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REGULARITY CRITERIA OF THE 4D NAVIER–STOKES EQUATIONS INVOLVING TWO VELOCITY FIELD COMPONENTS*

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Abstract. We study the Serrin-type regularity criteria for the solutions to the four-dimensional Navier–Stokes equations and magnetohydrodynamics system. We show that the sufficient condition for the solution to the four-dimensional Navier–Stokes equations to preserve its initial regularity for all time may be reduced in the following ways: from a bound on the four-dimensional velocity vector field to any two of its four components; from a bound on the gradient of the velocity vector field to the gradient of any two of its four components; and from a gradient of the pressure scalar field to any two of its partial derivatives. Results are further generalized to the magnetohydrodynamics system. These results may be seen as a four-dimensional extension of many analogous results that exist in the three-dimensional case and also component reduction results of many classical results.

Key words. Navier–Stokes equations, magnetohydrodynamics system, scaling-invariance, regularity criteria.

AMS subject classifications. 35B65, 35Q35, 35Q86.

1. Introduction

We study the N-dimensional ($N \geq 2$) Navier–Stokes equations (NSE) and magnetohydrodynamics (MHD) system defined respectively as follows:

$$\frac{du}{dt} + (u\cdot\nabla)u + \nabla\pi = \nu\Delta u, \tag{1.1a}$$

$$\nabla\cdot u = 0, \quad u(x,0) = u_0(x), \tag{1.1b}$$

$$\frac{du}{dt} + (u\cdot\nabla)u + \nabla\pi = \nu\Delta u + (b\cdot\nabla)b, \tag{1.2a}$$

$$\frac{db}{dt} + (u\cdot\nabla)b = \eta\Delta b + (b\cdot\nabla)u, \tag{1.2b}$$

$$\nabla\cdot u = \nabla\cdot b = 0, \quad (u,b)(x,0) = (u_0,b_0)(x), \tag{1.2c}$$

where $u = (u_1,\dots,u_N): \mathbb{R}^N \times \mathbb{R}^+ \mapsto \mathbb{R}^N, b = (b_1,\dots,b_N): \mathbb{R}^N \times \mathbb{R}^+ \mapsto \mathbb{R}^N, \pi: \mathbb{R}^N \times \mathbb{R}^+ \mapsto \mathbb{R}$ represent the velocity vector field, magnetic vector field, and pressure scalar field, respectively. We denote by the parameters $\nu,\eta \geq 0$ the viscosity and magnetic diffusivity respectively. Hereafter, we also denote $\frac{d}{dt}$ by ∂_t and $\frac{d}{dx_i}$ by $\partial_i, i = 1,\dots,N$. Further, we denote by $\nabla_{i,j}$ the gradient vector field with ∂_i,∂_j on the ith, jth component respectively and zero elsewhere and by $\Delta_{i,j}$ the sum of second derivatives in the ith and jth directions, e.g., $\nabla_{1,2} = (\partial_1,\partial_2,0,\dots,0), \Delta_{1,2} = \sum_{k=1}^2 \partial_{kk}^2$.

The importance and difficulty of the global regularity issue of the solution to these two systems are well known. In short, this is because the systems are both energy-supercritical in any dimension bigger than two, even with $\nu,\eta > 0$. Indeed, e.g., for the MHD system, taking L^2-inner products with (u,b) on the system (1.2), respectively, and integrating in time lead to

$$\sup_{t\in[0,T]} (\|u\|_{L^2}^2 + \|b\|_{L^2}^2)(t) + \int_0^T \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 d\tau \leq \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2. \tag{1.3}$$

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On the other hand, it can be shown that if $(u,b)(x,t)$ solves the system (1.2), then so does $(u_\lambda,b_\lambda)(x,t)\triangleq\lambda(u,b)(\lambda x,\lambda^2 t)$. A direct computation shows that

$$\|u_\lambda(x,t)\|_{L^2}^2+\|b_\lambda(x,t)\|_{L^2}^2=\lambda^{2-N}(\|u(x,\lambda^2 t)\|_{L^2}^2+\|b(x,\lambda^2 t)\|_{L^2}^2).$$

We call an equation with a scaling symmetry critical when the strongest norm for which an *a priori* estimate is available is scaling-invariant. Thus, it is standard to classify the two-dimensional NSE and the MHD system as energy-critical, while for any dimension higher, energy-supercritical; in fact, it can be considered that the supercriticality increases in dimension.

In the two-dimensional case with $\nu,\eta>0$, the authors in [23, 28] have shown the uniqueness of the solution to the NSE and the MHD system, respectively. In fact, in the two-dimensional case, due to the simplicity of the form after taking curls, when the dissipative and diffusive terms are replaced by fractional Laplacians, their powers may be reduced furthermore below one; we refer interested readers to [36] for the NSE with $\nu=0$, [6] and references found therein for the MHD system. In any dimension strictly higher than two, the problem concerning the global regularity of the strong solution and the uniqueness of the weak solution to both systems remain open and hence much effort has been devoted to provide criteria so that they hold. We now review some of them, emphasizing those of most relevance to the current manuscript.

Initiated by the author in [29], it has been established that, if a weak solution u of the NSE with $\nu>0$ satisfies

$$u\in L^r(0,T;L^p(\mathbb{R}^N)),\quad \frac{N}{p}+\frac{2}{r}\leq 1,\quad p\in(N,\infty], \tag{1.4}$$

then u is smooth (see [10, 12] for the endpoint case). In [2], the author showed that, if u solves the NSE (1.1) with $\nu>0$ and

$$\nabla u\in L^r(0,T;L^p(\mathbb{R}^N)),\ \ N\geq 3,\ \ \frac{N}{p}+\frac{2}{r}=2,\ \ 1<r\leq\min\{2,\frac{N}{N-2}\}, \tag{1.5}$$

then u is a regular solution. For the MHD system, the authors in [16, 39] independently showed that the sufficient condition for the regularity of the solution pair (u,b) to the MHD system (1.2) may be reduced to just u. For many more important results in this direction of research, all of which we cannot list here, we refer to the prominent work of [1, 15] and references found therein. We do mention that the author in [40] showed that only in the case of $N=3,4$, u, the solution to the NSE (1.1) with $\nu>0$, is regular and unique if

$$\nabla\pi\in L^r(0,T;L^p(\mathbb{R}^N)),\quad \frac{N}{p}+\frac{2}{r}\leq 3,\quad \frac{N}{3}\leq p\leq\infty. \tag{1.6}$$

We emphasize that the norm $\|\cdot\|_{L_T^rL_x^p}$ in Equation (1.4) is scaling invariant precisely when $\frac{N}{p}+\frac{2}{r}=1$; i.e.

$$\int_0^T\|u_\lambda(x,t)\|_{L^p}^r dt=\int_0^{\lambda^2 T}\|u(x,t)\|_{L^p}^r dt\ \text{ if and only if }\ \frac{N}{p}+\frac{2}{r}=1,$$

where $u_\lambda(x,t)=\lambda u(\lambda x,\lambda^2 t)$, and similarly for the norm in Equation (1.5) at the endpoint of 2.

We now survey some component reduction results of such criteria. The authors in [21] showed that, if u solves the NSE with $N=3,\nu>0$ and

$$u_3 \in L^r(0,T;L^p(\mathbb{R}^3)), \quad \frac{3}{p}+\frac{2}{r}\le \frac{5}{8}, \quad r\in[\frac{54}{23},\frac{18}{5}], \tag{1.7}$$
$$\text{or } \nabla u_3 \in L^r(0,T;L^p(\mathbb{R}^3)), \quad \frac{3}{p}+\frac{2}{r}\le \frac{11}{6}, \quad r\in[\frac{24}{5},\infty],$$

then the solution is regular (see also [3, 41] for similar results on $u_3,\nabla u_3$). For the MHD system, in particular the authors in [18] showed that if (u,b) solves the system (1.2) with $N=3,\nu,\eta>0$ and

$$u_3, b\in L^r(0,T;L^p(\mathbb{R}^3)), \quad \frac{3}{p}+\frac{2}{r}\le \frac{3}{4}+\frac{1}{2p}, \quad p>\frac{10}{3}, \tag{1.8}$$

then the solution pair (u,b) remains smooth for all time. In [31], the author reduced this constraint on u_3,b to u_3,b_1,b_2 in special cases making use of the special structure of the system (1.2). For more interesting component reduction results of the regularity criteria, we refer to, e.g., [4, 5, 13, 17, 22, 26, 30, 32, 38]. In particular, the authors in [7] obtained a regularity criterion for the three-dimensional NSE in terms of only u_3 in a scaling-invariant norm, although no longer $L_T^rL_x^p$-space (see also [8, 24, 35]). In relation to our discussion below, we already emphasize that every component reduction result listed here is of the case of $N=3$.

We now motivate the study of the systems (1.1) and (1.2) in the fourth dimension specifically. It has been realized by many mathematicians working in the research direction of the NSE that dimension four deserves special attention (see, e.g., [19, Section 4]). The significance of the fourth dimension for the NSE (and six-dimensional stationary NSE) has motivated much investigation in the research direction of partial regularity theory (see, e.g., [9, 11, 27]); we also recall Equation f(1.6), which holds only for $N=3,4$. In fact, the fourth dimension being a certain threshold to the component reduction regularity criteria can be seen clearly as follows. To the best of the author's knowledge, all such component reduction results to the systems (1.1) and (1.2) are obtained through an H^1-estimate. Due to Lemma 2.3, higher regularity follows once we show that the solution, e.g., u in the case of the NSE (1.1) satisfies $\int_0^T\|\nabla u\|_{L^N(\mathbb{R}^N)}^2 d\tau<\infty$. This implies that, because $H^1(\mathbb{R}^N)\hookrightarrow L^N(\mathbb{R}^N)$ only for $N=2,3,4$ but not $N>4$ by Sobolev embedding, H^1-bound, from which $u\in L^2(0,T;H^2(\mathbb{R}^N))$ follows from the dissipative term, is sufficient for higher regularity only if $N=2,3,4$. Thus, in dimension strictly higher than four, one needs to bound beyond H^1-norm; however, because the decomposition of the non-linear terms is the most important ingredient of component reduction results (see Proposition 3.1), this will complicate the proof significantly. To the best of the author's knowledge, component reduction results for dimension strictly larger than three does not exist in the literature.

Let us also discuss the two major obstacles in extending the component reduction results of regularity criteria from dimension three to dimension four. In the case of the NSE (1.1) with $N=3,\nu>0$, the standard procedure to obtain a criteria in terms of u_3 may be, e.g., to first estimate every partial derivative except the last and hence $\|\nabla_{1,2}u\|_{L^2}$ and in this process separate u_3 in the non-linear term

$$\int (u\cdot\nabla)u\cdot\Delta_{1,2}u dx \le c\int |u_3||\nabla u||\nabla\nabla_{1,2}u|dx, \tag{1.9}$$

where $\nabla_{1,2} = (\partial_1, \partial_2, 0), \Delta_{1,2} = \sum_{k=1}^{2} \partial_{kk}^2$ (cf. [21] Lemma 2.3). Thereafter, upon a full gradient and hence an H^1-estimate, on the non-linear term one separates $|\nabla_{1,2} u|$

$$\int (u \cdot \nabla) u \cdot \Delta u dx \leq c \int |\nabla_{1,2} u| |\nabla u|^2 dx \tag{1.10}$$

(cf. [41]) so that the $\|\nabla_{1,2} u\|_{L^2}$-estimate may be applied in Equation (1.10). In the case of $N=4$, it seems difficult to separate u_3 or even u_3 and u_4 in $\int (u \cdot \nabla) u \cdot \Delta_{1,2,3} u dx$. Our first key observation is that we can separate u_3, u_4 from $\int (u \cdot \nabla) u \cdot \Delta_{1,2} u dx$ (see Proposition 3.1). However, this leaves two other directions, namely x_3, x_4, instead of only one in contrast to the case of $N=3$ and prevents us from obtaining an inequality analogous to Equation (1.10) upon the full H^1-estimate due to a sum of this type

$$\sum_{j=1}^{4} \sum_{i,k=3}^{4} \int \partial_k u_i \partial_i u_j \partial_k u_j dx$$

(see Equation (3.24)). We observe that, in the three-dimensional case, i, j, and k sum up to only 3 so that, using $\nabla \cdot u = 0$ from Equation (1.1), one may deduce

$$\begin{aligned} \sum_{j=1}^{3} \sum_{i,k=3}^{3} \int \partial_k u_i \partial_i u_j \partial_k u_j dx &= \sum_{j=1}^{3} \int \partial_3 u_3 \partial_3 u_j \partial_3 u_j dx \\ &= -\sum_{j=1}^{3} \int (\partial_1 u_1 + \partial_2 u_2) \partial_3 u_j \partial_3 u_j dx \end{aligned}$$

and hence Equation (1.10) follows. However, in the four-dimensional case, there are cross-terms such as $\partial_3 u_4$, which prevents us from reaching Equation (1.10). Our second key observation is that the non-linear term may be seen as an operator as a sum of

$$u \cdot \nabla = \sum_{i=1}^{4} u_i \partial_i = \sum_{i=1}^{2} u_i \partial_i + \sum_{i=3}^{4} u_i \partial_i$$

so that in the first sum the $\nabla_{1,2}$-estimate may be applied while in the second we use our hypothesis on u_3, u_4 (see Equation (3.24) and also Equation (3.27)).

We now present our results

Theorem 1.1. *Let $N=4$ and let*

$$u \in C([0,T]; H^s(\mathbb{R}^4)) \cap L^2([0,T]; H^{s+1}(\mathbb{R}^4)) \tag{1.11}$$

be the solution to the NSE (1.1) for a given $u_0 \in H^s(\mathbb{R}^4), s > 4$. Suppose u_3, u_4 with their corresponding $p_i, r_i, i = 3,4$ satisfy the following roles of f:

$$\int_0^T \|f\|_{L^{p_i}}^{r_i} d\tau \leq c, \quad \frac{4}{p_i} + \frac{2}{r_i} \leq \frac{1}{p_i} + \frac{1}{2}, \quad 6 < p_i \leq \infty, \tag{1.12}$$

or $\sup_{t \in [0,T]} \|f(t)\|_{L^6}$ being sufficiently small. Then u remains in the same regularity class (1.11) on $[0,T']$ for some $T' > T$.

Theorem 1.2. *Let $N=4$ and let u be in the regularity class (1.11) of the solution to the NSE (1.1) for a given $u_0 \in H^s(\mathbb{R}^4), s > 4$. Suppose $\nabla u_3, \nabla u_4$ with their corresponding $p_i, r_i, i = 3,4$ satisfy the following roles of f:*

$$\int_0^T \|f\|_{L^{p_i}}^{r_i} d\tau \leq c, \quad \frac{4}{p_i} + \frac{2}{r_i} \leq \begin{cases} \frac{5}{4} + \frac{1}{p_i}, & \text{if } \frac{12}{5} < p_i \leq 4 \\ 1 + \frac{2}{p_i}, & \text{if } 4 < p_i \leq \infty \end{cases}, \tag{1.13}$$

or $\sup_{t\in[0,T]}\|f(t)\|_{L^{\frac{12}{5}}}$ *being sufficiently small. Then* u *remains in the same regularity class (1.11) on* $[0,T']$ *for some* $T'>T$.

Theorem 1.3. *Let* $N=4$ *and let*

$$u,b\in C([0,T);H^s(\mathbb{R}^4))\cap L^2([0,T);H^{s+1}(\mathbb{R}^4)) \tag{1.14}$$

be the solution pair to the MHD system (1.2) for a given $u_0,b_0\in H^s(\mathbb{R}^4), s>4$. *Suppose* u_3,u_4,b *with their corresponding* $p_i,r_i,i=3,4,\ldots,b$ *satisfy the following roles of* f:

$$\int_0^T \|f\|_{L^{p_i}}^{r_i}d\tau\le c,\quad \frac{4}{p_i}+\frac{2}{r_i}\le\frac{1}{p_i}+\frac{1}{2},\quad 6<p_i\le\infty, \tag{1.15}$$

or $\sup_{t\in[0,T]}\|f(t)\|_{L^6}$ *being sufficiently small. Then* u,b *remain in the same regularity class (1.14) on* $[0,T']$ *for some* $T'>T$.

Theorem 1.4. *Let* $N=4$ *and let* u,b *be in the regularity class (1.14) be the solution pair to the MHD system (1.2) for a given* $u_0,b_0\in H^s(\mathbb{R}^4), s>4$. *Suppose* $\nabla u_3,\nabla u_4,\nabla b$ *with their corresponding* $p_i,r_i,i=3,4,\ldots,b$ *satisfy the following roles of* f:

$$\int_0^T \|f\|_{L^{p_i}}^{r_i}d\tau\le c,\ \frac{4}{p_i}+\frac{2}{r_i}\le\begin{cases}\frac{5}{4}+\frac{1}{p_i}, & \text{if } \frac{12}{5}<p_i\le 4\\ 1+\frac{2}{p_i}, & \text{if } 4<p_i\le\infty\end{cases}, \tag{1.16}$$

or $\sup_{t\in[0,T]}\|f(t)\|_{L^{\frac{12}{5}}}$ *being sufficiently small. Then* u,b *remain in the same regularity class (1.14) on* $[0,T']$ *for some* $T'>T$.

Theorem 1.5. *Let* $N=4$ *and let* u *be in the regularity class (1.11) be the solution to the NSE (1.1) for a given* $u_0\in H^s(\mathbb{R}^4), s>4$. *Suppose* $\partial_3\pi,\partial_4\pi$ *with their corresponding* $p_i,r_i,i=3,4$ *satisfy the following roles of* f:

$$\int_0^T \|f\|_{L^{p_i}}^{r_i}d\tau\le c,\quad \frac{4}{p_i}+\frac{2}{r_i}<\frac{8}{3},\quad \frac{12}{7}<p_i<6. \tag{1.17}$$

Then u *remains in the same regularity class (1.11) on* $[0,T']$ *for some* $T'>T$.

Remark 1.1.

(1) In comparing Theorem 1.1 with Equation (1.4), Theorem 1.2 with Equation (1.5), and Theorem 1.5 with Equation (1.6), we may consider the results of this manuscript as component reduction of many previous works. Moreover, in comparing theorems 1.1 and 1.2 with Equation (1.7) and Theorem 1.3 with Equation (1.8), we may consider the results of this manuscript as a four-dimension extension of many previous work in three-dimension.

(2) Lemma 2.3 of [21] has found many applications, e.g., in the study on the anisotropic NSE (e.g. [37]). We note that our Proposition 3.1 can be readily generalized further to any $\mathbb{R}^N, N\ge 3$; we chose to state the case $N=4$ for the simplicity of presentation.

(3) In [34], the author showed that, for dimensions $N=3,4,5$, N-many component regularity criteria may be reduced to $(N-1)$-many components for the generalized MHD system following the method in [30]; the results in [34] and this manuscript do not cover each other. In [33] the author also obtained a regularity criteria of and N-dimensional porous media equation governed by Darcy's law in terms of one partial derivative of the scalar-valued solution. The method in [33] cannot be applied to the systems (1.1) and (1.2).

In Section 2, we set up notations and state key facts. Local theory is well-known (cf. [25]); hence, by the standard argument of continuation of local theory, we only need to obtain H^s-bounds. We present the proofs of theorems 1.3, 1.4 and 1.5. Because the NSE is the MHD system at $b\equiv 0$, the proofs of theorems 1.3, and 1.4 immediately deduce theorems 1.1 and 1.2, respectively. Thereafter, we conclude with a brief further discussion.

2. Preliminaries

Throughout the rest of the manuscript, we shall assume $\nu,\eta=1$ for simplicity. For brevity, we write $\int f$ for $\int_{\mathbb{R}^N} f(x)dx$ and $A\lesssim_{a,b} B$ when there exists a constant $c\geq 0$ of significant dependence only on a,b such that $A\leq cB$, and similarly $A\approx_{a,b} B$ in the case of $A=cB$. We denote the fractional Laplacian operator $\Lambda^s \triangleq (-\Delta)^{\frac{s}{2}}$ and

$$W(t)\triangleq(\|\nabla_{1,2}u\|_{L^2}^2+\|\nabla_{1,2}b\|_{L^2}^2)(t), \qquad X(t)\triangleq(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)(t),$$

$$Y(t)\triangleq(\|\nabla\nabla_{1,2}u\|_{L^2}^2+\|\nabla\nabla_{1,2}b\|_{L^2}^2)(t), \quad Z(t)\triangleq(\|\Delta u\|_{L^2}^2+\|\Delta b\|_{L^2}^2)(t).$$

The following is a special case of Troisi's inequality (cf. [14]). The proof of the case $N=3$ in the appendix of [5] can be readily generalized to the case $N=4$.

LEMMA 2.1. *Let $f\in C_0^\infty(\mathbb{R}^4)$. Then*

$$\|f\|_{L^4}\lesssim\|\partial_1 f\|_{L^2}^{\frac{1}{4}}\|\partial_2 f\|_{L^2}^{\frac{1}{4}}\|\partial_3 f\|_{L^2}^{\frac{1}{4}}\|\partial_4 f\|_{L^2}^{\frac{1}{4}}. \tag{2.1}$$

We will use the following elementary inequality frequently:

$$(a+b)^p\leq 2^p(a^p+b^p), \quad \text{for } 0\leq p<\infty \text{ and } a,b\geq 0. \tag{2.2}$$

We will also use the following commutator estimate to prove another lemma concerning higher regularity.

LEMMA 2.2. *(cf. [20]) Let f,g be smooth such that $\nabla f\in L^{p_1},\Lambda^{s-1}g\in L^{p_2},\Lambda^s f\in L^{p_3},g\in L^{p_4},p\in(1,\infty),\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}+\frac{1}{p_4},p_2,p_3\in(1,\infty),s>0$. Then*

$$\|\Lambda^s(fg)-f\Lambda^s g\|_{L^p}\lesssim(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{p_2}}+\|\Lambda^s f\|_{L^{p_3}}\|g\|_{L^{p_4}}).$$

An immediate application of Lemma 2.2 gives the following result:

LEMMA 2.3. *Let (u,b) be the solution to the MHD system (1.2) in $[0,T]$ with $u_0,b_0\in H^s(\mathbb{R}^N),N\geq 3,s>2+\frac{N}{2}$. If $\int_0^T\|\nabla u\|_{L^N}^2+\|\nabla b\|_{L^N}^2 d\tau\lesssim 1$, then*

$$\sup_{t\in[0,T]}(\|\Lambda^s u\|_{L^2}^2+\|\Lambda^s b\|_{L^2}^2)(t)+\int_0^T\|\Lambda^s\nabla u\|_{L^2}^2+\|\Lambda^s\nabla b\|_{L^2}^2 d\tau\lesssim 1.$$

Proof. This is a standard computation; we sketch it for completeness. We apply Λ^s on the system (1.2) and take L^2-inner products with $\Lambda^s u,\Lambda^s b$, respectively, to obtain

$$\begin{aligned}&\frac{1}{2}\partial_t(\|\Lambda^s u\|_{L^2}^2+\|\Lambda^s b\|_{L^2}^2)+\|\Lambda^s\nabla u\|_{L^2}^2+\|\Lambda^s\nabla b\|_{L^2}^2\\ =&-\int[\Lambda^s((u\cdot\nabla)u)-u\cdot\nabla\Lambda^s u]\cdot\Lambda^s u-\int[\Lambda^s((u\cdot\nabla)b)-u\cdot\nabla\Lambda^s b]\cdot\Lambda^s b\\ &+\int[\Lambda^s((b\cdot\nabla)b)-b\cdot\nabla\Lambda^s b]\cdot\Lambda^s u+\int[\Lambda^s((b\cdot\nabla)u)-b\cdot\nabla\Lambda^s u]\cdot\Lambda^s b\end{aligned}$$

$$\lesssim (\|\nabla u\|_{L^N} + \|\nabla b\|_{L^N})(\|\Lambda^s u\|_{L^2} + \|\Lambda^s b\|_{L^2})(\|\Lambda^s \nabla u\|_{L^2} + \|\Lambda^s \nabla b\|_{L^2})$$
$$\le \frac{1}{2}(\|\Lambda^s \nabla u\|_{L^2}^2 + \|\Lambda^s \nabla b\|_{L^2}^2) + c(\|\nabla u\|_{L^N}^2 + \|\nabla b\|_{L^N}^2)(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s b\|_{L^2}^2)$$

by Hölder's inequalities, Lemma 2.2, Sobolev embedding of $\dot{H}^1(\mathbb{R}^N) \hookrightarrow L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, Young's inequalities, and Equation (2.2). Thus, after absorbing, Gronwall's inequality completes the proof of Lemma 2.3. □

Due to Lemma 2.3, the proof of our theorems are complete once we obtain the H^1-bound.

3. Proof of Theorem 1.3

3.1. $\|\nabla_{1,2} u\|_{L^2}^2 + \|\nabla_{1,2} b\|_{L^2}^2$-estimate. We first prove an important decomposition which we present as a proposition.

PROPOSITION 3.1. *Let $N=4$ and (u,b) be the solution pair to the MHD system (1.2). Then*

$$\int (u\cdot\nabla)u\cdot\Delta_{1,2}u + (u\cdot\nabla)b\cdot\Delta_{1,2}b - (b\cdot\nabla)b\cdot\Delta_{1,2}u - (b\cdot\nabla)u\cdot\Delta_{1,2}b$$
$$\lesssim \int (|u_3|+|u_4|)|\nabla u||\nabla\nabla_{1,2}u| + |b|(|\nabla u|+|\nabla b|)(|\nabla\nabla_{1,2}u|+|\nabla\nabla_{1,2}b|). \tag{3.1}$$

Moreover,

$$\int (u\cdot\nabla)u\cdot\Delta_{1,2}u + (u\cdot\nabla)b\cdot\Delta_{1,2}b - (b\cdot\nabla)b\cdot\Delta_{1,2}u - (b\cdot\nabla)u\cdot\Delta_{1,2}b$$
$$\lesssim \int (|\nabla u_3|+|\nabla u_4|)|\nabla_{1,2}u||\nabla u| + |\nabla b||\nabla_{1,2}b||\nabla u|. \tag{3.2}$$

Proof. We write components-wise and integrate by parts to obtain

$$\begin{aligned}
&\int (u\cdot\nabla)u\cdot\Delta_{1,2}u = -\sum_{i,j=1}^{4}\sum_{k=1}^{2}\int \partial_k u_i \partial_i u_j \partial_k u_j \\
&= -\sum_{j=1}^{4}\sum_{i,k=1}^{2}\int \partial_k u_i \partial_i u_j \partial_k u_j - \sum_{i=3}^{4}\sum_{j=1}^{4}\sum_{k=1}^{2}\int \partial_k u_i \partial_i u_j \partial_k u_j \\
&= -\sum_{i,j,k=1}^{2}\int \partial_k u_i \partial_i u_j \partial_k u_j - \sum_{j=3}^{4}\sum_{i,k=1}^{2}\int \partial_k u_i \partial_i u_j \partial_k u_j - \sum_{i=3}^{4}\sum_{j=1}^{4}\sum_{k=1}^{2}\int \partial_k u_i \partial_i u_j \partial_k u_j.
\end{aligned} \tag{3.3}$$

For the second and third integrals of Equation (3.3), we integrate by parts to obtain

$$\begin{aligned}
&-\sum_{j=3}^{4}\sum_{i,k=1}^{2}\int \partial_k u_i \partial_i u_j \partial_k u_j - \sum_{i=3}^{4}\sum_{j=1}^{4}\sum_{k=1}^{2}\int \partial_k u_i \partial_i u_j \partial_k u_j \\
&= \sum_{j=3}^{4}\sum_{i,k=1}^{2}\int u_j \partial_i(\partial_k u_i \partial_k u_j) + \sum_{i=3}^{4}\sum_{j=1}^{4}\sum_{k=1}^{2}\int u_i \partial_k(\partial_i u_j \partial_k u_j)
\end{aligned}$$

$$\lesssim \int (|u_3|+|u_4|)|\nabla u||\nabla\nabla_{1,2}u|. \tag{3.4}$$

On the other hand, we write the first integral of Equation (3.3) explicitly

$$\begin{aligned}
&-\sum_{i,j,k=1}^{2}\int \partial_k u_i \partial_i u_j \partial_k u_j \\
=&-\int (\partial_1 u_1)^3+\partial_2 u_1\partial_1 u_1\partial_2 u_1+\partial_1 u_1\partial_1 u_2\partial_1 u_2+\partial_2 u_1\partial_1 u_2\partial_2 u_2 \\
&+\partial_1 u_2\partial_2 u_1\partial_1 u_1+\partial_2 u_2\partial_2 u_1\partial_2 u_1+\partial_1 u_2\partial_2 u_2\partial_1 u_2+(\partial_2 u_2)^3 \triangleq \sum_{i=1}^{8}\mathrm{I}_i.
\end{aligned} \tag{3.5}$$

We combine and use the incompressibility condition of u to obtain

$$\begin{aligned}
\mathrm{I}_1+\mathrm{I}_8 &= -\int (\partial_1 u_1)^3+(\partial_2 u_2)^3 \\
&= \int (\partial_1 u_1)^2\partial_2 u_2+(\partial_1 u_1)^2(\partial_3 u_3+\partial_4 u_4)+(\partial_2 u_2)^2\partial_1 u_1+(\partial_2 u_2)^2(\partial_3 u_3+\partial_4 u_4).
\end{aligned} \tag{3.6}$$

We combine the first and third terms to obtain

$$\int (\partial_1 u_1)^2\partial_2 u_2+(\partial_2 u_2)^2\partial_1 u_1 = -\int \partial_1 u_1\partial_2 u_2(\partial_3 u_3+\partial_4 u_4)$$

so that we may continue Equation (3.6) by

$$\begin{aligned}
\mathrm{I}_1+\mathrm{I}_8 &= -\int \partial_1 u_1\partial_2 u_2(\partial_3 u_3+\partial_4 u_4) \\
&\quad +\int (\partial_1 u_1)^2(\partial_3 u_3+\partial_4 u_4)+(\partial_2 u_2)^2(\partial_3 u_3+\partial_4 u_4) \\
&= \int u_3\partial_3(\partial_1 u_1\partial_2 u_2)+u_4\partial_4(\partial_1 u_1\partial_2 u_2) \\
&\quad -\int u_3\partial_3[(\partial_1 u_1)^2+(\partial_2 u_2)^2]+u_4\partial_4[(\partial_1 u_1)^2+(\partial_2 u_2)^2] \\
&\lesssim \int (|u_3|+|u_4|)|\nabla u||\nabla\nabla_{1,2}u|.
\end{aligned} \tag{3.7}$$

Similarly,

$$\begin{aligned}
\mathrm{I}_2+\mathrm{I}_6 &= -\int \partial_2 u_1\partial_1 u_1\partial_2 u_1+\partial_2 u_2\partial_2 u_1\partial_2 u_1 \\
&= \int (\partial_2 u_1)^2(\partial_3 u_3+\partial_4 u_4) \lesssim \int (|u_3|+|u_4|)|\nabla u||\nabla\nabla_{1,2}u|,
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
\mathrm{I}_3+\mathrm{I}_7 &= -\int \partial_1 u_1\partial_1 u_2\partial_1 u_2+\partial_1 u_2\partial_2 u_2\partial_1 u_2 \\
&= \int (\partial_1 u_2)^2(\partial_3 u_3+\partial_4 u_4) \lesssim \int (|u_3|+|u_4|)|\nabla u||\nabla\nabla_{1,2}u|,
\end{aligned} \tag{3.9}$$

$$\mathrm{I}_4+\mathrm{I}_5 = -\int \partial_2 u_1\partial_1 u_2\partial_2 u_2+\partial_1 u_2\partial_2 u_1\partial_1 u_1$$

$$=\int \partial_2 u_1 \partial_1 u_2 (\partial_3 u_3 + \partial_4 u_4) \lesssim \int (|u_3|+|u_4|)|\nabla u||\nabla\nabla_{1,2} u|. \tag{3.10}$$

Next, we may estimate the other three terms as follows:

$$\begin{aligned}
&\int (u\cdot\nabla)b\cdot\Delta_{1,2}b-(b\cdot\nabla)b\cdot\Delta_{1,2}u-(b\cdot\nabla)u\cdot\Delta_{1,2}b\\
&=-\sum_{i,j=1}^{4}\sum_{k=1}^{2}\int \partial_k u_i \partial_i b_j \partial_k b_j + \sum_{i,j=1}^{4}\sum_{k=1}^{2}\int \partial_k b_i \partial_i b_j \partial_k u_j + \partial_k b_i \partial_i u_j \partial_k b_j\\
&=\sum_{i,j=1}^{4}\sum_{k=1}^{2}\int \partial_k u_i b_j \partial_{ik}^2 b_j - \sum_{i,j=1}^{4}\sum_{k=1}^{2}\int \partial_k b_i b_j \partial_{ik}^2 u_j + b_i \partial_k(\partial_i u_j \partial_k b_j)\\
&\lesssim \int |b|(|\nabla u|+|\nabla b|)(|\nabla\nabla_{1,2}u|+|\nabla\nabla_{1,2}b|).
\end{aligned}\tag{3.11}$$

Applying Equations (3.7)–(3.10) in Equation (3.5), considering Equations (3.3), (3.4), and (3.11), we obtain Equation (3.1). Now we go back to Equation (3.3) and estimate the second and third integrals by

$$\begin{aligned}
&-\sum_{j=3}^{4}\sum_{i,k=1}^{2}\int \partial_k u_i \partial_i u_j \partial_k u_j - \sum_{i=3}^{4}\sum_{j=1}^{4}\sum_{k=1}^{2}\int \partial_k u_i \partial_i u_j \partial_k u_j\\
&\lesssim \sum_{j=3}^{4}\sum_{k=1}^{2}\int |\partial_k u||\nabla u_j||\partial_k u| + \sum_{i=3}^{4}\sum_{k=1}^{2}\int |\nabla u_i||\nabla u||\partial_k u|\\
&\lesssim \int (|\nabla u_3|+|\nabla u_4|)|\nabla_{1,2}u||\nabla u|,
\end{aligned}\tag{3.12}$$

whereas, continuing from Equation (3.7),

$$\begin{aligned}
\mathrm{I}_1+\mathrm{I}_8 &= -\int \partial_1 u_1 \partial_2 u_2 (\partial_3 u_3+\partial_4 u_4) + \left((\partial_1 u_1)^2+(\partial_2 u_2)^2\right)(\partial_3 u_3+\partial_4 u_4)\\
&\lesssim \int |\nabla_{1,2}u|^2(|\partial_3 u_3|+|\partial_4 u_4|);
\end{aligned}\tag{3.13}$$

continuing from Equation (3.8),

$$\mathrm{I}_2+\mathrm{I}_6 = \int (\partial_2 u_1)^2(\partial_3 u_3+\partial_4 u_4) \lesssim \int |\nabla_{1,2}u|^2(|\partial_3 u_3|+|\partial_4 u_4|); \tag{3.14}$$

continuing from Equation (3.9),

$$\mathrm{I}_3+\mathrm{I}_7 = \int (\partial_1 u_2)^2(\partial_3 u_3+\partial_4 u_4) \lesssim \int |\nabla_{1,2}u|^2(|\partial_3 u_3|+|\partial_4 u_4|); \tag{3.15}$$

and, continuing from Equation (3.10),

$$\mathrm{I}_4+\mathrm{I}_5 = \int \partial_2 u_1 \partial_1 u_2(\partial_3 u_3+\partial_4 u_4) \lesssim \int |\nabla_{1,2}u|^2(|\partial_3 u_3|+|\partial_4 u_4|). \tag{3.16}$$

Thus, considering Equations (3.12)–(3.16) in Equation (3.3), we have shown

$$\int (u\cdot\nabla)u\cdot\Delta_{1,2}u \lesssim \int (|\nabla u_3|+|\nabla u_4|)|\nabla_{1,2}u||\nabla u|. \tag{3.17}$$

Next, we estimate continuing from Equation (3.11)

$$\begin{aligned}&\int (u\cdot\nabla)b\cdot\Delta_{1,2}b-(b\cdot\nabla)b\cdot\Delta_{1,2}u-(b\cdot\nabla)u\cdot\Delta_{1,2}b\\ =&-\sum_{i,j=1}^{4}\sum_{k=1}^{2}\int \partial_k u_i\partial_i b_j\partial_k b_j-\partial_k b_i\partial_i b_j\partial_k u_j-\partial_k b_i\partial_i u_j\partial_k b_j\lesssim\int|\nabla b||\nabla_{1,2}b||\nabla u|. \qquad (3.18)\end{aligned}$$

Considering Equations (3.17) and (3.18), we obtain Equation (3.2). This completes the proof of Proposition 3.1. ∎

With this proposition, we now obtain our first estimate.

PROPOSITION 3.2. *Let $N=4$ and let (u,b) be the solution pair to the MHD system (1.2) that satisfies the hypothesis of Theorem 1.3. Then, for all $t\in(0,T]$ and $p_i\in[6,\infty]$,*

$$\begin{aligned}&\sup_{\tau\in[0,t]}W(\tau)+\int_0^t Y(\tau)d\tau\\ \leq& W(0)+c\sum_{i=3}^{4}\int_0^t\|u_i\|_{L^{p_i}}^{\frac{2p_i}{p_i-2}}X^{\frac{p_i-4}{p_i-2}}(\tau)Z^{\frac{2}{p_i-2}}(\tau)+\|b\|_{L^{p_b}}^{\frac{2p_b}{p_b-2}}X^{\frac{p_b-4}{p_b-2}}(\tau)Z^{\frac{2}{p_b-2}}(\tau)d\tau\end{aligned}$$

with the usual convention at the case $p_i=\infty, i=3,4,b$; i.e., $\frac{2p_i}{p_i-2}=2, \frac{p_i-4}{p_i-2}=1, \frac{2}{p_i-2}=0$.

Proof. We treat the case $6\leq p_i<\infty$ $i=3,4,\dots,b$ first. We take L^2-inner products on the system (1.2) with $-\Delta_{1,2}u,-\Delta_{1,2}b$, respectively, to obtain in sum

$$\begin{aligned}&\frac{1}{2}\partial_t W(t)+Y(t)\\ \lesssim&\sum_{i=3}^{4}\int|u_i||\nabla u||\nabla\nabla_{1,2}u|+|b|(|\nabla u|+|\nabla b|)(|\nabla\nabla_{1,2}u|+|\nabla\nabla_{1,2}b|)\triangleq \mathrm{II}_1+\mathrm{II}_2 \qquad (3.19)\end{aligned}$$

by Equation (3.1). Now we estimate

$$\begin{aligned}\mathrm{II}_1\approx&\sum_{i=3}^{4}\int|u_i||\nabla u||\nabla\nabla_{1,2}u|\lesssim\sum_{i=3}^{4}\|u_i\|_{L^{p_i}}\|\nabla u\|_{L^{\frac{2p_i}{p_i-2}}}\|\nabla\nabla_{1,2}u\|_{L^2}\\ \lesssim&\sum_{i=3}^{4}\|u_i\|_{L^{p_i}}\|\nabla u\|_{L^2}^{\frac{p_i-4}{p_i}}\|\nabla u\|_{L^4}^{\frac{4}{p_i}}\|\nabla\nabla_{1,2}u\|_{L^2}\\ \lesssim&\sum_{i=3}^{4}\|u_i\|_{L^{p_i}}\|\nabla u\|_{L^2}^{\frac{p_i-4}{p_i}}\|\nabla\nabla_{1,2}u\|_{L^2}^{\frac{2}{p_i}+1}\|\Delta u\|_{L^2}^{\frac{2}{p_i}}\\ \leq&\frac{1}{4}\|\nabla\nabla_{1,2}u\|_{L^2}^2+c\sum_{i=3}^{4}\|u_i\|_{L^{p_i}}^{\frac{2p_i}{p_i-2}}X^{\frac{p_i-4}{p_i-2}}(t)Z^{\frac{2}{p_i-2}}(t) \qquad (3.20)\end{aligned}$$

by Hölder's and interpolation inequalities, Equation (2.1), and Young's inequalities. Similarly,

$$\mathrm{II}_2\approx\int|b|(|\nabla u|+|\nabla b|)(|\nabla\nabla_{1,2}u|+|\nabla\nabla_{1,2}b|)$$

$$\begin{aligned}
&\lesssim \|b\|_{L^{p_b}}(\|\nabla u\|_{L^2}^{\frac{p_b-4}{p_b}}+\|\nabla b\|_{L^2}^{\frac{p_b-4}{p_b}})(\|\nabla u\|_{L^4}^{\frac{4}{p_b}}+\|\nabla b\|_{L^4}^{\frac{4}{p_b}})(\|\nabla\nabla_{1,2}u\|_{L^2}+\|\nabla\nabla_{1,2}b\|_{L^2})\\
&\lesssim \|b\|_{L^{p_b}}X^{\frac{p_b-4}{2p_b}}(t)(\|\nabla\nabla_{1,2}u\|_{L^2}^{\frac{2}{p_b}}\|\Delta u\|_{L^2}^{\frac{2}{p_b}}+\|\nabla\nabla_{1,2}b\|_{L^2}^{\frac{2}{p_b}}\|\Delta b\|_{L^2}^{\frac{2}{p_b}})Y^{\frac{1}{2}}(t)\\
&\le \frac{1}{4}Y(t)+c\|b\|_{L^{p_b}}^{\frac{2p_b}{p_b-2}}X^{\frac{p_b-4}{p_b-2}}(t)Z^{\frac{2}{p_b-2}}(t) \qquad (3.21)
\end{aligned}$$

by Hölder's and interpolation inequalities, Equations (2.2) and (2.1), and Young's inequality. In sum of Equations (3.20) and (3.21) in Equation (3.19), after absorbing and integrating over time $[0,t], t\in(0,T]$, we obtain the desired result in case $6\le p_i<\infty$. In case, $p_i=\infty$, the estimate is in fact simpler: we have

$$\begin{aligned}
&\mathrm{II}_1 \lesssim \sum_{i=3}^{4}\|u_i\|_{L^\infty}\|\nabla u\|_{L^2}\|\nabla\nabla_{1,2}u\|_{L^2}\le \frac{1}{4}\|\nabla\nabla_{1,2}u\|_{L^2}^2+c\sum_{i=3}^{4}\|u_i\|_{L^\infty}^2\|\nabla u\|_{L^2}^2,\\
&\mathrm{II}_2 \lesssim \|b\|_{L^\infty}(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2})(\|\nabla\nabla_{1,2}u\|_{L^2}+\|\nabla\nabla_{1,2}b\|_{L^2})\le \frac{1}{4}Y(t)+c\|b\|_{L^\infty}^2X(t).
\end{aligned}$$

Thus, in case $p_i=\infty$, Proposition 3.2 holds with $\frac{2p_i}{p_i-2}=2, \frac{p_i-4}{p_i-2}=1, \frac{2}{p_i-2}=0$. □

3.2. $\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2$-estimate. The next important step of the proof is to make use of the $\|\nabla_{1,2}u\|_{L^2}^2+\|\nabla_{1,2}b\|_{L^2}^2$-estimate to obtain the bound on $\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2$, which requires another key decomposition (see Equations (3.24) and (3.27)).

PROPOSITION 3.3. *Let $N=4$ and let (u,b) be the solution pair to the MHD system (1.2) that satisfies the hypothesis of Theorem 1.3. Then*

$$\sup_{t\in[0,T]}X(t)+\int_0^T Z(\tau)d\tau \lesssim 1.$$

Proof. Firstly, we assume $6\le p_i<\infty$ again. We take L^2-inner products on the system (1.2) with $(-\Delta u,-\Delta b)$, respectively, to obtain

$$\begin{aligned}
&\frac{1}{2}\partial_t X(t)+Z(t)\\
&=\int (u\cdot\nabla)u\cdot\Delta_{1,2}u+(u\cdot\nabla)u\cdot\Delta_{3,4}u+(u\cdot\nabla)b\cdot\Delta_{1,2}b+(u\cdot\nabla)b\cdot\Delta_{3,4}b\\
&\quad -(b\cdot\nabla)b\cdot\Delta_{1,2}u-(b\cdot\nabla)b\cdot\Delta_{3,4}u-(b\cdot\nabla)u\cdot\Delta_{1,2}b-(b\cdot\nabla)u\cdot\Delta_{3,4}b\triangleq\sum_{i=1}^{8}\mathrm{III}_i. \qquad (3.22)
\end{aligned}$$

From Equations (3.19)–(3.21), we already have the estimates of

$$\begin{aligned}
&\mathrm{III}_1+\mathrm{III}_3+\mathrm{III}_5+\mathrm{III}_7\lesssim \mathrm{II}_1+\mathrm{II}_2\\
&\lesssim \sum_{i=3}^{4}\|u_i\|_{L^{p_i}}\|\nabla u\|_{L^2}^{\frac{p_i-4}{p_i}}\|\nabla\nabla_{1,2}u\|_{L^2}^{\frac{2+p_i}{p_i}}\|\Delta u\|_{L^2}^{\frac{2}{p_i}}+\|b\|_{L^{p_b}}X^{\frac{p_b-4}{2p_b}}(t)Z^{\frac{1}{p_b}}(t)Y^{\frac{1}{p_b}+\frac{1}{2}}(t)\\
&\le \frac{1}{16}Z(t)+c\sum_{i=3}^{4}(\|u_i\|_{L^{p_i}}^{\frac{2p_i}{p_i-4}}+\|b\|_{L^{p_b}}^{\frac{2p_b}{p_b-4}})X(t) \qquad (3.23)
\end{aligned}$$

by Young's inequalities. Next, we work on III_2, which we first integrate by parts and decompose as follows:

$$\mathrm{III}_2=\int (u\cdot\nabla)u\cdot\Delta_{3,4}u=-\sum_{i,j=1}^{4}\sum_{k=3}^{4}\int \partial_k u_i\partial_i u_j\partial_k u_j$$

$$
\begin{aligned}
&= -\sum_{i=1}^{2}\sum_{j=1}^{4}\sum_{k=3}^{4}\int \partial_k u_i \partial_i u_j \partial_k u_j - \sum_{j=1}^{4}\sum_{i,k=3}^{4}\int \partial_k u_i \partial_i u_j \partial_k u_j \\
&= -\sum_{i=1}^{2}\sum_{j=1}^{4}\sum_{k=3}^{4}\int \partial_k u_i \partial_i u_j \partial_k u_j + \sum_{j=1}^{4}\sum_{i,k=3}^{4}\int u_i \partial_k (\partial_i u_j \partial_k u_j) \\
&\lesssim \int |\nabla u|^2 |\nabla_{1,2} u| + \sum_{i=3}^{4}\int |u_i||\nabla u||\nabla^2 u| \triangleq \mathrm{IV}_1 + \mathrm{IV}_2. \qquad (3.24)
\end{aligned}
$$

We estimate

$$
\begin{aligned}
\mathrm{IV}_1 &\approx \int |\nabla_{1,2} u||\nabla u|^2 \lesssim \|\nabla_{1,2} u\|_{L^2}\|\nabla u\|_{L^4}^2 \\
&\lesssim \|\nabla_{1,2} u\|_{L^2}\|\nabla\nabla_{1,2} u\|_{L^2}\|\Delta u\|_{L^2} \lesssim W^{\frac{1}{2}}(t) Y^{\frac{1}{2}}(t) Z^{\frac{1}{2}}(t) \qquad (3.25)
\end{aligned}
$$

by Hölder's inequalities and Equation (2.1). On the other hand,

$$
\begin{aligned}
\mathrm{IV}_2 &\lesssim \sum_{i=3}^{4} \|u_i\|_{L^{p_i}} \|\nabla u\|_{L^{\frac{2p_i}{p_i-2}}} \|\nabla^2 u\|_{L^2} \\
&\lesssim \sum_{i=3}^{4} \|u_i\|_{L^{p_i}} \|\nabla u\|_{L^2}^{1-\frac{4}{p_i}} \|\Delta u\|_{L^2}^{1+\frac{4}{p_i}} \le \frac{1}{16} Z(t) + c\sum_{i=3}^{4} \|u_i\|_{L^{p_i}}^{\frac{2p_i}{p_i-4}} X(t) \qquad (3.26)
\end{aligned}
$$

by Hölder's, Gagliardo–Nirenberg, and Young's inequalities. Next, again we carefully decompose

$$
\begin{aligned}
\mathrm{III}_4 &= \int (u\cdot\nabla) b \cdot \Delta_{3,4} b = -\sum_{i,j=1}^{4}\sum_{k=3}^{4}\int \partial_k u_i \partial_i b_j \partial_k b_j \\
&= -\sum_{i=1}^{2}\sum_{j=1}^{4}\sum_{k=3}^{4}\int \partial_k u_i \partial_i b_j \partial_k b_j - \sum_{j=1}^{4}\sum_{i,k=3}^{4}\int \partial_k u_i \partial_i b_j \partial_k b_j \\
&= -\sum_{i=1}^{2}\sum_{j=1}^{4}\sum_{k=3}^{4}\int \partial_k u_i \partial_i b_j \partial_k b_j + \sum_{j=1}^{4}\sum_{i,k=3}^{4}\int u_i \partial_k (\partial_i b_j \partial_k b_j) \\
&\lesssim \int |\nabla u||\nabla_{1,2} b||\nabla b| + \sum_{i=3}^{4}\int |u_i||\nabla b||\nabla^2 b| \triangleq \mathrm{IV}_3 + \mathrm{IV}_4. \qquad (3.27)
\end{aligned}
$$

We estimate

$$
\begin{aligned}
\mathrm{IV}_3 &\approx \int |\nabla u||\nabla_{1,2} b||\nabla b| \lesssim \|\nabla_{1,2} b\|_{L^2}\|\nabla u\|_{L^4}\|\nabla b\|_{L^4} \\
&\lesssim \|\nabla_{1,2} b\|_{L^2}\|\nabla\nabla_{1,2} u\|_{L^2}^{\frac{1}{2}}\|\nabla\nabla_{1,2} b\|_{L^2}^{\frac{1}{2}}\|\Delta u\|_{L^2}^{\frac{1}{2}}\|\Delta b\|_{L^2}^{\frac{1}{2}} \lesssim W^{\frac{1}{2}}(t) Y^{\frac{1}{2}}(t) Z^{\frac{1}{2}}(t) \qquad (3.28)
\end{aligned}
$$

by Hölder's inequalities, Equation (2.1), and Young's inequalities. On the other hand, we estimate, similarly to IV_2 in Equation (3.26),

$$
\mathrm{IV}_4 \lesssim \sum_{i=3}^{4} \|u_i\|_{L^{p_i}} \|\nabla b\|_{L^{\frac{2p_i}{p_i-2}}} \|\nabla^2 b\|_{L^2}
$$

$$\lesssim \sum_{i=3}^{4} \|u_i\|_{L^{p_i}} \|\nabla b\|_{L^2}^{1-\frac{4}{p_i}} \|\Delta b\|_{L^2}^{1+\frac{4}{p_i}} \leq \frac{1}{16} Z(t) + c \sum_{i=3}^{4} \|u_i\|_{L^{p_i}}^{\frac{2p_i}{p_i-4}} X(t) \tag{3.29}$$

by Hölder's, Gagliardo–Nirenberg, and Young's inequalities. Finally, similarly to IV_2 in Equation (3.26) again

$$\begin{aligned} \mathrm{III}_6 + \mathrm{III}_8 =& -\int (b\cdot\nabla) b \cdot \Delta_{3,4} u + (b\cdot\nabla) u \cdot \Delta_{3,4} b \\ \lesssim& \|b\|_{L^{p_b}} (\|\nabla b\|_{L^2}^{1-\frac{4}{p_b}} + \|\nabla u\|_{L^2}^{1-\frac{4}{p_b}})(\|\Delta u\|_{L^2}^{1+\frac{4}{p_b}} + \|\Delta b\|_{L^2}^{1+\frac{4}{p_b}}) \\ \leq& \frac{1}{16} Z(t) + c \|b\|_{L^{p_b}}^{\frac{2p_b}{p_b-4}} X(t) \end{aligned} \tag{3.30}$$

by Hölder's, Gagliardo–Nirenberg, and Young's inequalities. Thus, applying Equations (3.23)–(3.30) in Equation (3.22), we obtain after absorbing

$$\frac{1}{2}\partial_t X + \frac{1}{2} Z(t) \lesssim \sum_{i=3}^{4} (\|u_i\|_{L^{p_i}}^{\frac{2p_i}{p_i-4}} + \|b\|_{L^{p_b}}^{\frac{2p_b}{p_b-4}}) X(t) + W^{\frac{1}{2}}(t) Y^{\frac{1}{2}}(t) Z^{\frac{1}{2}}(t). \tag{3.31}$$

Now we assume $6 < p_i < \infty$. Integrating over $[0,t], t \in (0,T]$, we obtain

$$\begin{aligned} & X(t) + \int_0^t Z(\tau) d\tau \\ \leq & X(0) + c \sum_{i=3}^{4} \int_0^t (\|u_i\|_{L^{p_i}}^{\frac{2p_i}{p_i-4}} + \|b\|_{L^{p_b}}^{\frac{2p_b}{p_b-4}}) X(\tau) d\tau + c \int_0^t W^{\frac{1}{2}}(\tau) Y^{\frac{1}{2}}(\tau) Z^{\frac{1}{2}}(\tau) d\tau. \end{aligned}$$

We focus only on the last integral which we bound by a constant multiples of

$$\begin{aligned} & \sup_{\tau\in[0,t]} W^{\frac{1}{2}}(\tau) \left(\int_0^t Y(\tau) d\tau\right)^{\frac{1}{2}} \left(\int_0^t Z(\tau) d\tau\right)^{\frac{1}{2}} \\ \lesssim & \left(W(0) + \sum_{i=3}^{4} \int_0^t \|u_i\|_{L^{p_i}}^{\frac{2p_i}{p_i-2}} X^{\frac{p_i-4}{p_i-2}}(\tau) Z^{\frac{2}{p_i-2}}(\tau) + \|b\|_{L^{p_b}}^{\frac{2p_b}{p_b-2}} X^{\frac{p_b-4}{p_b-2}}(\tau) Z^{\frac{2}{p_b-2}}(\tau) d\tau \right) \\ & \times \left(\int_0^t Z(\tau) d\tau\right)^{\frac{1}{2}} \\ \lesssim & \left(\int_0^t Z(\tau) d\tau\right)^{\frac{1}{2}} + \sum_{i=3}^{4} \left(\int_0^t \|u_i\|_{L^{p_i}}^{\frac{2p_i}{p_i-4}} X(\tau) d\tau \right)^{\frac{p_i-4}{p_i-2}} \left(\int_0^t Z(\tau) d\tau\right)^{\frac{p_i+2}{2(p_i-2)}} \\ & + \left(\int_0^t \|b\|_{L^{p_b}}^{\frac{2p_b}{p_b-4}} X(\tau) d\tau \right)^{\frac{p_b-4}{p_b-2}} \left(\int_0^t Z(\tau) d\tau\right)^{\frac{p_b+2}{2(p_b-2)}} \\ \leq & \frac{1}{2} \int_0^t Z(\tau) d\tau + c(1 + \sum_{i=3}^{4} \left(\int_0^t \|u_i\|_{L^{p_i}}^{\frac{4p_i}{p_i-6}} X(\tau) d\tau \right) \left(\int_0^t X(\tau) d\tau \right)^{\frac{p_i-2}{p_i-6}} \\ & + \left(\int_0^t \|b\|_{L^{p_b}}^{\frac{4p_b}{p_b-6}} X(\tau) d\tau \right) \left(\int_0^t X(\tau) d\tau \right)^{\frac{p_b-2}{p_b-6}}) \\ \leq & \frac{1}{2} \int_0^t Z(\tau) d\tau + c \left(1 + \sum_{i=3}^{4} \int_0^t (\|u_i\|_{L^{p_i}}^{\frac{4p_i}{p_i-6}} + \|b\|_{L^{p_b}}^{\frac{4p_b}{p_b-6}}) X(\tau) d\tau \right) \end{aligned}$$

by Hölder's inequalities, Proposition 3.2, Young's inequalities, and Equation (1.3). After absorbing, Gronwall's inequality implies the desired result in the case of $6<p_i<\infty, r_i<\infty$.

We now consider the case $p_i=\infty$, assuming for the simplicity of presentation that $p_3=p_4=p_b=\infty$. Firstly, we could have computed in contrast to Equations (3.23), (3.24), (3.27), and (3.30), respectively

$$\begin{aligned}
&\mathrm{III}_1+\mathrm{III}_3+\mathrm{III}_5+\mathrm{III}_7\\
\lesssim&\mathrm{II}_1+\mathrm{II}_2\\
\lesssim&\sum_{i=3}^4\|u_i\|_{L^\infty}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}+\|b\|_{L^\infty}(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2})(\|\Delta u\|_{L^2}+\|\Delta b\|_{L^2})\\
\leq&\frac{1}{16}Z(t)+c\sum_{i=3}^4(\|u_i\|_{L^\infty}^2+\|b\|_{L^\infty}^2)X(t),
\end{aligned}\tag{3.32}$$

$$\begin{aligned}
\mathrm{III}_2\leq&\mathrm{IV}_1+\mathrm{IV}_2\\
\lesssim&\|\nabla_{1,2}u\|_{L^2}\|\nabla u\|_{L^4}^2+\sum_{i=3}^4\|u_i\|_{L^\infty}\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\\
\leq&\frac{1}{16}Z(t)+c\left(W^{\frac12}(t)Y^{\frac12}(t)Z^{\frac12}(t)+\sum_{i=3}^4\|u_i\|_{L^\infty}^2X(t)\right),
\end{aligned}\tag{3.33}$$

$$\begin{aligned}
\mathrm{III}_4\lesssim&\mathrm{IV}_3+\mathrm{IV}_4\\
\lesssim&\|\nabla u\|_{L^4}\|\nabla_{1,2}b\|_{L^2}\|\nabla b\|_{L^4}+\sum_{i=3}^4\|u_i\|_{L^\infty}\|\nabla b\|_{L^2}\|\Delta b\|_{L^2}\\
\lesssim&\|\nabla_{1,2}b\|_{L^2}\|\nabla\nabla_{1,2}u\|_{L^2}^{\frac12}\|\Delta u\|_{L^2}^{\frac12}\|\nabla\nabla_{1,2}b\|_{L^2}^{\frac12}\|\Delta b\|_{L^2}^{\frac12}+\sum_{i=3}^4\|u_i\|_{L^\infty}\|\nabla b\|_{L^2}\|\Delta b\|_{L^2}\\
\leq&\frac{1}{16}Z(t)+c\left(W^{\frac12}(t)Y^{\frac12}(t)Z^{\frac12}(t)+\sum_{i=3}^4\|u_i\|_{L^\infty}^2X(t)\right),
\end{aligned}\tag{3.34}$$

$$\begin{aligned}
\mathrm{III}_6+\mathrm{III}_8\lesssim&\|b\|_{L^\infty}(\|\nabla b\|_{L^2}+\|\nabla u\|_{L^2})(\|\nabla^2 u\|_{L^2}+\|\nabla^2 b\|_{L^2})\\
\leq&\frac{1}{16}Z(t)+c\|b\|_{L^\infty}^2X(t),
\end{aligned}\tag{3.35}$$

all by Hölder's and Young's inequalities and Equation (2.1) only in Equations (3.33) and (3.34). Thus, applying Equations (3.32)–(3.35) in Equation (3.22), absorbing and integrating in time $[0,t]$, we obtain

$$\begin{aligned}
&X(t)+\frac{3}{2}\int_0^t Z(\tau)d\tau\\
\leq&X(0)+c\sum_{i=3}^4\int_0^t(\|u_i\|_{L^\infty}^2+\|b\|_{L^\infty}^2)X(\tau)d\tau
\end{aligned}$$

$$
\begin{aligned}
&+c\sup_{\tau\in[0,t]}W^{\frac{1}{2}}(\tau)\left(\int_0^t Y(\tau)d\tau\right)^{\frac{1}{2}}\left(\int_0^t Z(\tau)d\tau\right)^{\frac{1}{2}}\\
\leq&\frac{1}{2}\int_0^t Z(\tau)d\tau+X(0)+c\sum_{i=3}^4\int_0^t(\|u_i\|_{L^\infty}^2+\|b\|_{L^\infty}^2)X(\tau)d\tau\\
&+c\left(W(0)+\sum_{i=3}^4\int_0^t(\|u_i\|_{L^\infty}^2+\|b\|_{L^\infty}^2)X(\tau)d\tau)\right)^2\\
\leq&\frac{1}{2}\int_0^t Z(\tau)d\tau\\
&+c\left(\sum_{i=3}^4\int_0^t(\|u_i\|_{L^\infty}^2+\|b\|_{L^\infty}^2)X(\tau)d\tau+1+\sum_{i=3}^4\int_0^t(\|u_i\|_{L^\infty}^4+\|b\|_{L^\infty}^4)X(\tau)d\tau\right)
\end{aligned}
$$

by Hölder's inequality, Proposition 3.2, Young's inequality, Equation (2.2), and Equation (1.3). This completes the proof in case $p_i=\infty$.

We now prove the second statement of Theorem 1.3, namely the smallness result when $p_i=6, r_i=\infty$. For simplicity of presentation, we assume $p_i=6$ for all $i=3,4,\dots,b$. We integrate in time on Equation (3.31) to obtain

$$
\begin{aligned}
&X(t)+\int_0^t Z(\tau)d\tau\\
\leq&X(0)+c(\sum_{i=3}^4\sup_{\tau\in[0,t]}(\|u_i\|_{L^6}^6+\|b\|_{L^6}^6)(\tau)\int_0^t X(\tau)d\tau+\sup_{\tau\in[0,t]}W^{\frac{1}{2}}(t)\left(\int_0^t Y(\tau)d\tau\right)^{\frac{1}{2}}\\
&\times\left(\int_0^t Z(\tau)d\tau\right)^{\frac{1}{2}})\\
\lesssim&1+\left(W(0)+\sum_{i=3}^4\int_0^t(\|u_i\|_{L^6}^3+\|b\|_{L^6}^3)X^{\frac{1}{2}}(\tau)Z^{\frac{1}{2}}(\tau)d\tau\right)\left(\int_0^t Z(\tau)d\tau\right)^{\frac{1}{2}}\\
\lesssim&1+\left(\int_0^t Z(\tau)d\tau\right)^{\frac{1}{2}}+\sum_{i=3}^4\sup_{\tau\in[0,t]}(\|u_i\|_{L^6}^3+\|b\|_{L^6}^3)(\tau)\left(\int_0^t X(\tau)d\tau\right)^{\frac{1}{2}}\left(\int_0^t Z(\tau)d\tau\right)\\
\leq&\frac{1}{2}\int_0^t Z(\tau)d\tau+c
\end{aligned}
$$

for $\sum_{i=3}^4\sup_{\tau\in[0,t]}(\|u_i\|_{L^6}^3+\|b\|_{L^6}^3)(\tau)$ sufficiently small, where we used Hölder's inequality, Proposition 3.2, Young's inequality, and Equation (1.3). Absorbing, Gronwall's inequality completes the proof of Theorem 1.3. □

4. Proof of Theorem 1.4

We assume for simplicity of presentation that for all $i=3,4,\dots,b$ and $p_i\in[\frac{12}{5},4]$ or $p_i\in[4,\infty]$. A combination of mixed cases can be obtained following the proofs below.

PROPOSITION 4.1. *Let $N=4$ and let (u,b) be the solution pair to the MHD system (1.2) that satisfies the hypothesis of Theorem 1.4. Then for all $t\in(0,T]$,*

$$\sup_{\tau\in[0,t]} W(\tau)+\int_0^t Y(\tau)d\tau$$
$$\leq \begin{cases} W(0)+c\sum_{i=3}^4\int_0^t \|\nabla u_i\|_{L^{p_i}}^{\frac{4p_i}{3p_i-4}} X^{\frac{4(p_i-2)}{3p_i-4}}(\tau)Z^{\frac{4-p_i}{3p_i-4}}(\tau) \\ \qquad\qquad +\|\nabla b\|_{L^{p_b}}^{\frac{4p_b}{3p_b-4}} X^{\frac{4(p_b-2)}{3p_b-4}}(\tau)Z^{\frac{4-p_b}{3p_b-4}}(\tau)d\tau, & \text{if } p_i\in[\frac{12}{5},4], \\ W(0)+c\sum_{i=3}^4\int_0^t (\|\nabla u_i\|_{L^{p_i}}^{\frac{p_i}{p_i-2}}+\|\nabla b\|_{L^{p_b}}^{\frac{p_b}{p_b-2}})X(\tau)d\tau, & \text{if } p_i\in[4,\infty], \end{cases}$$

with the usual convention at $p_i=\infty, i=3,4,\dots,b$; *i.e.* $\frac{p_i}{p_i-2}=1$.

Proof. We first assume $p_i\in\left[\frac{12}{5},4\right]$. We take L^2-inner products of the system (1.2) with $-\Delta_{1,2}u, -\Delta_{1,2}b$, respectively, and estimate

$$\frac{1}{2}\partial_t W(t)+Y(t)\lesssim\sum_{i=3}^4\int|\nabla u_i||\nabla_{1,2}u||\nabla u|+|\nabla b||\nabla_{1,2}b||\nabla u| \tag{4.1}$$

by Equation (3.2). Now we estimate

$$\begin{aligned}\sum_{i=3}^4\int|\nabla u_i||\nabla_{1,2}u||\nabla u| &\lesssim\sum_{i=3}^4\|\nabla u_i\|_{L^{p_i}}\|\nabla_{1,2}u\|_{L^4}\|\nabla u\|_{L^{\frac{4p_i}{3p_i-4}}} \\ &\lesssim\sum_{i=3}^4\|\nabla u_i\|_{L^{p_i}}\|\nabla\nabla_{1,2}u\|_{L^2}\|\nabla u\|_{L^2}^{2(\frac{p_i-2}{p_i})}\|\nabla u\|_{L^4}^{\frac{4-p_i}{p_i}} \\ &\lesssim\sum_{i=3}^4\|\nabla u_i\|_{L^{p_i}}\|\nabla\nabla_{1,2}u\|_{L^2}^{\frac{4+p_i}{2p_i}}\|\nabla u\|_{L^2}^{2(\frac{p_i-2}{p_i})}\|\Delta u\|_{L^2}^{\frac{4-p_i}{2p_i}} \\ &\leq\frac{1}{4}Y(t)+c\sum_{i=3}^4\|\nabla u_i\|_{L^{p_i}}^{\frac{4p_i}{3p_i-4}}X^{\frac{4(p_i-2)}{3p_i-4}}(t)Z^{\frac{4-p_i}{3p_i-4}}(t)\end{aligned} \tag{4.2}$$

by Hölder's inequalities, Sobolev embedding of $\dot{H}^1(\mathbb{R}^4)\hookrightarrow L^4(\mathbb{R}^4)$, interpolation inequality, Equation (2.1), and Young's inequality. Similarly, we obtain

$$\begin{aligned}\int|\nabla b||\nabla_{1,2}b||\nabla u| &\lesssim\|\nabla b\|_{L^{p_b}}\|\nabla\nabla_{1,2}b\|_{L^2}\|\nabla u\|_{L^{\frac{4p_b}{3p_b-4}}} \\ &\lesssim\|\nabla b\|_{L^{p_b}}Y^{\frac{1}{2}}(t)\|\nabla u\|_{L^2}^{2(\frac{p_b-2}{p_b})}\|\nabla u\|_{L^4}^{\frac{4-p_b}{p_b}} \\ &\lesssim\|\nabla b\|_{L^{p_b}}Y^{\frac{1}{2}}(t)X^{\frac{p_b-2}{p_b}}(t)\|\nabla\nabla_{1,2}u\|_{L^2}^{\frac{4-p_b}{2p_b}}\|\Delta u\|_{L^2}^{\frac{4-p_b}{2p_b}} \\ &\leq\frac{1}{4}Y(t)+c\|\nabla b\|_{L^{p_b}}^{\frac{4p_b}{3p_b-4}}X^{\frac{4(p_b-2)}{3p_b-4}}(t)Z^{\frac{4-p_b}{3p_b-4}}(t).\end{aligned} \tag{4.3}$$

With Equations (4.2) and (4.3) applied to Equation (4.1), absorbing and integrating in time lead to

$$\begin{aligned}&W(t)+\int_0^t Y(\tau)d\tau \\ &\leq W(0)+c\sum_{i=3}^4\int_0^t\|\nabla u_i\|_{L^{p_i}}^{\frac{4p_i}{3p_i-4}}X^{\frac{4(p_i-2)}{3p_i-4}}(\tau)Z^{\frac{4-p_i}{3p_i-4}}(\tau)+\|\nabla b\|_{L^{p_b}}^{\frac{4p_b}{3p_b-4}}X^{\frac{4(p_b-2)}{3p_b-4}}(\tau)Z^{\frac{4-p_b}{3p_b-4}}(\tau)d\tau.\end{aligned} \tag{4.4}$$

We now work on the case $4<p_i<\infty$

$$\begin{aligned}\sum_{i=3}^{4}\int|\nabla u_i||\nabla_{1,2}u||\nabla u| &\lesssim \sum_{i=3}^{4}\|\nabla u_i\|_{L^{p_i}}\|\nabla_{1,2}u\|_{L^{\frac{2p_i}{p_i-2}}}\|\nabla u\|_{L^2}\\ &\lesssim \sum_{i=3}^{4}\|\nabla u_i\|_{L^{p_i}}\|\nabla_{1,2}u\|_{L^2}^{\frac{p_i-4}{p_i}}\|\nabla_{1,2}u\|_{L^4}^{\frac{4}{p_i}}\|\nabla u\|_{L^2}\\ &\lesssim \sum_{i=3}^{4}\|\nabla u_i\|_{L^{p_i}}\|\nabla_{1,2}u\|_{L^2}^{\frac{p_i-4}{p_i}}\|\nabla\nabla_{1,2}u\|_{L^2}^{\frac{4}{p_i}}\|\nabla u\|_{L^2}\\ &\leq \frac{1}{4}Y(t)+c\sum_{i=3}^{4}\|\nabla u_i\|_{L^{p_i}}^{\frac{p_i}{p_i-2}}X(t)\end{aligned} \tag{4.5}$$

by Hölder's and interpolation inequalities, Sobolev embedding of $\dot{H}^1(\mathbb{R}^4)\hookrightarrow L^4(\mathbb{R}^4)$, and Young's inequality. Similarly, we estimate

$$\begin{aligned}\int|\nabla b||\nabla_{1,2}b||\nabla u| &\lesssim \|\nabla b\|_{L^{p_b}}\|\nabla_{1,2}b\|_{L^2}^{\frac{p_b-4}{p_b}}\|\nabla_{1,2}b\|_{L^4}^{\frac{4}{p_b}}\|\nabla u\|_{L^2}\\ &\lesssim \|\nabla b\|_{L^{p_b}}\|\nabla_{1,2}b\|_{L^2}^{\frac{p_b-4}{p_b}}\|\nabla\nabla_{1,2}b\|_{L^2}^{\frac{4}{p_b}}\|\nabla u\|_{L^2}\\ &\leq \frac{1}{4}Y(t)+c\|\nabla b\|_{L^{p_b}}^{\frac{p_b}{p_b-2}}X(t).\end{aligned} \tag{4.6}$$

We apply Equations (4.5) and (4.6) in Equation (4.1), absorb and integrate in time to obtain

$$W(t)+\int_0^t Y(\tau)d\tau \leq W(0)+c\sum_{i=3}^{4}\int_0^t\left(\|\nabla u_i\|_{L^{p_i}}^{\frac{p_i}{p_i-2}}+\|\nabla b\|_{L^{p_b}}^{\frac{p_b}{p_b-2}}\right)X(\tau)d\tau. \tag{4.7}$$

The case of $p_i=\infty$ requires only a standard modification as done in the proof of Theorem 1.3; that is,

$$\sum_{i=3}^{4}\int|\nabla u_i||\nabla_{1,2}u||\nabla u|\lesssim \sum_{i=3}^{4}\|\nabla u_i\|_{L^\infty}\|\nabla_{1,2}u\|_{L^2}\|\nabla u\|_{L^2}\lesssim \sum_{i=3}^{4}\|\nabla u_i\|_{L^\infty}X(t),$$

$$\int|\nabla b||\nabla_{1,2}b||\nabla u|\lesssim \|\nabla b\|_{L^\infty}\|\nabla_{1,2}b\|_{L^2}\|\nabla u\|_{L^2}\lesssim \|\nabla b\|_{L^\infty}X(t)$$

so that summing and integrating in time leads to the desired result. This completes the proof of Proposition 4.1. ∎

Proposition 4.2. *Let $N=4$ and let (u,b) be the solution pair to the MHD system (1.2) that satisfies the hypothesis of Theorem 1.4. Then*

$$\sup_{t\in[0,T]}X(t)+\int_0^T Z(\tau)d\tau\lesssim 1.$$

Proof. Similarly to the proof of Theorem 1.3, we estimate from Equation (3.22). For $p_i\in[\frac{12}{5},4]$, we continue our estimate from Equations (4.1), (4.2), and (4.3) to obtain

$$\mathrm{III}_1+\mathrm{III}_3+\mathrm{III}_5+\mathrm{III}_7\lesssim \sum_{i=3}^{4}\|\nabla u_i\|_{L^{p_i}}\|\nabla\nabla_{1,2}u\|_{L^2}^{\frac{4+p_i}{2p_i}}\|\nabla u\|_{L^2}^{2(\frac{p_i-2}{p_i})}\|\Delta u\|_{L^2}^{\frac{4-p_i}{2p_i}}$$

$$
\begin{aligned}
&+\|\nabla b\|_{L^{p_b}} Y^{\frac{4+p_b}{4p_b}}(t) X^{\frac{p_b-2}{p_b}}(t)\|\Delta u\|_{L^2}^{\frac{4-p_b}{2p_b}} \\
\leq &\frac{1}{16} Z(t)+c\sum_{i=3}^{4}\left(\|\nabla u_i\|_{L^{p_i}}^{\frac{p_i}{p_i-2}}+\|\nabla b\|_{L^{p_b}}^{\frac{p_b}{p_b-2}}\right) X(t) \qquad (4.8)
\end{aligned}
$$

by Young's inequality. We now decompose integrating by parts

$$
\begin{aligned}
\mathrm{III}_2 &= -\sum_{i,j=1}^{4}\sum_{k=3}^{4}\int \partial_k u_i \partial_i u_j \partial_k u_j \\
&= -\sum_{i=1}^{2}\sum_{j=1}^{4}\sum_{k=3}^{4}\int \partial_k u_i \partial_i u_j \partial_k u_j - \sum_{j=1}^{4}\sum_{i,k=3}^{4}\int \partial_k u_i \partial_i u_j \partial_k u_j \\
&\lesssim \int |\nabla u|^2 |\nabla_{1,2} u| + \sum_{i=3}^{4}\int |\nabla u_i||\nabla u|^2 \triangleq \mathrm{V}_1+\mathrm{V}_2 \qquad (4.9)
\end{aligned}
$$

where V_1 is estimated in an identical manner as IV_1 in Equation (3.25), while we estimate

$$
\begin{aligned}
\mathrm{V}_2 &\lesssim \sum_{i=3}^{4}\|\nabla u_i\|_{L^{p_i}}\|\nabla u\|_{L^{\frac{2p_i}{p_i-1}}}^2 \\
&\lesssim \sum_{i=3}^{4}\|\nabla u_i\|_{L^{p_i}}\|\nabla u\|_{L^2}^{2(\frac{p_i-2}{p_i})}\|\Delta u\|_{L^2}^{2(\frac{2}{p_i})} \leq \frac{1}{16}Z(t)+c\sum_{i=3}^{4}\|\nabla u_i\|_{L^{p_i}}^{\frac{p_i}{p_i-2}} X(t) \qquad (4.10)
\end{aligned}
$$

by Hölder's, Gagliardo–Nirenberg, and Young's inequalities. Next, we decompose

$$
\begin{aligned}
\mathrm{III}_4 &= -\sum_{i,j=1}^{4}\sum_{k=3}^{4}\int \partial_k u_i \partial_i b_j \partial_k b_j \\
&= -\sum_{i=1}^{2}\sum_{j=1}^{4}\sum_{k=3}^{4}\int \partial_k u_i \partial_i b_j \partial_k b_j - \sum_{j=1}^{4}\sum_{i,k=3}^{4}\int \partial_k u_i \partial_i b_j \partial_k b_j \\
&\lesssim \int |\nabla u||\nabla_{1,2} b||\nabla b| + \sum_{i=3}^{4}\int |\nabla u_i||\nabla b|^2 \triangleq \mathrm{V}_3+\mathrm{V}_4 \qquad (4.11)
\end{aligned}
$$

where we estimate V_3 as IV_3 in Equation (3.28), while the same estimate of V_2 in Equation (4.10) leads to

$$
\begin{aligned}
\mathrm{V}_4 &\lesssim \sum_{i=3}^{4}\|\nabla u_i\|_{L^{p_i}}\|\nabla b\|_{L^{\frac{2p_i}{p_i-1}}}^2 \\
&\lesssim \sum_{i=3}^{4}\|\nabla u_i\|_{L^{p_i}}\|\nabla b\|_{L^2}^{2(\frac{p_i-2}{p_i})}\|\Delta b\|_{L^2}^{2(\frac{2}{p_i})} \leq \frac{1}{16}Z(t)+c\sum_{i=3}^{4}\|\nabla u_i\|_{L^{p_i}}^{\frac{p_i}{p_i-2}} X(t). \qquad (4.12)
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\mathrm{III}_6+\mathrm{III}_8 \\
&= \sum_{i,j=1}^{4}\sum_{k=3}^{4}\int \partial_k b_i \partial_i b_j \partial_k u_j + \partial_k b_i \partial_i u_j \partial_k b_j
\end{aligned}
$$

$$\begin{aligned}
&\lesssim \int |\nabla b|^2 |\nabla u| \\
&\lesssim \|\nabla b\|_{L^{p_b}} \|\nabla b\|_{L^{\frac{2p_b}{p_b-1}}} \|\nabla u\|_{L^{\frac{2p_b}{p_b-1}}} \\
&\lesssim \|\nabla b\|_{L^{p_b}} \|\nabla b\|_{L^2}^{\frac{p_b-2}{p_b}} \|\Delta b\|_{L^2}^{\frac{2}{p_b}} \|\nabla u\|_{L^2}^{\frac{p_b-2}{p_b}} \|\Delta u\|_{L^2}^{\frac{2}{p_b}} \\
&\leq \frac{1}{16} Z(t) + c\|\nabla b\|_{L^{p_b}}^{\frac{p_b}{p_b-2}} X(t)
\end{aligned} \tag{4.13}$$

by Hölder's, Gagliardo–Nirenberg, and Young's inequalities. Thus, we obtain by applying Equations (4.8)–(4.13) in Equation (3.22), absorbing, and integrating in time

$$\begin{aligned}
&X(t) + \frac{3}{2}\int_0^t Z(\tau) d\tau \\
&\lesssim 1 + \sum_{i=3}^4 \int_0^t (\|\nabla u_i\|_{L^{p_i}}^{\frac{p_i}{p_i-2}} + \|\nabla b\|_{L^{p_b}}^{\frac{p_b}{p_b-2}}) X(\tau) d\tau \\
&\quad + \sup_{\tau \in [0,t]} W^{\frac{1}{2}}(\tau) \left(\int_0^t Y(\tau) d\tau \right)^{\frac{1}{2}} \left(\int_0^t Z(\tau) d\tau \right)^{\frac{1}{2}},
\end{aligned} \tag{4.14}$$

where we also used Hölder's inequality. Now we assume $p_i \in (\frac{12}{5}, 4]$. For the last term only, we bound it by a constant multiples of

$$\begin{aligned}
&\left(1 + \sum_{i=3}^4 \int_0^t \|\nabla u_i\|_{L^{p_i}}^{\frac{4p_i}{3p_i-4}} X^{\frac{4(p_i-2)}{3p_i-4}}(\tau) Z^{\frac{4-p_i}{3p_i-4}}(\tau) + \|\nabla b\|_{L^{p_b}}^{\frac{4p_b}{3p_b-4}} X^{\frac{4(p_b-2)}{3p_b-4}}(\tau) Z^{\frac{4-p_b}{3p_b-4}}(\tau) d\tau \right) \\
&\quad \times \left(\int_0^t Z(\tau) d\tau \right)^{\frac{1}{2}} \\
&\lesssim \left(\int_0^t Z(\tau) d\tau \right)^{\frac{1}{2}} + \sum_{i=3}^4 \left(\int_0^t \|\nabla u_i\|_{L^{p_i}}^{\frac{p_i}{p_i-2}} X(\tau) d\tau \right)^{\frac{4(p_i-2)}{3p_i-4}} \left(\int_0^t Z(\tau) d\tau \right)^{\frac{4+p_i}{2(3p_i-4)}} \\
&\quad + \left(\int_0^t \|\nabla b\|_{L^{p_b}}^{\frac{p_b}{p_b-2}} X(\tau) d\tau \right)^{\frac{4(p_b-2)}{3p_b-4}} \left(\int_0^t Z(\tau) d\tau \right)^{\frac{4+p_b}{2(3p_b-4)}} \\
&\leq \frac{1}{2} \int_0^t Z(\tau) d\tau \\
&\quad + c \left(1 + \sum_{i=3}^4 \left(\int_0^t \|\nabla u_i\|_{L^{p_i}}^{\frac{p_i}{p_i-2}} X(\tau) d\tau \right)^{\frac{8(p_i-2)}{5p_i-12}} + \left(\int_0^t \|\nabla b\|_{L^{p_b}}^{\frac{p_b}{p_b-2}} X(\tau) d\tau \right)^{\frac{8(p_b-2)}{5p_b-12}} \right) \\
&\leq \frac{1}{2} \int_0^t Z(\tau) d\tau + c \left(1 + \sum_{i=3}^4 \left(\int_0^t \|\nabla u_i\|_{L^{p_i}}^{\frac{8p_i}{5p_i-12}} X(\tau) d\tau \right) + \left(\int_0^t \|\nabla b\|_{L^{p_b}}^{\frac{8p_b}{5p_b-12}} X(\tau) d\tau \right) \right)
\end{aligned}$$

due to Proposition 4.1, Hölder's, and Young's inequalities and (1.3).

Next, we consider the case of $4 < p_i < \infty$. We restart from Equation (3.22), where we continue our estimates from Equations (4.1), (4.5), and (4.6) to obtain

$$\begin{aligned}
&\mathrm{III}_1 + \mathrm{III}_3 + \mathrm{III}_5 + \mathrm{III}_7 \\
&\lesssim \sum_{i=3}^4 \|\nabla u_i\|_{L^{p_i}} \|\nabla_{1,2} u\|_{L^2}^{\frac{p_i-4}{p_i}} \|\nabla \nabla_{1,2} u\|_{L^2}^{\frac{4}{p_i}} \|\nabla u\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&+\|\nabla b\|_{L^{p_b}}\|\nabla_{1,2}b\|_{L^2}^{\frac{p_b-4}{p_b}}\|\nabla\nabla_{1,2}b\|_{L^2}^{\frac{4}{p_b}}\|\nabla u\|_{L^2}\\
\le&\frac{1}{16}Z(t)+c\sum_{i=3}^{4}(\|\nabla u_i\|_{L^{p_i}}^{\frac{p_i}{p_i-2}}+\|\nabla b\|_{L^{p_b}}^{\frac{p_b}{p_b-2}})X(t)
\end{aligned}
\tag{4.15}
$$

by Young's inequality. The rest of the estimates of $\mathrm{III}_2,\mathrm{III}_4,\mathrm{III}_6,\mathrm{III}_8$ all go through as in the case $p_i\in[\frac{12}{5},4]$. Indeed, continuing from Equation (4.9), we bound $\mathrm{III}_2\lesssim \mathrm{V}_1+\mathrm{V}_2$, where V_1 is estimated as IV_1 in Equation (3.25) and V_2 is estimated identically as for Equation (4.10). The estimates of III_4 also go through as in Equation (4.11): $\mathrm{III}_4\lesssim \mathrm{V}_3+\mathrm{V}_4$, where V_3 is estimated as IV_3 in Equation (3.28) and V_4 in (4.12). Finally, we use the estimate of $\mathrm{III}_6+\mathrm{III}_8$ in Equation (4.13). Thus, in sum, after absorbing and integrating in time, we obtain

$$
\begin{aligned}
X(t)+\frac{3}{2}\int_0^t Z(\tau)d\tau\le X(0)+c\sum_{i=3}^{4}\int_0^t(\|\nabla u_i\|_{L^{p_i}}^{\frac{p_i}{p_i-2}}+\|\nabla b\|_{L^{p_b}}^{\frac{p_b}{p_b-2}})X(\tau)d\tau&\\
+c\sup_{\tau\in[0,t]}W^{\frac{1}{2}}(\tau)\left(\int_0^t Y(\tau)d\tau\right)^{\frac{1}{2}}\left(\int_0^t Z(\tau)d\tau\right)^{\frac{1}{2}}&
\end{aligned}
$$

by Hölder's inequality. We bound the last term by

$$
\begin{aligned}
&c(W(0)+\sum_{i=3}^{4}\int_0^t(\|\nabla u_i\|_{L^{p_i}}^{\frac{p_i}{p_i-2}}+\|\nabla b\|_{L^{p_b}}^{\frac{p_b}{p_b-2}})X(\tau)d\tau\left(\int_0^t Z(\tau)d\tau\right)^{\frac{1}{2}}\\
\le&\frac{1}{2}\int_0^t Z(\tau)d\tau+c\left(1+\sum_{i=3}^{4}\left(\int_0^t(\|\nabla u_i\|_{L^{p_i}}^{\frac{p_i}{p_i-2}}+\|\nabla b\|_{L^{p_b}}^{\frac{p_b}{p_b-2}})X(\tau)d\tau\right)^2\right)\\
\le&\frac{1}{2}\int_0^t Z(\tau)d\tau+c\left(1+\sum_{i=3}^{4}\int_0^t(\|\nabla u_i\|_{L^{p_i}}^{\frac{2p_i}{p_i-2}}+\|\nabla b\|_{L^{p_b}}^{\frac{2p_b}{p_b-2}})X(\tau)d\tau\right)
\end{aligned}
$$

by Proposition 4.1, Young's, and Hölder's inequalities and Equation (1.3). After absorbing, Gronwall's inequality implies the desired result. We now consider the case of $p_i=\infty$. For simplicity, we assume $p_i=\infty$ for all $i=3,4,\dots,b$. We continue from Equation (3.22), where we estimate in contrast to Equation (4.15)

$$
\begin{aligned}
&\mathrm{III}_1+\mathrm{III}_3+\mathrm{III}_5+\mathrm{III}_7\\
\lesssim&\sum_{i=3}^{4}\int|\nabla u_i||\nabla_{1,2}u||\nabla u|+|\nabla b||\nabla_{1,2}b||\nabla u|\lesssim\sum_{i=3}^{4}(\|\nabla u_i\|_{L^\infty}+\|\nabla b\|_{L^\infty})X(t)
\end{aligned}
$$

due to Equation (4.1) and Hölder's, and Young's inequalities. Moreover, from $\mathrm{III}_2\lesssim \mathrm{V}_1+\mathrm{V}_2$ of Equation (4.9), we estimate V_1 is estimated as IV_1 in Equation (3.25) and

$$
\mathrm{V}_2\approx\sum_{i=3}^{4}\int|\nabla u_i||\nabla u|^2\lesssim\sum_{i=3}^{4}\|\nabla u_i\|_{L^\infty}\|\nabla u\|_{L^2}^2.
$$

Moreover, from $\mathrm{III}_4\lesssim \mathrm{V}_3+\mathrm{V}_4$ of Equation (4.11), we have V_3 estimated as IV_3 in Equation (3.28), while

$$
\mathrm{V}_4\approx\sum_{i=3}^{4}\int|\nabla u_i||\nabla b|^2\lesssim\sum_{i=3}^{4}\|\nabla u_i\|_{L^\infty}\|\nabla b\|_{L^2}^2.
$$

Finally, continuing our estimate from Equation (4.13),

$$\mathrm{III}_6+\mathrm{III}_8 \lesssim \int |\nabla b|^2 |\nabla u| \lesssim \|\nabla b\|_{L^\infty} \|\nabla b\|_{L^2} \|\nabla u\|_{L^2} \lesssim \|\nabla b\|_{L^\infty} X(t).$$

In sum, integrating in time, we obtain

$$\begin{aligned}
& X(t)+2\int_0^t Z(\tau)d\tau \\
\lesssim & X(0)+\sum_{i=3}^4 \int (\|\nabla u_i\|_{L^\infty}+\|\nabla b\|_{L^\infty}) X(\tau)d\tau \\
& + \sup_{\tau\in[0,t]} W^{\frac{1}{2}}(\tau)\left(\int_0^t Y(\tau)d\tau\right)^{\frac{1}{2}} \left(\int_0^t Z(\tau)d\tau\right)^{\frac{1}{2}} \\
\lesssim & X(0)+\sum_{i=3}^4 \int (\|\nabla u_i\|_{L^\infty}+\|\nabla b\|_{L^\infty}) X(\tau)d\tau \\
& + \left(W(0)+\sum_{i=3}^4 \int_0^t (\|\nabla u_i\|_{L^\infty}+\|\nabla b\|_{L^\infty}) X(\tau)d\tau\right)\left(\int_0^t Z(\tau)d\tau\right)^{\frac{1}{2}} \\
\leq & \int_0^t Z(\tau)d\tau + c\left(1+\sum_{i=3}^4 (\int_0^t \|\nabla u_i\|_{L^\infty}^2 + \|\nabla b\|_{L^\infty}^2) X(\tau)d\tau)\right)
\end{aligned}$$

by Hölder's inequality, Proposition 4.1, Young's inequality, and Equation (1.3).

Finally, we prove the smallness result in the case $p_i=\frac{12}{5}, r_i=\infty$, for which for simplicity of presentation, we assume $r_i=\infty, p_i=\frac{12}{5}$, for all $i=3,4,\dots,b$. From Equation (4.14),

$$\begin{aligned}
& X(t)+\frac{3}{2}\int_0^t Z(\tau)d\tau \lesssim X(0)+\sum_{i=3}^4 \int_0^t (\|\nabla u_i\|_{L^{\frac{12}{5}}}^6 + \|\nabla b\|_{L^{\frac{12}{5}}}^6) X(\tau)d\tau \\
& + \left(W(0)+\sum_{i=3}^4 \int_0^t (\|\nabla u_i\|_{L^{\frac{12}{5}}}^3 + \|\nabla b\|_{L^{\frac{12}{5}}}^3) X^{\frac{1}{2}}(\tau) Z^{\frac{1}{2}}(\tau)d\tau\right)\left(\int_0^t Z(\tau)d\tau\right)^{\frac{1}{2}} \\
\leq & \frac{1}{4}\int_0^t Z(\tau)d\tau + c\sum_{i=3}^4 \int_0^t (\|\nabla u_i\|_{L^{\frac{12}{5}}}^6 + \|\nabla b\|_{L^{\frac{12}{5}}}^6) X(\tau)d\tau \\
& + c\left(1+\sum_{i=3}^4 (\int_0^t (\|\nabla u_i\|_{L^{\frac{12}{5}}}^3 + \|\nabla b\|_{L^{\frac{12}{5}}}^3) X^{\frac{1}{2}}(\tau) Z^{\frac{1}{2}}(\tau)d\tau)^2\right) \\
\leq & \frac{1}{4}\int_0^t Z(\tau)d\tau + c\sum_{i=3}^4 \sup_{\tau\in[0,t]} (\|\nabla u_i\|_{L^{\frac{12}{5}}}^6 + \|\nabla b\|_{L^{\frac{12}{5}}}^6)(\tau) \int_0^t X(\tau)d\tau \\
& + c\left(1+\sum_{i=3}^4 \sup_{\tau\in[0,t]} (\|\nabla u_i\|_{L^{\frac{12}{5}}}^6 + \|\nabla b\|_{L^{\frac{12}{5}}}^6)(\tau) \int_0^t X(\tau)d\tau \int_0^t Z(\tau)d\tau\right) \\
\leq & \frac{1}{2}\int_0^t Z(\tau)d\tau + c
\end{aligned}$$

for $\sum_{i=3}^4 \sup_{t\in[0,T]} (\|\nabla u_i\|_{L^{\frac{12}{5}}}^6 + \|\nabla b\|_{L^{\frac{12}{5}}}^6)(t)$ sufficiently small, where we used Hölder's inequality, Proposition 4.1, Young's inequality, Equation (2.2), and Equation (1.3). This completes the proof of Theorem 1.4. $\square$

5. Proof of Theorem 1.5

We fix $q_i \in (\frac{12}{7}, 6)$ and then $p_i = 6+\epsilon$ for $\epsilon > 0$ sufficiently small so that $\frac{2(6+\epsilon)}{(6+\epsilon)+1} < q_i$ and also $q_i < 6 < p_i$. This implies that, for all $\epsilon > 0$ sufficiently small, we have $q_i \in (\frac{2p_i}{p_i+1}, p_i)$. Now we multiply the ith component of the u-equation of the system (1.1) with $|u_i|^{p_i-2}u_i$ and integrate in space to obtain

$$\begin{aligned}
&\frac{1}{p_i}\partial_t \|u_i\|_{L^{p_i}}^{p_i} + c(p_i)\|u_i\|_{L^{2p_i}}^{p_i} \\
\lesssim & \|\partial_i \pi\|_{L^{q_i}} \|u_i\|_{L^{\frac{(p_i-1)q_i}{q_i-1}}}^{p_i-1} \\
\lesssim & \|\partial_i \pi\|_{L^{q_i}} \|u_i\|_{L^{p_i}}^{\frac{p_iq_i-2p_i+q_i}{q_i}} \|u_i\|_{L^{2p_i}}^{\frac{2(p_i-q_i)}{q_i}} \\
\leq & \frac{c(p_i)}{2}\|u_i\|_{L^{2p_i}}^{p_i} + c\|\partial_i\pi\|_{L^{q_i}}^{\frac{p_iq_i}{p_iq_i-2p_i+2q_i}} \|u_i\|_{L^{p_i}}^{p_i(\frac{p_iq_i-2p_i+q_i}{p_iq_i-2p_i+2q_i})},
\end{aligned}$$

where we used the lower bound estimate on the dissipative term of

$$c(p_i)\|u_i\|_{L^{2p_i}}^{p_i} \approx \| |u_i|^{\frac{p_i}{2}} \|_{L^4}^2 \lesssim \| |u_i|^{\frac{p_i}{2}} \|_{\dot{H}^1}^2 \approx \frac{(p_i-1)4}{p_i^2} \int |\nabla |u_i|^{\frac{p_i}{2}}|^2 = -\int \Delta u |u_i|^{p_i-2}u_i$$

for some constant $c(p_i)$ that depends on p_i, Hölder's inequality, interpolation, and Young's inequality. We absorb and obtain

$$\frac{1}{p_i}\partial_t\|u_i\|_{L^{p_i}}^{p_i} + \frac{c(p_i)}{2}\|u_i\|_{L^{2p_i}}^{p_i} \lesssim \|\partial_i\pi\|_{L^{q_i}}^{\frac{p_iq_i}{p_iq_i-2p_i+2q_i}}(1+\|u_i\|_{L^{p_i}}^{p_i})$$

by Young's inequality. By the hypothesis of Theorem 1.5 and Gronwall's inequality, for all $\epsilon > 0$ sufficiently small, we have $\sum_{i=3}^4 \sup_{t\in[0,T]} \|u_i\|_{L^{p_i}}(t) \lesssim 1$, where $p_i = 6+\epsilon$. By Theorem 1.1, the proof of Theorem 1.5 is complete.

6. Further Discussion

There are many results on the regularity criteria component reduction theory of the three-dimensional NSE and the MHD system that we may look forward to being generalized to the four-dimensional case, as done in our paper. We remark however that the results beside what we presented in this paper did not seem readily generalizable.

For this precise reason, although the authors in [7] showed that a bound on u_3 in a scaling-invariant norm suffices to obtain the uniqueness of the weak solution to the three-dimensional NSE (see also [8, 24, 35] for the case of the MHD system), it is not clear to the author at the time of writing whether these results in [7, 8, 24, 35] may be extended to a two-component regularity criteria in a scaling-invariant norm for the four-dimensional NSE or the MHD system. The proofs in [7, 8, 24, 35] are highly nontrivial, employing anisotropic Littlewood–Paley theory and some key identities due to the divergence-free property of the solution to the NSE and the MHD system. While the results in our paper rely heavily on the decomposition of the nonlinear terms in Proposition 3.1, that decomposition is not sufficient to improve our result to the scaling-invariant level.

Finally, we also note that, in order to reduce our two-component regularity criteria for the four-dimensional NSE to one component or to extend it to higher dimension such as five, it seems to require a completely new approach.

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