{2C}{M^2t_\star}\,2^{3k}\,\mathscr{V}_{k-1}^2.\end{aligned}$$

We choose $a\in(0,1)$ small enough and $M>0$ large enough, such that $a^k\mathscr{V}_0$ is a supersolution of this sequence of inequalities. If $\mathscr{V}_{k-1}\le a^{k-1}\mathscr{V}_0$ holds, then we get

$$\mathscr{V}_k\le\frac{2\mathscr{V}_0}{M^2t_\star a^2}\,\left(2^3a^2\right)^k\,\mathscr{V}_0.$$

Therefore, we can conclude that $\mathscr{V}_k$ is smaller than $a^k\mathscr{V}_0$ provided the following two conditions are fulfilled:

$$a\le\frac{1}{2^3},\qquad M\ge\left(\frac{2\mathscr{V}_0}{t_\star a^2}\right)^{1/2}.$$

Bearing in mind that $M_0=0$ (thus $\rho_p^{(0)}=\rho_p$) and $T_0=t_\star/2$, it remains to evaluate

$$\mathscr{V}_0=\sup_{t_\star/2\le t\le T}\frac12\int|\rho_p|^2(t,x)\,\mathrm{d}x+\int_{t_\star/2}^T\int|\nabla_x\rho_p(\tau,x)|^2\,\mathrm{d}x\,\mathrm{d}\tau.$$

To this end, we go back to the first equation in the system (3.6) with $q=2$ (energy inequality), integrated over $(t_\star/2,t)$:

$$\frac12\int|\rho_p(t,x)|^2\,\mathrm{d}x+\int_{t_\star/2}^t\int|\nabla_x\rho_p(\tau,x)|^2\,\mathrm{d}x\,\mathrm{d}\tau\le\frac12\int|\rho_p(t_\star/2,x)|^2\,\mathrm{d}x\le\frac{\mathscr{M}}{2}\left(1+\frac{2}{t_\star}\right),$$

where the last inequality uses Lemma 3.5. By the way, we bear in mind that the estimate in Lemma 3.5 relies on the Gagliardo–Nirenberg–Sobolev inequality and it assumes $N=2$: the restriction on the space dimension does not come from the De Giorgi argument in itself but from the need of an estimate on $\mathscr{V}_0$, which relies on Lemma 3.5. In other words, we have obtained

$$\mathscr{V}_0\le\frac{\mathscr{M}}{2}\left(1+\frac{2}{t_\star}\right).$$

Since $N=2$, we end up with the following bound from below for M:

$$M\ge\left(\frac{\mathscr{M}(1+2/t_\star)}{t_\star a^2}\right)^{1/2}.$$

In particular, notice that M behaves like $1/t_\star$ as $t_\star\to0$.

Fix $T>0$. Then, for any given $0<t_\star\ll1$, we can find M large enough to ensure that $\lim_{k\to\infty}\mathscr{V}_k=0$. Let us now consider the average over $[t_k,T]$

$$\frac{1}{T-t_k}\int_{t_k}^T\int|\rho_p^{(k)}(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t\le2\mathscr{V}_k.$$

However, for a.e. $(t,x)\in[0,\infty)\times\mathbb{R}^N$, we have

$$\lim_{k\to\infty}\left(\frac{|\rho_p^{(k)}(t,x)|^2}{T-t_k}\times\mathbf{1}_{t_k\le t\le T}\right)=\frac{|\rho_p^{(k)}(t,x)|^2}{T-t_\star}\times\mathbf{1}_{t_\star\le t\le T}\times\mathbf{1}_{\rho_p(t,x)\ge M}.$$

By virtue of Fatou's lemma, we conclude that

$$\frac{1}{T-t_\star}\int_{t_\star}^T\int|\rho_p(t,x)|^2\mathbf{1}_{\rho_p(t,x)\ge M}\,\mathrm{d}x\,\mathrm{d}t\le\lim_{k\to0}\mathscr{V}_k=0.$$

It implies that

$$\rho_p(t,x)\mathbf{1}_{\rho_p(t,x)\geq M}=0 \qquad \text{for a.e. } (t,x)\in(t_\star,T)\times\mathbb{R}^N$$

holds and, thus, $\rho_p(t,x)$ is dominated by M.

Once this bound is obtained for ρ_p, we proceed similarly for dealing with ρ_c. We use exactly the same notation, with $M_k=\mu(1-1/2^k)$, $\mu>0$ being the quantity to be determined, and $t_k:=t_{\star\star}(1-1/2^{k+1})$ where $t_{\star\star}=2t_\star$. In particular we now have $t_\star\leq t_k\leq t_{\star\star}$. We are led to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\rho_c^{(k)}|^2\,\mathrm{d}x+\int|\nabla_x\rho_c^{(k)}|^2\,\mathrm{d}x=\int\rho_p|\rho_c^{(k)}|^2\ \mathrm{d}x\leq M\int|\rho_c^{(k)}|^2\ \mathrm{d}x,$$

where M is the bound we have just obtained for ρ_p. We integrate over $[s,t]$, with $t_{k-1}<s<t_k<t<T$, and next we average over $s\in[t_{k-1},t_k]$. We obtain

$$\begin{aligned}
&\frac{1}{2}\int|\rho_c^{(k)}|^2(t,x)\,\mathrm{d}x+\int_{t_k}^t\int|\nabla_x\rho_c^{(k)}|^2\,\mathrm{d}x\\
&\leq\frac{1}{2}\frac{1}{t_k-t_{k-1}}\int_{t_{k-1}}^{t_k}\int|\rho_c^{(k)}(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s+M\int_{t_{k-1}}^T\int\left|\rho_c^{(k)}(\tau,x)\right|^2\mathrm{d}x\,\mathrm{d}\tau\\
&\leq\left(\frac{2^k}{t_{\star\star}}+M\right)\int_{t_{k-1}}^T\int|\rho_c^{(k)}(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

We now set

$$\mathscr{V}_k'=\sup_{t_k\leq t\leq T}\frac{1}{2}\int|\rho_c^{(k)}|^2(t,x)\,\mathrm{d}x+\int_{t_k}^t\int|\nabla_x\rho_c^{(k)}|^2\,\mathrm{d}x.$$

Repeating the arguments detailed above yields

$$\mathscr{V}_k'\leq\frac{2}{\mu^2}\ 2^{2k}\left(\frac{2^k}{t_{\star\star}}+M\right)(\mathscr{V}_k')^2\leq\frac{2}{\mu^2}\ \left(\frac{1}{t_{\star\star}}+M\right)\ 2^{3k}\ (\mathscr{V}_{k-1}')^2.$$

We apply the same reasoning as above, which leads us to impose

$$\mu\geq\left(\frac{2\mathscr{V}_0'}{a^2}\left(M+\frac{1}{t_{\star\star}}\right)\right)^{1/2}.$$

We need to estimate $\mathscr{V}_0'$. To this end, we go back to Equation (3.6),

$$\frac{\mathrm{d}}{\mathrm{d}t}\int|\rho_c|^2\,\mathrm{d}x+C_1\int|\nabla_x\rho_c|^2\,\mathrm{d}x\leq C_2\int\rho_p^3\,\mathrm{d}x\leq C_2Cm_p\int|\nabla_x\rho_p|^2\,\mathrm{d}x,$$

by using Equation (3.5) and mass conservation. Integrate over $(t_{\star\star}/2,t)=(t_\star,t)$ to obtain

$$\begin{aligned}
&\int|\rho_c(t,x)|^2\,\mathrm{d}x+C_1\int_{t_\star}^t\int|\nabla_x\rho_c(\tau,x)|^2\,\mathrm{d}x\,\mathrm{d}\tau\\
&\leq\int|\rho_c(t_\star,x)|^2\,\mathrm{d}x+C_2Cm_p\int_{t_\star}^t\int|\nabla_x\rho_p(\tau,x)|^2\,\mathrm{d}x\,\mathrm{d}\tau\\
&\leq\mathscr{M}\left(1+\frac{1}{t_\star}\right)+C_2Cm_p\mathscr{V}_0\\
&\leq\mathscr{M}\left(1+\frac{CC_2m_p}{2}\right)\left(1+\frac{2}{t_\star}\right).
\end{aligned}$$

It follows that

$$\mathscr{V}_0' \le \mathscr{M}\Big(1+\frac{CC_2m_p}{2}\Big)\Big(1+\frac{2}{t_\star}\Big).$$

Therefore, since $N=2$, we arrive at the condition

$$\mu \ge \left(\frac{2}{a^2}\Big(M+\frac{1}{2t_\star}\Big)\mathscr{M}\Big(1+\frac{CC_2m_p}{2}\Big)\Big(1+\frac{2}{t_\star}\Big)\right)^{1/2},$$

which behaves like $1/t_\star$ for small $t_\star$'s. We conclude that $\rho_c(t,x)\le\mu$ holds a.e. on $(t_{\star\star},T)\times\mathbb{R}^2$. We point out that both M and μ depend on $t_\star$ and, of course, they blow up as $t_\star\to 0$. What is interesting is to remark that the estimate is uniform over large times. □

3.4. Existence-uniqueness of solutions. We are going to obtain the solutions of the system (1.1)–(1.2) by means of a fixed point argument. The method is quite classical, and we only sketch the proof, pointing out some technical difficulties. We start by assuming that the initial data $(\rho_{p,0},\rho_{c,0})$ belongs to $L^1\cap L^\infty(\mathbb{R}^N)$. Let $0<T<\infty$. We consider two functions $\tilde\rho_p,\tilde\rho_c:(0,T)\times\mathbb{R}^N\to\mathbb{R}^N$ such that

$$\begin{aligned}
&0\le\tilde\rho_p(t,x)\le\|\rho_{p,0}\|_\infty, && 0\le\tilde\rho_c(t,x)\le\|\rho_{c,0}\|_\infty e^{T\|\rho_{p,0}\|_\infty},\\
&\int\tilde\rho_p(t,x)\,\mathrm{d}x=m_p, && \int\tilde\rho_c(t,x)\,\mathrm{d}x=m_c.
\end{aligned}\tag{3.9}$$

Let us denote by $\mathscr{C}_T$ the (convex) set of functions that fulfill Equation (3.9). The intermediate result is stated as follows (for $N\ge 3$ it is likely far from optimal; since the regularity analysis requires $N=2$, we do not elaborate more on this case here).

Proposition 3.1. *Let $(\rho_{p,0},\rho_{c,0})\in L^1\cap L^\infty(\mathbb{R}^N)$. Furthermore, we assume that $x\mapsto x^2\rho_{p,0}(x)$ and $x\mapsto x^2\rho_{c,0}(x)$ belong to $L^1(\mathbb{R}^N)$. Then, for any $T>0$, the system* (1.1)–(1.2) *with data $(\rho_{p,0},\rho_{c,0})$ admits a unique solution in $\mathscr{C}_T$.*

3.4.1. Preliminary observations. Given $(\tilde\rho_p,\tilde\rho_c)\in\mathscr{C}_T$, we define $\tilde\Phi_p,\tilde\Phi_c$ by solving

$$\Delta_x\tilde\Phi_p=-\alpha\tilde\rho_c,\qquad \Delta_x\tilde\Phi_c=\tilde\rho_p.$$

Lemma 3.4 tells us that $\nabla_x\tilde\Phi_p$ and $\nabla_x\tilde\Phi_c$ are bounded functions. Then, we can introduce the solutions of the linear equations

$$\begin{aligned}
&\partial_t\rho_p-\mathrm{div}_x(\rho_p\nabla_x\tilde\Phi_p+\nabla_x\rho_p)=0,\\
&\partial_t\rho_c-\mathrm{div}_x(\rho_c\nabla_x\tilde\Phi_c+\nabla_x\rho_c)=0,\\
&\rho_p\Big|_{t=0}=\rho_{p,0},\qquad \rho_c\Big|_{t=0}=\rho_{c,0},
\end{aligned}$$

By standard theory of parabolic equations (see, e.g., [6, Thm. X.9]), solutions are found in $C([0,T];L^2(\mathbb{R}^N))\cap L^2(0,T;H^1(\mathbb{R}^N))$. Repeating the derivation of the a priori estimates, we check that $(\rho_p,\rho_c)\in\mathscr{C}_T$.

Let $(\rho_p,\rho_c)=\mathscr{T}(\tilde\rho_p,\tilde\rho_c)$ and $(\mu_p,\mu_c)=\mathscr{T}(\tilde\mu_p,\tilde\mu_c)$. We denote by $(\tilde\Psi_p,\tilde\Psi_c)$ the potential associated to $(\tilde\mu_p,\tilde\mu_c)$. We obtain

$$\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{dt}}\int|\rho_p-\mu_p|^2\,\mathrm{d}x+\int|\nabla_x(\rho_p-\mu_p)|^2\,\mathrm{d}x\\
&=-\frac{\alpha}{2}\int\tilde\rho_c|\rho_p-\mu_p|^2\,\mathrm{d}x-\int\mu_p\nabla_x(\tilde\Phi_p-\tilde\Psi_p)\cdot\nabla_x(\rho_p-\mu_p)\,\mathrm{d}x\\
&\le\frac12\int|\nabla_x(\rho_p-\mu_p)|^2\,\mathrm{d}x+\frac12\int\mu_p^2|\nabla_x(\tilde\Phi_p-\tilde\Psi_p)|^2\,\mathrm{d}x.
\end{aligned}$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int|\rho_p-\mu_p|^2\,\mathrm{d}x+\int|\nabla_x(\rho_p-\mu_p)|^2\,\mathrm{d}x$$
$$\leq\left(\int\mu_p^{2q/(q-2)}\,\mathrm{d}x\right)^{(q-2)/q}\left(\int|\nabla_x(\tilde{\Phi}_p-\tilde{\Psi}_p)|^q\,\mathrm{d}x\right)^{2/q}.$$

When the space dimension N is larger than 2, we can use the following Hardy–Littlewood–Sobolev inequality [24, Thm. 4.3].

LEMMA 3.7. *Let $\alpha>1$. The operator defined by*

$$H:f\mapsto\int\frac{f(y)}{|x-y|^{N/\alpha}}\,\mathrm{d}y$$

is continuous from $L^p(\mathbb{R}^N)$ to $L^q(\mathbb{R}^N)$ for any $1<p<\frac{\alpha}{\alpha-1}$ and $1/q=1/p+1/\alpha-1$.

Let $N\geq 3$; we use Lemma 3.7 with $\alpha=\frac{N}{N-1}$ and $p=2$. It leads to $q=\frac{2N}{N-2}>2$, and we denote by $|||H|||$ the corresponding norm. As a matter of fact, we note that

$$\begin{aligned}\left(\int\rho_p^{2q/(q-2)}\,\mathrm{d}x\right)^{(q-2)/q}&=\|\rho_p\|^2_{2q/(q-2)}\leq\|\rho_p\|_\infty^{(q+2)/q}\|\rho_p\|_1^{(q-2)/q}\\&\leq\|\rho_{p,0}\|_\infty^{(q+2)/q}m_p^{(q-2)/q}=C_0.\end{aligned}$$

We are thus led to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int|\rho_p-\mu_p|^2\,\mathrm{d}x+\int|\nabla_x(\rho_p-\mu_p)|^2\,\mathrm{d}x\leq C_0|||H|||\int|\tilde{\rho}_c-\tilde{\mu}_c|^2\,\mathrm{d}x.$$

We proceed similarly for the chaser species, and we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int|\rho_c-\mu_c|^2\,\mathrm{d}x+\int|\nabla_x(\rho_c-\mu_c)|^2\,\mathrm{d}x$$
$$\leq\|\rho_{p,0}\|_\infty\int|\rho_c-\mu_c|^2\,\mathrm{d}x+\|\rho_{c,0}\|_\infty^{(q+2)/q}e^{T\|\rho_{p,0}\|_\infty(q+2)/q}m_c^{(q-2)/q}|||H|||\int|\tilde{\rho}_p-\tilde{\mu}_p|^2\,\mathrm{d}x.$$

We add these two inequalities and we apply the Grönwall lemma. It allows us to define a constant $\mathscr{K}(T)$, which depends on T and on the L^1 and L^∞ norms of the data, such that

$$\begin{aligned}&\int(|\rho_p-\mu_p|^2+|\rho_c-\mu_c|^2)(t,x)\,\mathrm{d}x\\&\leq e^{T\|\rho_{p,0}\|_\infty}\Bigg(\int(|\rho_{p,0}-\mu_{p,0}|^2+|\rho_{c,0}-\mu_{c,0}|^2)\,\mathrm{d}x\\&\quad+T\mathscr{K}(T)\sup_{0\leq s\leq T}\int(|\tilde{\rho}_p-\tilde{\mu}_p|^2+|\tilde{\rho}_c-\tilde{\mu}_c|^2)(s,x)\,\mathrm{d}x\Bigg).\end{aligned}\tag{3.10}$$

Relation (3.10) holds when $N\geq 3$, and it proves

- that $\mathscr{T}:(\tilde{\rho}_p,\tilde{\rho}_c)\mapsto(\rho_p,\rho_c)$ is continuous on $L^\infty(0,T;L^2(\mathbb{R}^N))$; actually, it defines a contraction mapping in this space when T is small enough, which implies the local existence-uniqueness of a solution in $\mathscr{C}_T$;

- whatever the choice of T, the uniqueness of the solutions in $\mathscr{C}_T$, by means of Grönwall's lemma, as well as the continuity of the solution with respect to the initial data.

When $N=2$, the argument to get uniqueness is more involved, as we shall see below. In what follows, we shall also use the following observation:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{x^2}{2}\rho_p\,\mathrm{d}x &= -\int x\cdot\nabla_x\rho_p - \int x\cdot\nabla_x\tilde{\Phi}_p\rho_p\,\mathrm{d}x\\ &\leq N\int \rho_p\,\mathrm{d}x + \|\nabla_x\tilde{\Phi}_p\|_\infty\left(\int\rho_p\,\mathrm{d}x\right)^{1/2}\left(\int x^2\rho_p\,\mathrm{d}x\right)^{1/2}\\ &\leq m_p\left(N+\frac{1}{2}\|\nabla_x\tilde{\Phi}_p\|_\infty^2\right)+\frac{1}{2}\int x^2\rho_p\,\mathrm{d}x.\end{aligned}$$

By Lemma 3.4, we have

$$\|\nabla_x\tilde{\Phi}_p\|_\infty \leq C_2\|\tilde{\rho}_c\|_1^{1/N}\|\tilde{\rho}_c\|_\infty^{1-1/N}\leq \Upsilon(T),$$

where $\Upsilon(T)$ depends on the L^1 and L^∞ norms of the data and has an exponential growth with respect to T. From now on, we use the generic notation $\Upsilon(T)$ for such a quantity, while the precise value of the constant might vary from one line to another. A similar computation holds for ρ_c. Applying Grönwall's lemma, we deduce that

$$\int x^2\rho_p\,\mathrm{d}x+\int x^2\rho_c\,\mathrm{d}x\leq\Upsilon(T) \tag{3.11}$$

holds. Finally, the analysis uses the following claim, the proof of which can be found in Appendix B for the sake of completeness.

LEMMA 3.8. *The operator* $f\mapsto\int_{\mathbb{R}^N}\frac{x-y}{|x-y|^N}\,f(y)\,\mathrm{d}y$ *is continuous and compact from* $L^1(\mathbb{R}^N)$ *to* $L^q(B(0,R))$ *for any* $1\leq q<\frac{N}{N-1}$ *and* $0<R<\infty$.

3.4.2. Global existence. Let us go back to the existence of solutions. We already know that $\mathscr{T}(\mathscr{C}_T)\subset\mathscr{C}_T$. We are going to prove that $\mathscr{T}$ is continuous for the $L^1((0,T)\times\mathbb{R}^N)$-norm. We consider $(\tilde{\rho}_{p,n},\tilde{\rho}_{c,n})\in\mathscr{C}_T$ which converges to $(\tilde{\rho}_p,\tilde{\rho}_c)$ in $L^1((0,T)\times\mathbb{R}^N)$. Of course, the limit belongs to $\mathscr{C}_T$. Reproducing the same manipulations as above, we find

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int|\rho_{p,n}-\rho_p|^2\,\mathrm{d}x+\int|\nabla_x\rho_{p,n}-\nabla_x\rho_p|^2\,\mathrm{d}x\\ &\leq-\alpha\int\tilde{\rho}_c|\rho_{p,n}-\rho_p|^2\,\mathrm{d}x+\int\rho_{p,n}^2|\nabla_x(\tilde{\Phi}_{p,n}-\tilde{\Phi}_p)|^2\,\mathrm{d}x.\end{aligned}$$

For any $0<R<\infty$, the last integral can be dominated by

$$\begin{aligned}&\|\rho_{p,n}\|_\infty^2\|\nabla_x(\tilde{\Phi}_{p,n}-\tilde{\Phi}_p)\|_\infty\int_{|x|\leq R}|\nabla_x(\tilde{\Phi}_{p,n}-\tilde{\Phi}_p)|\,\mathrm{d}x\\ &\qquad+\|\rho_{p,n}\|_\infty\|\nabla_x(\tilde{\Phi}_{p,n}-\tilde{\Phi}_n)\|_\infty^2\int_{|x|\geq R}\rho_{p,n}\,\mathrm{d}x.\end{aligned}$$

On the one hand, since $\rho_{p,n}$ lies in $\mathscr{C}_T$, we can find $\Upsilon(T)>0$ such that $\|\rho_{p,n}\|_\infty\leq\Upsilon(T)$, and we also have $\|\nabla_x\tilde{\Phi}_{p,n}\|_\infty\leq\Upsilon(T)$, $\|\nabla_x\tilde{\Phi}_p\|_\infty\leq\Upsilon(T)$. On the other hand, by

Equation (3.11), we get $\sup_n \int_{|x|\geq R} \rho_{p,n}\,\mathrm{d}x \leq \frac{\Upsilon(T)}{R^2}$. Similar observations hold for $\rho_{c,n}$. Therefore, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int (|\rho_{p,n}-\rho_p|^2+|\rho_{c,n}-\rho_c|^2)\,\mathrm{d}x+\int (|\nabla_x\rho_{p,n}-\nabla_x\rho_p|^2+|\nabla_x\rho_{c,n}-\nabla_x\rho_c|^2)\,\mathrm{d}x\\
&\leq \Upsilon(T)\Bigg(\int (|\rho_{p,n}-\rho_p|^2+|\rho_{c,n}-\rho_c|^2)\,\mathrm{d}x\\
&\qquad +\frac{1}{R^2}+\int_{|x|\leq R}(|\nabla_x(\tilde\Phi_{p,n}-\tilde\Phi_p)|+|\nabla_x(\tilde\Phi_{c,n}-\tilde\Phi_c)|)\,\mathrm{d}x\Bigg).
\end{aligned}$$

Let $\delta>0$. We apply Grönwall's lemma again. For $T>0$ fixed, we can find R large enough, depending on T and δ, such that

$$\begin{aligned}
&\int (|\rho_{p,n}-\rho_p|^2+|\rho_{c,n}-\rho_c|^2)\,\mathrm{d}x\\
&\leq \delta+\Upsilon(T)\int_0^T\int_{|x|\leq R}(|\nabla_x(\tilde\Phi_{p,n}-\tilde\Phi_p)|+|\nabla_x(\tilde\Phi_{c,n}-\tilde\Phi_c)|)\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

Going back to Lemma 3.8, we conclude that $(\rho_{p,n},\rho_{c,n})\to(\rho_p,\rho_c)$ in $L^\infty(0,T;L^2(\mathbb{R}^N))$ as $n\to\infty$. We deduce that the convergence also holds in $L^1((0,T)\times\mathbb{R}^N)$ since

$$\begin{aligned}
\int_0^T\int |\rho_{p,n}-\rho_p|\,\mathrm{d}x\,\mathrm{d}t &\leq T\sqrt{|B(0,R)|}\left(\sup_{0\leq t\leq T}\int |\rho_{p,n}-\rho_p|^2\,\mathrm{d}x\right)^{1/2}\\
&\quad +\frac{T}{R^2}\underbrace{\sup_n\sup_{0\leq t\leq T}\int x^2(\rho_{p,n}+\rho_p)\,\mathrm{d}x}_{\leq\Upsilon(T)},
\end{aligned}\tag{3.12}$$

where we choose R large enough and then let $n\to\infty$. A similar estimate applies for $\rho_{c,n}-\rho_c$.

Next, we establish that $\mathscr{T}$ is a compact mapping for the L^1 norm. Let $\big((\tilde\rho_{p,n},\tilde\rho_{c,n})\big)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{C}_T$. We already know that $\rho_{p,n}$ and $\rho_{c,n}$ are both bounded in $L^\infty(0,T;L^2(\mathbb{R}^N))\cap L^2(0,T;H^1(\mathbb{R}^N))$. Furthermore, $\partial_t\rho_{p,n}=\mathrm{div}_x(\nabla_x\rho_{p,n}+\rho_{p,n}\nabla_x\tilde\Phi_{p,n})$ is bounded in $L^2(0,T;H^{-1}(\mathbb{R}^N))$. The Aubin–Lions–Simon lemma [36, Sec. 8, Cor. 4] tells us that $\rho_{p,n}$ is compact in $L^2((0,T)\times B(0,R))$ for any $0<R<\infty$. Reasoning as in Equation (3.12), we deduce that $\rho_{p,n}$ is compact in $L^1((0,T)\times\mathbb{R}^2)$. A similar conclusion applies to $\rho_{c,n}$. The Schauder theorem ensures the existence of a fixed point $(\rho_p,\rho_c)=\mathscr{T}(\rho_p,\rho_c)\in\mathscr{C}_T$ and thus a solution of the system (1.1)–(1.2).

3.4.3. Uniqueness ($N=2$). It remains to discuss the uniqueness of the solutions in dimension $N=2$ (the case of higher dimension being treated through Equation (3.10)). To this end, our argument is inspired by [34] (note that the necessary adaptations are not fully detailed in [13] for the specific case of dimension $N=2$). The proof uses the following claims (we refer the reader for instance to [12, Lemma 1] and [8, Thm. 3.1.3], respectively).

LEMMA 3.9. *Assume $N=2$. Let $\rho\in L^1\cap L^2(\mathbb{R}^2)$ such that $x\mapsto|x|\rho(x)\in L^1(\mathbb{R}^2)$ and $\int\rho\,\mathrm{d}x=0$. Let $\Phi=\frac{1}{2\pi}\int\ln(|x-y|)\rho(y)\,\mathrm{d}y$. Then $\nabla_x\Phi$ belongs to $L^2(\mathbb{R}^2)$.*

LEMMA 3.10 (Calderón–Zygmung inequality). *There exists $K_\star>0$ such that for any $1<q_0\leq q<\infty$ and any $g\in L^q(\mathbb{R}^N)$, the function $V(x)=\int\frac{x-y}{|x-y|}\,\frac{g(y)}{|x-y|^{N-1}}\,\mathrm{d}y$ satisfies*

$$\|\nabla_x V\|_q\leq K_\star\, q\|g\|_q.$$

Let $\rho_{p,j},\rho_{c,j}$, with $j\in\{1,2\}$, be two solutions of the system (1.1)–(1.2) associated to the same initial data. We set $P=\rho_{p,1}-\rho_{p,2}$, $C=\rho_{c,1}-\rho_{c,2}$, $\Psi_P=\Phi_{p,1}-\Phi_{p,2}$, $\Psi_C=\Phi_{c,1}-\Phi_{c,2}$. We have

$$\begin{aligned}
&\partial_t P-\mathrm{div}_x(P\nabla_x\Phi_{p,1}+\rho_{p,2}\nabla_x\Psi_P)=\Delta_x P,\\
&\partial_t C-\mathrm{div}_x(C\nabla_x\Phi_{c,1}+\rho_{c,2}\nabla_x\Psi_C)=\Delta_x C,
\end{aligned}$$

with

$$-\Delta_x\Phi_P=\alpha C,\qquad \Delta_x\Phi_C=P.$$

The solutions constructed above are such that the non-negative functions $\rho_{p,j},\rho_{c,j}$ are bounded in $L^\infty(0,T;L^1\cap L^\infty(\mathbb{R}^2))$, with $x^2(\rho_{p,j}+\rho_{c,j})$ bounded in $L^\infty(0,T;L^1(\mathbb{R}^2))$. In particular, we have $P,C\in L^\infty(0,T;L^1\cap L^\infty(\mathbb{R}^2))$, with $x^2P,x^2C\in L^\infty(0,T;L^1(\mathbb{R}^2))$ and $\int P\,\mathrm{d}x=0=\int C\,\mathrm{d}x$. According to Lemma 3.9, we thus have $\nabla_x\Psi_P,\nabla_x\Psi_C\in L^\infty(0,T;L^2(\mathbb{R}^2))$. We compute

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}&\int\big(|\nabla_x\Psi_P|^2+|\nabla_x\Psi_C|^2\big)\,\mathrm{d}x=\alpha\int\Psi_P\partial_t C\,\mathrm{d}x-\int\Psi_C\partial_t P\,\mathrm{d}x\\
=-\alpha&\int\nabla_x\Psi_P\cdot(C\nabla_x\Phi_{c,1}+\rho_{c,2}\nabla_x\Psi_C)\,\mathrm{d}x-\alpha\int\nabla_x\Psi_P\cdot\nabla_x C\,\mathrm{d}x\\
&\quad+\int\nabla_x\Psi_C\cdot(P\nabla_x\Phi_{p,1}+\rho_{p,2}\nabla_x\Psi_P)\,\mathrm{d}x+\int\nabla_x\Psi_C\cdot\nabla_x P\,\mathrm{d}x\\
=\mathrm{I}+&\int\nabla_x\Psi_P\cdot\nabla_x\Psi_C(\rho_{p,2}-\alpha\rho_{c,2})\,\mathrm{d}x\\
&\quad-\alpha\int\nabla_x\Psi_P\cdot\nabla_x C\,\mathrm{d}x+\int\nabla_x\Psi_C\cdot\nabla_x P\,\mathrm{d}x
\end{aligned}$$

where we have set

$$\begin{aligned}
\mathrm{I}&=\int(-\alpha C\nabla_x\Phi_{c,1}\cdot\nabla_x\Psi_P+P\nabla_x\Phi_{p,1}\cdot\nabla_x\Psi_C)\,\mathrm{d}x\\
&=\int(\Delta_x\Psi_P\nabla_x\Phi_{c,1}\cdot\nabla_x\Psi_P+\Delta_x\Psi_C\nabla_x\Phi_{p,1}\cdot\nabla_x\Psi_C)\,\mathrm{d}x.
\end{aligned}$$

Using several integrations by parts, this integral can be recast as

$$\begin{aligned}
\mathrm{I}=&-\int D_x^2\Phi_{c,1}\nabla_x\Psi_P\cdot\nabla_x\Psi_P\,\mathrm{d}x+\frac{1}{2}\int\rho_{p,1}|\nabla_x\Psi_P|^2\,\mathrm{d}x\\
&-\int D_x^2\Phi_{p,1}\nabla_x\Psi_C\cdot\nabla_x\Psi_C\,\mathrm{d}x-\frac{\alpha}{2}\int\rho_{c,1}|\nabla_x\Psi_C|^2\,\mathrm{d}x.
\end{aligned}$$

Therefore, we arrive at the following estimate:

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\big(|\nabla_x\Psi_P|^2+|\nabla_x\Psi_C|^2\big)\,\mathrm{d}x\\
&\le\Upsilon(T)\int\big(|\nabla_x\Psi_P|^2+|\nabla_x\Psi_C|^2\big)\,\mathrm{d}x+\frac{1}{2}\int\big(|\nabla_x P|^2+|\nabla_x C|^2\big)\,\mathrm{d}x\\
&\qquad-\int D_x^2\Phi_{c,1}\nabla_x\Psi_P\cdot\nabla_x\Psi_P\,\mathrm{d}x-\int D_x^2\Phi_{p,1}\nabla_x\Psi_C\cdot\nabla_x\Psi_C\,\mathrm{d}x,
\end{aligned}$$

where, as above, $\Upsilon(T)>0$ depends on the L^1- and L^∞-norms of the initial data. We are going to combine this estimate to

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int P^2\,\mathrm{d}x+\int|\nabla_x P|^2\,\mathrm{d}x=-\int\nabla_x P\cdot(P\nabla_x\Phi_{p,1}+\rho_{p,2}\nabla_x\Psi_P)\,\mathrm{d}x\\&=-\frac{\alpha}{2}\int\rho_{c,1}P^2\,\mathrm{d}x-\int\rho_{p,2}\nabla_x\Psi_P\cdot\nabla_x P\,\mathrm{d}x\\&\le\frac{\alpha\|\rho_{c,1}\|_\infty}{2}\int P^2\,\mathrm{d}x+\frac{\|\rho_{p,2}\|_\infty^2}{2}\int|\nabla_x\Psi_P|^2\,\mathrm{d}x+\frac{1}{2}\int|\nabla_x P|^2\,\mathrm{d}x,\end{aligned}$$

and, similarly,

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int C^2\,\mathrm{d}x+\int|\nabla_x C|^2\,\mathrm{d}x\\&\le\frac{\|\rho_{p,1}\|_\infty}{2}\int C^2\,\mathrm{d}x+\frac{\|\rho_{c,2}\|_\infty^2}{2}\int|\nabla_x\Psi_C|^2\,\mathrm{d}x+\frac{1}{2}\int|\nabla_x C|^2\,\mathrm{d}x.\end{aligned}$$

Let us denote

$$\mathscr{E}(t)=\int(|C(t,x)|^2+|P(t,x)|^2)\,\mathrm{d}x+\int\left(|\nabla_x\Psi_P(t,x)|^2+|\nabla_x\Psi_C(t,x)|^2\right)\mathrm{d}x.$$

The previous manipulations allow us to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}\le\Upsilon(T)\mathscr{E}-\int D_x^2\Phi_{c,1}\nabla_x\Psi_P\cdot\nabla_x\Psi_P\,\mathrm{d}x-\int D_x^2\Phi_{p,1}\nabla_x\Psi_C\cdot\nabla_x\Psi_C\,\mathrm{d}x.$$

The last integrals can be dominated by using Hölder's inequality; we are led to

$$\begin{aligned}&\|D_x^2\Phi_{c,1}\|_q\left(\int|\nabla_x\Psi_P|^{2q'}\,\mathrm{d}x\right)^{1/q'}+\|D_x^2\Phi_{p,1}\|_q\left(\int|\nabla_x\Psi_C|^{2q'}\,\mathrm{d}x\right)^{1/q'}\\&\le\|D_x^2\Phi_{c,1}\|_q\|\nabla_x\Psi_P\|_\infty^{2/q}\left(\int|\nabla_x\Psi_P|^2\,\mathrm{d}x\right)^{1/q'}\\&\quad+\|D_x^2\Phi_{p,1}\|_q\|\nabla_x\Psi_C\|_\infty^{2/q}\left(\int|\nabla_x\Psi_C|^2\,\mathrm{d}x\right)^{1/q'}.\end{aligned}$$

The second derivatives can be controlled by appealing to Lemma 3.10. Note that

- on the one hand, both $\rho=\rho_{p,j}$ and $\rho=\rho_{c,j}$ satisfy the rough estimate $\|\rho\|_q\le\|\rho\|_1+\|\rho\|_\infty$;
- on the other hand, for any $q\ge2$, $\|\nabla_x\Psi_P\|_\infty^{2/q}\le1+\|\nabla_x\Phi_{p,1}\|_\infty+\|\nabla_x\Phi_{p,2}\|_\infty$ holds, as well as a similar estimate for $\nabla_x\Psi_C$.

We can thus find a constant $\Upsilon(T)>0$ which does not depend on $q\ge2$, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}\le\Upsilon(T)(\mathscr{E}+q\mathscr{E}^{1-1/q}).$$

We simply write $\mathscr{E}=(\frac{1}{q}\mathscr{E}^{1/q})\times q\mathscr{E}^{1-1/q}$, where $q\ge2$ and we already know that $t\mapsto\mathscr{E}(t)$ is bounded on $[0,T]$. We arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}\le q\Upsilon(T)\mathscr{E}^{1-1/q}.$$

We remind the reader that $\mathscr{E}(0)=0$. Pick $\eta>0$ and let $t\in[0,T]\mapsto z_\eta(t)$ be the solution of the ODE $\frac{\mathrm{d}}{\mathrm{d}t}z_\eta(t)=\Upsilon(T)q(\eta+z_\eta(t))^{1-1/q}$, with $z_\eta(0)=\eta$. We find $z_\eta(t)=((2\eta)^{1/q}+$

$\Upsilon(T)t)^q - \eta$. Clearly $\mathscr{E}(t) \le z_\eta(t)$ holds for any $\eta > 0$. Letting η go to 0, we deduce that $\mathscr{E} \le (\Upsilon(T)t)^q$ holds for any $2 \le q < \infty$. We now let q go to ∞, which yields $\mathscr{E}(t) = 0$ provided $0 \le t \le 1/\Upsilon(T)$. We repeat the argument on successive time intervals of length $1/\Upsilon(T)$, and we conclude that $\mathscr{E}$ vanishes on the whole interval $[0,T]$. It implies $\nabla_x \Psi_P = \nabla_x \Psi_C = 0$, $P = C = 0$.

3.4.4. Unbounded data. We detail in the case of $N = 2$ how to extend the existence result to unbounded data. If the initial data $(\rho_{p,0}, \rho_{c,0})$ lies in $C_c^\infty(\mathbb{R}^2)$, standard results about the regularity of solutions of parabolic equations can be used, and we can justify for these solutions the derivation of the a priori estimates. In particular, they are uniformly bounded. Finally, we wish to extend the set of initial data, considering possibly unbounded data. The regularity analysis provides a priori estimates in L^q for any $1 < q \le \infty$, depending only on the L^1 norm of the data, for any positive time. However, the estimates blow up as $t \to 0$, and the singularity is not integrable on $[0,T]$. Therefore, we are still facing the difficulty of defining the product $\rho \nabla_x \Phi$. For the Keller–Segel system (1.6), a symmetrisation trick can be used in order to compensate in dimension $N = 2$ (and $N = 1$) for the singularity of the convolution kernel (see the formulation in [12, 33, 35]). Due to the crossing in the coupling, this trick does not operate here. Moreover, we shall work by approximation from bounded data, and we are facing the difficulty of the lack of compactness in Lebesgue's spaces of sequences which are only bounded in L^1. For these reasons, we work with initial data in $L^{1+\delta}(\mathbb{R}^2)$, $\delta > 0$.

Let $\rho_{p,0}, \rho_{c,0}$ be in $L^{1+\delta}(\mathbb{R}^2)$. We take a sequence of smooth initial data $\rho_{p,0}^n, \rho_{c,0}^n \in C_c^\infty(\mathbb{R}^2)$ that converges to $\rho_{p,0}, \rho_{c,0}$ in $L^{1+\delta}(\mathbb{R}^2)$. As said above, the a priori estimates apply to the solution (ρ_p^n, ρ_c^n) associated with $(\rho_{p,0}^n, \rho_{c,0}^n)$: ρ_p^n, ρ_c^n are bounded in the space $L^\infty(0,T; L^{1+\delta}(\mathbb{R}^2))$, with $\nabla_x(\rho_p^n)^{(1+\delta)/2}$ and $\nabla_x(\rho_c^n)^{(1+\delta)/2}$ bounded in $L^2((0,T)\times\mathbb{R}^2)$. Owing to (3.5), we deduce that ρ_p^n and ρ_c^n are bounded in $L^{2+\delta}((0,T)\times\mathbb{R}^2)$. Since, by Lemma 3.8, $\nabla_x \Phi_p^n$ and $\nabla_x \Phi_c^n$ are bounded in $L^q_{\mathrm{loc}}((0,T)\times\mathbb{R}^2)$, for any $1 \le q < 2$, the products $\rho_p^n \nabla_x \Phi_p^n$ and $\rho_c^n \nabla_x \Phi_c^n$ lie in a bounded set of $L^1_{\mathrm{loc}}((0,T)\times\mathbb{R}^2)$. We are left with the task of passing to the limit in the non-linear terms. We only treat the prey equation, the chaser equation being treated in a similar way. We can assume, possibly at the price of extracting subsequences, that $\rho_p^n \rightharpoonup \rho_p$ weakly in $L^{2+\delta}((0,T)\times\mathbb{R}^2)$ and $\nabla_x \Phi_p^n \rightharpoonup \nabla_x \Phi_p$ weakly in $L^q((0,T)\times B(0,R))$ for any $1 \le q < 2$ and $0 < R < \infty$. Furthermore, on the one hand, by Lemma 3.8, we have the following "compactness property with respect to the space variable":

$$\lim_{|h|\to 0} \left(\sup_n \|\nabla_x \Phi_p^n(t, x+h) - \nabla_x \Phi_p^n(t,x)\|_{L^q((0,T)\times B(0,R))} \right) = 0.$$

On the other hand, $\partial_t \rho^n$ appears as the sum of the first and second derivatives of sequences bounded in $L^1((0,T)\times B(0,R))$. Directly applying [25, Lemma 5.1] allows us to conclude that $\rho_p^n \nabla_x \Phi_p^n \rightharpoonup \rho_p \nabla_x \Phi_p$ in the sense of distributions, as $n \to \infty$, with $\Delta_x \Phi_p = -\alpha \rho_c$. The estimates also imply that, for any $\varphi \in C_c^\infty(\mathbb{R}^2)$, the sequence $\int \rho_p^n(t,x)\varphi(x)\,\mathrm{d}x$ can be assumed to converge in $C([0,T])$ to $\int \rho_p(t,x)\varphi(x)\,\mathrm{d}x$. The uniform bound on the second-order moment also allows us to justify the mass conservation. This ends the proof of Theorem 2.2, once we use the improved regularity proven in Lemma 3.6 and 3.3.

3.5. Comments on space dimensions $N \neq 2$. Let us discuss the role of space dimension. To this end, let us go back to a more general form of the Gagliardo–

Niremberg–Sobolev inequality [29]

$$\left(\int_{\mathbb{R}^N} |\xi|^p \,\mathrm{d}x\right)^{1/p} \le C\left(\int_{\mathbb{R}^N} |\nabla\xi|^r \,\mathrm{d}x\right)^{a/r} \left(\int_{\mathbb{R}^N} |\xi|^q \,\mathrm{d}x\right)^{(1-a)/q},$$

which holds with

$$\frac{1}{p} = a\left(\frac{1}{r} - \frac{1}{N}\right) + \frac{1-a}{q}.$$

We naturally control the L^1 norm through the mass conservation property, which leads to select $q=1$. Besides, the De Giorgi analysis relies on the estimate on the L^2 norm, which leads to $p=2+1=3$ and $r=2$. With such a choice of parameters, we get $a=\frac{2}{3}\times\frac{2N}{N+2}$. We can absorb the gradient by using the dissipation induced by the diffusion as in the proof of Lemma 3.2 provided $\frac{pa}{r} = \frac{2N}{N+2} \le 1$, which thus restricts to dimensions $N=1$ or $N=2$. For $N=1$, we can thus establish the analog to Theorem 2.2.

THEOREM 3.1. *Theorem 2.2 also holds in dimension $N=1$.*

Proof. We sketch the proof and leave the details to the reader. Going back to the proof of Lemma 3.2, for $N=1$, the Gagliardo–Nirenberg–Sobolev inequality

$$\int \rho^3 \,\mathrm{d}x \le C\left(\int \left|\frac{\mathrm{d}}{\mathrm{d}x}\rho\right|^2 \mathrm{d}x\right)^{2/3} \left(\int \rho \,\mathrm{d}x\right)^{5/3}$$

yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \rho_c^2 \,\mathrm{d}x + 2\int \left|\frac{\mathrm{d}}{\mathrm{d}x}\rho_c\right|^2 \mathrm{d}x \\
&\le 2\sqrt{\delta}\int \rho_c^3 \,\mathrm{d}x + \frac{1}{\delta}\int \rho_p^3 \,\mathrm{d}x \\
&\le 2C\sqrt{\delta}\left(\int \left|\frac{\mathrm{d}}{\mathrm{d}x}\rho_c\right|^2 \mathrm{d}x\right)^{2/3} \left(\int \rho_c \,\mathrm{d}x\right)^{5/3} + \frac{C}{\delta}\left(\int \left|\frac{\mathrm{d}}{\mathrm{d}x}\rho_p\right|^2 \mathrm{d}x\right)^{2/3} \left(\int \rho_p \,\mathrm{d}x\right)^{5/3}.
\end{aligned}$$

Young's inequality and mass conservation permit us to find $a, b > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \rho_c^2 \,\mathrm{d}x + \int \left|\frac{\mathrm{d}}{\mathrm{d}x}\rho_c\right|^2 \mathrm{d}x \le a + b\int \left|\frac{\mathrm{d}}{\mathrm{d}x}\rho_p\right|^2 \mathrm{d}x.$$

Since the estimate on ρ_p is clear, see (3.3), we deduce that the analog of Lemma 3.2 in dimension $N=1$ provides an estimate with linear growth on $\int \rho_c^2(t,x)\,\mathrm{d}x$ and $\int_0^t \int |\frac{\mathrm{d}}{\mathrm{d}x}\rho_c|^2 \,\mathrm{d}x$ when $\rho_{p,0}$ and $\rho_{c,0}$ lie in $L^2(\mathbb{R})$.

Similarly, we turn to the adaptation of Lemma 3.5 for $N=1$. With $q=2$, (3.6) becomes

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \rho_p^2 \,\mathrm{d}x + 2\int \left|\frac{\mathrm{d}}{\mathrm{d}x}\rho_p\right|^2 \mathrm{d}x \le 0, \\
&\frac{\mathrm{d}}{\mathrm{d}t}\int \rho_c^2 \,\mathrm{d}x + C_1\int \left|\frac{\mathrm{d}}{\mathrm{d}x}\rho_c\right|^2 \mathrm{d}x \le C_2\int \rho_p^3 \,\mathrm{d}x + C_3.
\end{aligned} \tag{3.13}$$

We combine again the Gagliardo–Nirenberg–Sobolev inequality with the Cauchy–Schwarz inequality

$$\int \rho^2 \,\mathrm{d}x = \int \rho^{1/2}\rho^{3/2} \,\mathrm{d}x \le \left(\int \rho \,\mathrm{d}x\right)^{1/2} \left(\int \rho^3 \,\mathrm{d}x\right)^{1/2},$$

and we eventually arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{X}+\beta\mathscr{X}^3\le\alpha,$$

with

$$\mathscr{X}=\int\rho_p^2\,\mathrm{d}x+A\int\rho_c^2\,\mathrm{d}x$$

for some constants $A,\alpha,\beta>0$. Then, the argument in Appendix A justifies that Lemma 3.5 applies for $N=1$ as well. Then, we can reproduce the proof of Lemma 3.6 to conclude that the solution becomes instantaneously bounded. ∎

It is equally possible to elaborate further on the behaviour of the solutions when $N>2$, at the price of a suitable smallness condition on the data. The argument, directly inspired by [31, Section 5.2.2] and the references therein for the Keller–Segel system, also provides decay estimates.

Theorem 3.2. *Let $N>2$. There exists a constant κ_N such that, if the initial data $\rho_{p,0},\rho_{c,0}\in L^1(\mathbb{R}^N)$ satisfies*

$$\|\rho_{p,0}\|_{N/2}+\|\rho_{c,0}\|_{N/2}\le\kappa_N,$$

then the system (1.1)–(1.2) *admits a global weak solution such that ρ_p and ρ_c belong to $L^\infty(0,\infty;L^{N/2}(\mathbb{R}^N))$, with $\nabla\rho_p^{N/4}$ and $\nabla\rho_c^{N/4}$ in $L^2((0,\infty)\times\mathbb{R}^N)$. Furthermore, there exists a constant C_N such that*

$$\|\rho_p(t,\cdot)\|_{N/2}+\|\rho_c(t,\cdot)\|_{N/2}\le C_N\frac{1}{t^{(N-2)/2}}.$$

Proof. The generalization of Equation (3.5) to any space dimension reads

$$\int_{\mathbb{R}^N}|\xi|^{q+1}\,\mathrm{d}x\le C_q\int_{\mathbb{R}^N}|\nabla\xi^{q/2}|^2\,\mathrm{d}x\,\left(\int_{\mathbb{R}^N}|\xi|^{N/2}\,\mathrm{d}x\right)^{2/N}.$$

Of course, the difficulty relies on the fact we do not control naturally the $L^{N/2}$ norm, while the L^1 norm is preserved by the equation. We go back to Equations (3.3) and (3.4). The former tells us that ρ_p is bounded in $L^\infty(0,\infty;L^q(\mathbb{R}^N))$, with $\nabla\rho_b^{q/2}$ bounded in $L^2((0,\infty)\times\mathbb{R}^N)$ when $\rho_{p,0}\in L^q(\mathbb{R}^N)$, as noticed in Lemma 3.1. Proceeding as in the proof of Lemma 3.2, the latter becomes

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int\rho_c^q\,\mathrm{d}x+4\frac{q-1}{q}\int|\nabla\rho_c^{q/2}|^2\,\mathrm{d}x\\&\le q\int\rho_c^{q+1}\,\mathrm{d}x+\int\rho_p^{q+1}\,\mathrm{d}x\\&\le qC_q\left(\int\rho_c^{N/2}\,\mathrm{d}x\right)^{2/N}\int|\nabla\rho_c^{q/2}|^2\,\mathrm{d}x+C_q\left(\int\rho_p^{N/2}\,\mathrm{d}x\right)^{2/N}\int|\nabla\rho_p^{q/2}|^2\,\mathrm{d}x.\end{aligned}$$

We use this relation in the specific case $q=N/2>1$. Let $\Lambda>0$. We thus get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int\rho_c^{N/2}\,\mathrm{d}x+\Lambda\int\rho_p^{N/2}\,\mathrm{d}x\right\}+\left(4\frac{N-2}{N}-\frac{N}{2}C_{N/2}\left(\int\rho_c^{N/2}\,\mathrm{d}x\right)^{2/N}\right)\int|\nabla\rho_c^{N/4}|^2\,\mathrm{d}x\\&\quad+\left(4\frac{N-2}{N}\Lambda-\frac{N}{2}C_{N/2}\left(\int\rho_p^{N/2}\,\mathrm{d}x\right)^{2/N}\right)\int|\nabla\rho_p^{N/4}|^2\,\mathrm{d}x\\&\le 0.\end{aligned}$$

By Lemma 3.1, we know that $\int \rho_p^{N/2}\,\mathrm{d}x \le \int \rho_{p,0}^{N/2}\,\mathrm{d}x$. Hence, let us pick

$$\Lambda > \frac{N^2}{8(N-2)}\ C_{N/2}\left(\int \rho_{p,0}^{N/2}\,\mathrm{d}x\right)^{2/N}.$$

We are led to

$$\begin{aligned}
&\int \rho_c^{N/2}(t,x)\,\mathrm{d}x \le \int \rho_c^{N/2}(t,x)\,\mathrm{d}x + \Lambda \int \rho_p^{N/2}(t,x)\,\mathrm{d}x \\
&\le \int \rho_{c,0}^{N/2}(x)\,\mathrm{d}x + \Lambda \int \rho_{p,0}^{N/2}(x)\,\mathrm{d}x \\
&\quad + \int_0^t \left(\frac{N}{2} C_{N/2}\left(\int \rho_c^{N/2}(s,x)\,\mathrm{d}x\right)^{2/N} - 4\frac{N-2}{N}\right)\int |\nabla \rho_c^{N/4}(s,x)|^2\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

Finally, a simple continuity argument shows that, if initially

$$\left(\int \rho_{c,0}^{N/2}(x)\,\mathrm{d}x + \Lambda \int \rho_{p,0}^{N/2}(x)\,\mathrm{d}x\right)^{2/N} < 8\frac{N-2}{N^2 C_{N/2}}$$

holds, then, this property is preserved. It proves the uniform bound on the $L^{N/2}$-norm of the densities, under the smallness condition.

This analysis shows that $\|\rho_p(t,\cdot)\|_{N/2}$ and $\|\rho_c(t,\cdot)\|_{N/2}$ are uniformly bounded. More precisely, we can find a constant $\kappa > 0$ such that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int \rho_c^{N/2}\,\mathrm{d}x + \Lambda \int \rho_p^{N/2}\,\mathrm{d}x\right\} \\
&\le -\kappa\left(\int |\nabla \rho_c^{N/4}|^2\,\mathrm{d}x + \int |\nabla \rho_p^{N/4}|^2\,\mathrm{d}x\right) \\
&\le -\frac{\kappa}{C_{N/2}}\left(\left(\int \rho_c^{N/2}\,\mathrm{d}x\right)^{-2/N}\int \rho_c^{1+N/2}\,\mathrm{d}x + \left(\int \rho_p^{N/2}\,\mathrm{d}x\right)^{-2/N}\int \rho_p^{1+N/2}\,\mathrm{d}x\right)
\end{aligned}$$

by using the Gagliardo–Nirenberg–Sobolev inequality. Next, we use the simple interpolation inequality

$$\int \xi^{N/2}\,\mathrm{d}x \le \left(\int \xi\,\mathrm{d}x\right)^{2/N}\left(\int \xi^{1+N/2}\,\mathrm{d}x\right)^{(N-2)/N},$$

which, combined to the mass conservation, allows us to obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int \rho_c^{N/2}\,\mathrm{d}x + \Lambda \int \rho_p^{N/2}\,\mathrm{d}x\right\} \\
&\le -\frac{\kappa}{C_{N/2}}\left(m_c^{-2/(N-2)}\left(\int \rho_c^{N/2}\,\mathrm{d}x\right)^{N/(N-2)-2/N}\right. \\
&\qquad\qquad \left. + m_p^{-2/(N-2)}\left(\int \rho_p^{N/2}\,\mathrm{d}x\right)^{N/(N-2)-2/N}\mathrm{d}x\right).
\end{aligned}$$

Let us set

$$\mathscr{U}(t) = \int \rho_c^{N/2}(t,x)\,\mathrm{d}x + \Lambda \int \rho_p^{N/2}(t,x)\,\mathrm{d}x.$$

Owing to the elementary inequalities $\underline{C}_\theta(a^\theta+b^\theta)\leq(a+b)^\theta\leq\overline{C}_\theta(a^\theta+b^\theta)$, which hold for any $a,b\geq 0$, and $\theta\in(0,1)$, we obtain the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{U}(t)\leq -C\mathscr{U}(t)^{N/(N-2)-2/N}$$

for a certain $C>0$. We set $\alpha_N=\frac{4}{N(N-2)}>0$ so that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathscr{U}(t))^{-\alpha_N}\geq C\alpha_N.$$

We deduce the decay with a rate given by t^{-1/α_N} by integrating this ODE. ∎

4. Analysis of the kinetic model

As said in the Introduction, we adopt the functional framework introduced in [33]. Differences with the analysis in [33] are due to the following facts:

- We are dealing with a system of kinetic equations instead of considering a mere scalar unknown.
- The coupling crosses the influence of a population on the other; accordingly, the structure of the system changes and we cannot use important properties of the Vlasov–Poisson–Fokker–Planck system (like the compensation between the time evolution of the kinetic energy and the potential energy, etc.).

It is likely that our existence result is not optimal; it could certainly be improved by adapting the techniques in [5]. However, Theorem 2.3 provides a unified functional framework to handle the asymptotic issues in Theorem 2.4. It would be tempting to extend the latter by working with renormalisation methods, as in [9]. However, we are still facing the lack of estimates on the potentials (which in the present analysis lie in L^∞) and of energy/entropy structure as for the usual Vlasov–Poisson–Fokker–Planck system.

4.1. A priori estimates. What makes the norm $|||\cdot|||_q$ well-adapted to handle this problem can be recapped in the following statement.

LEMMA 4.1. *Let $f:\mathbb{R}^N\times\mathbb{R}^N\to[0,\infty]$ be an integrable function such that $|||f|||_q<\infty$ for some $N<q<\infty$. Then*

i) $f\in L^q(\mathbb{R}^N\times\mathbb{R}^N)$,

ii) For any $1\leq s\leq q$, *we have* $|||f|||_s^s\leq\|f\|_1+|||f|||_q^q$.

iii) $\rho(x)=\int f(x,v)\,\mathrm{d}v$ *lies in* $L^q(\mathbb{R}^N)$. *In fact, we have* $\|\rho\|_p\leq|||f|||_q$.

iv) We set $\Psi(x)=\int\frac{x-y}{|x-y|^N}\rho(y)\,\mathrm{d}y$. *If* $q>N$, *then* $\Psi\in L^\infty(\mathbb{R}^N)$. *Furthermore, there exists a constant* $C>0$ *such that*

$$\|\Psi\|_\infty\leq C\|\rho\|_q^\beta\|\rho\|_1^{1-\beta},\qquad \beta=\frac{q(N-1)}{(q-1)N},\qquad 1-\beta=\frac{q-N}{(q-1)N}.$$

Note that $\frac{2\beta}{q}<N$ *when* $q>N\geq 2$.

Proof. The first item comes from the obvious relation

$$\iint|f|^q\,\mathrm{d}v\,\mathrm{d}x=\iint\left(\frac{|f|}{M}\right)^q M\times M^{q-1}\,\mathrm{d}v\,\mathrm{d}x\leq\|M\|_\infty^{q-1}|||f|||_q,$$

where M is given by Equation (1.4). Property (ii) follows by interpolation: writing $s=\theta+(1-\theta)q$, $0\le\theta\le1$, we obtain

$$\begin{aligned}\iint\left|\frac{f}{M}\right|^s M\,\mathrm{d}v\,\mathrm{d}x &= \iint\left(\frac{|f|}{M}\right)^\theta M^\theta\times\left(\frac{|f|}{M}\right)^{(1-\theta)q} M^{1-\theta}\,\mathrm{d}v\,\mathrm{d}x\\ &\le\left(\iint|f|\,\mathrm{d}v\,\mathrm{d}x\right)^\theta\left(\iint\left|\frac{f}{M}\right|^q M\,\mathrm{d}v\,\mathrm{d}x\right)^{1-\theta}\end{aligned}$$

and we conclude by convexity. Next, Hölder's inequality yields

$$\int\rho^q\,\mathrm{d}x=\int\left(\int\frac{f}{M}M^{1/q}M^{1/q'}\,\mathrm{d}v\right)^q\mathrm{d}x\le\int\left(\int\left(\frac{f}{M}\right)^q M\,\mathrm{d}v\right)\left(\int M dv\right)^{q/q'}\mathrm{d}x.$$

Since $\int M\,\mathrm{d}v=1$, we are led to $\|\rho\|_q\le|||f|||_q$. Eventually, with $A>0$, we split

$$\Psi(x)=\int_{|x-y|\le A}\cdots\,\mathrm{d}y+\int_{|x-y|\ge A}\cdots\,\mathrm{d}y.$$

The second integral is dominated by $\frac{\|f\|_1}{A^{N-1}}$, while we get

$$\int_{|x-y|\le A}\frac{|\rho(y)|}{|x-y|^{N-1}}\,\mathrm{d}y\le\|\rho\|_q\left(|\mathbb{S}^{N-1}|\int_0^A\frac{\mathrm{d}r}{r^{(q'-1)(N-1)}}\right)^{1/q'}.$$

The right-hand side is finite since $q>N$ implies $(q'-1)(N-1)<1$. Optimizing with respect to A, we obtain the desired result. ∎

Let us start by establishing a priori estimates, following [33, Lemmas 2.3 & 3.1]. Let f be a solution of the Fokker–Planck equation

$$\partial_t f+\frac{1}{\epsilon}(v\cdot\nabla_x f-\nabla_x\Phi\cdot\nabla_v f)=\frac{1}{\epsilon^2}Lf, \tag{4.1}$$

where we assume, for the time being, that Φ is a given potential. Let $\mathscr{H}:[0,\infty)\to[0,\infty)$ be a convex function. We obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\iint\mathscr{H}\left(\frac{f}{M}\right)M\,\mathrm{d}v\,\mathrm{d}x+\frac{1}{2\epsilon^2}\iint\mathscr{H}''\left(\frac{f}{M}\right)\left|\nabla_v\left(\frac{f}{M}\right)\right|^2 M\,\mathrm{d}v\,\mathrm{d}x\\ &\le\frac{1}{2}\|\nabla_x\Phi\|_\infty^2\iint\mathscr{H}''\left(\frac{f}{M}\right)\left|\nabla_v\left(\frac{f}{M}\right)\right|^2 M\,\mathrm{d}v\,\mathrm{d}x.\end{aligned} \tag{4.2}$$

We use this relation with $\mathscr{H}(z):=z^q$, where $q\ge1$. We remark that $\mathscr{H}''(f/M)f^2/M=q(q-1)\mathscr{H}(f/M)M$. It allows us to conclude that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\iint\left(\frac{f}{M}\right)^q M\,\mathrm{d}v\,\mathrm{d}x+\frac{q(q-1)}{2\epsilon^2}\iint\left(\frac{f}{M}\right)^{q-2}\left|\nabla_v\left(\frac{f}{M}\right)\right|^2 M\,\mathrm{d}v\,\mathrm{d}x\\ &\le\frac{q(q-1)}{2}\|\nabla_x\Phi\|_\infty^2\iint\left(\frac{f}{M}\right)^q M\,\mathrm{d}v\,\mathrm{d}x\end{aligned} \tag{4.3}$$

holds. We obtain useful estimates by applying these observations to the solutions of the system (1.5). The following statement brings out that $\mathscr{C}_{q,T}$ is an adapted set to establish existence-uniqueness of solutions of the system (1.3) and to investigate the asymptotic behaviour of solutions of the system (1.5) as $\epsilon \to 0$.

LEMMA 4.2. *Let* $(f_p^\epsilon, f_c^\epsilon)$ *be a solution of the system* (1.5). *We assume that*

$$\sup_{\epsilon>0}\Big(|||f_{p,0}^\epsilon|||_q^q + |||f_{c,0}^\epsilon|||_q^q\Big) < \infty.$$

For any $T>0$ *small enough, there exists a constant* C_T *such that*

$$\sup_{0\le t\le T,\ \epsilon>0}\Big(|||f_p^\epsilon(t,\cdot)|||_q^q + |||f_c^\epsilon(t,\cdot)|||_q^q\Big) \le C_T,$$
$$\sup_{0\le t\le T,\ \epsilon>0}\Big(\|\nabla_x\Phi_p^\epsilon(t,\cdot)\|_\infty + \|\nabla_x\Phi_c^\epsilon(t,\cdot)\|_\infty\Big) \le C_T.$$

Proof. Estimate (4.3) and Lemma 4.1 apply for both the equations for f_p^ϵ and f_c^ϵ in the system (1.5). Let $q>N$. Let us set

$$Z(t) = |||f_c^\epsilon(t,\cdot)|||_q^q + |||f_c^\epsilon(t,\cdot)|||_q^q$$

We arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}Z(t) \le \mathscr{C}\ Z(t)^{1+2\beta/q}$$

with $\mathscr{C} = \frac{q(q-1)}{2}(m_c^{2(1-\beta)} + m_p^{2(1-\beta)})$. We can compare Z to the solution $y : t\mapsto y(t)$ of the non-linear ODE $y'(t) = \mathscr{C}y^{1+2\beta/q}(t)$, with $y(0) = \sup_{\epsilon>0}(|||f_{p,0}^\epsilon|||_q^q + |||f_{c,0}^\epsilon|||_q^q) \ge Z(0)$. Let $T_\star$ stand for the lifespan of this solution (note that it does not depend on ϵ). We conclude that $0\le Z(t)\le y(t)$ holds for every $t\in[0,T_\star)$. ∎

We shall need further estimates, which will be useful to control moments and entropy dissipation. To be more specific, we shall make use of the following claim.

LEMMA 4.3. *Let* $(f_p^\epsilon, f_c^\epsilon)$ *be a solution of the system* (1.5). *In addition to the hypothesis of Lemma 4.2, we assume that*

$$\sup_{\epsilon>0}\left(\iint f_{j,0}^\epsilon\left(|\ln(f_{j,0}^\epsilon)| + |x| + \frac{v^2}{2}\right)\mathrm{d}v\,\mathrm{d}x\right) < \infty$$

holds for $j\in\{p,c\}$. *Then, for any* $T>0$ *small enough (as in Lemma 4.2), there exists a constant* C_T *such that*

$$\sup_{0\le t\le T,\ \epsilon>0}\left(\iint f_j^\epsilon\left(|\ln(f_j^\epsilon)| + |x| + \frac{v^2}{2}\right)\mathrm{d}v\,\mathrm{d}x\right) \le C_T,$$
$$\sup_{\epsilon>0}\frac{1}{2\epsilon^2}\int_0^T\iint |v\sqrt{f_j^\epsilon} + 2\nabla_v\sqrt{f_j^\epsilon}|^2\,\mathrm{d}x\,\mathrm{d}x\,\mathrm{d}t \le C_T.$$

Proof. Let us go back to the generic Equation (4.1). We use (4.2) with $\mathscr{H}(z) = z\ln(z)$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\iint f\ln\left(\frac{f}{M}\right)\mathrm{d}v\,\mathrm{d}x + \frac{1}{2\epsilon^2}\iint\left|v\sqrt{f} + 2\nabla_v\sqrt{f}\right|^2\mathrm{d}v\,\mathrm{d}x\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\iint f\left(\ln(f) + \frac{v^2}{2}\right)\mathrm{d}v\,\mathrm{d}x + \frac{1}{2\epsilon^2}\iint\left|v\sqrt{f} + 2\nabla_v\sqrt{f}\right|^2\mathrm{d}v\,\mathrm{d}x\\
&\le \frac{1}{2}\|\nabla_x\Phi\|_\infty^2\iint f\,\mathrm{d}v\,\mathrm{d}x = \frac{m}{2}\|\nabla_x\Phi\|_\infty^2
\end{aligned}$$

with $m = \iint f_0 \,\mathrm{d}v\,\mathrm{d}x$, by mass conservation. We shall combine this estimate with the time evolution of the first space moment

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\iint |x| f \,\mathrm{d}v\,\mathrm{d}x &= \frac{1}{\epsilon}\iint \frac{x}{|x|}\cdot v f \,\mathrm{d}v\,\mathrm{d}x = \iint \frac{x}{|x|}\sqrt{f}\cdot\frac{v\sqrt{f}+2\nabla_v\sqrt{f}}{\epsilon}\,\mathrm{d}v\,\mathrm{d}x\\ &\le \frac{1}{2}\iint f\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2\epsilon^2}\iint \left|v\sqrt{f}+2\nabla_v\sqrt{f}\right|^2\mathrm{d}v\,\mathrm{d}x.\end{aligned}$$

These inequalities do not directly provide a useful estimate since $z\ln(z)$ changes sign. We use the decomposition

$$\begin{aligned}z|\ln(z)| &= z\ln(z) - 2z\ln(z)\mathbf{1}_{0\le z\le e^{-\omega}} - 2z\ln(z)\mathbf{1}_{e^{-\omega}<z\le 1}\\ &\le z\ln(z) + \frac{4}{e}e^{-\omega/2} + 2\omega z.\end{aligned}$$

With $\omega = \frac{1}{4}(\frac{v^2}{2}+|x|)$, we are led to

$$\begin{aligned}&\iint f\Big(|\ln(f)| + \frac{|x|}{2} + \frac{v^2}{4}\Big)\mathrm{d}v\,\mathrm{d}x + \frac{1}{2\epsilon^2}\int_0^t |v\sqrt{f}+2\nabla_v\sqrt{f}|^2\,\mathrm{d}x\,\mathrm{d}x\,\mathrm{d}s\\ &\le \iint f_0\Big(|\ln(f_0)| + |x| + \frac{v^2}{2}\Big)\mathrm{d}v\,\mathrm{d}x + \frac{\|\nabla_x\Phi\|_\infty^2}{2}m + \frac{4}{e}\iint e^{-v^2/16-|x|/8}\,\mathrm{d}v\,\mathrm{d}x.\end{aligned}$$

We readily adapt the argument to deal with the system (1.5). ∎

4.2. Asymptotics analysis. We are now dealing with the rescaled system (1.5). We are going to use the uniform estimates in Lemma 4.2 (where we remind the reader that $q > N$) and Lemma 4.3. In what follows, we consider $T > 0$ as given by Lemma 4.2.

LEMMA 4.4. *We can find $C_T > 0$ such that for $j \in \{p,c\}$,*

$$\frac{1}{\epsilon^2}\int_0^T \iint \left|\nabla_v\left(\frac{f_j^\epsilon}{M}\right)\right|^2 M\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \le C_T. \tag{4.4}$$

Moreover, we have

$$\lim_{\epsilon\to 0}\int_0^T \iint |f_j^\epsilon - \rho_j^\epsilon M|^2\,\frac{\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t}{M} = 0,$$

(precisely, it is of order $\mathscr{O}(\epsilon^2)$) where we have set $\rho_j^\epsilon = \int f_j^\epsilon\,\mathrm{d}v$.

Proof. Since $2 \le N < q$, we use Lemma 4.1-ii): we go back to Equation (4.3) for $\mathscr{H}(z) = z^2$, bearing in mind that $\iint |f_j^\epsilon|^2/M\,\mathrm{d}v\,\mathrm{d}x$ is uniformly bounded with respect to $\epsilon > 0$ and $t \in [0,T]$, by virtue of Lemma 4.2. Integrating Equation (4.3) over $[0,T]$ we conclude that Equation (4.4) holds. The next step follows by applying the following Sobolev inequality (see [4, Corollary 2.18 and Theorem 3.2]): there exists $\Lambda > 0$ such that, for any admissible function f¡

$$\int\left|f(v) - M(v)\int f(w)\,\mathrm{d}w\right|^2\frac{\mathrm{d}v}{M(v)} \le \Lambda\int\left|\nabla_v\left(\frac{f}{M}\right)\right|^2 M\,\mathrm{d}v.$$

∎

Having at hand these estimates, the asymptotic analysis is now understood by looking at the moment system, obtained by velocity averaging the equations. Integrate Equation (1.5) with respect to v. We obtain the following conservation equations:

$$\partial_t \rho_c^\epsilon + \mathrm{div}_x J_c^\epsilon = 0, \qquad \partial_t \rho_p^\epsilon + \epsilon^{-1} \mathrm{div}_x J_p^\epsilon = 0, \tag{4.5}$$

where we have set

$$J_j^\epsilon = \frac{1}{\epsilon} \int v f_j^\epsilon \,\mathrm{d}v.$$

Similarly, multiplying Equation (1.5) by v and integrating yield

$$\begin{aligned} &\epsilon^2 \partial_t J_c^\epsilon + \mathrm{Div}_x \int (v \otimes v) f_c^\epsilon \,\mathrm{d}v + \rho_c^\epsilon \nabla_x \Phi_c^\epsilon = -J_{c,\epsilon}, \\ &\epsilon^2 \partial_t J_p^\epsilon + \mathrm{Div}_x \int (v \otimes v) f_p^\epsilon \,\mathrm{d}v + \rho_p^\epsilon \nabla_x \Phi_p^\epsilon = -J_{p,\epsilon}. \end{aligned} \tag{4.6}$$

Lemma 4.5. *For $j \in \{p,c\}$, on the one hand, the sequence $\left(J_j^\epsilon\right)_{\epsilon>0}$ is bounded in $L^2((0,T)\times\mathbb{R}^N)$, and on the other hand, we can rewrite*

$$\int v \otimes v f_j^\epsilon \,\mathrm{d}v = \rho_j^\epsilon \mathbb{I} + \epsilon R_j^\epsilon,$$

where $\left(R_j^\epsilon\right)_{\epsilon>0}$ is bounded in $L^2((0,T)\times\mathbb{R}^N)$.

Proof. We write

$$\begin{aligned} \int_0^T \int |J_j^\epsilon|^2 \,\mathrm{d}x\,\mathrm{d}t &= \int_0^T \int \left| \int \frac{f_j^\epsilon}{M} \frac{(-\nabla_v M)}{\epsilon} \,\mathrm{d}v \right|^2 \mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \int \left| \int \frac{M}{\epsilon} \nabla_v \left(\frac{f_j^\epsilon}{M} \right) \mathrm{d}v \right|^2 \mathrm{d}x\,\mathrm{d}t \\ &\leq \frac{1}{\epsilon^2} \int_0^T \iint M \left| \nabla_v \left(\frac{f_j^\epsilon}{M} \right) \right|^2 \mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \leq C_T, \end{aligned}$$

and we conclude by going back to Equation (4.4). Next, R_j^ϵ is defined by

$$R_j^\epsilon = \int v \otimes v \sqrt{M}\, \frac{f_j^\epsilon - \rho_j^\epsilon M}{\epsilon \sqrt{M}} \,\mathrm{d}v,$$

so that

$$|R_j^\epsilon|^2 \leq \int |v|^4 M \,\mathrm{d}v \times \int \frac{|f_j^\epsilon - \rho_j^\epsilon M|^2}{\epsilon^2 M} \,\mathrm{d}v,$$

and we conclude by using the estimates that have led to Lemma 4.4. ∎

Possibly at the price of extracting subsequences, we can assume that

$$\begin{array}{ll} \rho_j^\epsilon \rightharpoonup \rho_j & \text{weakly in } L^q((0,T)\times\mathbb{R}^N, \\ J_j^\epsilon \rightharpoonup J_j & \text{weakly in } L^2((0,T)\times\mathbb{R}^N), \\ \nabla_x \Phi_j^\epsilon \rightharpoonup \nabla_x \Phi & \text{weakly}-\star \text{ in } L^\infty((0,T)\times\mathbb{R}^N). \end{array}$$

Hence, the only difficulty for passing to the limit in Equations (4.5)–(4.6) relies on the non-linear term $\rho_j^\epsilon \nabla_x \Phi_j^\epsilon$.

LEMMA 4.6. *Up to a subsequence, ρ_j^ϵ converges to ρ_j, for $j\in\{p,c\}$, strongly in $L^s(0,T;L^r(\mathbb{R}^N))$ for any $1\le s<\infty$, $1\le r<q$.*

Proof. The proof relies on a suitable application of the average lemma. Indeed, by Lemma 4.2 and Lemma 4.1, f_j^ϵ is bounded in $L^\infty(0,T;L^q(\mathbb{R}^N\times\mathbb{R}^N))$, with $q\ge N$, thus, by a mere interpolation argument, it is bounded also in $L^2((0,T)\times\mathbb{R}^N\times\mathbb{R}^N)$. It satisfies

$$(\epsilon\partial_t+v\cdot\nabla_x)f_j^\epsilon=\nabla_v\cdot g^\epsilon$$

with

$$g^\epsilon=f_j^\epsilon\nabla_x\Phi_j^\epsilon+\sqrt{M}\times\frac{\sqrt{M}}{\epsilon}\nabla_v\left(\frac{f_j^\epsilon}{M}\right).$$

It follows from Equation (4.4) that g^ϵ is bounded in $L^2((0,T)\times\mathbb{R}^N\times\mathbb{R}^N)$. Then, the average lemma (see for instance [11] or [26, Lemma 4.2]) tells us that

$$\lim_{|h|\to0}\left\{\sup_{\epsilon>0}\int_0^T\int_{B(0,R)}\left|\int f_j^\epsilon(t,x+h,v)\psi(v)\,\mathrm{d}v-\int f_j^\epsilon(t,x,v)\psi(v)\,\mathrm{d}v\right|^2\mathrm{d}x\,\mathrm{d}t\right\}=0$$

holds for any $\psi\in C_c^\infty(\mathbb{R}^N)$, $0<R<\infty$. Since the kinetic energy is uniformly bounded, we can work with smooth functions ψ not necessarily compactly supported. In particular, we have

$$\lim_{|h|\to0}\left\{\sup_{\epsilon>0}\int_0^T\int_{B(0,R)}\left|\rho_j^\epsilon(t,x+h)-\rho_j^\epsilon(t,x)\right|^2\mathrm{d}x\,\mathrm{d}t\right\}=0.$$

Furthermore $\partial_t\rho_j^\epsilon=-\nabla_x\cdot J_j^\epsilon$ is bounded in $L^2(0,T;H^{-1}(\mathbb{R}^N))$, by virtue of Lemma 4.5. We deduce that ρ_j^ϵ is relatively compact in $L^2((0,T)\times B(0,R))$ for any $0<R<\infty$ and thus in $L^1((0,T)\times B(0,R))$ too (see for instance Appendix B in [3]). Going back to Lemma 4.3, we see that $|x|\rho_j^\epsilon$ is bounded in $L^\infty(0,T;L^1(\mathbb{R}^N))$, which allows us to conclude that ρ_j^ϵ converges to ρ_j strongly in $L^1((0,T)\times B(0,R))$ (up to a subsequence). Finally, by lemmas 4.1 and 4.2, we know that ρ_j^ϵ is bounded in $L^\infty(0,T;L^q(\mathbb{R}^N))$. Interpolation estimates then tell us that the convergence holds in any $L^r(0,T;L^s(\mathbb{R}^N))$ for $1\le r<\infty$, $1\le s<q$. ∎

Combining Lemma 4.1 and Lemma 4.6, we can pass to the limit in the product $\rho_j^\epsilon\nabla_x\Phi_j^\epsilon$, say weakly in $L^r(0,T;L^s(\mathbb{R}^N))$, for $1\le r<\infty$, $1\le s<q$, at least for a suitable subsequence. Accordingly, we obtain

$$\partial_t\rho_j+\mathrm{div}_xJ_j=0,\qquad -J_j=\nabla_x\rho_j+\rho_j\nabla_x\Phi_j,$$

when we let ϵ go to 0 in Equations (4.5) and (4.6). We thus find the system (1.1)–(1.2). Finally, since $\partial_t\rho_j^\epsilon$ is bounded in $L^2(0,T;H^{-1}(\mathbb{R}^N))$, we can also assume that

$$\lim_{\epsilon\to0}\int\rho_j^\epsilon(t,x)\varphi(x)\,\mathrm{d}x=\int\rho_j(t,x)\varphi(x)\,\mathrm{d}x$$

uniformly on $[0,T]$ for any trial function $\varphi\in L^{q'}(\mathbb{R}^N)$, so that the initial data for the limiting equation also makes sense.

4.3. Existence of solutions. Let (f_c, f_p) be the solution of the linear system

$$\partial_t f_c + v\cdot\nabla_x f_c - \nabla_x\tilde{\Phi}_c\cdot\nabla_v f_c = L(f_c)$$
$$\partial_t f_p + v\cdot\nabla_x f_p - \nabla_x\tilde{\Phi}_p\cdot\nabla_v f_p = L(f_p),$$

with initial condition $f_{c,0}, f_{p,0} \geq 0$, where the potentials are given by the convolution formulae

$$\nabla_x\tilde{\Phi}_c = -C_N\int\tilde{\rho}_p(t,y)\frac{x-y}{|x-y|^N}\,\mathrm{d}y, \qquad \nabla_x\tilde{\Phi}_p = \alpha C_N\int\tilde{\rho}_c(t,y)\frac{x-y}{|x-y|^N}\,\mathrm{d}y.$$

Note that $f_c, f_p \geq 0$. We denote $\mathscr{S}(\tilde{\rho}_c,\tilde{\rho}_p) = (\rho_c,\rho_p)$, with $(\rho_c,\rho_p) = \int(f_c,f_p)\,\mathrm{d}v$. Owing to Lemma 4.1 and reproducing the estimates in the proof of Lemma 4.2, we readily find an invariant set for $\mathscr{S}$. To be more specific, let us set $\mathscr{R}_0 = |||f_{c,0}|||_q + |||f_{p,0}|||_q$. Let $\mathscr{R} > \mathscr{R}_0$. Suppose $\sup_{0\leq t\leq T}(\|\tilde{\rho}_c(t,\cdot)\|_q + \|\tilde{\rho}_p(t,\cdot)\|_q) \leq \mathscr{R}$, with $q > N$. By Equation (4.3), Lemma 4.1, and Grönwall's lemma, we get

$$\sup_{0\leq t\leq T}(|||f_c(t,\cdot)|||_q + |||f_p(t,\cdot)|||_q) \leq \mathscr{R}_0 e^{\bar{C}\mathscr{R}^{2\beta}T},$$

for a certain constant $\bar{C} > 0$, which depends on q, N, m_c, m_p. From now on, we can thus fix $0 < T < T_\star$ small enough such that $(\tilde{\rho}_c,\tilde{\rho}_p) \mapsto (f_c, f_p)$ leaves the ball with radius $\mathscr{R}$ in $\mathscr{C}_{q,T}$ invariant. Accordingly, the convex set

$$\mathscr{G} = \Bigg\{\rho_c,\rho_p : (0,T)\times\mathbb{R}^N \to [0,\infty],\ \int(\rho_c,\rho_p)\,\mathrm{d}x = (m_c,m_p),$$
$$\sup_{0\leq t\leq T}\Big(\|\rho_c(t,\cdot)\|_q + \|\rho_p(t,\cdot)\|_q \leq \mathscr{R}\Bigg\}$$

is left invariant by the mapping $\mathscr{S}$. Furthermore, reproducing the arguments of the proof of Lemma 4.3, we observe that

$$\sup_{0\leq t\leq T}\iint(v^2 + |x|)(f_c + f_p)\,\mathrm{d}v\,\mathrm{d}x \leq C(T,\mathscr{R}) \tag{4.7}$$

holds.

Next, let us pick two pairs $(\tilde{\rho}_{c,1},\tilde{\rho}_{p,1})$ and $(\tilde{\rho}_{c,2},\tilde{\rho}_{p,2})$ in this set $\mathscr{G}$, and consider the associated solutions $(f_{c,j}, f_{p,j})$. We define $(\delta_c,\delta_p) = (f_{c,2} - f_{c,1}, f_{p,2} - f_{p,1})$. We check that (δ_c,δ_p) verifies the system

$$\partial_t\delta_c + v\cdot\nabla_x\delta_c - \nabla_x\tilde{\Phi}_{c,1}\cdot\nabla_v\delta_c = L(\delta_c) + \nabla_x\left(\tilde{\Phi}_{c,2} - \tilde{\Phi}_{c,1}\right)\cdot\nabla_v f_{c,2},$$
$$\partial_t\delta_p + v\cdot\nabla_x\delta_p - \nabla_x\tilde{\Phi}_{p,1}\cdot\nabla_v\delta_p = L(\delta_p) + \nabla_x\left(\tilde{\Phi}_{p,2} - \tilde{\Phi}_{p,1}\right)\cdot\nabla_v f_{p,2}.$$

Repeating the manipulations that have led to Equation (4.3), we get

$$\begin{aligned}
&\tfrac{\mathrm{d}}{\mathrm{d}t}\textstyle\iint\mathscr{H}\left(\tfrac{\delta}{M}\right)M\,\mathrm{d}v\,\mathrm{d}x + \tfrac{1}{2}\textstyle\iint\mathscr{H}''\left(\tfrac{\delta}{M}\right)\left|\nabla_v\left(\tfrac{\delta}{M}\right)\right|^2 M\,\mathrm{d}v\,\mathrm{d}x\\
&\leq \frac{q(q-1)}{2}\left\|\nabla_x\tilde{\Phi}_1\right\|_\infty^2\iint\mathscr{H}\left(\frac{\delta}{M}\right)M\,\mathrm{d}v\,\mathrm{d}x\\
&\qquad + \iint\mathscr{H}''\left(\frac{\delta}{M}\right)f_2\nabla_x\left(\tilde{\Phi}_2 - \tilde{\Phi}_1\right)\nabla_v\left(\frac{\delta}{M}\right)\mathrm{d}v\,\mathrm{d}x
\end{aligned}$$

with δ (resp. $\tilde{\Phi}_j$, f_j) either δ_p or δ_c (resp. $\tilde{\Phi}_{p,j}$) or $\tilde{\Phi}_{c,j}$, $f_{p,j}$, or $f_{c,j}$). The additional term is dominated by using the Cauchy–Schwarz inequality

$$\begin{aligned}
&\iint \mathscr{H}''\left(\frac{\delta}{M}\right) f_2 \nabla_x \left(\tilde{\Phi}_2 - \tilde{\Phi}_1\right) \nabla_v \frac{\delta}{M} \,\mathrm{d}v\,\mathrm{d}x \\
&\leq \left\| \nabla_x \left(\tilde{\Phi}_2 - \tilde{\Phi}_1\right) \right\|_\infty \iint \mathscr{H}''\left(\frac{\delta}{M}\right) f_2 \left| \nabla_v \left(\frac{\delta}{M}\right) \right| \mathrm{d}v\,\mathrm{d}x \\
&\leq \left\| \nabla_x \left(\tilde{\Phi}_2 - \tilde{\Phi}_1\right) \right\|_\infty \left(\iint \mathscr{H}''\left(\frac{\delta}{M}\right) \left| \nabla_v \left(\frac{\delta}{M}\right) \right|^2 M \,\mathrm{d}v\,\mathrm{d}x \right)^{1/2} \\
&\qquad \times \left(\iint \mathscr{H}''\left(\frac{\delta}{M}\right) \frac{f_2^2}{M} \,\mathrm{d}v\,\mathrm{d}x \right)^{1/2} \\
&\leq \frac{1}{2} \iint \mathscr{H}''\left(\frac{\delta}{M}\right) \left| \nabla_v \left(\frac{\delta}{M}\right) \right|^2 M \,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2} \left\| \nabla_x \left(\tilde{\Phi}_2 - \tilde{\Phi}_1\right) \right\|_\infty^2 \iint \mathscr{H}''\left(\frac{\delta}{M}\right) \frac{f_2^2}{M} \,\mathrm{d}v\,\mathrm{d}x.
\end{aligned}$$

We use this relation with $\mathscr{H}(z) = z^s$, with $1 < s < q$. The last integral can be estimated by using Hölder's inequality as follows:

$$\begin{aligned}
&\iint \mathscr{H}''\left(\frac{\delta}{M}\right) \frac{f_2^2}{M} \,\mathrm{d}v\,\mathrm{d}x = s(s-1) \iint \left(\frac{\delta}{M}\right)^{s-2} \left(\frac{f_2}{M}\right)^2 M \,\mathrm{d}v\,\mathrm{d}x \\
&\leq s(s-1) \left(\iint \left(\frac{\delta}{M}\right)^s M \,\mathrm{d}v\,\mathrm{d}x \right)^{1-2/s} \left(\iint \left(\frac{f_2}{M}\right)^s M \,\mathrm{d}v\,\mathrm{d}x \right)^{2/s} \\
&\leq s(s-1) \left(\left(1 - \frac{2}{s}\right) \iint \left(\frac{\delta}{M}\right)^s M \,\mathrm{d}v\,\mathrm{d}x + \frac{2}{s} \iint \left(\frac{f_2}{M}\right)^s M \,\mathrm{d}v\,\mathrm{d}x \right).
\end{aligned}$$

We arrive at

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \iint \left(\frac{\delta}{M}\right)^s M \,\mathrm{d}v\,\mathrm{d}x \\
&\leq \frac{s(s-1)}{2} \left\| \nabla_x \tilde{\Phi}_1 \right\|_\infty^2 \iint \left(\frac{\delta}{M}\right)^s M \,\mathrm{d}v\,\mathrm{d}x + \frac{s(s-1)}{2} \left\| \nabla_x \left(\tilde{\Phi}_2 - \tilde{\Phi}_1\right) \right\|_\infty^2 \\
&\quad \times \left(\left(1 - \frac{2}{s}\right) \iint \left(\frac{\delta}{M}\right)^s M \,\mathrm{d}v\,\mathrm{d}x + \frac{2}{s} \iint \left(\frac{f_2}{M}\right)^s M \,\mathrm{d}v\,\mathrm{d}x \right).
\end{aligned}$$

Then, using the definition of $|||\cdot|||_s$ together with Grönwall's lemma, we obtain

$$\begin{aligned}
|||\delta(t,\cdot)|||_s^s &\leq (s-1) \int_0^t \left\| \nabla_x \left(\tilde{\Phi}_2 - \tilde{\Phi}_1\right)(\tau,\cdot) \right\|_\infty^2 |||f_2(\tau,\cdot)|||_s^s \,\mathrm{d}\tau \\
&\quad \times \exp\left\{ \int_0^t \frac{(s-1)(s-2)}{2} \left\| \nabla_x \left(\tilde{\Phi}_2 - \tilde{\Phi}_1\right)(\tau,\cdot) \right\|_\infty^2 + \frac{s(s-1)}{2} \left\| \nabla_x \tilde{\Phi}_1(\tau,\cdot) \right\|_\infty^2 \mathrm{d}\tau \right\}.
\end{aligned}$$

Going back to Lemma 4.1, we can find a constant $\tilde{C}(T,\mathscr{R})$ such that

$$|||\delta(t,\cdot)|||_s^s \leq T\tilde{C}(T,\mathscr{R}) \sup_{0\leq t\leq T} \left\| \nabla_x \left(\tilde{\Phi}_2 - \tilde{\Phi}_1\right)(\tau,\cdot) \right\|_\infty^2,$$

for any $0 \leq t \leq T$. Using the estimates in Lemma 4.1 again, we obtain

$$\left\| \nabla_x \left(\tilde{\Phi}_2 - \tilde{\Phi}_1\right)(\tau,\cdot) \right\|_\infty^2 \leq C \|(\tilde{\rho}_2 - \tilde{\rho}_1)(\tau,\cdot)\|_s^{2\beta} \|(\tilde{\rho}_2 - \tilde{\rho}_1)(\tau,\cdot)\|_1^{2-2\beta}.$$

It allows to conclude that the following continuity property holds: if $\big((\tilde\rho_{c,n},\tilde\rho_{p,n})\big)_{n\in\mathbb{N}}$ is a sequence of elements of $\mathscr{G}$ which converges to $(\tilde\rho_c,\tilde\rho_p)$ in $L^{2\beta}(0,T;L^s(\mathbb{R}^N))$, then $\big((f_{c,n},f_{p,n})\big)_{n\in\mathbb{N}}$ converges to (f_c,f_p) strongly in $\mathscr{C}_{s,T}$, and therefore $\big((\rho_{c,n},\rho_{p,n})\big)_{n\in\mathbb{N}}$ converges to (ρ_c,ρ_p) in $L^{2\beta}(0,T;L^s(\mathbb{R}^N))$. (We remind the reader that $1<2\beta<s$.)

Finally, let us consider a sequence $\big((\tilde\rho_{c,n},\tilde\rho_{p,n})\big)_{n\in\mathbb{N}}$ in $\mathscr{G}$. We have seen that $(f_{c,n},f_{p,n})$ is bounded in $\mathscr{C}_{q,T}$, and consequently, for $j\in\{c,p\}$,

- $f_{j,n}$ is bounded in $L^\infty(0,T;L^q(\mathbb{R}^N\times\mathbb{R}^N))$ since

$$\iint |f|^q\,\mathrm{d}v\,\mathrm{d}x=\iint\Big|\frac{f}{M}\Big|^q M\times M^{q-1}\,\mathrm{d}v\,\mathrm{d}x\le\|M\|_\infty^{q-1}|||f|||_q;$$

- by interpolation $\sup_{0\le t\le T}|||f_{j,n}(t,\cdot)|||_r$ is bounded for any $1\le r\le q$.

Going back to (4.3) with $q=2$, we deduce that

$$\int_0^T\iint\Big|\nabla_v\frac{f_{j,n}}{M}\Big|^2 M\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\le C_T$$

is uniformly bounded. It follows that

$$(\partial_t+v\cdot\nabla_x)f_{j,n}=\mathrm{div}_v g_{j,n}$$

where

$$g_{j,n}=f_{j,n}\nabla_x\tilde\Phi_{j,n}+\sqrt{M}\times\sqrt{M}\nabla_v\left(\frac{f_{j,n}}{M}\right)$$

is bounded in $L^\infty(0,T;L^2(\mathbb{R}^N\times\mathbb{R}^N))$. The standard average lemma tells us that the integrals $\int f_{j,n}(t,x,v)\psi(v)\,\mathrm{d}v$ belong to a compact in $L^2_{\mathrm{loc}}((0,T)\times\mathbb{R}^N)$ for any $\psi\in C_c^\infty(\mathbb{R}^N)$. Owing to Equation (4.7), we deduce that $\rho_{j,n}$ is compact in $L^r(0,T;L^s(\mathbb{R}^N))$ for any $1\le r<\infty$ and $1\le s<q$. It allows us to apply the Schauder theorem in order to justify the existence of a fixed point of the mapping $\mathscr{S}$.

Appendix A. A comparison Lemma.

Lemma A.1. *Let $X:[0,T]\to(0,\infty)$, for $0\le T\le\infty$, which satisfies for any $t\in[0,T]$,*

$$X'(t)+aX^\gamma(t)\le b$$

for some given $a,b>0$ and $\gamma>1$. Then, we can find $C>0$, which depends on a,b,γ, such that

$$X(t)\le C\Big(1+\frac{1}{t^{1/(\gamma-1)}}\Big)$$

holds for any $t\in[0,T]$.

Proof. The estimate is directly inspired by [2, Appendix A], where a more intricate statement is proved. Let

$$Z(t)=A\Big(1+\frac{1}{t^{1/(\gamma-1)}}\Big).$$

We observe that $t\mapsto Z(t)$ is non-increasing and thus bounded from below by $A=\lim_{t\to\infty}Z(t)$ Next, we compute

$$Z'(t)=-A(\gamma-1)\Big(\frac{Z(t)}{A}-1\Big)^\gamma<0$$

since $\gamma > 1$. Therefore, it follows that

$$Z'(t) + aZ^\gamma(t) \geq A\big(aA^{\gamma-1} - (\gamma-1)\big)\Big(\frac{Z(t)}{A}\Big)^\gamma \geq A\big(aA^{\gamma-1} - (\gamma-1)\big)$$

which can be made larger than b by choosing A large enough.

Since $\lim_{t\to 0} Z(t) = \infty$, we have $Z(t) > X(t)$ at least on some interval $[0, T_\star)$. Let us set $t_0 = \sup\{t > 0,\ Z(s) > X(s) \text{ on } 0 \leq s \leq t\}$. Suppose $t_0 < T$: we can find $t_1 \in (t_0, T)$ such that $Z(t_1) = X(t_1)$. By definition of t_0, we can find two sequences $t^{(k)}$ and $s^{(k)}$ such that

$$\begin{aligned}
&t_0 < t^{(k+1)} \leq t^{(k)} \leq t_1, && t_0 \leq s^{(k+1)} \leq s^{(k)} < t_1,\\
&t_0 < s^{(k)} < t^{(k)} \leq t_1, &&\\
&\lim_{k\to\infty} t^{(k)} = t_0 = \lim_{k\to\infty} s^{(k)}, &&\\
&X(t^{(k)}) > Z(s^{(k)}) = X(s^{(k)}), && X(t) > Z(t) \text{ for } s^{(k)} < t < t^{(k)}.
\end{aligned}$$

(We might have $t^{(k)} = t_1$ and $s^{(k)} = t_0$.) We write

$$\int_{s^{(k)}}^{t^{(k)}} X'(t)\,\mathrm{d}t = X(t^{(k)}) - X(s^{(k)}) > Z(t^{(k)}) - Z(s^{(k)}) = \int_{s^{(k)}}^{t^{(k)}} Z'(t)\,\mathrm{d}t.$$

By the mean value theorem, we can find $\zeta^{(k)} \in (s^{(k)}, t^{(k)})$—which implies $X(\zeta^{(k)}) > Z(\zeta^{(k)})$—such that $X'(\zeta^{(k)}) > Z'(\zeta^{(k)})$. This contradicts the fact that, for any $t \in [0,T]$, we have $X'(t) + aX^\gamma(t) \leq b \leq Z'(t) + aZ^\gamma(t)$, which yields $Z'(t) - X'(t) \geq a(X^\gamma(t) - Z^\gamma(t))$. $\square$

Appendix B. Proof of Lemma 3.8. In fact, we simply discuss the analog of Lemma 3.8 for the operator

$$f \longmapsto Tf(x) = \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^\lambda}\,\mathrm{d}y.$$

Let $R > 0$. We pick $M > 0$ and split the integral

$$\begin{aligned}
\int_{|x|\leq R} |Tf(x)|^q\,\mathrm{d}x &= \int_{|x|\leq R} \left| \int_{|x-y|\leq M} \cdots\,\mathrm{d}y + \int_{|x-y|\geq M} \cdots\,\mathrm{d}y \right|^q \mathrm{d}x\\
&\leq 2^{q-1} \int_{|x|\leq R} \left(\int_{|x-y|\leq M} |f(y)|\,\mathrm{d}y \right)^{q-1} \left(\int_{|x-y|\leq M} \frac{|f(y)|}{|x-y|^{\lambda q}}\,\mathrm{d}y \right) \mathrm{d}x\\
&\quad + \frac{2^{q-1}}{M^{q\lambda}} \int_{|x|\leq R} \left(\int_{|x-y|\geq M} |f(y)|\,\mathrm{d}y \right)^q \mathrm{d}x\\
&\leq 2^{q-1} \|f\|_1^{q-1} \int_{|z|\leq M} \frac{1}{|z|^{\lambda q}} \left(\int_{|x|\leq R} |f(x-z)|\,\mathrm{d}x \right) \mathrm{d}z + \frac{2^{q-1} B(0,R)}{M^{q\lambda}} \|f\|_1^q\\
&\leq 2^{q-1} \|f\|_1^q |\mathbb{S}^{N-1}| \int_0^M \frac{\mathrm{d}r}{r^{\lambda q - N + 1}} + \frac{2^{q-1} B(0,R)}{M^{q\lambda}} \|f\|_1^q.
\end{aligned}$$

The integral over $[0, M]$ is finite when $1 \leq q < N/\lambda$. In this case, we end up with

$$\int_{|x|\leq R} |Tf(x)|^q\,\mathrm{d}x \leq C\|f\|_1^q$$

with C depending on N, λ, R, and M. It proves that T is a bounded operator from $L^1(\mathbb{R}^N)$ to $L^q(B(0,R))$.

The same estimate, together with Lebesgue's theorem, shows that

$$T_\delta f(x) = \int_{\mathbb{R}^N} \frac{f(y)}{\delta + |x-y|^\lambda}\,\mathrm{d}y$$

converges to Tf as $\delta \to 0$ in $L^q(B(0,R))$. The convergence is uniform over the unit ball of $L^1(\mathbb{R}^N)$. Next, let us consider a sequence $\big(f_n\big)_{n\in\mathbb{N}}$ of integrable functions, with $\|f_n\|_1 = 1$. We readily check that, for any $\delta > 0$, $T_\delta f_n$ fulfils the hypothesis of the Arzelà–Ascoli theorem. Therefore, $\{T_\delta f_n,\ n \in \mathbb{N}\}$ is relatively compact in $C(B(0,R))$, and thus in $L^q(B(0,R))$ as well. We conclude that T is a compact operator.

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