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UNIQUENESS OF WEAK SOLUTIONS OF THE FULL COUPLED NAVIER–STOKES AND Q-TENSOR SYSTEM IN 2D*

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Abstract. This paper is devoted to the full system of incompressible liquid crystals, as modeled in the Q-tensor framework. The main purpose is to establish the uniqueness of weak solutions in a two-dimensional setting, without imposing an extra regularity on the solutions themselves. This result only requires the initial data to fulfill the features which allow the existence of a weak solution. Thus, we also revisit the global existence result in dimensions two and three.

Key words. Nematic liquid crystal fluids, Navier–Stokes equations, global wellposedness.

AMS subject classifications. 35Q30, 76A05, 76A15.

1. Introduction

The main aim of this article is to prove the uniqueness of weak solutions for a type of coupled Navier–Stokes and Q-tensor systems proposed in [5] and studied numerically and analytically in [1,14,16–18,36]. This type of system models nematic liquid crystals and provides in a certain sense an extension of the classical Ericksen–Leslie model [14], whose uniqueness of weak solutions was proved in [38]. In the remainder of this introduction we will briefly present the equations and state our main result.

The system models the evolution of liquid crystal molecules together with the underlying flow, through a parabolic-type system coupling an incompressible Navier–Stokes system with a nonlinear convection-diffusion system. The local orientation of the molecules is described through a function Q taking values from $\mathbb{R}_+\times\Omega\subset\mathbb{R}_+\times\mathbb{R}^d, d=2,3$ into the set of so-called d-dimensional Q-tensors, that is

$$S_0^{(d)} \stackrel{\text{def}}{=} \big\{Q\in\mathbb{M}^{d\times d}; Q_{ij}=Q_{ji}, \operatorname{tr}(Q)=0, i,j=1,\dots,d\big\}$$

(the most relevant physical situations being $d=2,3$). The evolution of the Qs is driven by a gradient flow of the free energy of the molecules as well as the transport, distortion and alignment effects caused by the flow. The flow field $u:\mathbb{R}_+\times\Omega\to\mathbb{R}^d$ satisfies a forced incompressible Navier–Stokes system, with the forcing provided by the additional, non-Newtonian stress caused by the molecules orientations, thus expressed in terms of Q. We restrict ourselves to the case $\Omega=\mathbb{R}^d$ and work with non-dimensional quantities. The evolution of Q is given by

$$\partial_t Q + u\cdot\nabla Q - S(\nabla u, Q) = -\Gamma\frac{\partial\mathcal{F}_e}{\partial Q} \tag{1.1}$$

with $\Gamma>0$. Here

$$\mathcal{F}_e(Q)=\int_{\mathbb{R}^d}\frac{L}{2}|\nabla Q|^2+\Big(\frac{a}{2}\operatorname{tr}(Q^2)-\frac{b}{3}\operatorname{tr}(Q^3)+\frac{c}{4}\operatorname{tr}^2(Q^2)\Big)\,dx \tag{1.2}$$

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is the free energy of the liquid crystal molecules and $\frac{\partial \mathcal{F}_e}{\partial Q}$ denotes the variational derivative. The L,a,b,c constants are specific to the material with

$$L>0 \text{ and } a,b,c\in\mathbb{R}, c>0. \tag{1.3}$$

If $u=0$ the Q-tensor equation would simply be a gradient flow of the free energy. For $u\neq 0$ the molecules are transported by the flow (as indicated by the convective derivative $\partial_t + u\cdot\nabla$) as well as being tumbled and aligned by the flow, a fact described by the term

$$S(\nabla u,Q)\stackrel{\text{def}}{=}(\xi D+\Omega)(Q+\frac{1}{d}\mathrm{Id})+(Q+\frac{1}{d}\mathrm{Id})(\xi D-\Omega)-2\xi(Q+\frac{1}{d}\mathrm{Id})\mathrm{tr}(Q\nabla u) \tag{1.4}$$

where $D\stackrel{def}{=}\frac{1}{2}\left(\nabla u+(\nabla u)^T\right)$ and $\Omega\stackrel{def}{=}\frac{1}{2}\left(\nabla u-(\nabla u)^T\right)$ are, respectively, the symmetric part and the antisymmetric part of the velocity gradient matrix ∇u. The constant ξ is specific to the liquid crystal material.

The flow satisfies the forced Navier–Stokes system

$$\begin{aligned}\partial_t u+(u\cdot\nabla)u&=\nu\Delta u+\nabla p+\lambda\nabla\cdot(\tau+\sigma)\\ \nabla\cdot u&=0\end{aligned}$$

where $\nu,\lambda>0$ with λ measuring the ratio of the elastic effects (produced by the liquid crystal molecules) to that of the diffusive effects. The forcing is provided by the additional stress caused by the presence of the liquid crystal molecules. More specifically we have the symmetric part of the additional stress tensor

$$\tau\stackrel{\text{def}}{=}\Big[-\xi\Big(Q+\frac{1}{d}\mathrm{Id}\Big)H-\xi H\big(Q+\frac{1}{d}\mathrm{Id}\big)+2\xi\big(Q+\frac{1}{d}\mathrm{Id}\big)QH-L\nabla Q\odot\nabla Q\Big] \tag{1.5}$$

and the antisymmetric part

$$\sigma\stackrel{\text{def}}{=}QH-HQ, \tag{1.6}$$

where we denoted

$$H\stackrel{\text{def}}{=}-\frac{\partial\mathcal{F}_e}{\partial Q}=L\Delta Q-aQ+b[Q^2-\frac{\mathrm{tr}(Q^2)}{d}\mathrm{Id}]-cQ\mathrm{tr}(Q^2). \tag{1.7}$$

Summarising, we have the following coupled system:

$$\begin{aligned}&(\partial_t+u\cdot\nabla)Q-S(\nabla u,Q)\\ =&\Gamma(L\Delta Q-aQ+b[Q^2-\frac{\mathrm{tr}(Q^2)}{d}\mathrm{Id}]-cQ\mathrm{tr}(Q^2))\partial_t u+(u\cdot\nabla)u\\ =&\nu\Delta u+\nabla p+\lambda\nabla\cdot(QH-HQ)\\ &+\lambda\nabla\cdot\Big[-\xi\Big(Q+\frac{1}{d}\mathrm{Id}\Big)H-\xi H\big(Q+\frac{1}{d}\mathrm{Id}\big)+2\xi\big(Q+\frac{1}{d}\mathrm{Id}\big)QH-L\nabla Q\odot\nabla Q\Big]\\ &\nabla\cdot u=0,\end{aligned} \tag{1.8}$$

where $\Gamma,L,\nu,c>0$ and $a,b\in\mathbb{R}$. Let us observe that this is a slight extension of the system considered in [36], where $\lambda=1$. However, this does not create any major difficulties compared to equations in [36], but it is more relevant from a physical point of view.

The main result of the paper is the uniqueness of weak solutions, which are defined in a rather standard manner.

Definition 1.1. *A pair (Q,u) is called a weak solution of the system (1.8), subject to initial data*

$$Q(0,x)=\bar{Q}(x)\in H^1(\mathbb{R}^d;S_0^{(d)}),\, u(0,x)=\bar{u}(x)\in L^2(\mathbb{R}^d;\mathbb{R}^d), \nabla\cdot\bar{u}=0 \text{ in } \mathcal{D}'(\mathbb{R}^d) \tag{1.9}$$

if $Q\in L^\infty_{loc}(\mathbb{R}_+;H^1)\cap L^2_{loc}(\mathbb{R}_+;H^2)$, $u\in L^\infty_{loc}(\mathbb{R}_+;L^2)\cap L^2_{loc}(\mathbb{R}_+;H^1)$ *and if for every compactly supported* $\varphi\in C^\infty([0,\infty)\times\mathbb{R}^d;S_0^{(d)})$, $\psi\in C^\infty([0,\infty)\times\mathbb{R}^d;\mathbb{R}^d)$ *with* $\nabla\cdot\psi=0$ *we have*

$$\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}^d}(-Q\cdot\partial_t\varphi-\Gamma L\Delta Q\cdot\varphi)-Q\cdot u\nabla_x\varphi dx\,dt\\
&\quad-\int_0^\infty\int_{\mathbb{R}^d}(\xi D+\Omega)(Q+\frac{1}{d}\mathrm{Id})\cdot\varphi+(Q+\frac{1}{d}\mathrm{Id})(\xi D-\Omega)\cdot\varphi dxdt\\
&\quad-2\xi\int_0^\infty\int_{\mathbb{R}^d}(Q+\frac{1}{d}\mathrm{Id})\,tr(Q\nabla u)\cdot\varphi\,dx\,dt\\
&=\int_{\mathbb{R}^d}\bar{Q}(x)\cdot\varphi(0,x)\,dx+\Gamma\int_0^\infty\int_{\mathbb{R}^d}\Big\{-aQ+b[Q^2-\frac{tr(Q^2)}{d}\mathrm{Id}]-cQ\,tr(Q^2)\Big\}\cdot\varphi\,dx\,dt
\end{aligned} \tag{1.10}$$

and

$$\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}^d}-u\partial_t\psi-u_\alpha u_\beta\partial_\alpha\psi_\beta+\nu\nabla u\nabla\psi\,dt\,dx-\int_{\mathbb{R}^d}\bar{u}(x)\psi(0,x)\,dx\\
&=L\lambda\int_0^\infty\int_{\mathbb{R}^d}Q_{\gamma\delta,\alpha}Q_{\gamma\delta,\beta}\psi_{\alpha,\beta}-Q_{\alpha\gamma}\Delta Q_{\gamma\beta}\psi_{\alpha,\beta}+\Delta Q_{\alpha\gamma}Q_{\gamma\beta}\psi_{\alpha,\beta}\,dx\,dt\\
&\quad+\xi\lambda\int_0^\infty\int_{\mathbb{R}^d}\left(Q_{\alpha\gamma}+\frac{\delta_{\alpha\gamma}}{d}\right)H_{\gamma\beta}\psi_{\alpha,\beta}+H_{\alpha\gamma}\left(Q_{\gamma\beta}+\frac{\delta_{\gamma\beta}}{d}\right)\psi_{\alpha,\beta}\,dx\,dt\\
&\quad-2\xi\lambda\int_0^\infty\int_{\mathbb{R}^d}(Q_{\alpha\beta}+\frac{\delta_{\alpha\beta}}{d})Q_{\gamma\delta}H_{\gamma\delta}\psi_{\alpha,\beta}\,dx\,dt.
\end{aligned} \tag{1.11}$$

We can now state our main result, which is the existence and uniqueness of weak solution

Theorem 1.1. *Let $d=2,3$ and take*

$$Q(0,x)=\bar{Q}(x)\in H^1(\mathbb{R}^d;S_0^{(d)}),\, u(0,x)=\bar{u}(x)\in L^2(\mathbb{R}^d), \nabla\cdot\bar{u}=0 \text{ in } \mathcal{D}'(\mathbb{R}^d).$$

Then the system (1.8) *admits a global weak solution. Moreover, if $d=2$, then uniqueness holds.*

Remark 1.1. With minor modifications to the proof, that are left to the interested reader, the result also holds when the system is $d=2$ in the domain but $d=3$ in the target, which physically corresponds to a situation where there is no dependence in one of the three spatial directions.

The main part of the theorem is about uniqueness, as the existence part is just a fairly straightforward revisiting of the arguments in [36]. The main difficulties associated with treating the system (1.8) are related to the presence of the Navier–Stokes part. One can essentially think of the system as a highly non-trivial perturbation of a Navier–Stokes system. It is known that for Navier–Stokes alone the uniqueness of weak solutions in 2D can be achieved through rather standard arguments, while in 3D it is a major open problem.

The extended system that we deal with has an intermediary position, as the perturbation produced by the presence of the additional stress-tensor generates significant technical difficulties related in the first place to the weak norms available for the u term. A rather common way of dealing with this issue is by using a weak norm for estimating the difference between the two weak solutions, a norm that is below the natural spaces in which the weak solutions are defined. This approach was used before in the context of the related Leslie–Ericksen model [20] as well as for the usual Navier–Stokes system in [15] and [29].

In our case, for technical convenience, we use a homogeneous Sobolev space, namely $\dot{H}^{-\frac{1}{2}}$. The fact that the initial data for the difference is zero (i.e. $(\delta u, \delta Q)_{t=0} = 0$) helps in controlling the difference in such a low regularity space. However, one of the main reasons for choosing the homogeneous setting is a specific product law (see Theorem A.1 in Appendix A). The mentioned theorem shows that the product is a bounded operator from $\dot{H}^s(\mathbb{R}^2) \times \dot{H}^t(\mathbb{R}^2)$ into $\dot{H}^{s+t-1}(\mathbb{R}^2)$ for any $|s|, |t| \leq 1$, such that $s+t$ is positive. We note that evaluating the difference at regularity level $s=0$ (i.e. in L^2), would only allow to prove a weak-strong uniqueness result, along the lines of [35]. Working in a negative Sobolev space, $\dot{H}^s$ with $s \in (-1,0)$, allows us to capture the uniqueness of weak solutions. We expect that a similar proof would work in any $\dot{H}^s$ with $s \in (-1,0)$. Our choice of $s = -\frac{1}{2}$ is just for convenience.

Our main work is to obtain the delicate double-logarithmic type estimates that lead to an Osgood lemma, a generalization of Gronwall's inequality (see [2, Lemma 3.4]). Indeed the uniqueness reduces to an estimate of the following type:

$$\Phi'(t) \leq \chi(t) \Big\{ \Phi(t) + \Phi(t) \ln\Big(1+e+\frac{1}{\Phi(t)}\Big) + \Phi(t) \ln\Big(1+e+\frac{1}{\Phi(t)}\Big) \ln\ln\Big(1+e+\frac{1}{\Phi(t)}\Big) \Big\},$$

where $\Phi(t)$ is a norm of the difference between two solutions and χ is a priori in L^1_{loc}.

In addition to these, there are some difficulties that are specific to this system. These are of two different types, being related to the following:

- controlling the "extraneous" maximal derivatives: that is the highest derivatives in u that appear in the Q equation and the highest derivatives in Q that appear in the u equation, and
- controlling the high powers of Q , such as $Q\mathrm{tr}(Q^2)$ in particular those that interact with u terms (such as $Q\mathrm{tr}(Q\nabla u)$).

The first difficulty is dealt with by taking into account the specific feature of the coupling that allows for the *cancellation of the worst terms* when considering certain physically meaningful combinations of terms. This feature is explored in the next section where we revisit and revise the existence proof from [36]. In what concerns the second difficulty, this is overcome by delicate harmonic analysis arguments leading to the double-logarithmic estimates mentioned before.

The paper is organised as follows: In the next section we revisit the existence arguments done in cite [36], providing a slight adaptation to our case and a minor correction to one of the estimates used there. The main benefit of this section is that it exhibits in a simple setting a number of cancellations that are crucial later-on for the uniqueness argument. In the third section we start by introducing a number of technical harmonic analysis tools related to the Littlewood–Paley theory and then use them in the proof of our main result. Some standard but perhaps less-known tools, together with some more technical estimates, are postponed into the appendices.

Notations and conventions. Let $S_0^{(d)}\subset\mathbb{M}^{d\times d}$ denote the space of Q-tensors in dimension d, i.e.

$$S_0^{(d)}\stackrel{\text{def}}{=}\left\{Q\in\mathbb{M}^{d\times d};Q_{ij}=Q_{ji},\text{tr}(Q)=0,i,j=1,\dots,d\right\}.$$

We use the Einstein summation convention, that is we assume summation over repeated indices.

We define the Frobenius norm of a matrix $|Q|\stackrel{\text{def}}{=}\sqrt{\text{tr}Q^2}=\sqrt{Q_{\alpha\beta}Q_{\alpha\beta}}$ and define Sobolev spaces of Q-tensors in terms of this norm. For instance $H^1(\mathbb{R}^d,S_0^{(d)})\stackrel{\text{def}}{=}\{Q:\mathbb{R}^d\to S_0^{(d)},\int_{\mathbb{R}^d}|\nabla Q(x)|^2+|Q(x)|^2\,dx<\infty\}$ where $|\nabla Q|^2(x)\stackrel{\text{def}}{=}Q_{\alpha\beta,\gamma}(x)Q_{\alpha\beta,\gamma}(x)$ with $Q_{\alpha\beta,\gamma}\stackrel{\text{def}}{=}\partial_\gamma Q_{\alpha\beta}$. For $A,B\in S_0^{(d)}$ we denote $A:B=\text{tr}(AB)$, $|A|=\sqrt{\text{tr}(A^2)}$, and $\|(A,B)\|_X=\|A\|_X+\|B\|_X$, for any suitable Banach space X. We also denote $\Omega_{\alpha\beta}\stackrel{\text{def}}{=}\frac{1}{2}(\partial_\beta u_\alpha-\partial_\alpha u_\beta),u_{\alpha,\beta}\stackrel{\text{def}}{=}\partial_\beta u_\alpha$ and $(\nabla Q\odot\nabla Q)_{ij}=Q_{\alpha\beta,i}Q_{\alpha\beta,j}$.

2. The energy decay, a priori estimates, and scaling

In the absence of the flow, when $u=0$ in the system (1.8), the free energy is a Lyapunov functional of the system. If $u\neq 0$ we still have a Lyapunov functional for the system (1.8) but this time one that includes the kinetic energy of the system. These estimates provide as usual the basis for obtaining a priori estimates for the system. The propositions in this section show this and their proofs follow closely the ones of the similar propositions in [36], where they were done for the case $\lambda=1$. The reason for including them is to display in a relatively simple setting the cancellations that will appear again in the proof of the uniqueness theorem but in a much more complicated framework.

PROPOSITION 2.1. *The system* (1.8) *has a Lyapunov functional:*

$$\begin{aligned}E(t)\stackrel{\text{def}}{=}&\frac{1}{2}\int_{\mathbb{R}^d}|u|^2(t,x)\mathrm{d}x+\int_{\mathbb{R}^d}\frac{L\lambda}{2}|\nabla Q|^2(t,x)\mathrm{d}x\\&+\lambda\int_{\mathbb{R}^d}(\frac{a}{2}tr(Q^2(t,x))-\frac{b}{3}tr(Q^3(t,x))+\frac{c}{4}tr^2(Q^2(t,x)))\mathrm{d}x.\end{aligned}\tag{2.1}$$

If $d=2,3$ and (Q,u) is a smooth solution of (1.8) such that $Q\in L^\infty(0,T;H^1(\mathbb{R}^d))\cap L^2(0,T;H^2(\mathbb{R}^d))$ and $u\in L^\infty(0,T;L^2(\mathbb{R}^d))\cap L^2(0,T;H^1(\mathbb{R}^d))$, then, for all $t<T$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(t)=&-\nu\int_{\mathbb{R}^d}|\nabla u|^2\,dx\\&-\Gamma\lambda\int_{\mathbb{R}^d}\text{tr}\left(L\Delta Q-aQ+b[Q^2-\frac{tr(Q^2)}{d}\text{Id}]-cQtr(Q^2)\right)^2\mathrm{d}x\le 0.\end{aligned}\tag{2.2}$$

Proof. We multiply the first equation in the system (1.8) on the right by $-\lambda H$, take the trace, integrate over $\mathbb{R}^d$ by parts and sum with the second equation multiplied by u and integrated over $\mathbb{R}^d$ by parts (let us observe that, because of our assumptions on Q and u, we do not have boundary terms, when integrating by parts). We obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\frac{1}{2}|u|^2+\frac{L\lambda}{2}|\nabla Q|^2+\lambda(\frac{a}{2}\text{tr}(Q^2)-\frac{b}{3}\text{tr}(Q^3)+\frac{c}{4}\text{tr}^2(Q^2))\mathrm{d}x\\&+\nu\int_{\mathbb{R}^d}|\nabla u|^2\mathrm{d}x+\Gamma\lambda\int_{\mathbb{R}^d}\text{tr}\left(L\Delta QL-aQ+b[Q^2-\frac{\text{tr}(Q^2)}{d}\text{Id}]-cQ\text{tr}(Q^2)\right)^2\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
=&\underbrace{\lambda\int_{\mathbb{R}^d} u\cdot\nabla Q_{\alpha\beta}\left(-aQ_{\alpha\beta}+b[Q_{\alpha\gamma}Q_{\gamma\beta}-\frac{\delta_{\alpha\beta}}{d}\mathrm{tr}(Q^2)]-cQ_{\alpha\beta}\mathrm{tr}(Q^2))\right)\mathrm{d}x}_{\stackrel{\mathrm{def}}{=}\mathcal{I}}\\
&+\underbrace{\lambda\int_{\mathbb{R}^d}(-\Omega_{\alpha\gamma}Q_{\gamma\beta}+Q_{\alpha\gamma}\Omega_{\gamma\beta})\left(-aQ_{\alpha\beta}+b[Q_{\alpha\delta}Q_{\delta\beta}-\frac{\delta_{\alpha\beta}}{d}\mathrm{tr}(Q^2)]-cQ_{\alpha\beta}\mathrm{tr}(Q^2))\right)\mathrm{d}x}_{\stackrel{\mathrm{def}}{=}\mathcal{II}}\\
&-\lambda\xi\underbrace{\int_{\mathbb{R}^d}(Q_{\alpha\gamma}+\frac{\delta_{\alpha\gamma}}{d})D_{\gamma\beta}H_{\alpha\beta}\mathrm{d}x}_{\stackrel{\mathrm{def}}{=}\mathcal{J}_1}-\lambda\xi\underbrace{\int_{\mathbb{R}^d}D_{\alpha\gamma}(Q_{\gamma\beta}+\frac{\delta_{\gamma\beta}}{d})H_{\alpha\beta}\mathrm{d}x}_{\stackrel{\mathrm{def}}{=}\mathcal{J}_2}\\
&+2\lambda\xi\underbrace{\int_{\mathbb{R}^d}(Q_{\alpha\beta}+\frac{\delta_{\alpha\beta}}{d})H_{\alpha\beta}\mathrm{tr}(Q\nabla u)\mathrm{d}x}_{\stackrel{\mathrm{def}}{=}\mathcal{J}_3}+L\lambda\underbrace{\int_{\mathbb{R}^d}u_\gamma Q_{\alpha\beta,\gamma}\Delta Q_{\alpha\beta}\mathrm{d}x}_{\stackrel{\mathrm{def}}{=}\mathcal{A}}\\
&-\frac{L\lambda}{2}\underbrace{\int_{\mathbb{R}^d}u_{\alpha,\gamma}Q_{\gamma\beta}\Delta Q_{\alpha\beta}\mathrm{d}x}_{\stackrel{\mathrm{def}}{=}\mathcal{B}}+\frac{L\lambda}{2}\underbrace{\int_{\mathbb{R}^d}u_{\gamma,\alpha}Q_{\gamma\beta}\Delta Q_{\alpha\beta}\mathrm{d}x}_{\stackrel{\mathrm{def}}{=}\mathcal{C}}\\
&+\frac{L\lambda}{2}\underbrace{\int_{\mathbb{R}^d}Q_{\alpha\gamma}u_{\gamma,\beta}\Delta Q_{\alpha\beta}\mathrm{d}x}_{\mathcal{C}}-\frac{L\lambda}{2}\underbrace{\int_{\mathbb{R}^d}Q_{\alpha\gamma}u_{\beta,\gamma}\Delta Q_{\alpha\beta}\mathrm{d}x}_{\mathcal{B}}\\
&+L\lambda\underbrace{\int_{\mathbb{R}^d}Q_{\gamma\delta,\alpha}Q_{\gamma\delta,\beta}u_{\alpha,\beta}\mathrm{d}x}_{\stackrel{\mathrm{def}}{=}\mathcal{AA}}-L\lambda\underbrace{\int_{\mathbb{R}^d}Q_{\alpha\gamma}\Delta Q_{\gamma\beta}u_{\alpha,\beta}\mathrm{d}x}_{\stackrel{\mathrm{def}}{=}\mathcal{CC}}\\
&+L\lambda\underbrace{\int_{\mathbb{R}^d}\Delta Q_{\alpha\gamma}Q_{\gamma\beta}u_{\alpha,\beta}\mathrm{d}x}_{\stackrel{\mathrm{def}}{=}\mathcal{BB}}+\lambda\xi\underbrace{\int_{\mathbb{R}^d}(Q_{\alpha\gamma}+\frac{\delta_{\alpha\gamma}}{d})H_{\gamma\beta}u_{\alpha,\beta}\mathrm{d}x}_{\stackrel{\mathrm{def}}{=}\mathcal{JJ}_1}\\
&+\lambda\xi\underbrace{\int_{\mathbb{R}^d}H_{\alpha\gamma}(Q_{\gamma\beta}+\frac{\delta_{\gamma\beta}}{d})u_{\alpha,\beta}\mathrm{d}x}_{\stackrel{\mathrm{def}}{=}\mathcal{JJ}_2}-2\lambda\xi\underbrace{\int_{\mathbb{R}^d}(Q_{\alpha\beta}+\frac{\delta_{\alpha\beta}}{d})u_{\alpha,\beta}\mathrm{tr}(QH)\mathrm{d}x}_{\stackrel{\mathrm{def}}{=}\mathcal{JJ}_3}\\
=&-L\lambda\underbrace{\int_{\mathbb{R}^d}u_{\alpha,\gamma}Q_{\gamma\beta}\Delta Q_{\alpha\beta}\mathrm{d}x}_{2\mathcal{B}}+L\lambda\underbrace{\int_{\mathbb{R}^d}u_{\gamma,\alpha}Q_{\gamma\beta}\Delta Q_{\alpha\beta}\mathrm{d}x}_{2\mathcal{C}}\\
&-L\lambda\underbrace{\int_{\mathbb{R}^d}Q_{\alpha\gamma}\Delta Q_{\gamma\beta}u_{\alpha,\beta}\mathrm{d}x}_{\mathcal{CC}}+L\lambda\underbrace{\int_{\mathbb{R}^d}\Delta Q_{\alpha\gamma}Q_{\gamma\beta}u_{\alpha,\beta}\mathrm{d}x}_{\mathcal{BB}}=0, \qquad (2.3)
\end{aligned}
$$

where $\mathcal{I}=0$ (since $\nabla\cdot u=0$), $\mathcal{II}=0$ (since $Q_{\alpha\beta}=Q_{\beta\alpha}$), and for the second equality we used

$$
\begin{aligned}
&\underbrace{\int_{\mathbb{R}^d}u_\gamma Q_{\alpha\beta,\gamma}\Delta Q_{\alpha\beta}\mathrm{d}x}_{\mathcal{A}}+\underbrace{\int_{\mathbb{R}^d}Q_{\gamma\delta,\alpha}Q_{\gamma\delta,\beta}u_{\alpha,\beta}\mathrm{d}x}_{\mathcal{AA}}\\
&=\int_{\mathbb{R}^d}u_\gamma Q_{\alpha\beta,\gamma}\Delta Q_{\alpha\beta}\mathrm{d}x-\int_{\mathbb{R}^d}Q_{\gamma\delta,\alpha}Q_{\gamma\delta,\beta\beta}u_\alpha\mathrm{d}x-\int_{\mathbb{R}^d}Q_{\gamma\delta,\alpha\beta}Q_{\gamma\delta,\beta}u_\alpha\mathrm{d}x\\
&=\int_{\mathbb{R}^d}\frac{1}{2}Q_{\gamma\delta,\beta}Q_{\gamma\delta,\beta}u_{\alpha,\alpha}\mathrm{d}x=0
\end{aligned}
$$

together with $Q_{\alpha\alpha}=H_{\alpha\alpha}=u_{\alpha,\alpha}=0$, $\mathcal{J}_3=\mathcal{J}\mathcal{J}_3$ and

$$\begin{aligned}
&\mathcal{J}_1+\mathcal{J}_2\\
&=\int_{\mathbb{R}^d}\frac{1}{2}Q_{\alpha\gamma}u_{\gamma,\beta}H_{\alpha\beta}\\
&\quad+\frac{1}{2}Q_{\alpha\gamma}u_{\beta,\gamma}H_{\alpha\beta}\mathrm{d}x+\int_{\mathbb{R}^d}\frac{1}{2}u_{\alpha,\gamma}Q_{\gamma\beta}H_{\alpha\beta}+\frac{1}{2}u_{\gamma,\alpha}Q_{\gamma\beta}H_{\alpha\beta}\mathrm{d}x+\frac{2}{d}\int_{\mathbb{R}^d}D_{\alpha\beta}H_{\alpha\beta}\mathrm{d}x\\
&=\int_{\mathbb{R}^d}\frac{1}{2}\big(Q_{\alpha\gamma}u_{\gamma,\beta}H_{\alpha\beta}+u_{\gamma,\alpha}Q_{\gamma\beta}H_{\alpha\beta}\big)+\frac{1}{2}\big(Q_{\alpha\gamma}u_{\beta,\gamma}H_{\alpha\beta}+u_{\alpha,\gamma}Q_{\gamma\beta}H_{\alpha\beta}\big)\mathrm{d}x\\
&\quad+\frac{1}{d}\int_{\mathbb{R}^d}(u_{\alpha,\beta}+u_{\beta,\alpha})H_{\alpha\beta}\mathrm{d}x\\
&=\int_{\mathbb{R}^d}H_{\beta\alpha}Q_{\alpha\gamma}u_{\gamma,\beta}+Q_{\gamma\alpha}H_{\alpha\beta}u_{\beta,\gamma}\mathrm{d}x+\frac{2}{d}\int_{\mathbb{R}^d}u_{\alpha,\beta}H_{\alpha\beta}\mathrm{d}x=\mathcal{J}\mathcal{J}_1+\mathcal{J}\mathcal{J}_2.
\end{aligned}$$

Finally, the last equality in the computation (2.3) is a consequence of the straightforward identities $2\mathcal{B}+\mathcal{B}\mathcal{B}=2\mathcal{C}+\mathcal{C}\mathcal{C}=0$. ∎

It can be easily checked that the system has a scaling; namely we have the following lemma.

LEMMA 2.1. *Let (Q,u,p) be a solution of the system (1.8). Then letting*

$$u_\delta(t,x)\stackrel{\text{def}}{=}\delta u(\delta x,\delta^2 t),\quad Q_\delta(t,x)\stackrel{\text{def}}{=}Q(\delta x,\delta^2 t),\quad p_\delta(t,x)\stackrel{\text{def}}{=}\delta^2 p(\delta x,\delta^2) \tag{2.4}$$

we have that $(Q_\delta,u_\delta,p_\delta)$ satisfy the system (1.8) with $F(Q)=-aQ+b[Q^2-\frac{tr(Q^2)}{d}\mathrm{Id}]-cQtr(Q^2)$ replaced by $F_\delta(Q_\delta)=\delta^2\left[-aQ_\delta+b[(Q_\delta)^2-\frac{tr(Q_\delta^2)}{d}-cQ_\delta tr(Q_\delta)^2\right]$. We note that, in dimension two, the space $\dot H^1(\mathbb{R}^2)\times L^2(\mathbb{R}^2)$ is invariant by the scaling.

In the following we assume that there exists a smooth solution of the system (1.8) and obtain estimates on the behaviour of various norms.

PROPOSITION 2.2. *Let (Q,u) be a smooth solution of the system* (1.8) *in dimension $d=2$ or $d=3$, with restriction* (1.3), *and smooth initial data $(\bar Q(x),\bar u(x))$, which decays fast enough at infinity so that we can integrate by parts in space (for any $t\ge 0$) without boundary terms. We assume that $|\xi|<\xi_0$, where ξ_0 is an explicitly computable constant, scale invariant, depending on $a,b,c,d,\Gamma,\nu,\lambda$.*

For $(\bar Q,\bar u)\in H^1\times L^2_x$, we have

$$\|Q(t,\cdot)\|_{H^1}\le C_1+\bar C_1 e^{\bar C_1 t}\|\bar Q\|_{H^1},\forall t\ge 0 \tag{2.5}$$

with $C_1,\bar C_1$ depending on $(a,b,c,d,\Gamma,L,\nu,\bar Q,\bar u)$. Moreover

$$\|u(t,\cdot)\|^2_{L^2_x}+\nu\int_0^t\|\nabla u\|^2_{L^2_x}\le C_1. \tag{2.6}$$

Proof. We denote

$$X_{\alpha\beta}\stackrel{\text{def}}{=}L\Delta Q_{\alpha\beta}-cQ_{\alpha\beta}\mathrm{tr}(Q^2),\ \alpha,\beta=1,2,3.$$

Then Equation (2.2) becomes

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}E(t)+\nu\|\nabla u\|_{L_x^2}^2+\Gamma\lambda L^2\|\Delta Q\|_{L_x^2}^2+\Gamma\lambda c^2\|Q\|_{L^6}^6\\
&\quad -2cL\Gamma\lambda\int_{\mathbb{R}^d}\Delta Q_{\alpha\beta}Q_{\alpha\beta}\mathrm{tr}(Q^2)\mathrm{d}x+a^2\Gamma\lambda\|Q\|_{L_x^2}^2+b^2\Gamma\lambda\int_{\mathbb{R}^d}\mathrm{tr}\left(Q^2-\frac{\mathrm{tr}(Q^2)}{d}\right)^2\mathrm{d}x\\
&\le 2a\Gamma\lambda\underbrace{\int_{\mathbb{R}^d}\mathrm{tr}(XQ)\mathrm{d}x}_{\stackrel{\mathrm{def}}{=}\mathcal{I}}-2b\Gamma\lambda\underbrace{\int_{\mathbb{R}^d}\mathrm{tr}(XQ^2)\mathrm{d}x}_{\stackrel{\mathrm{def}}{=}\mathcal{J}}+2ab\Gamma\lambda\int_{\mathbb{R}^d}\mathrm{tr}(Q^3)\mathrm{d}x.
\end{aligned}\tag{2.7}$$

Integrating by parts, we have

$$\begin{aligned}
&-2cL\Gamma\lambda\int_{\mathbb{R}^d}\Delta Q_{\alpha\beta}Q_{\alpha\beta}\mathrm{tr}(Q^2)\mathrm{d}x\\
&=2cL\Gamma\lambda\int_{\mathbb{R}^d}Q_{\alpha\beta,k}Q_{\alpha\beta,k}\mathrm{tr}(Q^2)\mathrm{d}x+2cL\Gamma\lambda\int_{\mathbb{R}^d}Q_{\alpha\beta,k}Q_{\alpha\beta}\partial_k\left(\mathrm{tr}(Q^2)\right)\mathrm{d}x\\
&=2cL\Gamma\lambda\int_{\mathbb{R}^d}|\nabla Q|^2\mathrm{tr}(Q^2)\mathrm{d}x+cL\Gamma\lambda\int_{\mathbb{R}^d}|\nabla\left(\mathrm{tr}(Q^2)\right)|^2\mathrm{d}x\ge 0
\end{aligned}\tag{2.8}$$

(where for the last inequality we used the assumption (1.3) and $L,\Gamma,\lambda>0$). One can easily see that

$$\mathcal{I}=-\frac{L}{2}\|\nabla Q\|_{L_x^2}^2-c\|Q\|_{L^4}^4.\tag{2.9}$$

On the other hand, for any $\varepsilon>0$ and $\tilde{C}=\tilde{C}(\varepsilon,c)$ an explicitly computable constant, we have

$$\begin{aligned}
\mathcal{J}&=L\int_{\mathbb{R}^d}Q_{\alpha\beta,kk}Q_{\alpha\gamma}Q_{\gamma\beta}\mathrm{d}x-c\int_{\mathbb{R}^d}\mathrm{tr}(Q^2)\mathrm{tr}(Q^3)\mathrm{d}x\\
&\le -L\int_{\mathbb{R}^d}Q_{\alpha\beta,k}Q_{\alpha\gamma,k}Q_{\gamma\beta}\mathrm{d}x-L\int_{\mathbb{R}^d}Q_{\alpha\beta,k}Q_{\alpha\gamma}Q_{\gamma\beta,k}\mathrm{d}x\\
&\qquad\qquad +\int_{\mathbb{R}^d}\mathrm{tr}(Q^2)\left(\frac{\tilde{C}}{\varepsilon}\mathrm{tr}(Q^2)+\varepsilon\mathrm{tr}^2(Q^2)\right)\mathrm{d}x\\
&\le L\varepsilon\int_{\mathbb{R}^d}|\nabla Q|^2\mathrm{tr}(Q^2)\mathrm{d}x+\frac{\tilde{C}}{\varepsilon}\|\nabla Q\|_{L_x^2}^2+\int_{\mathbb{R}^d}\mathrm{tr}(Q^2)\left(\frac{\tilde{C}}{\varepsilon}\mathrm{tr}(Q^2)+\varepsilon\mathrm{tr}^2(Q^2)\right)\mathrm{d}x.
\end{aligned}$$

Using the last three relations in Equation (2.7), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}E(t)+\nu\|\nabla u\|_{L_x^2}^2+\Gamma\lambda L^2\|\Delta Q\|_{L_x^2}^2+c^2\Gamma\lambda\|Q\|_{L^6}^6+a^2\Gamma\lambda\|Q\|_{L_x^2}^2\\
&\quad +2cL\Gamma\lambda\int_{\mathbb{R}^d}|\nabla Q|^2\mathrm{tr}(Q^2)\mathrm{d}x+cL\Gamma\lambda\int_{\mathbb{R}^d}|\nabla\left(\mathrm{tr}(Q^2)\right)|^2\mathrm{d}x\\
&\le 2|a|\Gamma\lambda(\frac{L}{2}\|\nabla Q\|_{L_x^2}^2+c\|Q\|_{L^4}^4)+2|b|\Gamma\lambda L\varepsilon\int_{\mathbb{R}^d}|\nabla Q|^2\mathrm{tr}(Q^2)\mathrm{d}x\\
&\quad +2|b|\Gamma\lambda\frac{\tilde{C}}{\varepsilon}\|\nabla Q\|_{L_x^2}^2+2|b|\Gamma\lambda\int_{\mathbb{R}^d}\mathrm{tr}(Q^2)\left(\frac{\tilde{C}}{\varepsilon}\mathrm{tr}(Q^2)+\varepsilon\mathrm{tr}^2(Q^2)\right)\mathrm{d}x\\
&\quad +2|ab|\Gamma\lambda(\varepsilon\|Q\|_{L_x^2}^2+\frac{\tilde{C}}{\varepsilon}\|Q\|_{L^4}^4).
\end{aligned}$$

Taking ε small enough, we can absorb all the terms with an epsilon coefficient on the right into the left-hand side, and we are left with

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}E(t)+\nu\|\nabla u\|_{L^2_x}^2+\Gamma\lambda L^2\|\Delta Q\|_{L^2_x}^2+\Gamma\lambda c^2\|Q\|_{L^6}^6\\&\quad+\Gamma\lambda a^2\|Q\|_{L^2_x}^2+2cL\Gamma\lambda\int_{\mathbb{R}^d}|\nabla Q|^2\mathrm{tr}(Q^2)\mathrm{d}x+cL\Gamma\lambda\int_{\mathbb{R}^d}|\nabla\left(\mathrm{tr}(Q^2)\right)|^2\mathrm{d}x\\&\leq\bar{C}\left(\|\nabla Q\|_{L^2_x}^2+\|Q\|_{L^4}^4\right),\end{aligned}\tag{2.10}$$

with $\bar{C}=\bar{C}(a,b,c)$.

The last relation is not yet enough because the Q terms without derivatives in $E(t)$ are not summing to a positive number. However, let us note that, if $a>0$ we obtain the a priori estimates by using the inequality $\mathrm{tr}(Q^3)\leq\frac{3}{8}\mathrm{tr}(Q^2)+\mathrm{tr}(Q^2)^2$. If $a\leq 0$ we have to estimate separately $\|Q\|_{L^2_x}$ and this asks for a smallness condition for ξ.

We need to control in some sense low frequencies of Q. To this end, we multiply the first equation in the system (1.8) by Q, take the trace, integrate over $\mathbb{R}^d$ by parts, and we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}|Q|^2(t,x)\mathrm{d}x\\&=\Gamma\Big(-L\int_{\mathbb{R}^d}|\nabla Q|^2\mathrm{d}x-a\int_{\mathbb{R}^d}|Q(x)|^2\mathrm{d}x+b\int_{\mathbb{R}^d}\mathrm{tr}(Q^3)\mathrm{d}x-c\int_{\mathbb{R}^d}|Q|^4\mathrm{d}x\Big)+\underbrace{\int_{\mathbb{R}^d}\mathrm{tr}(\Omega Q^2-Q\Omega Q)\mathrm{d}x}_{\stackrel{\mathrm{def}}{=}\mathcal{I}}\\&\quad+\underbrace{\xi\int_{\mathbb{R}^d}D_{\alpha\gamma}(Q_{\gamma\beta}+\frac{\delta_{\gamma\beta}}{d})Q_{\alpha\beta}+(Q_{\alpha\gamma}+\frac{\delta_{\alpha\gamma}}{d})D_{\gamma\beta}Q_{\alpha\beta}-2(Q_{\alpha\beta}+\frac{\delta_{\alpha\beta}}{d})Q_{\alpha\beta}\mathrm{tr}(Q\nabla u)\mathrm{d}x}_{\stackrel{\mathrm{def}}{=}\mathcal{II}}.\end{aligned}$$

Recalling that Q is symmetric we have $\mathcal{I}=0$. Also:

$$\begin{aligned}|\mathcal{II}|=&|2\xi||\int_{\mathbb{R}^d}\frac{1}{d}D_{\alpha\beta}Q_{\alpha\beta}+D_{\alpha\gamma}Q_{\gamma\beta}Q_{\beta\alpha}-Q_{\alpha\beta}Q_{\alpha\beta}\mathrm{tr}(Q\nabla u)\mathrm{d}x|\\&\leq C(d)\int_{\mathbb{R}^d}\varepsilon|\nabla u|^2\mathrm{d}x+\int_{\mathbb{R}^d}\frac{|\xi|^2}{\varepsilon}(|Q|^2+|Q|^6)\mathrm{d}x.\end{aligned}$$

Thus, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}|Q|^2\mathrm{d}x\leq C(d)\varepsilon\int_{\mathbb{R}^d}|\nabla u|^2\mathrm{d}x+\frac{|\xi|^2}{\varepsilon}\int_{\mathbb{R}^d}|Q|^2+|Q|^6\mathrm{d}x+\hat{C}\int_{\mathbb{R}^d}|Q|^2+|Q|^4\mathrm{d}x\tag{2.11}$$

with $\hat{C}=\hat{C}(a,b)>0$.

Let us observe now that there exists $M=M(a,b,c)$ large enough, so that

$$\frac{M}{2}\mathrm{tr}(Q^2)+\frac{c}{8}\mathrm{tr}^2(Q^2)\leq(M+\frac{a}{2})\mathrm{tr}(Q^2)-\frac{b}{3}\mathrm{tr}(Q^3)+\frac{c}{4}\mathrm{tr}^2(Q^2)\tag{2.12}$$

for any $Q\in S_0$.

Multiplying Equation (2.11) by M and summing with Equation (2.10), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(E(t)+M\|Q\|_{L^2_x}^2)+\nu\|\nabla u\|_{L^2_x}^2+\Gamma\lambda L^2\|\Delta Q\|_{L^2_x}^2+\Gamma\lambda c^2\|Q\|_{L^6}^6+a^2\|Q\|_{L^2_x}^2\\&\quad+2cL\Gamma\lambda\int_{\mathbb{R}^d}|\nabla Q|^2\mathrm{tr}(Q^2)\mathrm{d}x+cL\Gamma\lambda\int_{\mathbb{R}^d}|\nabla\left(\mathrm{tr}(Q^2)\right)|^2\mathrm{d}x\end{aligned}$$

$$\leq \bar{C}\left(\|\nabla Q\|_{L_x^2}^2+\|Q\|_{L^4}^4\right)+MC(d)\varepsilon\int_{\mathbb{R}^d}|\nabla u|^2\mathrm{d}x$$
$$+\frac{M|\xi|^2}{\varepsilon}\int_{\mathbb{R}^d}|Q|^2+|Q|^6\mathrm{d}x+M\hat{C}\int_{\mathbb{R}^d}|Q|^2+|Q|^4\mathrm{d}x. \tag{2.13}$$

We chose ε small enough so that $MC(d)\varepsilon<\nu$. Finally we make the assumption that $|\xi|$ is small enough, depending on a,b,c,d,ν, so that

$$\frac{M|\xi|^2}{\varepsilon}\leq \Gamma\lambda c^2.$$

Then taking into account Equation (2.12), we obtain the claimed relation (2.5). □

We note that the ξ small hypothesis is necessary because we are in infinite domain, for example, in the periodic domain, we can add a constant to the functional and get the a priori L^p estimates without any smallness condition on ξ.

3. The existence of weak solutions

The next proposition follows closely the similar result in [36] where it was done for $\lambda=1$. The purpose for including it here is to provide an alternative approximation system thus correcting the proof in [36] and also to show how the cancellations that appeared previously in the derivation of the energy law still survive at the approximate level but with some differences, a phenomenon which will appear in a much more complex setting in the proof of uniqueness in the next section.

Proposition 3.1. *For $d=2,3$ there exists a weak solution (Q,u) of the system* (1.8) *subject to initial conditions* (1.9). *The solution (Q,u) is such that $Q\in L^\infty_{loc}(\mathbb{R}_+;H^1)\cap L^2_{loc}(\mathbb{R}_+;H^2)$ and $u\in L^\infty_{loc}(\mathbb{R}_+;L^2)\cap L^2_{loc}(\mathbb{R}_+;H^1)$.*

Proof. As the first step of the construction of weak solutions for the system (1.8), we construct for any fixed $\varepsilon>0$ a global weak solution

$$Q_\varepsilon\in L^\infty_{loc}(\mathbb{R}_+;H^1)\cap L^2_{loc}(\mathbb{R}_+,H^2),\quad u_\varepsilon\in L^\infty_{loc}(\mathbb{R}_+,L^2)\cap L^2_{loc}(\mathbb{R}_+,H^1)$$

for the modified system obtained by mollifying the coefficients of the equation for the Q tensor and by adding to the equation of the velocity a regularizing term. This term is needed in order to estimate some "bad" terms which do not disappear in an energy estimate. For the simplicity of the notations, we drop the indices ε and denote the solution $(Q_\varepsilon,u_\varepsilon)$ by (Q,u).

$$\begin{cases}\partial_t Q+(R_\varepsilon u)\nabla Q-\Big(\big(R_\varepsilon(\xi D+\Omega)\big)(Q+\frac{1}{d}\mathrm{Id})\Big)-\Big((Q+\frac{1}{d}\mathrm{Id})R_\varepsilon(\xi D-\Omega)\Big)\\ \quad +2\xi\Big((Q+\frac{1}{d}\mathrm{Id})\mathrm{tr}\big(Q\nabla R_\varepsilon u\big)\Big)=\Gamma H\\ \partial_t u+(R_\varepsilon u)\nabla u-\nu\Delta u+\nabla p=-\varepsilon\mathcal{P}R_\varepsilon\left(\sum_{l,m=1}^d\nabla Q_{lm}\left(R_\varepsilon u\cdot\nabla Q_{lm}\right)|R_\varepsilon u\nabla Q|\right)\\ \quad +\varepsilon\mathcal{P}\nabla\cdot R_\varepsilon\left(\nabla R_\varepsilon u|\nabla R_\varepsilon u|^2\right)-\lambda\xi\nabla\cdot R_\varepsilon\left(\left(Q+\frac{1}{d}\mathrm{Id}\right)H\right)-\xi\mathcal{P}\nabla\cdot R_\varepsilon\left(H\left(Q+\frac{1}{d}\mathrm{Id}\right)\right)\\ \quad +2\lambda\xi\nabla\cdot R_\varepsilon\left(\left(Q+\frac{1}{d}\mathrm{Id}\right)(QH)\right)-L\lambda R_\varepsilon(\nabla\cdot\mathrm{tr}(\nabla Q\nabla Q))\\ \quad +L\lambda\mathcal{P}\nabla\cdot R_\varepsilon\left(Q\Delta Q-\Delta QQ\right)\\ (Q,u)|_{t=0}=(R_\varepsilon\overline{Q},R_\varepsilon\overline{u}),\end{cases}$$

where R_ε is the convolution operator with the kernel $\epsilon^{-d}\chi(\epsilon^{-1}\cdot)$.

In order to construct the global weak solution for this system, we use the classical Friedrich's scheme. We define the mollifying operator

$$\widehat{J_n f}(\xi) \stackrel{\text{def}}{=} 1_{\{2^{-n}\leq|\xi|\leq 2^n\}}\hat{f}(\xi).$$

We consider the approximating system

$$\begin{cases} \partial_t Q^{(n)} + J_n\Big(R_\varepsilon J_n u^n \nabla J_n Q^{(n)}\Big) - J_n\Big((\xi J_n R_\varepsilon D^{(n)} + J_n R_\varepsilon \Omega^{(n)})(J_n Q^{(n)} + \frac{1}{d}\mathrm{Id})\Big) \\ \quad - J_n\Big((J_n Q^{(n)} + \frac{1}{d}\mathrm{Id})(\xi J_n R_\varepsilon D^{(n)} - J_n R_\varepsilon \Omega^{(n)})\Big) \\ \quad + 2\xi J_n\Big((J_n Q^{(n)} + \frac{1}{d}\mathrm{Id})\mathrm{tr}\big(J_n Q^{(n)} \nabla J_n R_\varepsilon u^{(n)}\big)\Big) = \Gamma \tilde{H}^{(n)} \\ \partial_t u^n + \mathcal{P} J_n(\mathcal{P} J_n R_\varepsilon u^n \nabla \mathcal{P} J_n u^n) - \nu \Delta \mathcal{P} J_n u^{(n)} \\ = -\varepsilon \mathcal{P} J_n R_\varepsilon \Big(\sum_{l,m=1}^d \nabla J_n Q^{(n)}_{lm} \Big(R_\varepsilon J_n u^n \cdot \nabla J_n Q^{(n)}_{lm}\Big) |R_\varepsilon J_n u^n \nabla J_n Q^{(n)}| \Big) \\ \quad + \varepsilon \mathcal{P} \nabla \cdot J_n R_\varepsilon \Big(\nabla R_\varepsilon J_n u^{(n)} |\nabla R_\varepsilon J_n u^{(n)}|^2 \Big) \\ \quad - \lambda \xi \mathcal{P} \nabla \cdot J_n \Big(\big(J_n Q^{(n)} + \frac{1}{d}\mathrm{Id}\big) \tilde{H}^{(n)} \Big) - \lambda \xi \mathcal{P} \nabla \cdot J_n \Big(\tilde{H}^{(n)} \big(J_n Q^{(n)} + \frac{1}{d}\mathrm{Id}\big) \Big) \\ \quad + 2\lambda \xi \mathcal{P} \nabla \cdot J_n \Big(\big(J_n Q^{(n)} + \frac{1}{d}\mathrm{Id}\big) \big(J_n Q^{(n)} \tilde{H}^{(n)}\big) \Big) - L\lambda \mathcal{P} J_n (\nabla \cdot \mathrm{tr}(J_n Q^{(n)} \nabla J_n Q^{(n)})) \\ \quad + L\lambda \mathcal{P} \nabla \cdot J_n \big(J_n Q^{(n)} \Delta J_n Q^{(n)} - \Delta J_n Q^{(n)} J_n Q^{(n)} \big), \end{cases} \tag{3.1}$$

where $\mathcal{P}$ denotes the Leray projector onto divergence-free vector fields, M is a positive constant, and $\tilde{H}^{(n)} \stackrel{def}{=} L\Delta J_n Q^{(n)} - aJ_n Q^{(n)} + bJ_n[(J_n Q^{(n)} J_n Q^{(n)}) - \frac{\mathrm{tr}(J_n Q^{(n)} J_n Q^{(n)})}{d}\mathrm{Id}]$ $-cJ_n\Big(J_n Q^{(n)}\big|J_n Q^{(n)}\big|^2\Big)$. We take as initial data $(J_n R_\varepsilon \bar{Q}, J_n R_\varepsilon \bar{u})$.

The system above can be regarded as an ordinary differential equation in L^2 verifying the conditions of the Cauchy–Lipschitz theorem. Thus it admits a unique maximal solution $(Q^{(n)}, u^{(n)}) \in C^1([0,T_n); L^2(\mathbb{R}^d; \mathbb{R}^{d\times d}) \times L^2(\mathbb{R}^d, \mathbb{R}^d))$. As we have $(\mathcal{P} J_n)^2 = \mathcal{P} J_n$ and $J_n^2 = J_n$, the pair $(J_n Q^{(n)}, \mathcal{P} J_n u^{(n)})$ is also a solution of the system (3.1). By uniqueness we have $(J_n Q^{(n)}, \mathcal{P} J_n u^{(n)}) = (Q^{(n)}, u^{(n)})$, hence $(Q^{(n)}, u^{(n)}) \in C^1([0,T_n), H^\infty)$ and $(Q^{(n)}, u^{(n)})$ satisfy the system

$$\begin{cases} \partial_t Q^{(n)} + J_n\Big(R_\varepsilon u^n \nabla Q^{(n)}\Big) - J_n\Big((\xi R_\varepsilon D^{(n)} + R_\varepsilon \Omega^{(n)})(Q^{(n)} + \frac{1}{d}\mathrm{Id})\Big) \\ \quad - J_n\Big((Q^{(n)} + \frac{1}{d}\mathrm{Id})(\xi R_\varepsilon D^{(n)} - R_\varepsilon \Omega^{(n)})\Big) + 2\xi J_n\Big((Q^{(n)} + \frac{1}{d}\mathrm{Id})\mathrm{tr}\big(Q^{(n)} \nabla R_\varepsilon u^n\big)\Big) = \Gamma \bar{H}^{(n)} \\ \partial_t u^n + \mathcal{P} J_n(R_\varepsilon u^n \nabla u^n) - \nu \Delta u^{(n)} \\ = -\varepsilon \mathcal{P} J_n \Big(\sum_{l,m=1}^d \nabla Q^{(n)}_{lm} \Big(R_\varepsilon u^n \cdot \nabla Q^{(n)}_{lm}\Big) |R_\varepsilon u^n \nabla Q^{(n)}| \Big) \\ \quad + \varepsilon \mathcal{P} \nabla \cdot J_n R_\varepsilon \Big(\nabla R_\varepsilon u^{(n)} |\nabla R_\varepsilon u^{(n)}|^2 \Big) \\ \quad - \lambda \xi \mathcal{P} \nabla \cdot J_n \Big(\Big(Q^{(n)} + \frac{1}{d}\mathrm{Id}\Big) \bar{H}^{(n)} \Big) - \lambda \xi \mathcal{P} \nabla \cdot J_n \Big(\bar{H}^{(n)} \Big(Q^{(n)} + \frac{1}{d}\mathrm{Id}\Big) \Big) \\ \quad + 2\lambda \xi \mathcal{P} \nabla \cdot J_n \Big(\big(Q^{(n)} + \frac{1}{d}\mathrm{Id}\big) \big(Q^{(n)} \bar{H}^{(n)}\big) \Big) \\ \quad - L\lambda \mathcal{P} J_n (\nabla \cdot \mathrm{tr}(\nabla Q^{(n)} \nabla Q^{(n)})) + L\lambda \mathcal{P} \nabla \cdot J_n \Big(Q^{(n)} \Delta Q^{(n)} - \Delta Q^{(n)} Q^{(n)} \Big), \end{cases} \tag{3.2}$$

where

$$\bar{H}^{(n)} \stackrel{def}{=} L\Delta Q^{(n)} - aQ^{(n)} + bJ_n[(Q^{(n)} Q^{(n)}) - \frac{\mathrm{tr}(J_n(Q^{(n)} Q^{(n)}))}{d}\mathrm{Id}] - cJ_n(Q^{(n)}|Q^{(n)}|^2).$$

The initial data is $(J_n\bar{Q}, J_n\bar{u})$. We recall now a few properties of J_n.

LEMMA 3.1. *The operators $\mathcal{P}$ and J_n are selfadjoint in L^2. Moreover, J_n and $\mathcal{P}J_n$ are also idempotent, and J_n commutes with distributional derivatives.*

We proceed in a manner analogous to the proof of Proposition 2.1 and multiply the first equation in the system (3.2) by $-\lambda\bar{H}^{(n)}$, take the trace, integrate over $\mathbb{R}^d$ by parts, and add the second equation multiplied by $u^{(n)}$. Let us observe that almost all the cancellations in the proof of Equation (2.1) hold, except for a few terms that need to be estimated separately. We also have some more new terms that we added in the regularization, terms that control the ones which do not cancel. Thus we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\frac{1}{2}|u^n|^2+\frac{L\lambda}{2}|\nabla Q^{(n)}|^2+\lambda\big(\frac{a}{2}|Q^{(n)}|^2-\frac{b}{3}\mathrm{tr}(Q^{(n)})^3+\frac{c}{4}|Q^{(n)}|^4\big)\mathrm{d}x\\
&\quad+\nu\int_{\mathbb{R}^d}|\nabla u^n|^2\mathrm{d}x+\Gamma\lambda\int_{\mathbb{R}^d}\mathrm{tr}\Big[J_n\Big(L\Delta Q^{(n)}-aQ^{(n)}+b[(Q^{(n)})^2\\
&\quad-\frac{\mathrm{tr}((Q^{(n)})^2)}{3}\mathrm{Id}]-cQ^{(n)}\big|Q^{(n)}\big|^2\Big)\Big]^2\mathrm{d}x+\varepsilon\int_{\mathbb{R}^d}|R_\varepsilon u\nabla Q^{(n)}|^3\mathrm{d}x+\varepsilon\int_{\mathbb{R}^d}|R_\varepsilon\nabla u^n|^4\mathrm{d}x\\
&\le\lambda\int_{\mathbb{R}^d}J_n\left(R_\varepsilon u^n\cdot\nabla Q^{(n)}_{\alpha\beta}\right)J_n\left(bQ^{(n)}_{\alpha\gamma}Q^{(n)}_{\gamma\beta}-cQ^{(n)}_{\alpha\beta}\big|Q^{(n)}\big|^2\right)\mathrm{d}x\\
&\quad+\lambda\int_{\mathbb{R}^d}J_n\left(-R_\varepsilon\Omega^{(n)}_{\alpha\gamma}Q^{(n)}_{\gamma\beta}+Q^{(n)}_{\alpha\gamma}R_\varepsilon\Omega^{(n)}_{\gamma\beta}\right)J_n\left(bQ^{(n)}_{\alpha\delta}Q^{(n)}_{\delta\beta}-cQ^{(n)}_{\alpha\beta}\big|Q^{(n)}\big|^2\right)\mathrm{d}x. \qquad (3.3)
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\frac{1}{2}|u^n|^2+\frac{L\lambda}{2}|\nabla Q^{(n)}|^2+\lambda\big(\frac{a}{2}|Q^{(n)}|^2-\frac{b}{3}\mathrm{tr}(Q^{(n)})^3\\
&\quad+\frac{c}{4}|Q^{(n)}|^4\big)\mathrm{d}x+\nu\int_{\mathbb{R}^d}|\nabla u^n|^2\mathrm{d}x+\Gamma\lambda\int_{\mathbb{R}^d}L^2|\Delta Q^{(n)}|^2\mathrm{d}x+\Gamma\lambda a^2\int_{\mathbb{R}^d}|Q^{(n)}|^2\mathrm{d}x\\
&\quad+C(b^2,d,\Gamma,\lambda)\int_{\mathbb{R}^d}|Q^{(n)}|^4\mathrm{d}x+\Gamma\lambda c^2\int_{\mathbb{R}^d}\big|J_n(Q^{(n)}|Q^{(n)}|^2)\big|^2\mathrm{d}x\\
&\qquad+\varepsilon\int_{\mathbb{R}^d}|R_\varepsilon u\nabla Q^{(n)}|^3\mathrm{d}x+\varepsilon\int_{\mathbb{R}^d}|R_\varepsilon\nabla u^n|^4\mathrm{d}x\\
&\le\underbrace{2\Gamma\lambda c\int_{\mathbb{R}^d}L\Delta Q^{(n)}\cdot Q^{(n)}|Q^{(n)}|^2\mathrm{d}x}_{\stackrel{\mathrm{def}}{=}\mathcal{I}}\\
&\quad-2\Gamma\lambda\int_{\mathbb{R}^d}L\Delta Q^{(n)}\cdot\big(-aQ^{(n)}+bJ_n\Big[(Q^{(n)})^2-\frac{\mathrm{tr}(Q^{(n)})^2}{d}\mathrm{Id})\Big]\big)\mathrm{d}x\\
&\quad-2\Gamma\lambda\int_{\mathbb{R}^d}cQ^{(n)}|Q^{(n)}|^2\cdot\left(aQ^{(n)}-bJ_n\Big[(Q^{(n)})^2-\frac{\mathrm{tr}(Q^{(n)})^2}{d}\mathrm{Id}\Big]\right)\mathrm{d}x\\
&\quad+\lambda\underbrace{\int_{\mathbb{R}^d}J_n\left(R_\varepsilon u^n\cdot\nabla Q^{(n)}_{\alpha\beta}\right)J_n\left(bQ^{(n)}_{\alpha\gamma}Q^{(n)}_{\gamma\beta}-cQ^{(n)}_{\alpha\beta}\big|Q^{(n)}\big|^2\right)\mathrm{d}x}_{\stackrel{\mathrm{def}}{=}\mathcal{II}}\\
&\qquad+C\int_{\mathbb{R}^d}|R_\varepsilon\nabla u^n|^2|Q^{(n)}|^2\mathrm{d}x+\frac{\Gamma c^2}{8}\int_{\mathbb{R}^d}|J_n(Q^{(n)}|Q^{(n)}|^2)|^2\mathrm{d}x+C\int_{\mathbb{R}^d}|Q^{(n)}|^4\mathrm{d}x.
\end{aligned}
$$

We have that

$$
\mathcal{II}=\int_{\mathbb{R}^d}\left(R_\varepsilon u^n\cdot\nabla Q^{(n)}_{\alpha\beta}\right)J_n\left(bQ^{(n)}_{\alpha\gamma}Q^{(n)}_{\gamma\beta}-cQ^{(n)}_{\alpha\beta}\big|Q^{(n)}\big|^2\right)\mathrm{d}x
$$

$$
\begin{aligned}
&\le \left(\frac{4}{\Gamma c^2}+\frac{1}{4C(b^2,d,\Gamma)}\right)\int_{\mathbb{R}^d}|R_\varepsilon u^n\cdot\nabla Q^{(n)}|^2\mathrm{d}x\\
&\quad+\frac{C(b^2,d,\Gamma)}{2}\|Q^{(n)}\|_{L^4}^4+\frac{\Gamma c^2}{8}\int_{\mathbb{R}^d}|J_n(Q^{(n)}|Q^{(n)}|^2)|^2\mathrm{d}x\\
&\le \frac{\varepsilon}{2}\int_{\mathbb{R}^d}|R_\varepsilon u^n\cdot\nabla Q^{(n)}|^3\mathrm{d}x+C(\varepsilon,b^2,c^2,d,\Gamma)\int_{\mathbb{R}^d}\sum_{l,m=1}^d|R_\varepsilon u^n\cdot\nabla Q^{(n)}_{lm}|\mathrm{d}x\\
&\quad+\frac{C(b^2,c^2,d,\Gamma)}{2}\|Q^{(n)}\|_{L^4}^4+\frac{\Gamma c^2}{8}\int_{\mathbb{R}^d}|J_n(Q^{(n)}|Q^{(n)}|^2)|^2\mathrm{d}x\\
&\le \frac{\varepsilon}{2}\int_{\mathbb{R}^d}|R_\varepsilon u^n\cdot\nabla Q^{(n)}|^3\mathrm{d}x+C_1(\varepsilon,b^2,c^2,d,\Gamma)\int_{\mathbb{R}^d}|u^n|^2\mathrm{d}x\\
&\quad+C_2(\varepsilon,b,c,d^2,\Gamma)\int_{\mathbb{R}^d}|\nabla Q^{(n)}|^2\mathrm{d}x+\frac{C(b^2,d,\Gamma)}{2}\|Q^{(n)}\|_{L^4}^4\\
&\quad+\frac{\Gamma c^2}{8}\int_{\mathbb{R}^d}|J_n(Q^{(n)}|Q^{(n)}|^2)|^2\,dx. \qquad (3.4)
\end{aligned}
$$

Using the fact that $\mathcal{I}\le 0$ and the estimate for $\mathcal{II}$ shown before, we replace in Equation (3.3) and obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\frac{1}{2}|u^n|^2+\frac{L\lambda}{2}|\nabla Q^{(n)}|^2+\lambda\Big(\frac{a}{2}|Q^{(n)}|^2-\frac{b}{3}\mathrm{tr}(Q^{(n)})^3+\frac{c}{4}|Q^{(n)}|^4\Big)\mathrm{d}x\\
&+\nu\int_{\mathbb{R}^d}|\nabla u^n|^2\mathrm{d}x+\Gamma\lambda\int_{\mathbb{R}^d}L^2|\Delta Q^{(n)}|^2\mathrm{d}x+\frac{\varepsilon}{2}\int_{\mathbb{R}^d}|R_\varepsilon u^n\cdot\nabla Q^{(n)}|^3\mathrm{d}x+\frac{\varepsilon}{2}\int_{\mathbb{R}^d}|\nabla R_\varepsilon u^n|^4\mathrm{d}x\\
&\le\int_{\mathbb{R}^d}|Q^{(n)}|^2+|Q^{(n)}|^4\mathrm{d}x+C\int_{\mathbb{R}^d}|\nabla Q^{(n)}|^2\mathrm{d}x+C(\varepsilon)\int_{\mathbb{R}^d}|u^{(n)}|^2\mathrm{d}x.
\end{aligned}
$$

This estimate does not readily provide bounds on $Q^{(n)}$ because the term $\frac{a}{2}|Q^{(n)}|^2-\frac{b}{3}\mathrm{tr}(Q^{(n)})^3+\frac{c}{4}|Q^{(n)}|^4$ could be negative. In order to obtain H^1 estimates,we proceed as in the proof of Proposition 2.2. We put the proof in Appendix B by Proposition B.1. We can continue to proceed as in the proof of Proposition 2.2; in fact, in this case, because of the first two regularizing terms on the right-hand side of the u^n equation in the system (3.2) we do not need the ξ small assumption. These estimates allow us to conclude that $T_n=\infty$ and we also get the following a priori bounds:

$$
\begin{aligned}
\sup_n\|\nabla R_\varepsilon u^n\|_{L^4(0,T;L^4)},\sup_n\|R_\varepsilon u^n\cdot\nabla Q^{(n)}\|_{L^3(0,T;L^3)}&\le C(\varepsilon)\\
\sup_n\|Q^{(n)}\|_{L^2(0,T;H^2)\cap L^\infty(0,T;H^1)}&<\infty, \qquad (3.5)\\
\sup_n\|u^n\|_{L^\infty(0,T;L^2)\cap L^2(0,T;H^1)}&<\infty
\end{aligned}
$$

for any $T<\infty$. By the bounds which can be obtained by using the equation on $\partial_t(Q^{(n)},u^n)$ in some $L^\infty_{loc}(H^{-N})$ for large enough N, we get, by classical local compactness Aubin–Lions lemma, on a subsequence that

$$
\begin{aligned}
&Q^{(n)}\rightharpoonup Q \text{ in } L^2(0,T;H^2) \text{ and } Q^{(n)}\to Q \text{ in } L^2(0,T;H^{2-\delta}_{loc}),\ \forall\delta>0\\
&Q^{(n)}(t)\rightharpoonup Q(t) \text{ in } H^1 \text{ for all } t\in\mathbb{R}_+\\
&u^n\rightharpoonup u \text{ in } L^2(0,T;H^1) \text{ and } u^n\to u \text{ in } L^2(0,T;H^{1-\delta}_{loc}),\ \forall\delta>0\\
&u^n(t)\rightharpoonup u(t) \text{ in } L^2 \text{ for all } t\in\mathbb{R}_+.
\end{aligned}
$$

Thus we can pass to the limit and obtain a weak solution of the approximating system

$$\begin{cases} \partial_t Q^{(\varepsilon)} + R_\varepsilon u^\varepsilon \nabla Q^{(\varepsilon)} - (\xi R_\varepsilon D^\varepsilon + R_\varepsilon \Omega^\varepsilon)(Q^{(\varepsilon)} + \frac{1}{d}\mathrm{Id})\Big) + \Big((Q^{(\varepsilon)} + \frac{1}{d}\mathrm{Id})(\xi R_\varepsilon D^\varepsilon - R_\varepsilon \Omega^\varepsilon)\Big) \\ \quad -2\xi\Big((Q^{(\varepsilon)} + \frac{1}{d}\mathrm{Id})\mathrm{tr}\big(Q^{(\varepsilon)}\nabla u^\varepsilon\big)\Big) = \Gamma H^\varepsilon \\ \partial_t u^\varepsilon + \mathcal{P} R_\varepsilon u^\varepsilon \nabla u^\varepsilon = -\varepsilon \mathcal{P} R_\varepsilon \Big(\sum_{l,m=1}^d \nabla Q_{lm} \left(R_\varepsilon u \cdot \nabla Q_{lm}\right) |R_\varepsilon u \nabla Q|\Big) \\ \quad +\varepsilon \mathcal{P}\nabla\cdot R_\varepsilon \Big(R_\varepsilon \nabla u |R_\varepsilon \nabla u|^2\Big) - \lambda\xi\mathcal{P}\nabla\cdot R_\varepsilon\Big(\Big(Q^{(\varepsilon)} + \frac{1}{d}\mathrm{Id}\Big) H^\varepsilon\Big) \\ \quad -\lambda\xi\mathcal{P}\nabla\cdot R_\varepsilon\Big(H^\varepsilon\Big(Q^{(\varepsilon)} + \frac{1}{d}\mathrm{Id}\Big)\Big) + 2\lambda\xi\mathcal{P}\nabla\cdot R_\varepsilon\Big(\big(Q^{(\varepsilon)} + \frac{1}{d}\big)\big(Q^{(\varepsilon)} H^\varepsilon\big)\Big) \\ \quad -L\lambda\mathcal{P}(\nabla\cdot R_\varepsilon \mathrm{tr}(\nabla Q^{(\varepsilon)} \odot \nabla Q^{(\varepsilon)})) + L\lambda\mathcal{P}\nabla\cdot R_\varepsilon\Big(Q^{(\varepsilon)}\Delta Q^{(\varepsilon)} - \Delta Q^{(\varepsilon)} Q^{(\varepsilon)}\Big) + \nu\Delta u^\varepsilon \end{cases} \tag{3.6}$$

where we recall that $H = L\Delta Q^{(\varepsilon)} - aQ^{(\varepsilon)} + b[(Q^{(\varepsilon)})^2 - \frac{\mathrm{tr}((Q^{(\varepsilon)})^2))}{d}\mathrm{Id}] - cQ^{(\varepsilon)}\mathrm{tr}((Q^{(\varepsilon)})^2)$. The initial data for the limit system is $(R_\varepsilon \bar{Q}, R_\varepsilon \bar{u})$.

One can easily see that the solutions of the sytem (3.6) are smooth, first by obtaining C^∞ regularity for the first Q equations, by bootstrapping the regularity improvement provided by the linear heat equation, and then the regularity for the u equation, by bootstrapping the regularity improvement provided by a linear advection equation. For this system we can proceed as in the case of a priori estimates and obtain the same estimates, independent of ε because the solutions are smooth and all the cancellations that were used in the a priori estimates also hold here. In particular we obtain

$$\begin{aligned} &\sup_\varepsilon \|Q^{(\varepsilon)}\|_{L^\infty(0,T;H^1)\cap L^2(0,T;H^2)} < \infty, \\ &\sup_\varepsilon \|u^\varepsilon\|_{L^\infty(0,T;L^2)\cap L^2(0,T;H^1)} < \infty \end{aligned} \tag{3.7}$$

for any $T < \infty$. Taking into account those bounds and also the bounds which can be obtained by using the equation on $\partial_t(Q^\varepsilon, u^\varepsilon)$ in some $L^p_{loc}(H^{-N})$ for large enough N, we get, by classical local compactness Aubin–Lions lemma and by weak convergence arguments, that there exists a $Q \in L^\infty_{loc}(\mathbb{R}_+;H^1)\cap L^2_{loc}(\mathbb{R}_+;H^2)$ and a $u \in L^\infty_{loc}(\mathbb{R}_+;L^2)\cap L^2_{loc}(\mathbb{R}_+;H^1)$ so that, on a subsequence, we have

$$\begin{aligned} &Q^{(\varepsilon)} \rightharpoonup Q \text{ in } L^2(0,T;H^2) \text{ and } Q^{(\varepsilon)} \to Q \text{ in } L^2(0,T;H^{2-\delta}_{loc}), \forall \delta > 0 \\ &Q^{(\varepsilon)}(t) \rightharpoonup Q(t) \text{ in } H^1 \text{ for all } t \in \mathbb{R}_+ \\ &u^\varepsilon \rightharpoonup u \text{ in } L^2(0,T;H^1) \text{ and } u^\varepsilon \to u \text{ in } L^2(0,T;H^{1-\delta}_{loc}), \forall \delta > 0 \\ &u^\varepsilon(t) \rightharpoonup u(t) \text{ in } L^2 \text{ for all } t \in \mathbb{R}_+ \end{aligned} \tag{3.8}$$

These convergences allow us to the pass to the limit in the weak solutions of the system (3.6) to obtain a weak solution of the system (1.8), namely Equations (1.10) and (1.11). Of all the terms, there is only one type that is slightly difficult to treat in passing to the limit, namely

$$\begin{aligned} &L\int_0^\infty \int_{\mathbb{R}^d} \partial_\beta \left(Q^{(\varepsilon)}_{\alpha\gamma}\Delta Q^{(\varepsilon)}_{\gamma\beta} - \Delta Q^{(\varepsilon)}_{\alpha\gamma} Q^{(\varepsilon)}_{\gamma\beta}\right)\psi_\alpha \mathrm{d}x\mathrm{d}t \\ &= -L\int_0^\infty \int_{\mathbb{R}^d} \left(Q^{(\varepsilon)}_{\alpha\gamma}\Delta Q^{(\varepsilon)}_{\gamma\beta} - \Delta Q^{(\varepsilon)}_{\alpha\gamma} Q^{(\varepsilon)}_{\gamma\beta}\right)\cdot \psi_{\alpha,\beta}\mathrm{d}x\mathrm{d}t. \end{aligned}$$

Taking into account that ψ is compactly supported and the convergences (3.8), one can easily pass to the limit the terms $\psi_{\alpha,\beta} Q^{(\varepsilon)}_{\alpha\gamma}$ and $\psi_{\alpha,\beta} Q^{(\varepsilon)}_{\gamma\beta}$ strongly in $L^2(0,T;L^2)$.

Relations (3.8) give that $\Delta Q^{(\varepsilon)}_{\gamma\beta}$, $\Delta Q^{(\varepsilon)}_{\alpha\gamma}$ converges weakly in $L^2(0,T;L^2)$. Thus we get convergence to the limit term

$$\begin{aligned}&L\int_0^\infty\int_{\mathbb{R}^d}\partial_\beta(Q_{\alpha\gamma}\Delta Q_{\gamma\beta})\psi_\alpha \mathrm{d}x\mathrm{d}t - L\int_0^\infty\int_{\mathbb{R}^d}\partial_\beta(\Delta Q_{\alpha\gamma})Q_{\gamma\beta})\psi_\alpha \mathrm{d}x\mathrm{d}t\\ &= -L\int_0^T\int_{\mathbb{R}^d}(\Delta Q_{\gamma\beta})(\partial_\beta\psi_\alpha Q_{\alpha\gamma})\mathrm{d}x\mathrm{d}t + L\int_0^T\int_{\mathbb{R}^d}(\Delta Q_{\alpha\gamma})(\partial_\beta\psi_\alpha Q_{\gamma\beta})\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Using also the uniform bound of $\varepsilon\|R_\varepsilon u^\varepsilon\nabla Q^\varepsilon\|^3_{L^3}$, it is easy to check that

$$\varepsilon\int|R_\varepsilon u^\varepsilon\nabla Q^\varepsilon|^2\nabla Q^\varepsilon\cdot R_\varepsilon\mathcal{P}\varphi dxdt$$

converges to zero. A similar observation holds for the ε-regularisation term $\varepsilon\mathcal{P}\nabla\cdot\Big(R_\varepsilon\nabla u|R_\varepsilon\nabla u|^2\Big)$. ◻

4. The uniqueness of weak solutions

We start with a number of technical tools that are crucial for our proof.

4.1. Littlewood–Paley theory. We define $\mathcal{C}$ to be the ring of center 0, of small radius 1/2, and great radius 2. There exist two nonnegative radial functions χ and φ belonging respectively to $\mathcal{D}(B(0,1))$ and to $\mathcal{D}(\mathcal{C})$ so that

$$\chi(\xi)+\sum_{q\geq 0}\varphi(2^{-q}\xi)=1,\forall\xi\in\mathbb{R}^d \tag{4.1}$$

$$|p-q|\geq 5\Rightarrow \operatorname{Supp}\varphi(2^{-q}\cdot)\cap\operatorname{Supp}\varphi(2^{-p}\cdot)=\emptyset. \tag{4.2}$$

For instance, one can take $\chi\in\mathcal{D}(B(0,1))$ such that $\chi\equiv 1$ on $B(0,1/2)$ and take

$$\varphi(\xi)=\chi(\xi/2)-\chi(\xi).$$

Then, we are able to define the Littlewood–Paley decomposition. Let us denote by $\mathcal{F}$ the Fourier transform on $\mathbb{R}^d$. Let h, $\tilde{h}$, $\dot{\Delta}_q,\dot{S}_q$ $(q\in\mathbb{Z})$ be defined as follows:

$$\begin{aligned}h=\mathcal{F}^{-1}\varphi \quad\text{and}\quad \tilde{h}=\mathcal{F}^{-1}\chi,\\ \dot{\Delta}_q u=\mathcal{F}^{-1}(\varphi(2^{-q}\xi)\mathcal{F}u)=2^{qd}\int h(2^q y)u(x-y)dy,\\ \dot{S}_q u=\mathcal{F}^{-1}(\chi(2^{-q}\xi)\mathcal{F}u)=2^{qd}\int \tilde{h}(2^q y)u(x-y)dy.\end{aligned}$$

We recall that, for two appropriately smooth functions a and b, we have the Bony's paraproduct decomposition [4]

$$ab=\dot{T}_a b+\dot{T}_b a+\dot{R}(a,b),$$

where

$$\dot{T}_a b=\sum_{q\in\mathbb{Z}}\dot{S}_{q-1}a\dot{\Delta}_q b,\quad \dot{T}_b a=\sum_{q\in\mathbb{Z}}\dot{S}_{q-1}b\dot{\Delta}_q a,\quad\text{and}\quad \dot{R}(a,b)=\sum_{\substack{q\in\mathbb{Z},\\ i\in\{0,\pm1\}}}\dot{\Delta}_q a\dot{\Delta}_{q+i}b.$$

Then we have

$$\dot{\Delta}_q(ab) = \dot{\Delta}_q\dot{T}_a b + \dot{\Delta}_q\dot{T}_b a + \dot{\Delta}_q\dot{R}(a,b) = \dot{\Delta}_q\dot{T}_a b + \dot{\Delta}_q\tilde{R}(a,b),$$

where $\tilde{R}(a,b) = \dot{T}_b a + \dot{R}(a,b) = \sum_{q\in\mathbb{Z}}\dot{S}_{q+2}b\dot{\Delta}_q a$. Moreover:

$$\begin{aligned}
\dot{\Delta}_q(ab) &= \sum_{|q'-q|\le 5}\dot{\Delta}_q(\dot{S}_{q'-1}a\dot{\Delta}_{q'}b) + \sum_{q'>q-5}\dot{\Delta}_q(\dot{S}_{q'+2}b\dot{\Delta}_{q'}a)\\
&= \sum_{|q'-q|\le 5}[\dot{\Delta}_q,\dot{S}_{q'-1}a]\dot{\Delta}_{q'}b + \sum_{|q'-q|\le 5}\dot{S}_{q'-1}a\dot{\Delta}_q\dot{\Delta}_{q'}b + \sum_{q'>q-5}\dot{\Delta}_q(S_{q'+2}b\dot{\Delta}_{q'}a)\\
&= \sum_{|q'-q|\le 5}[\dot{\Delta}_q,\dot{S}_{q'-1}a]\dot{\Delta}_{q'}b + \sum_{|q'-q|\le 5}(\dot{S}_{q'-1}a - \dot{S}_{q-1}a)\dot{\Delta}_q\dot{\Delta}_{q'}b\\
&\quad + \sum_{q'>q-5}\dot{\Delta}_q(\dot{S}_{q'+2}b\dot{\Delta}_{q'}a) + \underbrace{\sum_{|q'-q|\le 5}\dot{S}_{q-1}a\dot{\Delta}_q\dot{\Delta}_{q'}b}_{=\dot{S}_{q-1}a\dot{\Delta}_q b}. \qquad (4.3)
\end{aligned}$$

In terms of this decomposition, we can express the Sobolev norm of an element u in the (nonhomogeneous!) space H^s as

$$\|u\|_{H^s} = \big(\|\dot{S}_0 u\|_{L^2}^2 + \sum_{q\in\mathbb{N}} 2^{2qs}\|\dot{\Delta}_q u\|_{L^2}^2\big)^{1/2}.$$

These are a particular case of the general nonhomogeneous Besov spaces $B^s_{p,r}$, for $s\in\mathbb{R}, p,r\in[1,\infty]^2$ consisting of all tempered distributions u such that

$$\|u\|_{B^s_{p,r}} \stackrel{def}{=} \begin{cases} \|(\|\dot{S}_0 u\|_{L^p}^r + \sum_{q\in\mathbb{N}} 2^{rqs}\|\dot{\Delta}_q u\|_{L^p}^r)^{\frac{1}{r}} & \text{if } r<\infty\\ \max(\|\dot{S}_0 u\|_{L^p}, \sup_{q\in\mathbb{N}} 2^{qs}\|\dot{\Delta}_q u\|_{L^p}) & \text{if } r=\infty\end{cases}$$

which reduces to the nonhomogeneous Sobolev space H^s for $p=r=2$.

Similarly we also have the norm of the *homogeneous* Sobolev spaces $\dot{H}^s$

$$\|u\|_{\dot{H}^s} = \big(\sum_{q\in\mathbb{Z}} 2^{2qs}\|\dot{\Delta}_q u\|_{L^2}^2\big)^{1/2}$$

and the homogeneous Besov spaces $\dot{B}^s_{p,r}$ for $s\in\mathbb{R}, p,r\in[1,\infty]^2$ consisting of all the homogeneous tempered distributions u such that:

$$\|u\|_{\dot{B}^s_{p,r}} \stackrel{def}{=} \begin{cases} \|(\sum_{q\in\mathbb{Z}} 2^{rqs}\|\dot{\Delta}_q u\|_{L^p}^r)^{\frac{1}{r}} & \text{if } r<\infty\\ \sup_{q\in\mathbb{Z}} 2^{qs}\|\dot{\Delta}_q u\|_{L^p} & \text{if } r=\infty\end{cases}$$

which reduces to the homogeneous Sobolev space $\dot{H}^s$ for $p=r=2$.

Let us note that the homogeneous Besov spaces have somewhat better product rules, and this specificity encoded in Theorem A.1 will be very useful in our subsequent estimates.

Furthermore we will need the following characterisation of the homogeneous norms, in terms of operators $\dot{S}_q u$.

LEMMA 4.1 (Prop. 2.33 , [2]). *Let $s<0$ and $p,r\in[1,\infty]^2$. A tempered distribution u belongs to $\dot{B}^s_{p,r}$ if and only if*

$$(2^{qs}\|\dot{S}_q u\|_{L^p})_{q\in\mathbb{Z}} \in l^r$$

and, for some constant C depending only on the dimension d, we have

$$C^{-|s|+1}\|u\|_{\dot{B}^s_{p,r}} \le \|(2^{qs}\|\dot{S}_q u\|_{L^p})_q\|_{l^r} \le C(1+\frac{1}{|s|})\|u\|_{\dot{B}^s_{p,r}}.$$

We will use the following well-known estimates.

LEMMA 4.2 ([12,13]).

(i) *(Bernstein inequalities)*

$$2^{-q}\|\nabla \dot{S}_q u\|_{L^p_x} \le C\|u\|_{L^p_x}, \forall 1\le p\le \infty$$

$$c\|\dot{\Delta}_q u\|_{L^p_x} \le 2^{-q}\|\dot{\Delta}_q \nabla u\|_{L^p_x} \le C\|\dot{\Delta}_q u\|_{L^p_x}, \forall 1\le p\le \infty$$

(ii) *(Bernstein inequalities)*

$$\|\dot{\Delta}_q u\|_{L^b_x} \le 2^{d(\frac{1}{a}-\frac{1}{b})q}\|\dot{\Delta}_q u\|_{L^a_x},\ \text{for } b\ge a\ge 1$$

$$\|\dot{S}_q u\|_{L^b_x} \le 2^{d(\frac{1}{a}-\frac{1}{b})q}\|\dot{S}_q u\|_{L^a_x},\ \text{for } b\ge a\ge 1$$

(iii) *(commutator estimate)*

$$\|[\dot{\Delta}_q,u]v\|_{L^p_x} \le C2^{-q}\|\nabla u\|_{L^r_x}\|v\|_{L^s_x} \tag{4.4}$$

with $\frac{1}{p}=\frac{1}{r}+\frac{1}{s}$. The constant C depends only on the function φ used in defining $\dot{\Delta}_q$ but not on p,r,s.

Proof. For the commutator estimate, we begin by writing

$$\begin{aligned}
[\dot{\Delta}_q,u]v(x) &= \dot{\Delta}_q(uv)(x)-u(x)\Delta_q v(x) = 2^{qd}\int h(2^q y)(u(x-y)-u(x))v(x-y)\mathrm{d}y \\
&= 2^{qd}\int_{\mathbb{R}^d}\int_0^1 \frac{\partial}{\partial\tau}\Big\{h(2^q y)u(x-\tau y)v(x-y)\mathrm{d}y\Big\}\mathrm{d}\tau \\
&= -2^{qd}\int_0^1\int_{\mathbb{R}^d} h(2^q y)y\cdot\nabla u(x-\tau y)v(x-y)\mathrm{d}y\mathrm{d}\tau \\
&= -2^{-q}\int_0^1\int_{\mathbb{R}^d} \tilde{h}_{2^q}(y)\cdot\nabla u(x-\tau y)v(x-y)\mathrm{d}y\mathrm{d}\tau,
\end{aligned}$$

where $\tilde{h}(y):=yh(y)\in\mathcal{S}(R^d)^d$ and $\tilde{h}_\lambda(y):=\lambda^d\tilde{h}(\lambda y)$. Using the Cauchy–Schwartz inequality and a change of variables, we get

$$\begin{aligned}
|[\dot{\Delta}_q,u]v(x)| &\le 2^{-q}\int_0^1\Big(\int_{\mathbb{R}^d}|\tilde{h}_{2^q}(y)||\nabla u(x-\tau y)|^{\frac{r}{p}}\mathrm{d}y\Big)^{\frac{p}{r}}\mathrm{d}\tau\Big(\int_{\mathbb{R}^d}|\tilde{h}_{2^q}(y)||v(x-y)|^{\frac{s}{p}}\mathrm{d}y\Big)^{\frac{p}{s}} \\
&= 2^{-q}\int_0^1\Big(\int_{\mathbb{R}^d}\frac{|\tilde{h}_{2^q\tau^{-1}}(y)|}{\tau^d}|\nabla u(x-y)|^{\frac{r}{p}}\mathrm{d}y\Big)^{\frac{p}{r}}\mathrm{d}\tau\Big(\int_{\mathbb{R}^d}|\tilde{h}_{2^q}(y)||v(x-y)|^{\frac{s}{p}}\mathrm{d}y\Big)^{\frac{p}{s}} \\
&= 2^{-q}\int_0^1\Big(\frac{|\tilde{h}_{2^q\tau^{-1}}|}{\tau^d}*|\nabla u|^{\frac{r}{p}}(x)\Big)^{\frac{p}{r}}\mathrm{d}\tau\Big(|\tilde{h}_{2^q}|*|v|^{\frac{s}{p}}(x)\Big)^{\frac{p}{s}}.
\end{aligned}$$

Taking the L^p norm in the x variable, using the Cauchy–Schwartz inequality in the x variable and convolution estimates, we obtain

$$\begin{aligned}
\|[\dot{\Delta}_q, u]v\|_{L^p_x} &\leq 2^{-q}\int_0^1 \left\|\left(\frac{|\tilde{h}_{2^q\tau^{-1}}|}{\tau^d} * |\nabla u|^{\frac{r}{p}}(x)\right)^{\frac{p}{r}}\right\|_{L^r_x} \mathrm{d}\tau \left\|\left(|\tilde{h}_{2^q}| * |v|^{\frac{s}{p}}(x)\right)^{\frac{p}{s}}\right\|_{L^s_x} \\
&\leq 2^{-q}\left(\int_0^1 \|\frac{|\tilde{h}_{2^q\tau^{-1}}|}{\tau^d} * |\nabla u|^{\frac{r}{p}}\|_{L^p_x}^{\frac{p}{r}} \mathrm{d}\tau\right) \||\tilde{h}_{2^q}| * |v|^{\frac{s}{p}}\|_{L^p_x}^{\frac{p}{s}} \\
&\leq 2^{-q}\int_0^1 \frac{\|\tilde{h}_{2^q\tau^{-1}}\|_{L^1_x}^{\frac{p}{r}}}{\tau^d} \mathrm{d}\tau \|\nabla u\|_{L^r_x} \|\tilde{h}_{2^q}\|_{L^1_x}^{\frac{p}{s}} \|v\|_{L^s_x} \\
&\leq 2^{-q}\|\tilde{h}_{2^{-q}}\|_{L^1}^{\frac{p}{r}} \|\tilde{h}_{2^{-q}}\|_{L^1}^{\frac{p}{s}} \|\nabla u\|_{L^r_x} \|v\|_{L^s_x}.
\end{aligned}$$

Now, since

$$\|\tilde{h}_{2^{-q}}\|_{L^1_x} = \int_{\mathbb{R}^d} 2^{-qd}|\tilde{h}(2^{-q}x)|\mathrm{d}x = \int_{\mathbb{R}^d} |\tilde{h}(y)|\mathrm{d}y = \|\tilde{h}\|_{L^1_x},$$

we finally obtain

$$\|[\dot{\Delta}_q, u]v\|_{L^p_x} \leq 2^{-q}\|\tilde{h}\|_{L^1_x}^{\frac{p}{r}} \|\tilde{h}\|_{L^1}^{\frac{p}{s}} \|\nabla u\|_{L^r_x} \|v\|_{L^s_x} = \|\tilde{h}\|_{L^1_x} 2^{-q}\|\nabla u\|_{L^r_x}\|v\|_{L^s_x},$$

so the constant in the inequality is $C = \|\tilde{h}\|_{L^1}$ and it does not depend on p, r, s. ∎

We will also make use of a Bernstein-type inequality evolving the operator $\dot{S}_q$.

LEMMA 4.3. *There exist two positive constants $\tilde{c}$ and $\tilde{C}$ such that*

$$\tilde{c}\|(\dot{S}_q - \dot{S}_{q'})u\|_{L^p_x} \leq 2^{-q}\|(\dot{S}_q - \dot{S}_{q'})\nabla u\|_{L^p_x} \leq \tilde{C}\|(\dot{S}_q - \dot{S}_{q'})u\|_{L^p_x}, \forall 1 \leq p \leq \infty$$

for any integers q and q' with $|q - q'| \leq 5$.

Proof. First, we consider new localizer functions as follows:

$$\tilde{\varphi}_q(\xi) := \frac{1}{10}\sum_{|q-j|\leq 10} \varphi_j(\xi) \quad \text{and} \quad \tilde{\chi}(\xi) := \begin{cases} \sum_{q\leq -1}\tilde{\varphi}_q(\xi) & \text{if} \quad \xi \neq 0, \\ 1 & \text{otherwise,} \end{cases}$$

so that Equations (4.1) and (4.2) are satisfied with $\tilde{\varphi}$ and $\tilde{\chi}$ instead of φ and χ. Then defining the new homogeneous dyadic block $\dot{\tilde{\Delta}}_q$ in the same line of $\dot{\Delta}_q$, we have

$$\dot{\tilde{\Delta}}_q(\dot{S}_q - \dot{S}_{q'})u = \frac{1}{10}\sum_{|q-j|\leq 10} \dot{\Delta}_j(\dot{S}_q - \dot{S}_{q'})(u) = \frac{1}{10}(\dot{S}_q - \dot{S}_{q'})(u).$$

Then the inequality turns out from **(i)** of Lemma 4.2, making use of $\dot{\tilde{\Delta}}_q$ instead of $\dot{\Delta}_q$. ∎

4.2. The proof of the uniqueness. In this section we provide the proof of the uniqueness result for the weak solutions of system (1.8). The main idea is to evaluate the difference between two weak solutions in a functional space which is less regular than L^2_x such as $\dot{H}^{-\frac{1}{2}}$. Such a strategy is not new in literature; for instance we recall [15] and [29]. We now provide the uniqueness part of the proof of Theorem 1.1.

Proof. Let us consider two weak solutions (u_1, Q_1) and (u_2, Q_2) of system (1.8). We denote $\delta u := u_1 - u_2$ and $\delta Q := Q_1 - Q_2$ while $\delta S(Q, \nabla u)$ stands for $S(Q_1, \nabla u_1) -$

$S(Q_2, \nabla u_2)$. Similarly, we define $\delta H(Q)$, $\delta F(Q)$, $\delta\tau$, and $\delta\sigma$. Thus $(\delta u, \delta Q)$ is a weak solution of

$$\begin{cases} \partial_t \delta Q - L\Delta\delta Q = \delta S(Q,\nabla u) + \Gamma\delta H(Q) - \delta u\cdot\nabla Q_1 - u_2\cdot\nabla\delta Q & \mathbb{R}_+\times\mathbb{R}^2,\\ \partial_t\delta u - \Delta\delta u + \nabla\delta\Pi = L\,\mathrm{div}\{\delta\tau+\delta\sigma\} - \delta u\cdot\nabla u_1 - u_2\cdot\nabla\delta u & \mathbb{R}_+\times\mathbb{R}^2,\\ \mathrm{div}\,\delta u = 0 & \mathbb{R}_+\times\mathbb{R}^2,\\ (\delta u,\delta Q)_{t=0} = (0,0) & \mathbb{R}^2. \end{cases} \tag{4.5}$$

First, let us explicitly state $\delta S(Q,\nabla u)$, $\delta F(Q)$, $\delta\tau$, and $\delta\sigma$ in terms of δQ and δu, namely

$$\begin{aligned} \delta S(Q,\nabla u) = &+(\xi\delta D+\delta\Omega)\delta Q + (\xi\delta D+\delta\Omega)\Big(Q_2+\frac{\mathrm{Id}}{2}\Big) + (\xi D_2+\Omega_2)\delta Q + \delta Q(\xi\delta D-\delta\Omega)\\ &+\Big(Q_2+\frac{\mathrm{Id}}{2}\Big)(\xi\delta D-\delta\Omega) + \delta Q(\xi D_2-\Omega_2) - 2\xi\delta Q\,\mathrm{tr}(\delta Q\nabla\delta u) - 2\xi\delta Q\,\mathrm{tr}(\delta Q\nabla u_2)\\ &-2\xi\delta Q\,\mathrm{tr}(Q_2\nabla\delta u) - 2\xi\Big(Q_2+\frac{\mathrm{Id}}{2}\Big)\mathrm{tr}(\delta Q\nabla\delta u) - 2\xi\delta Q\,\mathrm{tr}(Q_2\nabla u_2)\\ &-2\xi\Big(Q_2+\frac{\mathrm{Id}}{2}\Big)\mathrm{tr}(\delta Q\nabla u_2) - 2\xi\Big(Q_2+\frac{\mathrm{Id}}{2}\Big)\mathrm{tr}(Q_2\nabla\delta u), \end{aligned}$$

$$\begin{aligned} \delta F(Q) = &-a\delta Q + b\big(Q_1\delta Q+\delta Q Q_2\big) - b\,\mathrm{tr}\{\delta Q Q_1 + Q_2\delta Q\}\frac{\mathrm{Id}}{2}\\ &-c\big[\delta Q\mathrm{tr}\{Q_1^2\} + Q_2\mathrm{tr}\{\delta Q Q_1 + Q_2\delta Q\}\big] \end{aligned}$$

$$\delta H(Q) = \delta F(Q) + L\Delta\delta Q.$$

$$\begin{aligned} \delta\tau = &-\xi\delta Q\,F(Q_1) - \xi\Big(Q_2+\frac{\mathrm{Id}}{2}\Big)\delta F(Q) - L\xi\delta Q\,\Delta\delta Q - L\xi\delta Q\,\Delta Q_2 - L\xi\Big(Q_2+\frac{\mathrm{Id}}{2}\Big)\Delta\delta Q\\ &-\xi F(Q_1)\delta Q - \xi\delta F(Q)\Big(Q_2+\frac{\mathrm{Id}}{2}\Big) - L\xi\Delta\delta Q\delta Q - L\xi\Delta Q_2\delta Q - L\xi\Delta\delta Q\Big(Q_2+\frac{\mathrm{Id}}{2}\Big)\\ &+2\xi\delta Q\mathrm{tr}\{Q_1F(Q_1)\} + 2\xi Q_2\mathrm{tr}\{\delta Q F(Q_1)\}\\ &+2\xi Q_2\mathrm{tr}\{Q_2\delta F(Q)\} + 2L\xi\delta Q\mathrm{tr}\{\delta Q\Delta\delta Q\} + 2L\xi\delta Q\mathrm{tr}\{\delta Q\Delta Q_2\} + 2L\xi\delta Q\mathrm{tr}\{Q_2\Delta\delta Q\}\\ &+2L\xi\Big(Q_2+\frac{\mathrm{Id}}{2}\Big)\mathrm{tr}\{\delta Q\Delta\delta Q\} + 2L\xi\delta Q\mathrm{tr}\{Q_2\Delta Q_2\} + 2L\xi\Big(Q_2+\frac{\mathrm{Id}}{2}\Big)\mathrm{tr}\{\delta Q\Delta Q_2\}\\ &+2L\xi\Big(Q_2+\frac{\mathrm{Id}}{2}\Big)\mathrm{tr}\{Q_2\Delta\delta Q\} - L\nabla\delta Q\odot\nabla Q_1 - L\nabla Q_2\odot\nabla\delta Q\\ &-L\frac{\mathrm{Id}}{2}\mathrm{tr}\{\delta Q Q_1\} - L\frac{\mathrm{Id}}{2}\mathrm{tr}\{Q_2\delta Q\} \end{aligned}$$

$$\begin{aligned} \delta\sigma = \delta Q F(Q_1) + Q_2\delta F(Q) - F(Q_1)\delta Q - \delta F(Q)Q_2 + L\delta Q\Delta\delta Q + LQ_2\Delta\delta Q + L\delta Q\Delta Q_2&\\ - L\Delta\delta Q\delta Q - L\Delta Q_2\delta Q - L\Delta\delta Q Q_2&. \end{aligned}$$

Taking the inner product in $\dot H^{-1/2}$ of the first equation with $-L\lambda\Delta\delta Q$ and adding to it the scalar product in $\dot H^{-1/2}$ of the second one by $\frac{1}{\lambda}\delta u$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\frac{1}{2\lambda}\|\delta u\|^2_{\dot H^{-\frac12}} + L\|\nabla\delta Q\|^2_{\dot H^{-\frac12}}\Big] + \frac{\nu}{\lambda}\|\nabla\delta u\|^2_{\dot H^{-\frac12}} + \Gamma L^2\|\Delta\delta Q\|^2_{\dot H^{-\frac12}}$$

$$= -L\langle(\xi\delta D+\delta\Omega)\delta Q,\Delta\delta Q\rangle \underbrace{-L\xi\langle\delta D\, Q_2,\Delta\delta Q\rangle}_{\mathcal{A}_1} \underbrace{-L\langle\delta\Omega\, Q_2,\Delta\delta Q\rangle}_{\mathcal{B}_1} \underbrace{-L\xi\langle\frac{\delta D}{2},\Delta\delta Q\rangle}_{\mathcal{C}_1}$$

$$\underbrace{-L\langle\frac{\delta\Omega}{2},\Delta\delta Q\rangle}_{\mathcal{D}_1} -L\langle(\xi D_2+\Omega_2)\delta Q,\Delta\delta Q\rangle - L\langle\delta Q(\xi\delta D-\delta\Omega),\Delta\delta Q\rangle \underbrace{-L\xi\langle Q_2\delta D,\Delta\delta Q\rangle}_{\mathcal{A}_2}$$

$$\underbrace{+L\langle Q_2\delta\Omega,\Delta\delta Q\rangle}_{\mathcal{B}_2} \underbrace{-L\xi\langle\frac{\delta D}{2},\Delta\delta Q\rangle}_{\mathcal{C}_2} \underbrace{+L\langle\frac{\delta\Omega}{2},\Delta\delta Q\rangle}_{\mathcal{D}_2} -L\langle\delta Q(\xi D_2-\Omega_2),\Delta\delta Q\rangle$$

$$+2L\xi\langle\delta Q\,\mathrm{tr}(\delta Q\nabla\delta u),\Delta\delta Q\rangle+2L\xi\langle\delta Q\,\mathrm{tr}(\delta Q\nabla u_2),\Delta\delta Q\rangle+2L\xi\langle\delta Q\,\mathrm{tr}(Q_2\nabla\delta u),\Delta\delta Q\rangle$$

$$+2L\xi\langle Q_2\,\mathrm{tr}(\delta Q\nabla\delta u),\Delta\delta Q\rangle \underbrace{+2L\xi\langle\frac{\mathrm{Id}}{2}\mathrm{tr}(\delta Q\nabla\delta u),\Delta\delta Q\rangle}_{=0} +2L\xi\langle\delta Q\,\mathrm{tr}(Q_2\nabla u_2),\Delta\delta Q\rangle$$

$$+2L\xi\langle Q_2\,\mathrm{tr}(\delta Q\nabla u_2),\Delta\delta Q\rangle \underbrace{+2L\xi\langle\frac{\mathrm{Id}}{2}\mathrm{tr}(\delta Q\nabla u_2),\Delta\delta Q\rangle}_{=0} \underbrace{+2L\xi\langle Q_2\,\mathrm{tr}(Q_2\nabla\delta u),\Delta\delta Q\rangle}_{\mathcal{E}_1}$$

$$\underbrace{+2L\xi\langle\frac{\mathrm{Id}}{2}\mathrm{tr}(Q_2\nabla\delta u),\Delta\delta Q\rangle}_{=0} +La\Gamma\langle\delta Q,\Delta\delta Q\rangle-Lb\Gamma\langle Q_1\delta Q+\delta QQ_2,\Delta\delta Q\rangle$$

$$\underbrace{+Lb\Gamma\langle\mathrm{tr}\{\delta QQ_1+Q_2\delta Q\}\frac{\mathrm{Id}}{2},\Delta\delta Q\rangle}_{=0} +Lc\Gamma\langle\delta Q\mathrm{tr}\{Q_1^2\},\Delta\delta Q\rangle$$

$$+Lc\Gamma\langle Q_2\mathrm{tr}\{\delta QQ_1+Q_2\delta Q\},\Delta\delta Q\rangle+L\langle\delta u\cdot\nabla Q_1,\Delta\delta Q\rangle$$

$$+L\langle u_2\cdot\nabla\delta Q,\Delta\delta Q\rangle-a\xi\langle\delta QQ_1,\nabla\delta u\rangle+b\xi\langle\delta QQ_1^2,\nabla\delta u\rangle-b\xi\langle\delta Q\mathrm{tr}(Q_1^2)\frac{\mathrm{Id}}{2},\nabla\delta u\rangle$$

$$-c\xi\langle\delta Q\mathrm{tr}(Q_1^2)Q_1,\nabla\delta u\rangle-a\xi\langle(Q_2+\frac{\mathrm{Id}}{2})\delta Q,\nabla\delta u\rangle$$

$$+b\xi\langle(Q_2+\frac{\mathrm{Id}}{2})(Q_1\delta Q+\delta QQ_2),\nabla\delta u\rangle-b\xi\langle\frac{Q_2}{2}\mathrm{tr}\{\delta QQ_1+Q_2\delta Q\},\nabla\delta u\rangle$$

$$\underbrace{-b\xi\langle\mathrm{tr}\{\delta QQ_1+Q_2\delta Q\}\frac{\mathrm{Id}}{9},\nabla\delta u\rangle}_{=0} -c\xi\langle(Q_2+\frac{\mathrm{Id}}{2})\delta Q\mathrm{tr}\{Q_1^2\},\nabla\delta u\rangle$$

$$-c\xi\langle(Q_2+\frac{\mathrm{Id}}{2})Q_2\mathrm{tr}\{\delta QQ_1+Q_2\delta Q\},\nabla\delta u\rangle+L\xi\langle\delta Q\Delta\delta Q,\nabla\delta u\rangle+L\xi\langle\delta Q\Delta Q_2,\nabla\delta u\rangle$$

$$\underbrace{+L\xi\langle Q_2\Delta\delta Q,\nabla\delta u\rangle}_{\mathcal{A}_3} \underbrace{+L\xi\langle\frac{\Delta\delta Q}{2},\nabla\delta u\rangle}_{\mathcal{C}_3} -a\xi\langle Q_1\delta Q,\nabla\delta u\rangle$$

$$+b\xi\langle(Q_1^2-\mathrm{tr}\{Q_1^2\}\frac{\mathrm{Id}}{2})\delta Q,\nabla\delta u\rangle-c\xi\langle Q_1^2\mathrm{tr}\{Q_1^2\}\delta Q,\nabla\delta u\rangle-a\xi\langle\delta Q(Q_2+\frac{\mathrm{Id}}{2}),\nabla\delta u\rangle$$

$$+b\xi\langle(Q_1\delta Q+\delta QQ_2)(Q_2+\frac{\mathrm{Id}}{2}),\nabla\delta u\rangle-b\xi\langle\mathrm{tr}\{\delta QQ_1+Q_2\delta Q\}\frac{Q_2}{2},\nabla\delta u\rangle$$

$$\underbrace{-b\xi\langle\mathrm{tr}\{\delta QQ_1+Q_2\delta Q\}\frac{\mathrm{Id}}{9},\nabla\delta u\rangle}_{=0} -c\xi\langle\delta Q\mathrm{tr}\{Q_1^2\}(Q_2+\frac{\mathrm{Id}}{2}),\nabla\delta u\rangle$$

$$-c\xi\langle Q_2\mathrm{tr}\{\delta Q\delta Q_1+Q_2\delta Q\}(Q_2+\frac{\mathrm{Id}}{2}),\nabla\delta u\rangle+L\xi\langle\Delta\delta Q\,\delta Q,\nabla\delta u\rangle+L\xi\langle\Delta Q_2\delta Q,\nabla\delta u\rangle$$

$$\underbrace{+L\xi\langle\Delta\delta QQ_2,\nabla\delta u\rangle}_{\mathcal{A}_4} \underbrace{+L\xi\langle\frac{\Delta\delta Q}{2},\nabla\delta u\rangle}_{\mathcal{C}_4} +2a\xi\langle\delta Q\mathrm{tr}\{Q_1^2\},\nabla\delta u\rangle-2b\xi\langle\delta Q\mathrm{tr}\{Q_1^3\},\nabla\delta u\rangle$$

$$\underbrace{+2b\xi\langle\frac{\delta Q}{2}\mathrm{tr}\{Q_1\}\mathrm{tr}\{Q_1^2\},\nabla\delta u\rangle}_{=0}+2c\xi\langle\delta Q\mathrm{tr}\{Q_1^2\}^2,\nabla\delta u\rangle+2a\xi\langle Q_2\mathrm{tr}\{\delta QQ_1\},\nabla\delta u\rangle$$
$$-2b\xi\langle Q_2\mathrm{tr}\{\delta QQ_1^2\},\nabla\delta u\rangle\underbrace{+2b\xi\langle\frac{Q_2}{2}\mathrm{tr}\{\delta Q\}\mathrm{tr}\{Q_1^2\},\nabla\delta u\rangle}_{=0}+2c\xi\langle Q_2\mathrm{tr}\{\delta QQ_1\}\mathrm{tr}\{Q_1^2\},\nabla\delta u\rangle$$
$$+2a\xi\langle Q_2\mathrm{tr}\{Q_2\delta Q\},\nabla\delta u\rangle-2b\xi\langle Q_2\mathrm{tr}\{Q_2(Q_1\delta Q+\delta QQ_2)\},\nabla\delta u\rangle$$
$$\underbrace{+2b\xi\langle Q_2\mathrm{tr}\{\frac{Q_2}{2}\}\mathrm{tr}\{\delta QQ_1+Q_2\delta Q\},\nabla\delta u\rangle}_{=0}+2c\xi\langle Q_2\mathrm{tr}\{Q_2\delta Q\}\mathrm{tr}\{Q_1^2\},\nabla\delta u\rangle$$
$$+2c\xi\langle Q_2\mathrm{tr}\{Q_2^2\}\mathrm{tr}\{\delta QQ_1+Q_2\delta Q\},\nabla\delta u\rangle-2L\xi\langle\delta Q\mathrm{tr}\{\delta Q\Delta\delta Q\},\nabla\delta u\rangle$$
$$-2L\xi\langle\delta Q\mathrm{tr}\{\delta Q\Delta Q_2\},\nabla\delta u\rangle-2L\xi\langle\delta Q\mathrm{tr}\{Q_2\Delta\delta Q\},\nabla\delta u\rangle$$
$$-2L\xi\langle Q_2\mathrm{tr}\{\delta Q\Delta\delta Q\},\nabla\delta u\rangle\underbrace{-2L\xi\langle\frac{\mathrm{Id}}{2}\mathrm{tr}\{\delta Q\Delta\delta Q\},\nabla\delta u\rangle}_{=0}-2L\xi\langle\delta Q\mathrm{tr}\{Q_2\Delta Q_2\},\nabla\delta u\rangle$$
$$-2L\xi\langle Q_2\mathrm{tr}\{\delta Q\Delta Q_2\},\nabla\delta u\rangle\underbrace{-2L\xi\langle\frac{\mathrm{Id}}{2}\mathrm{tr}\{\delta Q\Delta Q_2\},\nabla\delta u\rangle}_{=0}\underbrace{-2L\xi\langle Q_2\mathrm{tr}\{Q_2\Delta\delta Q\},\nabla\delta u\rangle}_{\mathcal{E}_2}$$
$$\underbrace{-2L\xi\langle\frac{\mathrm{Id}}{2}\mathrm{tr}\{Q_2\Delta\delta Q\},\nabla\delta u\rangle}_{=0}+L\langle\nabla\delta Q\odot\nabla Q_1,\nabla\delta u\rangle+L\langle\nabla Q_2\odot\nabla\delta Q_1,\nabla\delta u\rangle$$
$$\underbrace{+L\langle\frac{\mathrm{Id}}{2}\mathrm{tr}\{\delta QQ_1\},\nabla\delta u\rangle}_{=0}\underbrace{+L\langle\frac{\mathrm{Id}}{2}\mathrm{tr}\{Q_2\delta Q\},\nabla\delta u\rangle}_{=0}+La\langle\delta QQ_1,\nabla\delta u\rangle$$
$$-Lb\langle\delta Q(Q_1^2-\mathrm{tr}\{Q_1^2\}\frac{\mathrm{Id}}{2}),\nabla\delta u\rangle+Lc\langle\delta QQ_1\mathrm{tr}\{Q_1^2\},\nabla\delta u\rangle$$
$$+a\langle Q_2\delta Q,\nabla\delta u\rangle-b\langle Q_2(Q_1\delta Q+\delta QQ_2),\nabla\delta u\rangle+b\langle Q_2\mathrm{tr}\{\delta QQ_1+Q_2\delta Q\}\frac{\mathrm{Id}}{2},\nabla\delta u\rangle$$
$$+c\langle Q_2\delta Q\mathrm{tr}\{Q_1^2\},\nabla\delta u\rangle\underbrace{+c\langle Q_2^2\mathrm{tr}\{\delta QQ_1+Q_2\delta Q\},\nabla\delta u\rangle}_{\mathcal{F}_1}$$
$$-a\langle Q_1\delta Q,\nabla\delta u\rangle+b\langle(Q_1^2-\mathrm{tr}\{Q_1^2\}\frac{\mathrm{Id}}{2})\delta Q,\nabla\delta u\rangle-c\langle Q_1\mathrm{tr}\{Q_1^2\}\delta Q,\nabla\delta u\rangle$$
$$-a\langle\delta QQ_2,\nabla\delta u\rangle+b\langle(Q_1\delta Q+\delta QQ_2)Q_2,\nabla\delta u\rangle$$
$$-b\langle\mathrm{tr}\{\delta QQ_1+Q_2\delta Q\}\frac{\mathrm{Id}}{2}Q_2,\nabla\delta u\rangle-c\langle\delta Q\mathrm{tr}\{Q_1^2\}Q_2,\nabla\delta u\rangle$$
$$\underbrace{-c\langle Q_2\mathrm{tr}\{\delta QQ_1+Q_2\delta Q\}Q_2,\nabla\delta u\rangle}_{\mathcal{F}_2}-L\langle\delta Q\Delta\delta Q,\nabla\delta u\rangle\underbrace{-L\langle Q_2\Delta\delta Q,\nabla\delta u\rangle}_{\mathcal{B}_3}$$
$$-L\langle\delta Q\Delta Q_2,\nabla\delta u\rangle+L\langle\Delta\delta Q\delta Q,\nabla\delta u\rangle\underbrace{+L\langle\Delta\delta QQ_2,\nabla\delta u\rangle}_{\mathcal{B}_4}$$
$$+L\langle\Delta Q_2\delta Q,\nabla\delta u\rangle-\langle u_2\cdot\nabla\delta u,\delta u\rangle-\langle\delta u\cdot\nabla u_1,\delta u\rangle. \tag{4.6}$$

Denoting

$$\Phi(t):=1/(2\lambda)\|\delta u(t)\|^2_{\dot H^{-1/2}}+L\|\nabla\delta Q(t)\|^2_{\dot H^{-1/2}},$$

we will aim to show that Φ satisfies the inequality

$$\Phi'(t)\le\chi(t)\mu(\Phi(t)), \tag{4.7}$$

where μ is an Osgood modulus of continuity (see [2], Definition 3.1) given by

$$\mu(r):=r+r\ln\Big(1+e+\frac{1}{r}\Big)+r\ln\Big(1+e+\frac{1}{r}\Big)\ln\ln\Big(1+e+\frac{1}{r}\Big) \tag{4.8}$$

with $\chi\in L^1_{loc}$ a priori. We are going to find a double-logarithmic estimate, hence thanks to the Osgood Lemma (see [2, Lemma 3.4]) and since $\Phi(0)$ is null, we get that $\Phi\equiv 0$, which yields the uniqueness of the solution for system (1.8).

First, let us observe following simplifications of Equation (4.6):

$$0=\mathcal{C}_1+\mathcal{C}_2+\mathcal{C}_3+\mathcal{C}_4=\mathcal{D}_1+\mathcal{D}_2=\mathcal{F}_1+\mathcal{F}_2.$$

The key method we use to obtain the desired estimates is the para-differential calculus decomposition summarized in the following.

Remark 4.1. Let q be an integer and A,B be $d\times d$ matrices whose components are homogeneous temperate distributions. We use the following notation:

$$\mathcal{J}^1_q(A,B):=\textstyle\sum_{|q-q'|\le 5}[\dot\Delta_q,\dot S_{q'-1}A]\dot\Delta_{q'}B, \qquad \mathcal{J}^3_q(A,B):=\dot S_{q-1}A\dot\Delta_q B,$$

$$\mathcal{J}^2_q(A,B):=\textstyle\sum_{|q-q'|\le 5}(\dot S_{q'-1}A-\dot S_{q-1}A)\dot\Delta_q\dot\Delta_{q'}B,\ \mathcal{J}^4_q(A,B):=\textstyle\sum_{q'\ge q-5}\dot\Delta_q(\dot\Delta_{q'}A\dot S_{q'+2}B).$$

Then we can decompose the product AB as follows:

$$\dot\Delta_q(AB)=\mathcal{J}^1_q(A,B)+\mathcal{J}^2_q(A,B)+\mathcal{J}^3_q(A,B)+\mathcal{J}^4_q(A,B) \tag{4.9}$$

for any integer q, thanks to Equation (4.3).

Moreover, from now on, we will use the notation $\lesssim$ as follows: for any non-negative real numbers a and b, we denote $a\lesssim b$ if and only if there exists a positive constant C (independent of a and b) such that $a\le Cb$.

4.2.1. Estimate of $\mathcal{A}_1+\mathcal{A}_2+\mathcal{A}_3+\mathcal{A}_4$. Let us begin analyzing the terms $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, and $\mathcal{A}_4$ of Equation (4.6). First, we observe that

$$\begin{aligned}\mathcal{A}_2&=-L\xi\sum_{q\in\mathbb{Z}}2^{-q}\langle\dot\Delta_q(Q_2\delta D),\dot\Delta_q\Delta\delta Q\rangle_{L^2_x}\\&=-L\xi\sum_{q\in\mathbb{Z}}2^{-q}\sum_{i=1}^4\langle\mathcal{J}^i_q(Q_2,\delta D),\dot\Delta_q\Delta\delta Q\rangle_{L^2_x}\end{aligned}$$

Now, when $i=1$, we have

$$\begin{aligned}&2^{-q}\langle\mathcal{J}^1_q(Q_2,\delta D),\dot\Delta_q\Delta\delta Q\rangle_{L^2_x}\\&=\sum_{|q-q'|\le 5}2^{-q}\langle[\dot\Delta_q,\dot S_{q'-1}Q_2]\dot\Delta_{q'}\delta D,\dot\Delta_q\Delta\delta Q\rangle_{L^2_x}\\&\lesssim\sum_{|q-q'|\le 5}2^{-q}\|[\dot\Delta_q,\dot S_{q'-1}Q_2]\dot\Delta_{q'}\delta D\|_{L^2_x}\|\dot\Delta_q\Delta\delta Q\|_{L^2_x}\\&\lesssim\sum_{|q-q'|\le 5}2^{-2q}\|\dot S_{q'-1}\nabla Q_2\|_{L^4_x}\|\dot\Delta_{q'}\delta D\|_{L^4_x}\|\dot\Delta_q\Delta\delta Q\|_{L^2_x}\\&\lesssim\sum_{|q-q'|\le 5}2^{-q}\|\dot S_{q'-1}\nabla Q_2\|^{\frac12}_{L^2_x}\|\dot S_{q'-1}\Delta Q_2\|^{\frac12}_{L^2_x}\|\dot\Delta_{q'}\delta u\|_{L^4_x}\|\dot\Delta_q\Delta\delta Q\|_{L^2_x}\\&\lesssim\sum_{|q-q'|\le 5}\|\nabla Q_2\|^{\frac12}_{L^2_x}\|\Delta Q_2\|^{\frac12}_{L^2_x}\|\dot\Delta_{q'}\delta u\|_{L^2_x}2^{-\frac q2}\|\dot\Delta_q\Delta\delta Q\|_{L^2_x}\end{aligned} \tag{4.10}$$

for every $q\in\mathbb{Z}$. Hence, we get

$$\begin{aligned}
&-L\xi\sum_{q\in\mathbb{Z}}2^{-q}\langle\mathcal{J}^1_q(Q_2,\delta D),\dot{\Delta}_q\Delta\delta Q\rangle_{L^2_x}\\
&\lesssim\|\nabla Q_2\|_{L^2_x}^{\frac12}\|\Delta Q_2\|_{L^2_x}^{\frac12}\|\delta u\|_{L^2_x}\|\Delta\delta Q\|_{\dot H^{-\frac12}}\\
&\lesssim\|\nabla Q_2\|_{L^2_x}^{\frac12}\|\Delta Q_2\|_{L^2_x}^{\frac12}\|\delta u\|_{\dot H^{-\frac12}}^{\frac12}\|\nabla\delta u\|_{\dot H^{-\frac12}}^{\frac12}\|\Delta\delta Q\|_{\dot H^{-\frac12}}\\
&\lesssim\|\nabla Q_2\|_{L^2_x}^{2}\|\Delta Q_2\|_{L^2_x}^{2}\|\delta u\|_{\dot H^{-\frac12}}^{2}+C_\nu\|\nabla\delta u\|_{\dot H^{-\frac12}}^{2}+C_{\Gamma,L}\|\Delta\delta Q\|_{\dot H^{-\frac12}}^{2},
\end{aligned}\tag{4.11}$$

where we have used the following interpolation inequality:

$$\|\delta u\|_{L^2_x}\le\|\delta u\|_{\dot H^{-\frac12}}^{\frac12}\|\delta u\|_{\dot H^{\frac12}}^{\frac12}=\|\delta u\|_{\dot H^{-\frac12}}^{\frac12}\|\nabla\delta u\|_{\dot H^{-\frac12}}^{\frac12}.$$

When $i=2$, the following inequalities are fulfilled:

$$\begin{aligned}
&2^{-q}\langle\mathcal{J}^2_q(Q_2,\delta D),\dot{\Delta}_q\Delta\delta Q\rangle_{L^2_x}\\
&=\sum_{|q-q'|\le5}2^{-q}\langle(\dot S_{q'-1}Q_2-\dot S_{q-1}Q_2)\dot\Delta_q\dot\Delta_{q'}\delta D,\dot\Delta_q\Delta\delta Q\rangle_{L^2_x}\\
&\lesssim\sum_{|q-q'|\le5}2^{-q}\|(\dot S_{q'-1}Q_2-\dot S_{q-1}Q_2)\|_{L^\infty_x}\|\dot\Delta_q\dot\Delta_{q'}\delta D\|_{L^2_x}\|\dot\Delta_q\Delta\delta Q\|_{L^2_x}\\
&\lesssim\sum_{|q-q'|\le5}2^{-2q}\|(\dot S_{q'-1}\Delta Q_2-\dot S_{q-1}\Delta Q_2)\|_{L^2_x}\|\dot\Delta_q\dot\Delta_{q'}\delta D\|_{L^2_x}\|\dot\Delta_q\Delta\delta Q\|_{L^2_x}\\
&\lesssim\sum_{|q-q'|\le5}2^{-2q}\|\Delta Q_2\|_{L^2_x}\|\dot\Delta_{q'}\delta D\|_{L^2_x}\|\dot\Delta_q\Delta\delta Q\|_{L^2_x}\\
&\lesssim\sum_{|q-q'|\le5}2^{-\frac{q'}{2}}\|\dot\Delta_{q'}\delta u\|_{L^2_x}2^{-\frac q2}\|\dot\Delta_q\Delta\delta Q\|_{L^2_x}\|\Delta Q_2\|_{L^2_x}
\end{aligned}\tag{4.12}$$

for any $q\in\mathbb{Z}$. Thus, it turns out that

$$-L\xi\sum_{q\in\mathbb{Z}}2^{-q}\langle\mathcal{J}^2_q(Q_2,\delta D),\dot\Delta_q\Delta\delta Q\rangle_{L^2_x}\lesssim\|\Delta Q_2\|_{L^2_x}^2\|\delta u\|_{\dot H^{-\frac12}}^2+C_{\Gamma,L}\|\Delta\delta Q\|_{\dot H^{-\frac12}}^2.\tag{4.13}$$

The term corresponding to $i=3$ cannot be estimated as before. We will see that this challenging term will be simplified. Finally, when $i=4$, we have

$$\begin{aligned}
&2^{-q}\langle\mathcal{J}^4_q(Q_2,\delta D),\dot\Delta_q\Delta\delta Q\rangle_{L^2_x}\\
&=L2^{-q}\sum_{q-q'\le5}\langle\dot\Delta_q\big[\dot\Delta_{q'}Q_2\dot S_{q'+2}\delta D\big],\dot\Delta_q\Delta\delta Q\rangle_{L^2_x}\\
&\lesssim2^{-q}\sum_{q-q'\le5}\|\dot\Delta_{q'}Q_2\|_{L^\infty_x}\|\dot S_{q'+2}\delta D\|_{L^2_x}\|\dot\Delta_q\Delta\delta Q\|_{L^2_x}\\
&\lesssim2^{-q}\sum_{q-q'\le5}2^{-q'}\|\dot\Delta_{q'}\Delta Q_2\|_{L^2_x}\|\dot S_{q'+2}\delta D\|_{L^2_x}2^q\|\dot\Delta_q\nabla\delta Q\|_{L^2_x}\\
&\lesssim\sum_{q-q'\le5}2^{\frac{q-q'}{2}}\|\dot\Delta_{q'}\Delta Q_2\|_{L^2_x}2^{-\frac{q'+2}{2}}\|\dot S_{q'+2}\delta D\|_{L^2_x}2^{-\frac q2}\|\dot\Delta_q\nabla\delta Q\|_{L^2_x}\\
&\lesssim\|\Delta Q_2\|_{L^2_x}2^{-\frac q2}\|\dot\Delta_q\nabla\delta Q\|_{L^2_x}\sum_{q-q'\le5}2^{\frac{q-q'}{2}}2^{-\frac{q'+2}{2}}\|\dot S_{q'+2}\delta D\|_{L^2_x},
\end{aligned}\tag{4.14}$$

for any $q\in\mathbb{Z}$. Hence,

$$\begin{aligned}
&-L\xi\sum_{q\in\mathbb{Z}}2^{-q}\langle\mathcal{J}_q^4(Q_2,\delta D),\dot{\Delta}_q\Delta\delta Q\rangle_{L^2_x}\\
&\lesssim\|\Delta Q_2\|_{L^2_x}\sum_{q\in\mathbb{Z}}2^{-\frac{q}{2}}\|\dot{\Delta}_q\nabla\delta Q\|_{L^2_x}\sum_{q'\in\mathbb{Z}}2^{\frac{q-q'}{2}}1_{(-\infty,5]}(q-q')2^{-\frac{q'+2}{2}}\|\dot{S}_{q'+2}\delta D\|_{L^2_x}\\
&\lesssim\|\Delta Q_2\|_{L^2_x}\|\nabla\delta Q\|_{\dot{H}^{-\frac{1}{2}}}\Big(\sum_{q\in\mathbb{Z}}\big|\sum_{q'\in\mathbb{Z}}2^{q-q'}1_{(-\infty,5]}(q-q')2^{-\frac{q'+2}{2}}\|\dot{S}_{q'+2}\delta D\|_{L^2_x}\big|^2\Big)^{\frac{1}{2}},
\end{aligned}$$

and by convolution

$$\begin{aligned}
&\Big(\sum_{q\in\mathbb{Z}}\big|\sum_{q'\in\mathbb{Z}}2^{q-q'}1_{(-\infty,5]}(q-q')2^{-\frac{q'+2}{2}}\|\dot{S}_{q'+2}\delta D\|_{L^2_x}\big|^2\Big)^{\frac{1}{2}}\\
&\lesssim\big(\sum_{q\leq 5}2^q\big)\Big(\sum_{q\in\mathbb{Z}}2^{-q}\|\dot{S}_q\delta D\|^2_{L^2_x}\Big)^{\frac{1}{2}}\Big)\lesssim\|\nabla\delta u\|_{\dot{H}^{-\frac{1}{2}}},
\end{aligned}$$

so that

$$\begin{aligned}
&-L\xi\sum_{q\in\mathbb{Z}}2^{-q}\langle\mathcal{J}_q^4(Q_2,\delta D),\dot{\Delta}_q\Delta\delta Q\rangle_{L^2_x}\\
&\lesssim\|\Delta Q_2\|_{L^2_x}\|\nabla\delta Q\|_{\dot{H}^{-\frac{1}{2}}}\|\nabla\delta u\|_{\dot{H}^{-\frac{1}{2}}}\\
&\lesssim\|\Delta Q_2\|^2_{L^2_x}\|\nabla\delta Q\|^2_{\dot{H}^{-\frac{1}{2}}}+C_\nu\|\nabla\delta u\|^2_{\dot{H}^{-\frac{1}{2}}}.
\end{aligned}\tag{4.15}$$

Summarising, it remains to control

$$\mathcal{A}_1+\mathcal{A}_3+\mathcal{A}_4-L\xi\sum_{q\in\mathbb{Z}}2^{-q}\langle\mathcal{J}_q^3(Q_2,\delta D),\dot{\Delta}_q\Delta\delta Q\rangle_{L^2_x}.$$

Now, observing that

$$\begin{aligned}
\mathcal{A}_1=-L\xi\sum_{q\in\mathbb{Z}}2^{-q}\langle\dot{\Delta}_q(\delta DQ_2),\dot{\Delta}_q\Delta\delta Q\rangle_{L^2_x}&=-L\xi\sum_{q\in\mathbb{Z}}2^{-q}\langle\dot{\Delta}_q{}^{\mathrm{t}}(\delta DQ_2),\dot{\Delta}_q{}^{\mathrm{t}}\Delta\delta Q\rangle_{L^2_x}\\
&=-L\xi\sum_{q\in\mathbb{Z}}2^{-q}\langle\dot{\Delta}_q(Q_2\delta D),\dot{\Delta}_q\Delta\delta Q\rangle_{L^2_x}=\mathcal{A}_2,
\end{aligned}$$

we estimate $\mathcal{A}_1$ with the previous inequalities, so that it remains to control

$$\begin{aligned}
&\mathcal{A}_3+\mathcal{A}_4-2L\xi\sum_{q\in\mathbb{Z}}2^{-q}\mathcal{J}_q^3(Q_2,\delta D)\\
=&\mathcal{A}_3+\mathcal{A}_4-2L\xi\sum_{q\in\mathbb{Z}}2^{-q}\int_{\mathbb{R}^2}\mathrm{tr}\big\{\dot{S}_{q-1}Q_2\dot{\Delta}_q\delta D\dot{\Delta}_q\Delta\delta Q\big\}.
\end{aligned}$$

Now, let us consider $\mathcal{A}_3=L\xi\langle Q_2\Delta\delta Q,\nabla\delta u\rangle$. We proceed along the lines used before, namely we use the decomposition given by Equation (4.9):

$$\mathcal{A}_3=L\xi\sum_{q\in\mathbb{Z}}2^{-q}\langle\dot{\Delta}_q(Q_2\Delta\delta Q),\dot{\Delta}_q\nabla\delta u\rangle_{L^2_x}=L\xi\sum_{q\in\mathbb{Z}}2^{-q}\sum_{i=1}^{4}\langle\mathcal{J}_q^i(Q_2,\Delta\delta Q),\dot{\Delta}_q\nabla\delta u\rangle_{L^2_x}.$$

When $i=1$, proceeding as for Equation (4.10), we have

$$\begin{aligned}&2^{-q}\langle \mathcal{J}^1_q(Q_2,\Delta\delta Q),\dot{\Delta}_q\nabla\delta u\rangle_{L^2_x}\\ &\lesssim \|\nabla Q_2\|_{L^2_x}^{\frac12}\|\Delta Q_2\|_{L^2_x}^{\frac12}\sum_{|q-q'|\le 5}\|\dot{\Delta}_{q'}\nabla\delta Q\|_{L^2_x}2^{-\frac{q}{2}}\|\dot{\Delta}_q\nabla\delta u\|_{L^2_x}.\end{aligned}$$

Thus considering the sum over $q\in\mathbb{Z}$ as in Equation (4.11), we deduce that

$$\begin{aligned}&L\xi\sum_{q\in\mathbb{Z}}2^{-q}\langle \mathcal{J}^1_q(Q_2,\Delta\delta Q),\dot{\Delta}_q\nabla\delta u\rangle_{L^2_x}\\ &\lesssim \|\nabla Q_2\|_{L^2_x}^2\|\Delta Q_2\|_{L^2_x}^2\|\nabla\delta Q\|^2_{\dot{H}^{-\frac12}}+C_\nu\|\nabla\delta u\|^2_{\dot{H}^{-\frac12}}+C_{\Gamma,L}\|\Delta\delta Q\|^2_{\dot{H}^{-\frac12}}.\end{aligned}\tag{4.16}$$

Proceeding as when proving Equation (4.12), when $i=2$, we get

$$2^{-q}\langle \mathcal{J}^2_q(Q_2,\Delta\delta Q),\dot{\Delta}_q\nabla\delta u\rangle_{L^2_x}\lesssim \sum_{|q-q'|\le 5}2^{-\frac{q}{2}}\|\dot{\Delta}_{q'}\delta u\|_{L^2_x}2^{-\frac{q}{2}}\|\dot{\Delta}_q\Delta\delta Q\|_{L^2_x}\|\Delta Q_2\|_{L^2_x}$$

for every $q\in\mathbb{Z}$. Thus, as in Equation (4.13), it turns out that

$$L\xi\sum_{q\in\mathbb{Z}}2^{-q}\langle \mathcal{J}^2_q(Q_2,\Delta\delta Q),\dot{\Delta}_q\nabla\delta u\rangle_{L^2_x}\lesssim \|\Delta Q_2\|^2_{L^2_x}\|\delta u\|^2_{\dot{H}^{-\frac12}}+C_{\Gamma,L}\|\Delta\delta Q\|^2_{\dot{H}^{-\frac12}}.\tag{4.17}$$

Finally, with the same strategy as for Equation (4.14), we observe that

$$\begin{aligned}&2^{-q}\langle \mathcal{J}^4_q(Q_2,\Delta\delta Q),\dot{\Delta}_q\nabla\delta u\rangle_{L^2_x}\\ &\lesssim \|\Delta Q_2\|_{L^2_x}2^{-q}\|\dot{\Delta}_q\delta u\|^2_{L^2_x}\sum_{q-q'\le 5}2^{\frac{q-q'}{2}}2^{-\frac{q'+2}{2}}\|\dot{S}_{q'+2}\Delta\delta Q\|_{L^2_x},\end{aligned}$$

hence, as for Equation (4.15), we obtain

$$\begin{aligned}L\xi\sum_{q\in\mathbb{Z}}2^{-q}\langle \mathcal{J}^4_q(Q_2,\Delta\delta Q),\dot{\Delta}_q\nabla\delta u\rangle_{L^2_x}&\lesssim \|\Delta Q_2\|_{L^2_x}\|\delta u\|_{\dot{H}^{-\frac12}}\|\Delta\delta Q\|_{\dot{H}^{-\frac12}}\\ &\lesssim \|\Delta Q_2\|^2_{L^2_x}\|\delta u\|^2_{\dot{H}^{-\frac12}}+C_{\Gamma,L}\|\Delta\delta Q\|^2_{\dot{H}^{-\frac12}}.\end{aligned}\tag{4.18}$$

Summarising all the previous considerations, we note that it remains to control

$$\begin{aligned}&\mathcal{A}_4+L\xi\sum_{q\in\mathbb{Z}}2^{-q}\Big[\langle \mathcal{J}^3_q(Q_2,\Delta\delta Q),\dot{\Delta}_q\nabla\delta u\rangle_{L^2_x}-2\int_{\mathbb{R}^2}\mathrm{tr}\big\{\dot{S}_{q-1}Q_2\dot{\Delta}_q\delta D\dot{\Delta}_q\Delta\delta Q\big\}\Big]\\ &=\mathcal{A}_4+L\xi\sum_{q\in\mathbb{Z}}2^{-q}\int_{\mathbb{R}^2}\Big[\mathrm{tr}\{\dot{S}_{q-1}Q_2\dot{\Delta}_q\Delta\delta Q\dot{\Delta}_q\nabla\delta u\}-2\mathrm{tr}\big\{\dot{S}_{q-1}Q_2\dot{\Delta}_q\delta D\dot{\Delta}_q\Delta\delta Q\big\}\Big].\end{aligned}$$

We handle the last term $\mathcal{A}_4$ arguing as for $\mathcal{A}_3$, since $\mathcal{A}_4$ is given by

$$\begin{aligned}L\xi\langle\dot{\Delta}_q(\Delta\delta Q Q_2),\dot{\Delta}_q\nabla\delta u\rangle_{\dot{H}^{-\frac12}}&=L\xi\langle\dot{\Delta}_q(Q_2\Delta\delta Q),{}^{\mathrm{t}}\dot{\Delta}_q\nabla\delta u\rangle_{\dot{H}^{-\frac12}}\\ &=L\xi\sum_{q\in\mathbb{Z}}2^{-q}\sum_{i=1}^4\langle \mathcal{J}^i_q(Q_2,\Delta\delta Q),\dot{\Delta}_q{}^{\mathrm{t}}\nabla\delta u\rangle_{L^2_x}.\end{aligned}$$

The terms related to $i=1,2,4$ are estimated similarly as $\mathcal{A}_3$. Hence it remains to evaluate

$$
\begin{aligned}
&L\xi\sum_{q\in\mathbb{Z}}2^{-q}\Big\{\langle\mathcal{J}_q^3(Q_2,\Delta\delta Q),\dot{\Delta}_q{}^{\mathrm{t}}\nabla\delta u\rangle_{L^2_x}\\
&\quad+\int_{\mathbb{R}^2}\big[\mathrm{tr}\{\dot{S}_{q-1}Q_2\dot{\Delta}_q\Delta\delta Q\dot{\Delta}_q\nabla\delta u\}-2\mathrm{tr}\big\{\dot{S}_{q-1}Q_2\dot{\Delta}_q\delta D\dot{\Delta}_q\Delta\delta Q\big\}\big]\Big\}\\
=&2L\xi\sum_{q\in\mathbb{Z}}2^{-q}\int_{\mathbb{R}^2}\Big[\mathrm{tr}\{\dot{S}_{q-1}Q_2\dot{\Delta}_q\Delta\delta Q\dot{\Delta}_q\delta D\}-\mathrm{tr}\big\{\dot{S}_{q-1}Q_2\dot{\Delta}_q\delta D\dot{\Delta}_q\Delta\delta Q\big\}\Big]=0,
\end{aligned}
$$

which is a null series since the trace acts on symmetric matrices, so that we can permute their order.

4.2.2. Estimate of $\mathcal{B}_1+\mathcal{B}_2+\mathcal{B}_3+\mathcal{B}_4$. Now we want to estimate $\mathcal{B}_1+\mathcal{B}_2+\mathcal{B}_3+\mathcal{B}_4$, namely

$$
-L\langle\delta\Omega Q_2-Q_2\delta\Omega,\Delta\delta Q\rangle_{\dot{H}^{-\frac12}}-L\langle Q_2\Delta\delta Q-\Delta\delta QQ_2,\nabla\delta u\rangle_{\dot{H}^{-\frac12}}.
$$

First, let us consider

$$
\begin{aligned}
\mathcal{B}_2&=L\langle Q_2\delta\Omega,\Delta\delta Q\rangle_{\dot{H}^{-\frac12}}\\
&=L\sum_{q\in\mathbb{Z}}2^{-q}\langle\dot{\Delta}_q(\delta\Omega Q_2),\dot{\Delta}_q\Delta\delta Q\rangle_{L^2_x}=L\sum_{q\in\mathbb{Z}}2^{-q}\sum_{i=1}^4\langle\mathcal{J}_q^i(Q_2,\delta\Omega),\dot{\Delta}_q\Delta\delta Q\rangle_{L^2_x}.
\end{aligned}
$$

Proceeding exactly as for proving Equations (4.11), (4.13) and (4.15), with $\delta\Omega$ instead of δD, the following estimates are obtained:

$$
\begin{aligned}
&L\sum_{q\in\mathbb{Z}}2^{-q}\langle\mathcal{J}_q^1(Q_2,\delta\Omega),\dot{\Delta}_q\Delta\delta Q\rangle_{L^2_x}\\
&\lesssim\|\nabla Q_2\|_{L^2_x}^2\|\Delta Q_2\|_{L^2_x}^2\|\delta u\|_{\dot{H}^{-\frac12}}^2+C_\nu\|\nabla\delta u\|_{\dot{H}^{-\frac12}}^2+C_{\Gamma,L}\|\Delta\delta Q\|_{\dot{H}^{-\frac12}}^2,
\end{aligned}
$$

$$
L\sum_{q\in\mathbb{Z}}2^{-q}\langle\mathcal{J}_q^2(Q_2,\delta\Omega),\dot{\Delta}_q\Delta\delta Q\rangle_{L^2_x}\lesssim\|\Delta Q_2\|_{L^2_x}^2\|\delta u\|_{\dot{H}^{-\frac12}}^2+C_{\Gamma,L}\|\Delta\delta Q\|_{\dot{H}^{-\frac12}}^2,
$$

and

$$
L\sum_{q\in\mathbb{Z}}2^{-q}\langle\mathcal{J}_q^4(Q_2,\delta\Omega),\dot{\Delta}_q\Delta\delta Q\rangle_{L^2_x}\lesssim\|\Delta Q_2\|_{L^2_x}^2\|\nabla\delta Q\|_{\dot{H}^{-\frac12}}^2+C_\nu\|\nabla\delta u\|_{\dot{H}^{-\frac12}}^2.
$$

Now observing that

$$
\mathcal{B}_1=-L\langle\delta\Omega Q_2,\Delta\delta Q\rangle_{\dot{H}^{-\frac12}}=-L\langle{}^{\mathrm{t}}(\delta\Omega Q_2),{}^{\mathrm{t}}\Delta\delta Q\rangle_{\dot{H}^{-\frac12}}=L\langle Q_2\delta\Omega,\Delta\delta Q\rangle_{\dot{H}^{-\frac12}}=\mathcal{B}_2,
$$

it remains to control

$$
\begin{aligned}
&\mathcal{B}_3+\mathcal{B}_4+2L\sum_{q\in\mathbb{Z}}2^{-q}\langle\mathcal{J}_q^3(Q_2,\delta\Omega),\dot{\Delta}_q\Delta\delta Q\rangle_{L^2_x}\\
=&\mathcal{B}_3+\mathcal{B}_4+2L\sum_{q\in\mathbb{Z}}2^{-q}\int_{\mathbb{R}^2}\mathrm{tr}\{\dot{S}_{q-1}Q_2\dot{\Delta}_q\delta\Omega\dot{\Delta}_q\Delta\delta Q\}.
\end{aligned}
$$

Now, we turn to $\mathcal{B}_3$:

$$-\mathcal{B}_3 = L\langle Q_2\Delta\delta Q, \nabla\delta u\rangle_{\dot H^{-\frac12}} = L\sum_{q\in\mathbb{Z}} 2^{-q}\langle \dot\Delta_q(Q_2\Delta\delta Q), \dot\Delta_q\nabla\delta u\rangle_{L^2_x}$$
$$= L\sum_{q\in\mathbb{Z}} 2^{-q}\sum_{i=1}^4 \langle \mathcal{J}^i_q(Q_2,\Delta\delta Q), \dot\Delta_q\nabla\delta u\rangle_{L^2_x}.$$

We remark that $\mathcal{B}_3 = -\mathcal{A}_3/\xi$, hence the terms related to $i=1,2,4$ are estimated as in Equations (4.16), (4.17), and (4.18). Thus it remains to control

$$\mathcal{B}_4 + L\sum_{q\in\mathbb{Z}} 2^{-q}\big[\langle \mathcal{J}^3_q(Q_2,\Delta\delta Q), \dot\Delta_q\nabla\delta u\rangle_{L^2_x} + 2\int_{\mathbb{R}^2}\mathrm{tr}\{\dot S_{q-1}Q_2\dot\Delta_q\delta\Omega\dot\Delta_q\Delta\delta Q\}\big]$$
$$= \mathcal{B}_4 + L\sum_{q\in\mathbb{Z}} 2^{-q}\int_{\mathbb{R}^2}\big[\mathrm{tr}\{\dot S_{q-1}Q_2\dot\Delta_q\Delta\delta Q\dot\Delta_q\nabla\delta u\} + 2\mathrm{tr}\{\dot S_{q-1}Q_2\dot\Delta_q\delta\Omega\dot\Delta_q\Delta\delta Q\}\big].$$

Observing that $\mathcal{B}_4 = -\mathcal{A}_4/\xi$ we argue as for $\mathcal{B}_3$; hence it remains to evaluate

$$L\sum_{q\in\mathbb{Z}} 2^{-q}\Big\{\langle \mathcal{J}^3_q(Q_2,\Delta\delta Q)\dot\Delta_q{}^{t}\nabla\delta u\rangle_{L^2_x}$$
$$+\int_{\mathbb{R}^2}\big[\mathrm{tr}\{\dot S_{q-1}Q_2\dot\Delta_q\Delta\delta Q\dot\Delta_q\nabla\delta u\} + 2\mathrm{tr}\{\dot S_{q-1}Q_2\dot\Delta_q\delta\Omega\dot\Delta_q\Delta\delta Q\}\big]\Big\}$$
$$= 2L\sum_{q\in\mathbb{Z}} 2^{-q}\int_{\mathbb{R}^2}\big[\mathrm{tr}\{\dot S_{q-1}Q_2\dot\Delta_q\Delta\delta Q\dot\Delta_q\delta\Omega\} + \mathrm{tr}\{\dot S_{q-1}Q_2\dot\Delta_q\delta\Omega\dot\Delta_q\Delta\delta Q\}\big] = 0,$$

where for the cancellation we used that $\dot S_{q-1}Q_2$ and $\dot\Delta_q\Delta\delta Q$ are symmetric while $\dot\Delta_q\delta\Omega$ is skew-symmetric.

4.2.3. One-logarithmic estimates. In this subsection, we evaluate the terms of Equation (4.6) which are related to the single-logarithmic term of the equality (4.7).

Estimate of $\langle \delta Q\mathrm{tr}\{Q_2\nabla u_2\}, \Delta\delta Q\rangle$. Let us fix a positive real number $N>0$ and split the considered term into two parts, the high and the low frequencies:

$$\langle \delta Q\mathrm{tr}\{Q_2\nabla u_2\}, \Delta\delta Q\rangle_{\dot H^{-\frac12}}$$
$$= \langle \delta Q\mathrm{tr}\{(\dot S_N Q_2)\nabla u_2\}, \Delta\delta Q\rangle_{\dot H^{-\frac12}} + \langle \delta Q\mathrm{tr}\{(\sum_{q\geq N}\dot\Delta_q Q_2)\nabla u_2\}, \Delta\delta Q\rangle_{\dot H^{-\frac12}}.$$

At first we deal with the low frequencies, observing that

$$\langle \delta Q\mathrm{tr}\{(\dot S_N Q_2)\nabla u_2\}, \Delta\delta Q\rangle_{\dot H^{-\frac12}} \lesssim \|\delta Q\mathrm{tr}\{(\dot S_N Q_2)\nabla u_2\}\|_{\dot H^{-\frac12}}\|\Delta\delta Q\|_{\dot H^{-\frac12}}$$
$$\lesssim \|\delta Q\|_{\dot H^{\frac12}}\|(\dot S_N Q_2)\nabla u_2\|_{L^2_x}\|\Delta\delta Q\|_{\dot H^{-\frac12}}$$
$$\lesssim \|\nabla\delta Q\|_{\dot H^{-\frac12}}\|\dot S_N Q_2\|_{L^\infty_x}\|\nabla u_2\|_{L^2_x}\|\Delta\delta Q\|_{\dot H^{-\frac12}}.$$

Hence by Theorem A.2, we get

$$\langle \delta Q\mathrm{tr}\{(\dot S_N Q_2)\nabla u_2\}, \Delta\delta Q\rangle_{\dot H^{-\frac12}}$$
$$\lesssim \|\nabla\delta Q\|_{\dot H^{-\frac12}}(\|Q_2\|_{L^2_x} + \sqrt{N}\|\nabla Q_2\|_{L^2_x})\|\nabla u_2\|_{L^2_x}\|\Delta\delta Q\|_{\dot H^{-\frac12}}$$

$$\lesssim (1+N)\|\nabla\delta Q\|^2_{\dot H^{-\frac12}}\|(Q_2,\nabla Q_2)\|^2_{L^2_x}\|\nabla u_2\|^2_{L^2_x}+C_\Gamma\|\Delta\delta Q\|^2_{\dot H^{-\frac12}}.$$

For the high frequencies, we proceed as follows:

$$\begin{aligned}
&\langle\delta Q\mathrm{tr}\{(\sum_{q\ge N}\dot\Delta_q Q_2)\nabla u_2\},\Delta\delta Q\rangle_{\dot H^{-\frac12}}\\
\lesssim&\|\delta Q\mathrm{tr}\{(\sum_{q\ge N}\dot\Delta_q Q_2)\nabla u_2\}\|_{\dot H^{-\frac12}}\|\Delta\delta Q\|_{\dot H^{-\frac12}}\\
\lesssim&\|\delta Q\|_{\dot H^{\frac34}}\|(\sum_{q\ge N}\dot\Delta_q Q_2)\nabla u_2\|_{\dot H^{-\frac14}}\|\Delta\delta Q\|_{\dot H^{-\frac12}}\\
\lesssim&\|(Q_1,Q_2)\|^{\frac14}_{L^2_x}\|\nabla(Q_1,Q_2)\|^{\frac34}_{L^2_x}\|\sum_{q\ge N}\dot\Delta_q Q_2\|_{\dot H^{\frac34}}\|\nabla u_2\|_{L^2_x}\|\Delta\delta Q\|_{\dot H^{-\frac12}}\\
\lesssim&\|(Q_1,Q_2)\|^{\frac14}_{L^2_x}\|\nabla(Q_1,Q_2)\|^{\frac34}_{L^2_x}(\sum_{q\ge N}2^{\frac34 q}\|\dot\Delta_q Q_2\|_{L^2_x})\|\nabla u_2\|_{L^2_x}\|\Delta\delta Q\|_{\dot H^{-\frac12}}\\
\lesssim&\|(Q_1,Q_2)\|^{\frac14}_{L^2_x}\|\nabla(Q_1,Q_2)\|^{\frac34}_{L^2_x}(\sum_{q\ge N}2^{-\frac q4}\|\dot\Delta_q \nabla Q_2\|_{L^2_x})\|\nabla u_2\|_{L^2_x}\|\Delta\delta Q\|_{\dot H^{-\frac12}}\\
\lesssim&\|(Q_1,Q_2)\|^{\frac14}_{L^2_x}\|\nabla(Q_1,Q_2)\|^{\frac34}_{L^2_x}(\sum_{q\ge N}2^{-\frac q4})\|\nabla Q_2\|_{L^2_x}\|\nabla u_2\|_{L^2_x}\|\Delta\delta Q\|_{\dot H^{-\frac12}}\\
\lesssim&2^{-\frac N4}\|(Q_1,Q_2)\|^{\frac14}_{L^2_x}\|\nabla(Q_1,Q_2)\|^{\frac34}_{L^2_x}\|\nabla Q_2\|_{L^2_x}\|\nabla u_2\|_{L^2_x}\|\Delta\delta Q\|_{\dot H^{-\frac12}}.
\end{aligned}$$

Now, fixing $t>0$ arbitrary and taking $N=N(t):=\lceil\ln(1+e+1/\Phi(t))\rceil>0$, where $\lceil\cdot\rceil$ is the ceiling function, we get

$$\begin{aligned}
&\langle\delta Q(t)\mathrm{tr}\{Q_2(t)\nabla u_2(t)\},\Delta\delta Q(t)\rangle_{\dot H^{-\frac12}}\\
\lesssim&\|(Q_2,\nabla Q_2)(t)\|^2_{L^2_x}\|\nabla u_2(t)\|^2_{L^2_x}\Phi(t)\ln\Big(1+e+\frac{1}{\Phi(t)}\Big)\\
&+\|(Q_1,Q_2)(t)\|^{\frac12}_{L^2_x}\|\nabla(Q_1,Q_2)(t)\|^{\frac32}_{L^2_x}\|\nabla Q_2(t)\|^2_{L^2_x}\|\nabla u_2(t)\|^2_{L^2_x}\Phi(t)+C_\Gamma\|\Delta\delta Q(t)\|^2_{\dot H^{-\frac12}}.
\end{aligned}$$

Thus we have obtained a one-logarithmic term of Equation (4.7). Similarly, we can handle the estimate of the following elements:

$$\begin{aligned}
&+2L\xi\langle\delta Q\,\mathrm{tr}(\delta Q\nabla\delta u),\Delta\delta Q\rangle_{\dot H^{-\frac12}}+2L\xi\langle\delta Q\,\mathrm{tr}(\delta Q\nabla u_2),\Delta\delta Q\rangle_{\dot H^{-\frac12}}\\
&+2L\xi\langle\delta Q\,\mathrm{tr}(Q_2\nabla\delta u),\Delta\delta Q\rangle_{\dot H^{-\frac12}}+2L\xi\langle Q_2\,\mathrm{tr}(\delta Q\nabla\delta u),\Delta\delta Q\rangle_{\dot H^{-\frac12}}\\
&+2L\xi\langle Q_2\,\mathrm{tr}(\delta Q\nabla u_2),\Delta\delta Q\rangle_{\dot H^{-\frac12}}-2L\xi\langle\delta Q\mathrm{tr}\{\delta Q\Delta\delta Q\},\nabla\delta u\rangle_{\dot H^{-\frac12}}\\
&-2L\xi\langle\delta Q\mathrm{tr}\{\delta Q\Delta Q_2\},\nabla\delta u\rangle_{\dot H^{-\frac12}}-2L\xi\langle\delta Q\mathrm{tr}\{Q_2\Delta\delta Q\},\nabla\delta u\rangle_{\dot H^{-\frac12}}\\
&-2L\xi\langle Q_2\mathrm{tr}\{\delta Q\Delta\delta Q\},\nabla\delta u\rangle_{\dot H^{-\frac12}}-2L\xi\langle\delta Q\mathrm{tr}\{Q_2\Delta Q_2\},\nabla\delta u\rangle_{\dot H^{-\frac12}}\\
&-2L\xi\langle Q_2\mathrm{tr}\{\delta Q\Delta Q_2\},\nabla\delta u\rangle_{\dot H^{-\frac12}}.
\end{aligned}$$

4.2.4. Double-logarithmic estimates. In this subsection, we perform the most challenging estimate. Now, we want to control $\mathcal{E}_1+\mathcal{E}_2$, namely

$$\begin{aligned}\mathcal{E}_1+\mathcal{E}_2&=2L\xi\big(\langle Q_2\mathrm{tr}\{Q_2\nabla\delta u\},\Delta\delta Q\rangle_{\dot H^{-\frac12}}-\langle Q_2\mathrm{tr}\{Q_2\Delta\delta Q\},\nabla\delta u\rangle_{\dot H^{-\frac12}}\big)\\&=2L\xi\sum_{q\in\mathbb{Z}}2^{-q}\int_{\mathbb{R}^2}\mathrm{tr}\big\{\dot\Delta_q(Q_2\mathrm{tr}\{Q_2\nabla\delta u\})\dot\Delta_q\Delta\delta Q-\dot\Delta_q(Q_2\mathrm{tr}\{Q_2\Delta\delta Q\})\dot\Delta_q\nabla\delta u\big\}\\&=2L\xi\sum_{i=1}^4\sum_{q\in\mathbb{Z}}2^{-q}\int_{\mathbb{R}^2}\mathrm{tr}\big\{\mathcal{J}_q^i(Q_2,\mathrm{tr}\{Q_2\nabla\delta u\}\mathrm{Id})\dot\Delta_q\Delta\delta Q-\mathcal{J}_q^i(Q_2,\mathrm{tr}\{Q_2\Delta\delta Q\}\mathrm{Id})\dot\Delta_q\nabla\delta u\big\}.\end{aligned}\tag{4.19}$$

We we will see that there are elements inside this decomposition that generate the double-logarithmic term in Equation (4.7). We proceed by considering the indices $i=1,2,3,4$, step by step.

Estimate of $\mathcal{J}_q^1$. We start with the term of Equation (4.19) related to $i=1$, passing trough the following decomposition:

$$\begin{aligned}\sum_{j=1}^4\sum_{|q-q'|\le5}\int_{\mathbb{R}_2}&\mathrm{tr}\big\{\big([\dot\Delta_q,\dot S_{q'-1}Q_2]\mathrm{tr}\{\mathcal{J}_{q'}^j(Q_2,\nabla\delta u)\}\mathrm{Id}\big)\dot\Delta_q\Delta\delta Q\\&-\big([\dot\Delta_q,\dot S_{q'-1}Q_2]\mathrm{tr}\{\mathcal{J}_{q'}^j(Q_2,\Delta\delta Q)\}\mathrm{Id}\big)\dot\Delta_q\nabla\delta u\big\}.\end{aligned}\tag{4.20}$$

When $j=1$, we have

$$\begin{aligned}\mathcal{I}_1^1(q,q',q''):=\int_{\mathbb{R}_2}&\Big\{\Big([\dot\Delta_q,\dot S_{q'-1}Q_2]\mathrm{tr}\{[\dot\Delta_{q'},\dot S_{q''-1}Q_2]\dot\Delta_{q''}\nabla\delta u)\}\mathrm{Id}\Big)\dot\Delta_q\Delta\delta Q\\&-\Big([\dot\Delta_q,\dot S_{q'-1}Q_2]\mathrm{tr}\{[\dot\Delta_{q'},\dot S_{q''-1}Q_2]\dot\Delta_{q''}\Delta\delta Q)\}\mathrm{Id}\Big)\dot\Delta_q\nabla\delta u\Big\}\\\lesssim&\,2^{-q}\|\dot S_{q'-1}\nabla Q_2\|_{L^\infty_x}2^{-q'}\|\dot S_{q''-1}\nabla Q_2\|_{L^\infty_x}\|\dot\Delta_{q''}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\\lesssim&\,2^{-q-q'}2^{\frac{q'}{2}}\|\dot S_{q'-1}\nabla Q_2\|_{L^4_x}2^{\frac{q''}{2}}\|\dot S_{q''-1}\nabla Q_2\|_{L^4_x}2^{q''}\\&\times\|\dot\Delta_{q''}(\delta u,\nabla\delta Q)\|_{L^2_x}2^q\|\dot\Delta_q(\delta u,\nabla\delta Q)\|_{L^2_x},\end{aligned}$$

which yields

$$\mathcal{I}_1^1(q,q',q'')\lesssim2^{\frac{3q''}{2}-\frac{q'}{2}}\|\nabla Q_2\|_{L^2_x}\|\Delta Q_2\|_{L^2_x}\|\dot\Delta_{q''}(\delta u,\nabla\delta Q)\|_{L^2_x}\|\dot\Delta_q(\delta u,\nabla\delta Q)\|_{L^2_x}.\tag{4.21}$$

Hence, taking the sum, we deduce that

$$\begin{aligned}&2L\xi\sum_{q\in\mathbb{Z}}\sum_{|q-q'|\le5}\sum_{|q'-q''|\le5}2^{-q}\mathcal{I}_1^1(q,q',q'')\\&\lesssim\sum_{q\in\mathbb{Z}}\sum_{|q-q'|\le5}\sum_{|q'-q''|\le5}2^{-q}2^{\frac{3q''}{2}-\frac{q'}{2}}\|\nabla Q_2\|_{L^2_x}\|\Delta Q_2\|_{L^2_x}\\&\qquad\qquad\times\|\dot\Delta_{q''}(\delta u,\nabla\delta Q)\|_{L^2_x}\|\dot\Delta_q(\delta u,\nabla\delta Q)\|_{L^2_x}\\&\lesssim\|\nabla Q_2\|_{L^2_x}\|\Delta Q_2\|_{L^2_x}\sum_{q\in\mathbb{Z}}\sum_{|q-q''|\le10}\|\dot\Delta_{q''}(\delta u,\nabla\delta Q)\|_{L^2_x}\|\dot\Delta_q(\delta u,\nabla\delta Q)\|_{L^2_x}\\&\lesssim\|\nabla Q_2\|_{L^2_x}\|\Delta Q_2\|_{L^2_x}\|(\delta u,\nabla\delta Q)\|_{L^2_x}^2\\&\lesssim\|\nabla Q_2\|_{L^2_x}\|\Delta Q_2\|_{L^2_x}\|(\delta u,\nabla\delta Q)\|_{\dot H^{-\frac12}}\|(\nabla\delta u,\Delta\delta Q)\|_{\dot H^{-\frac12}}\end{aligned}$$

$$\lesssim \|\nabla Q_2\|_{L^2_x}^2 \|\Delta Q_2\|_{L^2_x}^2 \|(\delta u, \nabla\delta Q)\|_{\dot H^{-\frac12}}^2 + C_\nu \|\nabla\delta u\|_{\dot H^{-\frac12}}^2 + C_{\Gamma,L}\|\Delta\delta Q\|_{\dot H^{-\frac12}}^2. \qquad (4.22)$$

Now, when $j=2$ in Equation (4.20), we remark that

$$\begin{aligned}
&\mathcal{I}_2^1(q,q',q'')\\
&:= \int_{\mathbb{R}_2} \mathrm{tr}\Big\{ \Big([\dot\Delta_q, \dot S_{q'-1}Q_2]\mathrm{tr}\{(\dot S_{q''-1}Q_2 - \dot S_{q'-1}Q_2)\dot\Delta_{q'}\dot\Delta_{q''}\nabla\delta u)\}\mathrm{Id}\Big)\dot\Delta_q\Delta\delta Q\\
&\qquad + \Big([\dot\Delta_q, \dot S_{q'-1}Q_2]\mathrm{tr}\{(\dot S_{q''-1}Q_2 - \dot S_{q'-1}Q_2)\dot\Delta_{q'}\dot\Delta_{q''}\Delta\delta Q)\}\mathrm{Id}\Big)\dot\Delta_q\nabla\delta u\Big\}\\
&\lesssim 2^{-q}\|\dot S_{q'-1}\nabla Q_2\|_{L^\infty_x}\|\dot S_{q''-1}Q_2 - \dot S_{q'-1}Q_2\|_{L^\infty_x}\\
&\qquad \times \|\dot\Delta_q\dot\Delta_{q''}(\nabla\delta u, \Delta\delta Q)\|_{L^2_x}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim 2^{-q}\|\dot S_{q'-1}\nabla Q_2\|_{L^\infty_x} 2^{-q'}\|\dot S_{q''-1}\nabla Q_2 - \dot S_{q'-1}\nabla Q_2\|_{L^\infty_x}\\
&\qquad \times \|\dot\Delta_{q''}(\nabla\delta u, \Delta\delta Q)\|_{L^2_x}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim 2^{-q-q'}2^{\frac{q'}{2}}\|\dot S_{q'-1}\nabla Q_2\|_{L^4_x}2^{\frac{q''}{2}}\|\dot S_{q''-1}\nabla Q_2\|_{L^4_x}\\
&\qquad \times 2^{q''}\|\dot\Delta_{q''}(\delta u, \nabla\delta Q)\|_{L^2_x}2^q\|\dot\Delta_q(\delta u,\nabla\delta Q)\|_{L^2_x}\\
&\lesssim 2^{\frac{3q''}{2}-\frac{q'}{2}}\|\nabla Q_2\|_{L^2_x}\|\Delta Q_2\|_{L^2_x}\|\dot\Delta_{q''}(\delta u,\nabla\delta Q)\|_{L^2_x}\|\dot\Delta_q(\delta u,\nabla\delta Q)\|_{L^2_x},
\end{aligned}$$

which is equivalent to Equation (4.21). Hence, proceeding as in Equation (4.22), we get

$$\begin{aligned}
&2L\xi\sum_{q\in\mathbb{Z}}\sum_{\substack{|q-q'|\le 5\\ |q'-q''|\le 5}} 2^{-q}\mathcal{I}_2^1(q,q',q'')\\
&\lesssim \|\nabla Q_2\|_{L^2_x}^2\|\Delta Q_2\|_{L^2_x}^2\|(\delta u,\nabla\delta Q)\|_{\dot H^{-\frac12}}^2 + C_\nu\|\nabla\delta u\|_{\dot H^{-\frac12}}^2 + C_{\Gamma,L}\|\Delta\delta Q\|_{\dot H^{-\frac12}}^2.
\end{aligned}$$

Concerning the term of Equation (4.20) related to $j=4$, we have

$$\begin{aligned}
&\mathcal{I}_4^1(q,q',q'') := \int_{\mathbb{R}_2}\mathrm{tr}\Big\{\Big([\dot\Delta_q,\dot S_{q'-1}Q_2]\mathrm{tr}\{\dot\Delta_{q'}(\dot\Delta_{q''}Q_2\dot S_{q''+2}\nabla\delta u)\}\mathrm{Id}\Big)\dot\Delta_q\Delta\delta Q\\
&\qquad - \Big([\dot\Delta_q,\dot S_{q'-1}Q_2]\mathrm{tr}\{\dot\Delta_{q'}(\dot\Delta_{q''}Q_2\dot S_{q''+2}\Delta\delta Q)\}\mathrm{Id}\Big)\dot\Delta_q\nabla\delta u\Big\}\\
&\lesssim 2^{-q}\|\dot S_{q'-1}\nabla Q_2\|_{L^\infty_x}\|\dot\Delta_q(\dot\Delta_{q''}Q_2\dot S_{q''+2}\nabla\delta u, \dot\Delta_{q''}Q_2\dot S_{q''+2}\Delta\delta Q)\|_{L^2_x}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim 2^{-q}\|\dot S_{q'-1}\nabla Q_2\|_{L^\infty_x}2^q\\
&\qquad \times\|\dot\Delta_q(\dot\Delta_{q''}Q_2\dot S_{q''+2}\nabla\delta u, \dot\Delta_{q''}Q_2\dot S_{q''+2}\Delta\delta Q)\|_{L^1_x}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim \|\dot S_{q'-1}\nabla Q_2\|_{L^\infty_x}\|\dot\Delta_{q''}Q_2\|_{L^2_x}\|\dot S_{q''+2}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim \|\dot S_{q'-1}\nabla Q_2\|_{L^\infty_x}2^{-2q''}\|\dot\Delta_{q''}\Delta Q_2\|_{L^2_x}\|\dot S_{q''+2}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim 2^{-2q''}2^{q'}\|\dot S_{q'-1}\nabla Q_2\|_{L^2_x}\|\dot\Delta_{q''}\Delta Q_2\|_{L^2_x}2^{q''}\|\dot S_{q''+2}(\delta u,\nabla\delta Q)\|_{L^2_x}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim 2^{q'-q''}\|\nabla Q_2\|_{L^2_x}\|\Delta Q_2\|_{L^2_x}\|\dot S_{q''+2}(\delta u,\nabla\delta Q)\|_{L^2_x}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}. \qquad (4.23)
\end{aligned}$$

Hence

$$\begin{aligned}
&2L\xi\sum_{q\in\mathbb{Z}}\sum_{|q-q'|\le 5}\sum_{q''\ge q'-5}2^{-q}\mathcal{I}_4^1(q,q',q'')\\
&\lesssim \|\nabla Q_2\|_{L^2_x}\|\Delta Q_2\|_{L^2_x}\sum_{q\in\mathbb{Z}}2^{-\frac{q}{2}}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad \times \sum_{[q-q'|\le 5} 2^{\frac{q'-q}{2}} \sum_{q''\ge q'-5} 2^{\frac{q'-q''}{2}} 2^{-\frac{q''}{2}} \|\dot S_{q''+2}(\delta u, \nabla\delta Q)\|_{L^2_x}\\
&\lesssim \|\nabla Q_2\|_{L^2_x}\|\Delta Q_2\|_{L^2_x} \sum_{q\in\mathbb{Z}} 2^{-\frac{q}{2}} \|\dot\Delta_q(\nabla\delta u, \Delta\delta Q)\|_{L^2_x}\\
&\qquad\qquad \times \sum_{q''\ge q-10} 2^{\frac{q-q''}{2}} 2^{-\frac{q''}{2}} \|\dot S_{q''+2}(\delta u, \nabla\delta Q)\|_{L^2_x}\\
&\lesssim \|\nabla Q_2\|_{L^2_x}\|\Delta Q_2\|_{L^2_x} \|(\nabla\delta u, \Delta\delta Q)\|_{\dot H^{-\frac12}}\\
&\qquad\qquad \times \Big[\sum_{q\in\mathbb{Z}} \Big| \sum_{q-q''\le 10} 2^{\frac{q-q''}{2}} 2^{-\frac{q''+2}{2}} \|\dot S_{q''+2}(\delta u, \nabla\delta Q)\|_{L^2_x}\Big|^2\Big]^{\frac12}\\
&\lesssim \|\nabla Q_2\|_{L^2_x}\|\Delta Q_2\|_{L^2_x} \|(\nabla\delta u, \Delta\delta Q)\|_{\dot H^{-\frac12}} \Big(\sum_{q\le 10} 2^{\frac{q}{2}}\Big)\Big(\sum_{q\in\mathbb{Z}} 2^{-q}\|\dot S_q(\delta u, \nabla\delta Q)\|_{L^2_x}^2\Big)^{\frac12}\\
&\lesssim \|\nabla Q_2\|_{L^2_x}\|\Delta Q_2\|_{L^2_x} \|(\nabla\delta u, \Delta\delta Q)\|_{\dot H^{-\frac12}} \|(\delta u, \nabla\delta Q)\|_{\dot H^{-\frac12}}\\
&\lesssim \|\nabla Q_2\|_{L^2_x}^2\|\Delta Q_2\|_{L^2_x}^2 \|(\delta u, \nabla\delta Q)\|_{\dot H^{-\frac12}}^2 + C_\nu\|\nabla\delta u\|_{\dot H^{-\frac12}}^2 + C_{\Gamma,L}\|\Delta\delta Q\|_{\dot H^{-\frac12}}^2. \qquad (4.24)
\end{aligned}
$$

Concerning Equation (4.20), it remains to control the term related to $j=3$. We fix $0<\varepsilon\le 5/6$ and consider the low frequencies $q\le N$ for some suitable positive $N\ge 1$ (so that $1+\sqrt{N}<2\sqrt{N}$):

$$
\begin{aligned}
\mathcal{I}_3^1(q,q') :=& \int_{\mathbb{R}_2} \mathrm{tr}\Big\{\Big([\dot\Delta_q, \dot S_{q'-1}Q_2]\mathrm{tr}\{\dot S_{q'-1}Q_2\,\dot\Delta_{q'}\nabla\delta u\}\mathrm{Id}\Big)\dot\Delta_q\Delta\delta Q\\
&\qquad - \Big([\dot\Delta_q, \dot S_{q'-1}Q_2]\mathrm{tr}\{\dot S_{q'-1}Q_2\,\dot\Delta_{q'}\Delta\delta Q\}\mathrm{Id}\Big)\dot\Delta_q\nabla\delta u\Big\}\\
\lesssim& 2^{-q}\|\dot S_{q'-1}\nabla Q_2\|_{L^{\frac{2}{\varepsilon}}_x} \|\dot S_{q'-1}Q_2\,\dot\Delta_{q'}\nabla\delta u, \dot S_{q'-1}Q_2\,\dot\Delta_{q'}\Delta\delta Q\|_{L^{\frac{2}{1-\varepsilon}}_x} \|\dot\Delta_q(\nabla\delta u, \Delta\delta Q)\|_{L^2_x}\\
\lesssim& \|\dot S_{q'-1}\nabla Q_2\|_{L^{\frac{2}{\varepsilon}}_x} \|\dot S_{q'-1}Q_2\|_{L^\infty_x} 2^{-q}\|\dot\Delta_{q'}(\nabla\delta u, \Delta\delta Q)\|_{L^{\frac{2}{1-\varepsilon}}_x} \|\dot\Delta_q(\nabla\delta u, \Delta\delta Q)\|_{L^2_x}.
\end{aligned}
$$

Thanks to Theorem A.2, we get

$$
\begin{aligned}
\|\dot S_{q'-1}Q_2\|_{L^\infty_x} &\lesssim (1+\sqrt{q'-1})\|(Q_2,\nabla Q_2)\|_{L^2_x}\\
&\lesssim (1+\sqrt{N})\|(Q_2,\nabla Q_2)\|_{L^2_x} \lesssim \sqrt{N}\|(Q_2,\nabla Q_2)\|_{L^2_x}.
\end{aligned}
$$

Hence $\mathcal{I}_3^1(q,q')$ is bounded by

$$
\mathcal{I}_3^1(q,q') \lesssim \sqrt{N}\|(Q_2,\nabla Q_2)\|_{L^{\frac{2}{\varepsilon}}_x}\|(Q_2,\nabla Q_2)\|_{L^2_x}\|\dot\Delta_{q'}(\delta u, \nabla\delta Q)\|_{L^{\frac{2}{1-\varepsilon}}_x}\|\dot\Delta_q(\nabla\delta u, \Delta\delta Q)\|_{L^2_x}. \qquad (4.25)
$$

Now, we will need the following inequality, which will finally lead to the delicate double-logarithmic estimate:

$$
\|(Q_2,\nabla Q_2)\|_{L^{\frac{2}{\varepsilon}}_x} \le \frac{C}{\sqrt{\varepsilon}}\|(Q_2,\nabla Q_2)\|_{L^2_x}^{\varepsilon}\|(\nabla Q_2,\Delta Q_2)\|_{L^2_x}^{1-\varepsilon},
$$

This is a consequence of Lemma A.1, imposing $p=1/\varepsilon$, where C is a positive constant independent of ε and Q_2. We will see that the double-logarithmic term comes out of a suitable choice of ε in terms of N. Again, using Lemma A.1 we have

$$
\|\dot\Delta_{q'}(\delta u, \nabla\delta Q)\|_{L^{\frac{2}{1-\varepsilon}}_x} \le \frac{C}{1-\varepsilon}\|\dot\Delta_{q'}(\delta u, \nabla\delta Q)\|_{L^2_x}^{1-\varepsilon}\|\dot\Delta_{q'}(\nabla\delta u, \Delta\delta Q)\|_{L^2_x}^{\varepsilon}
$$

$$\leq 6C\|\dot{\Delta}_{q'}(\delta u,\nabla\delta Q)\|_{L^2_x}^{1-\varepsilon}\|\dot{\Delta}_{q'}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}^{\varepsilon},$$

since $\varepsilon\leq 5/6$. Hence Equation (4.25) becomes

$$\mathcal{I}_3^1(q,q')\lesssim\sqrt{\frac{N}{\varepsilon}}\|(Q_2,\nabla Q_2)\|_{L^2_x}^{1+\varepsilon}\|(\nabla Q_2,\Delta Q_2)\|_{L^2_x}^{1-\varepsilon} \times\|\dot{\Delta}_{q'}(\delta u,\nabla\delta Q)\|_{L^2_x}^{1-\varepsilon}\|\dot{\Delta}_{q'}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}^{\varepsilon}\|\dot{\Delta}_{q}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}. \quad (4.26)$$

Thus, since $ab\leq a^{2/(1-\varepsilon)}+b^{2/(1+\varepsilon)}$, we deduce

$$\begin{aligned}\mathcal{I}_3^1(q,q')&\lesssim\Big(\frac{N}{\varepsilon}\Big)^{\frac{1}{1-\varepsilon}}\|(Q_2,\nabla Q_2)\|_{L^2_x}^{\frac{2(1+\varepsilon)}{1-\varepsilon}}\|(\nabla Q_2,\Delta Q_2)\|_{L^2_x}^2\|\dot{\Delta}_{q'}(\delta u,\nabla\delta Q)\|_{L^2_x}^2\\&\quad+\min\{C_\nu,C_\Gamma\}\|\dot{\Delta}_{q'}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}^{\frac{2\varepsilon}{1+\varepsilon}}\|\dot{\Delta}_{q}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}^{\frac{2}{1+\varepsilon}}\\&\lesssim\Big(\frac{N}{\varepsilon}\Big)^{\frac{1}{1-\varepsilon}}\|(Q_2,\nabla Q_2)\|_{L^2_x}^{\frac{2(1+\varepsilon)}{1-\varepsilon}}\|(\nabla Q_2,\Delta Q_2)\|_{L^2_x}^2\|\dot{\Delta}_{q'}(\delta u,\nabla\delta Q)\|_{L^2_x}^2\\&\quad+\min\{C_\nu,C_\Gamma\}\Big(\|\dot{\Delta}_{q'}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}^2+\|\dot{\Delta}_{q}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}^2\Big).\end{aligned}$$

Imposing $\varepsilon=(1+\ln N)^{-1}$ and observing that $\frac{1}{1-\varepsilon}=1+1/\ln N$

$$N^{\frac{1}{1-\varepsilon}}=N\,N^{\frac{1}{\ln N}}=eN,\quad \varepsilon^{-\frac{1}{1-\varepsilon}}=\varepsilon^{-1}\varepsilon^{-\frac{\varepsilon}{1-\varepsilon}}=(1+\ln N)e^{\frac{\varepsilon}{1-\varepsilon}\ln\frac{1}{\varepsilon}}\lesssim(1+\ln N),$$

we obtain

$$\begin{aligned}\mathcal{I}_3^1(q,q')&\lesssim N(1+\ln N)\max\big\{\|(Q_2,\nabla Q_2)\|_{L^2_x}^6,1\big\}\|(\nabla Q_2,\Delta Q_2)\|_{L^2_x}^2\|\dot{\Delta}_{q'}(\delta u,\nabla\delta Q)\|_{L^2_x}^2\\&\quad+\min\Big\{C_\nu,C_{\Gamma,L}\Big\}\Big(\|\dot{\Delta}_{q'}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}^2+\|\dot{\Delta}_{q}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}^2\Big),\end{aligned}$$

which yields

$$\begin{aligned}&\sum_{q\leq N}\sum_{|q-q'|\leq 5}2^{-q}\mathcal{I}_3^1(q,q')\\&\lesssim N(1+\ln N)\max\big\{\|(Q_2,\nabla Q_2)\|_{L^2_x}^6,1\big\}\|(\nabla Q_2,\Delta Q_2)\|_{L^2_x}^2\|(\delta u,\nabla\delta Q)\|_{\dot H^{-\frac12}}^2\\&\quad+C_\nu\|\nabla\delta u\|_{\dot H^{-\frac12}}^2+C_{\Gamma,L}\|\Delta\delta Q\|_{\dot H^{-\frac12}}^2.\end{aligned}$$

For the high frequencies, namely for $q>N\geq 1$, we proceed as follows:

$$\begin{aligned}\mathcal{I}_3^1(q,q')&\lesssim 2^{-q}\|\dot S_{q'-1}\nabla Q_2\|_{L^\infty_x}\\&\quad\times\|\dot S_{q'-1}Q_2\,\dot\Delta_{q'}\nabla\delta u,\dot S_{q'-1}Q_2\,\dot\Delta_{q'}\Delta\delta Q\|_{L^2_x}\|\dot{\Delta}_{q}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\&\lesssim 2^{-q}(1+\sqrt{q'})\|(\nabla Q_2,\Delta Q_2)\|_{L^2_x}\|\dot S_{q'-1}Q_2\|_{L^\infty_x}\\&\quad\times\|\dot{\Delta}_{q'}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\|\dot{\Delta}_{q}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\&\lesssim 2^{-q}(1+\sqrt{q'})^2\|(\nabla Q_2,\Delta Q_2)\|_{L^2_x}\|(Q_2,\nabla Q_2)\|_{L^2_x}\\&\quad\times\|\dot{\Delta}_{q'}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\|\dot{\Delta}_{q}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\&\lesssim q'\|(\nabla Q_2,\Delta Q_2)\|_{L^2_x}\|(Q_2,\nabla Q_2)\|_{L^2_x}\|\dot{\Delta}_{q'}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\|\dot{\Delta}_{q}(\delta u,\nabla\delta Q)\|_{L^2_x},\end{aligned}$$

which implies

$$\begin{aligned}
&\sum_{q>N}\sum_{|q-q'|\le 5} 2^{-q}\mathcal{I}_3^1(q,q')\\
&\lesssim \sum_{q>N}\sum_{|q-q'|\le 5} 2^{-2q}q'\|(\nabla Q_2,\Delta Q_2)\|_{L^2_x}\|(Q_2,\nabla Q_2)\|_{L^2_x}\\
&\qquad\qquad \times\|\dot\Delta_{q'}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\|\dot\Delta_q(\delta u,\nabla\delta Q)\|_{L^2_x}\\
&\lesssim \|(\nabla Q_2,\Delta Q_2)\|_{L^2_x}\|(Q_2,\nabla Q_2)\|_{L^2_x}\\
&\qquad\qquad \times\|(\delta u,\nabla\delta Q)\|_{L^2_x}\|(\nabla\delta u,\Delta\delta Q)\|_{\dot H^{-\frac12}}\sum_{q>N}\sum_{|q-q'|\le 5}2^{-\frac32 q+\frac12 q'}q'\\
&\lesssim 2^{-\frac N2}\|(\nabla Q_2,\Delta Q_2)\|_{L^2_x}\|(Q_2,\nabla Q_2)\|_{L^2_x}\|(\delta u,\nabla\delta Q)\|_{L^2_x}\|(\nabla\delta u,\Delta\delta Q)\|_{\dot H^{-\frac12}}.
\end{aligned}$$

Summarising, we get

$$\begin{aligned}
&\sum_{q\in\mathbb{Z}}\sum_{|q-q'|\le 5} 2^{-q}\mathcal{I}_3^1(q,q')\\
&\lesssim N(1+\ln N)\max\big\{\|(Q_2,\nabla Q_2)\|_{L^2_x}^6,1\big\}\|(\nabla Q_2,\Delta Q_2)\|_{L^2_x}^2\|(\delta u,\nabla\delta Q)\|_{\dot H^{-\frac12}}^2\\
&\quad +2^{-N}\|(\nabla Q_2,\Delta Q_2)\|_{L^2_x}^2\|(Q_2,\nabla Q_2)\|_{L^2_x}^2\|(u_1,u_2,\nabla Q_1,\nabla Q_2)\|_{L^2_x}^2\\
&\qquad\qquad +C_\nu\|\nabla\delta u\|_{\dot H^{-\frac12}}^2+C_{\Gamma,L}\|\Delta\delta Q\|_{\dot H^{-\frac12}}^2. \qquad (4.27)
\end{aligned}$$

Choosing $N=N(t):=\lceil\ln(1+e+1/\Phi(t))\rceil$ (thus $\varepsilon<1/(1+\ln\ln\{1+e\})<5/6$) where with $\lceil\cdot\rceil$ we denote the ceiling function, relation (4.27) implies

$$\begin{aligned}
&\sum_{q\in\mathbb{Z}}\sum_{|q-q'|\le 5} 2^{-q}\mathcal{I}_1^3(q,q')\\
&\lesssim \max\big\{\|(Q_2,\nabla Q_2)\|_{L^2_x}^6,1\big\}\\
&\quad\times\|(\nabla Q_2,\Delta Q_2)\|_{L^2_x}^2\|(\delta u,\nabla\delta Q)\|_{\dot H^{-\frac12}}^2\ln\Big(e+\frac{1}{\Phi(t)}\Big)\Big(1+\ln\ln\Big(e+\frac{1}{\Phi(t)}\Big)\Big)\\
&\qquad+\|(\nabla Q_2,\Delta Q_2)\|_{L^2_x}^2\|(Q_2,\nabla Q_2)\|_{L^2_x}^2\|(u_1,u_2,\nabla Q_1,\nabla Q_2)\|_{L^2_x}^2\\
&\qquad\qquad+\Phi(t)+C_\nu\|\nabla\delta u\|_{\dot H^{-\frac12}}^2+C_{\Gamma,L}\|\Delta\delta Q\|_{\dot H^{-\frac12}}^2. \qquad (4.28)
\end{aligned}$$

Estimate of $\mathcal{J}_q^2$. Now, we handle the term of Equation (4.19) related to $i=2$, namely

$$\begin{aligned}
\sum_{j=1}^4\sum_{|q-q'|\le 5}\int_{\mathbb{R}_2}\mathrm{tr}\big\{&(\dot S_{q'-1}Q_2-\dot S_{q-1}Q_2)\mathrm{tr}\{\dot\Delta_q\mathcal{J}^j_{q'}(Q_2,\nabla\delta u)\}\dot\Delta_q\Delta\delta Q\\
&(\dot S_{q'-1}Q_2-\dot S_{q-1}Q_2)\mathrm{tr}\{\dot\Delta_q\mathcal{J}^j_{q'}(Q_2,\Delta\delta Q)\}\dot\Delta_q\nabla\delta u\big\}. \qquad (4.29)
\end{aligned}$$

When $j=1$, we have

$$\begin{aligned}
\mathcal{I}_1^2(q,q',q''):=\int_{\mathbb{R}_2}\mathrm{tr}\Big\{&(\dot S_{q'-1}Q_2-\dot S_{q-1}Q_2)\mathrm{tr}\{\dot\Delta_q\big([\dot\Delta_{q'},\dot S_{q''-1}Q_2]\dot\Delta_{q''}\nabla\delta u\big)\}\dot\Delta_q\Delta\delta Q\\
&-(\dot S_{q'-1}Q_2-\dot S_{q-1}Q_2)\mathrm{tr}\{\dot\Delta_q\big([\dot\Delta_{q'},\dot S_{q''-1}Q_2]\dot\Delta_{q''}\Delta\delta Q\big)\}\dot\Delta_q\nabla\delta u\Big\}\\
\lesssim\|\dot S_{q'-1}Q_2-\dot S_{q-1}Q_2\|_{L^\infty_x}&
\end{aligned}$$

$$
\begin{aligned}
&\qquad\times\|\dot\Delta_q\big([\dot\Delta_{q'},\dot S_{q''-1}Q_2]\dot\Delta_{q''}(\nabla\delta u,\Delta\delta Q)\big)\|_{L^2_x}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim 2^{-\frac{q}{2}}\|\dot S_{q'-1}\nabla Q_2-\dot S_{q-1}\nabla Q_2\|_{L^4_x}2^{\frac{q}{2}}\\
&\qquad\times\|\dot\Delta_q\big([\dot\Delta_{q'},\dot S_{q''-1}Q_2]\dot\Delta_{q''}(\nabla\delta u,\Delta\delta Q)\big)\|_{L^{\frac{4}{3}}_x}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim\|\nabla Q_2\|_{L^4_x}2^{-q'}\|\dot S_{q''-1}\nabla Q_2\|_{L^4_x}\|\dot\Delta_{q''}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim 2^{-q'+q''+q}\|\nabla Q_2\|_{L^2_x}\|\Delta Q_2\|_{L^2_x}\|\dot\Delta_{q''}(\delta u,\nabla\delta Q)\|_{L^2_x}\|\dot\Delta_q(\delta u,\nabla\delta Q)\|_{L^2_x}.
\end{aligned}
$$

Since $|q-q'|\le 5$ and $|q'-q''|\le 5$, we get that $-q'+q''+q\simeq 3q''/2-q'/2$, so that the last inequality is equivalent to Equation (4.21). Hence, proceeding as in Equation (4.22), we get

$$
\begin{aligned}
&2L\xi\sum_{q\in\mathbb{Z}}\sum_{\substack{|q-q'|\le 5\\|q'-q''|\le 5}}2^{-q}\mathcal{I}_1^2(q,q',q'')\\
&\lesssim\|\nabla Q_2\|^2_{L^2_x}\|\Delta Q_2\|^2_{L^2_x}\|(\delta u,\nabla\delta Q)\|^2_{\dot H^{-\frac12}}+C_\nu\|\nabla\delta u\|^2_{\dot H^{-\frac12}}+C_{\Gamma,L}\|\Delta\delta Q\|^2_{\dot H^{-\frac12}}.
\end{aligned}
$$

When $j=2$, we observe that

$$
\begin{aligned}
&\mathcal{I}_2^2(q,q',q'')\\
&:=\int_{\mathbb{R}_2}\operatorname{tr}\Big\{(\dot S_{q'-1}Q_2-\dot S_{q-1}Q_2)\operatorname{tr}\{(\dot S_{q''-1}Q_2-\dot S_{q'-1}Q_2)\dot\Delta_{q'}\dot\Delta_{q''}\nabla\delta u\}\dot\Delta_q\Delta\delta Q\\
&\qquad-(\dot S_{q'-1}Q_2-\dot S_{q-1}Q_2)\operatorname{tr}\{(\dot S_{q''-1}Q_2-\dot S_{q'-1}Q_2)\dot\Delta_{q'}\dot\Delta_{q''}\Delta\delta Q\}\dot\Delta_q\nabla\delta u\Big\}\\
&\lesssim\|\dot S_{q'-1}Q_2-\dot S_{q-1}Q_2\|_{L^\infty_x}\|\dot S_{q''-1}Q_2-\dot S_{q'-1}Q_2\|_{L^\infty_x}\\
&\qquad\times\|\dot\Delta_{q'}\dot\Delta_{q''}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\|_{L^2_x}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim 2^{-\frac{q+q'}{2}}\|\dot S_{q'-1}\nabla Q_2-\dot S_{q-1}\nabla Q_2\|_{L^4_x}\|\dot S_{q''-1}\nabla Q_2-\dot S_{q'-1}\nabla Q_2\|_{L^4_x}\\
&\qquad\times\|\dot\Delta_{q''}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim 2^{\frac{q'}{2}+\frac{q}{2}}\|\nabla Q_2\|^2_{L^4_x}\|\dot\Delta_{q''}(\delta u,\nabla\delta Q)\|_{L^2_x}\|\dot\Delta_q(\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim 2^{\frac{q'}{2}+\frac{q}{2}}\|\nabla Q_2\|_{L^2_x}\|\Delta Q_2\|_{L^2_x}\|\dot\Delta_{q'}(\delta u,\nabla\delta Q)\|_{L^2_x}\|\dot\Delta_q(\delta u,\nabla\delta Q)\|_{L^2_x}.
\end{aligned}
$$

Since $|q-q'|\le 5$ and $|q'-q''|\le 5$ we get that $q'/2+q/2\simeq 3q''/2-q'/2$, so that the last inequality is equivalent to Equation (4.21). Hence, proceeding as in Equation (4.22), we get

$$
\begin{aligned}
&2L\xi\sum_{q\in\mathbb{Z}}\sum_{\substack{|q-q'|\le 5\\|q'-q''|\le 5}}2^{-q}\mathcal{I}_2^2(q,q',q'')\\
&\lesssim\|\nabla Q_2\|^2_{L^2_x}\|\Delta Q_2\|^2_{L^2_x}\|(\delta u,\nabla\delta Q)\|^2_{\dot H^{-\frac12}}+C_\nu\|\nabla\delta u\|^2_{\dot H^{-\frac12}}+C_{\Gamma,L}\|\Delta\delta Q\|^2_{\dot H^{-\frac12}}.
\end{aligned}
$$

When $j=4$

$$
\begin{aligned}
\mathcal{I}_4^2(q,q',q'')&:=\int_{\mathbb{R}_2}\Big\{(\dot S_{q'-1}Q_2-\dot S_{q-1}Q_2)\operatorname{tr}\{\dot\Delta_{q'}(\dot\Delta_{q''}Q_2\dot S_{q''+2}\nabla\delta u)\}\dot\Delta_q\Delta\delta Q\\
&\qquad-(\dot S_{q'-1}Q_2-\dot S_{q-1}Q_2)\operatorname{tr}\{\dot\Delta_{q'}(\dot\Delta_{q''}Q_2\dot S_{q''+2}\Delta\delta Q)\}\dot\Delta_q\nabla\delta u\Big\}\\
&\lesssim\|\dot S_{q'-1}Q_2-\dot S_{q-1}Q_2\|_{L^\infty_x}\|\dot\Delta_{q'}(\dot\Delta_{q''}Q_2\dot S_{q''+2}(\nabla\delta u,\Delta\delta Q))\|_{L^2_x}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim 2^{q'}\|\dot S_{q'-1}\nabla Q_2-\dot S_{q-1}\nabla Q_2\|_{L^2_x}\\
&\qquad\times\|\dot\Delta_{q'}(\dot\Delta_{q''}Q_2\dot S_{q''+2}(\nabla\delta u,\Delta\delta Q))\|_{L^1_x}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim 2^{q'-q''}\|\nabla Q_2\|_{L^2_x}\|\dot\Delta_{q''}\Delta Q_2\|_{L^2_x}\|\dot S_{q''+2}(\delta u,\nabla\delta Q)\|_{L^2_x}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim 2^{q'-q''}\|\nabla Q_2\|_{L^2_x}\|\Delta Q_2\|_{L^2_x}\|\dot S_{q''+2}(\delta u,\nabla\delta Q)\|_{L^2_x}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x},
\end{aligned}
$$

which is equivalent to the last inequality of Equation (4.23). Thus, arguing as in Equation (4.24), we deduce

$$
\begin{aligned}
&2L\xi\sum_{q\in\mathbb{Z}}\sum_{\substack{|q-q'|\le 5\\ q''\ge q'-5}}2^{-q}\mathcal{I}_4^2(q,q',q'')\\
&\lesssim\|\nabla Q_2\|_{L^2_x}^2\|\Delta Q_2\|_{L^2_x}^2\|(\delta u,\nabla\delta Q)\|_{\dot H^{-\frac12}}^2+C_\nu\|\nabla\delta u\|_{\dot H^{-\frac12}}^2+C_{\Gamma,L}\|\Delta\delta Q\|_{\dot H^{-\frac12}}^2.
\end{aligned}
$$

When $j=3$ we fix a real number $N>1$ and we consider the low frequencies $q'\le N$ as follows:

$$
\begin{aligned}
\mathcal{I}_3^2(q,q'):=&\int_{\mathbb{R}_2}\mathrm{tr}\Big\{(\dot S_{q'-1}Q_2-\dot S_{q-1}Q_2)\mathrm{tr}\{\dot\Delta_q(\dot S_{q'-1}Q_2\dot\Delta_{q'}\nabla\delta u)\}\dot\Delta_q\Delta\delta Q\\
&\qquad-(\dot S_{q'-1}Q_2-\dot S_{q-1}Q_2)\mathrm{tr}\{\dot\Delta_q(\dot S_{q'-1}Q_2\dot\Delta_{q'}\Delta\delta Q)\}\dot\Delta_q\nabla\delta u\Big\}\\
\lesssim&\|\dot S_{q'-1}Q_2-\dot S_{q-1}Q_2\|_{L^\infty_x}\|\dot\Delta_q(\dot S_{q'-1}Q_2\dot\Delta_{q'}(\nabla\delta u,\Delta\delta Q))\}\|_{L^2_x}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
\lesssim&2^{-q}\|\dot S_{q'-1}\Delta Q_2-\dot S_{q-1}\Delta Q_2\|_{L^2_x}\|\dot S_{q'-1}Q_2\|_{L^\infty_x}\|\dot\Delta_{q'}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
\lesssim&\|\dot S_{q'-1}\Delta Q_2-\dot S_{q-1}\Delta Q_2\|_{L^2_x}\|\dot S_{q'-1}Q_2\|_{L^\infty_x}\|\dot\Delta_{q'}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\|\dot\Delta_q(\delta u,\nabla\delta Q)\|_{L^2_x}.
\end{aligned}\tag{4.30}
$$

If $q'\le 1$ then $\|\dot S_{q'-1}Q_2\|_{L^\infty_x}\lesssim 2^{\frac{q'}{2}}\|\dot S_{q'-1}Q_2\|_{L^2_x}\le\|Q_2\|_{L^2_x}$, while if $1<q'\le N$ we have

$$
\|\dot S_{q'-1}Q_2\|_{L^\infty_x}\lesssim(\|Q_2\|_{L^2_x}+\sqrt{q'-1}\|\nabla Q_2\|_{L^2_x})\lesssim(\|Q_2\|_{L^2_x}+\sqrt{N}\|\nabla Q_2\|_{L^2_x}),
$$

thanks to Theorem A.2. Therefore, we deduce that

$$
\begin{aligned}
\mathcal{I}_3^2(q,q')&\lesssim\|\Delta Q_2\|_{L^2_x}(\|Q_2\|_{L^2_x}+\sqrt{N}\|\nabla Q_2\|_{L^2_x})\|\dot\Delta_{q'}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\|\dot\Delta_q(\delta u,\nabla\delta Q)\|_{L^2_x}\\
&\lesssim(1+N)\|\Delta Q_2\|_{L^2_x}^2\|(Q_2,\nabla Q_2)\|_{L^2_x}^2\|\dot\Delta_q(\delta u,\nabla\delta Q)\|_{L^2_x}^2\\
&\qquad+C_\nu\|\dot\Delta_{q'}\nabla\delta u\|_{L^2_x}^2+C_{\Gamma,L}\|\dot\Delta_{q'}\Delta\delta Q\|_{L^2_x}^2.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\sum_{q'\le N}\sum_{|q'-q|\le 5}2^{-q}\mathcal{I}_3^2(q,q')\\
&\lesssim(1+N)\|\Delta Q_2\|_{L^2_x}^2\|(Q_2,\nabla Q_2)\|_{L^2_x}^2\|(\delta u,\nabla\delta Q)\|_{\dot H^{-\frac12}}^2+C_\nu\|\nabla\delta u\|_{\dot H^{-\frac12}}^2+C_{\Gamma,L}\|\Delta\delta Q\|_{\dot H^{-\frac12}}^2.
\end{aligned}\tag{4.31}
$$

For the high frequencies $q'>N$, we get

$$
\begin{aligned}
&\mathcal{I}_3^2(q,q')\\
&\lesssim\|\dot S_{q'-1}Q_2-\dot S_{q-1}Q_2\|_{L^\infty_x}\|\dot\Delta_q(\dot S_{q'-1}Q_2\dot\Delta_{q'}(\nabla\delta u,\Delta\delta Q))\}\|_{L^2_x}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim 2^{-q}\|\dot S_{q'-1}\Delta Q_2-\dot S_{q-1}\Delta Q_2\|_{L^2_x}\|\dot S_{q'-1}Q_2\|_{L^\infty_x}\|\dot\Delta_{q'}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}
\end{aligned}
$$

$$\lesssim 2^{\frac{q'-q}{2}}\|\Delta Q_2\|_{L^2_x}(1+\sqrt{q'-1})\|(Q_2,\nabla Q_2)\|_{L^2_x}\|\dot\Delta_{q'}(\delta u,\nabla\delta Q)\|_{L^2_x}\|(\nabla\delta u,\Delta\delta Q)\|_{\dot H^{-\frac12}}$$
$$\lesssim (1+\sqrt{q'-1})\|\Delta Q_2\|_{L^2_x}\|(Q_2,\nabla Q_2)\|_{L^2_x}\|(\delta u,\ \delta Q)\|_{L^2_x}\|(\nabla\delta u,\Delta\delta Q)\|_{\dot H^{-\frac12}}. \tag{4.32}$$

Therefore

$$\sum_{q'>N}\sum_{|q-q'|\le5}2^{-q}\mathcal{I}_3^2(q,q')$$
$$\lesssim 2^{-N}\|\Delta Q_2\|_{L^2_x}\|\nabla Q_2\|_{L^2_x}\|(\delta u,\ \delta Q)\|_{L^2_x}\|(\nabla\delta u,\Delta\delta Q)\|_{\dot H^{-\frac12}}$$
$$\lesssim 2^{-2N}\|\Delta Q_2\|^2_{L^2_x}\|\nabla Q_2\|^2_{L^2_x}\|(\delta u,\ \delta Q)\|^2_{L^2_x}+C_\nu\|\nabla\delta u\|^2_{\dot H^{-\frac12}}+C_{\Gamma,L}\|\Delta\delta Q\|^2_{\dot H^{-\frac12}}. \tag{4.33}$$

Summarising Equations (4.31) and (4.33), we get

$$\sum_{q'\in\mathbb{Z}}\sum_{|q'-q|\le5}2^{-q}\mathcal{I}_3^2(q,q')$$
$$\lesssim(1+N)\|\Delta Q_2\|^2_{L^2_x}\|(Q_2,\nabla Q_2)\|^2_{L^2_x}\|(\delta u,\nabla\delta Q)\|^2_{\dot H^{-\frac12}}$$
$$+2^{-2N}\|\Delta Q_2\|^2_{L^2_x}\|\nabla Q_2\|^2_{L^2_x}\|(\delta u,\ \delta Q)\|^2_{L^2_x}+C_\nu\|\nabla\delta u\|^2_{\dot H^{-\frac12}}+C_{\Gamma,L}\|\Delta\delta Q\|^2_{\dot H^{-\frac12}}. \tag{4.34}$$

Now we define $N:=\lceil\ln\{e+1/\Phi(t)\}/2\rceil$, obtaining

$$\sum_{q\in\mathbb{Z}}\sum_{|q'-q|\le5}2^{-q}\mathcal{I}_3^2(q,q')$$
$$\lesssim\|\Delta Q_2(t)\|^2_{L^2_x}\|(Q_2,\nabla Q_2)(t)\|^2_{L^2_x}\|(\delta u,\nabla\delta Q(t))\|^2_{\dot H^{-\frac12}}+C_\nu\|\nabla\delta u(t)\|^2_{\dot H^{-\frac12}}$$
$$+C_{\Gamma,L}\|\Delta\delta Q(t)\|^2_{\dot H^{-\frac12}}+\|\Delta Q_2(t)\|^2_{L^2_x}\|\nabla Q_2(t)\|^2_{L^2_x}\|(\delta u,\ \delta Q)(t)\|^2_{L^2_x}\Big(1+\ln\Big(e+\frac{1}{\Phi(t)}\Big)\Big). \tag{4.35}$$

Estimate of $\mathcal{J}_q^3$. Now, let us deal with the term of Equation (4.19) related to $i=3$, namely

$$\int_{\mathbb{R}_2}\mathrm{tr}\big\{\dot S_{q-1}Q_2\mathrm{tr}\{\dot\Delta_q(Q_2\nabla\delta u)\}\dot\Delta_q\Delta\delta Q-\dot S_{q-1}Q_2]\mathrm{tr}\{\dot\Delta_q(Q_2\Delta\delta Q)\}\dot\Delta_q\nabla\delta u\big\}$$
$$=\sum_{j=1}^4\int_{\mathbb{R}_2}\mathrm{tr}\big\{\dot S_{q-1}Q_2\mathrm{tr}\{\mathcal{J}^j_{q'}(Q_2,\nabla\delta u)\}\dot\Delta_q\Delta\delta Q-\dot S_{q-1}Q_2\mathrm{tr}\{\mathcal{J}^j_{q'}(Q_2,\Delta\delta Q)\}\dot\Delta_q\nabla\delta u\big\}. \tag{4.36}$$

Let us consider $j=1$ and define

$$\mathcal{I}_1^3(q,q'):=\int_{\mathbb{R}_2}\mathrm{tr}\Big\{\dot S_{q-1}Q_2\mathrm{tr}\{[\dot\Delta_q,\dot S_{q'-1}Q_2]\dot\Delta_{q'}\nabla\delta u)\}\dot\Delta_q\Delta\delta Q$$
$$-\dot S_{q-1}Q_2\mathrm{tr}\{[\dot\Delta_q,\dot S_{q'-1}Q_2]\dot\Delta_{q'}\Delta\delta Q)\}\dot\Delta_q\nabla\delta u\Big\}.$$

We proceed as when proving Equation (4.26): we fix a positive real $\varepsilon\in(0,5/6]$ and we consider the low frequencies $q\le N$ for a suitable positive $N\ge1$.

$$\mathcal{I}_1^3(q,q')=\int_{\mathbb{R}_2}\mathrm{tr}\Big\{\dot S_{q-1}Q_2\mathrm{tr}\{[\dot\Delta_q,\dot S_{q'-1}Q_2]\dot\Delta_{q'}\nabla\delta u\}\dot\Delta_q\Delta\delta Q$$

$$
\begin{aligned}
&\qquad\qquad - \dot S_{q-1} Q_2 \mathrm{tr}\{[\dot\Delta_q, \dot S_{q'-1}Q_2]\dot\Delta_{q'}\Delta\delta Q\}\dot\Delta_q \nabla\delta u\Big\}\\
&\lesssim 2^{-q'}\|\dot S_{q-1}Q_2\|_{L^\infty_x}\|\dot S_{q'-1}\nabla Q_2\|_{L^{\frac{2}{\varepsilon}}_x}\|\dot\Delta_{q'}(\nabla\delta u,\Delta\delta Q)\|_{L_x^{\frac{2}{1-\varepsilon}}}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim (1+\sqrt{N})2^{q-q'}\|(Q_2,\nabla Q_2)\|_{L^2_x}\|\dot S_{q'-1}\nabla Q_2\|_{L^{\frac{2}{\varepsilon}}_x}\\
&\qquad\qquad \times\|\dot\Delta_{q'}(\delta u,\nabla\delta Q)\|_{L_x^{\frac{2}{1-\varepsilon}}}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim \sqrt{\frac{N}{\varepsilon}}\|(Q_2,\nabla Q_2)\|_{L^2_x}\|\dot S_{q'-1}\nabla Q_2\|_{L^2_x}^{\varepsilon}\|\dot S_{q'-1}\Delta Q_2\|_{L^2_x}^{1-\varepsilon}\\
&\qquad\qquad \times\|\dot\Delta_{q'}(\delta u,\nabla\delta Q)\|_{L^2_x}^{1-\varepsilon}\|\dot\Delta_{q'}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}^{\varepsilon}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x},
\end{aligned}
$$

which is equivalent to the last inequality of Equation (4.26). Hence, arguing as for proving Equation (4.28), we get

$$
\begin{aligned}
&\sum_{q\in\mathbb{Z}}\sum_{|q-q'|\le 5} 2^{-q}\mathcal{I}_1^3(q,q')\\
&\lesssim \max\big\{\|(Q_2,\nabla Q_2)\|_{L^2_x}^6,1\big\}\\
&\qquad \times\|(\nabla Q_2,\Delta Q_2)\|_{L^2_x}^2\|(\delta u,\nabla\delta Q)\|^2_{\dot H^{-\frac12}}\ln\Big(1+e+\frac{1}{\Phi(t)}\Big)\Big(1+\ln\ln\Big(1+e+\frac{1}{\Phi(t)}\Big)\Big)\\
&\qquad\qquad +\|(\nabla Q_2,\Delta Q_2)\|_{L^2_x}^2\|(Q_2,\nabla Q_2)\|^2_{L^2_x}\|(u_1,u_2,\nabla Q_1,\nabla Q_2)\|^2_{L^2_x}\Phi(t)\\
&\qquad\qquad\quad +C_\nu\|\nabla\delta u\|^2_{\dot H^{-\frac12}}+C_{\Gamma,L}\|\Delta\delta Q\|^2_{\dot H^{-\frac12}}.
\end{aligned}
$$

Further on, when $j=2$ in Equation (4.36), let us consider the low frequencies $q\le N$

$$
\begin{aligned}
\mathcal{I}_2^3(q,q'):=&\int_{\mathbb{R}_2}\mathrm{tr}\Big\{\dot S_{q-1}Q_2\mathrm{tr}\{(\dot S_{q'-1}Q_2-\dot S_{q-1}Q_2)\dot\Delta_q\dot\Delta_{q'}\nabla\delta u)\}\dot\Delta_q\Delta\delta Q\\
&\qquad -\dot S_{q-1}Q_2\mathrm{tr}\{(\dot S_{q'-1}Q_2-\dot S_{q-1}Q_2)\dot\Delta_{q'}\dot\Delta_{q'}\Delta\delta Q)\}\dot\Delta_q\nabla\delta u\Big\}\\
\lesssim&\|\dot S_{q-1}Q_2\|_{L^\infty_x}\|\dot S_{q'-1}Q_2-\dot S_{q-1}Q_2\|_{L^\infty_x}\|\dot\Delta_q\dot\Delta_{q'}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
\lesssim&\|\dot S_{q-1}Q_2\|_{L^\infty_x}\|\dot S_{q'-1}\Delta Q_2-\dot S_{q-1}\Delta Q_2\|_{L^2_x}\|\dot\Delta_{q'}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\|\dot\Delta_q(\delta u,\nabla\delta Q)\|_{L^2_x},
\end{aligned}
$$

which is as the last inequalities of Equation (4.30) (recalling that $q\sim q'$). Moreover when the high frequencies $q>N$

$$
\begin{aligned}
&\mathcal{I}_2^3(q,q')\\
&\lesssim\|\dot S_{q-1}Q_2\|_{L^\infty_x}\|\dot S_{q'-1}Q_2-\dot S_{q-1}Q_2\|_{L^\infty_x}\|\dot\Delta_q\dot\Delta_{q'}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim(1+\sqrt{q-1})\|(Q_2,\nabla Q_2)\|_{L^2_x}2^{-q}\|\dot S_{q'-1}\Delta Q_2-\dot S_{q-1}\Delta Q_2\|_{L^2_x}\\
&\qquad\qquad\times\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim(1+\sqrt{q-1})\|\Delta Q_2\|_{L^2_x}\|(Q_2,\nabla Q_2)\|_{L^2_x}\|(\delta u,\nabla\delta Q)\|_{L^2_x}\|(\nabla\delta u,\Delta\delta Q)\|_{\dot H^{-\frac12}},
\end{aligned}
$$

which is the equivalent to the last inequality (4.32). Hence, arguing as for proving Equation (4.35), we get

$$
\begin{aligned}
&\sum_{q\in\mathbb{Z}}\sum_{|q'-q|\le 5} 2^{-q}\mathcal{I}_2^3(q,q')\\
&\lesssim\|\Delta Q_2(t)\|^2_{L^2_x}\|(Q_2,\nabla Q_2)(t)\|^2_{L^2_x}\times\|(\delta u,\nabla\delta Q(t))\|^2_{\dot H^{-\frac12}}
\end{aligned}
$$

$$+C_\nu\|\nabla\delta u(t)\|^2_{\dot H^{-\frac12}}+C_{\Gamma,L}\|\Delta\delta Q(t)\|^2_{\dot H^{-\frac12}}$$
$$+\|\Delta Q_2(t)\|^2_{L^2_x}\|\nabla Q_2(t)\|^2_{L^2_x}\|(\delta u,\ \delta Q)(t)\|^2_{L^2_x}\Phi(t)\big(1+\ln\Big(1+e+\frac{1}{\Phi(t)}\Big)\big).$$

Now, when $j=3$ in Equation (4.36), we observe that

$$\mathcal{I}_3^3(q):=\int_{\mathbb{R}_2}\Big\{\mathrm{tr}\{\dot S_{q-1}Q_2\dot\Delta_q\nabla\delta u\}\mathrm{tr}\{\dot S_{q-1}Q_2\dot\Delta_q\Delta\delta Q\}$$
$$-\mathrm{tr}\{\dot S_{q-1}Q_2\dot\Delta_q\Delta\delta Q\}\mathrm{tr}\{\dot S_{q-1}Q_2\dot\Delta_q\nabla\delta u\}\Big\}=0,$$

for any $q\in\mathbb{Z}$.

Thus it remains to control the $j=4$ term, namely

$$\mathcal{I}_4^3(q,q'):=\int_{\mathbb{R}_2}\Big\{\dot S_{q-1}Q_2\mathrm{tr}\{(\dot\Delta_q(\dot\Delta_{q'}Q_2\dot S_{q'+2}\nabla\delta u)\}\dot\Delta_q\Delta\delta Q$$
$$-\dot S_{q-1}Q_2\mathrm{tr}\{(\dot\Delta_q(\dot\Delta_{q'}Q_2\dot S_{q'+2}\Delta\delta Q)\}\dot\Delta_q\nabla\delta u\Big\}$$
$$\lesssim\|\dot S_{q-1}Q_2\|_{L^\infty_x}\|\dot\Delta_q(\dot\Delta_{q'}Q_2\dot S_{q'+2}(\nabla\delta u,\Delta\delta Q))\|_{L^2_x}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}.$$

At first let us consider the low frequencies $q\le N$, with $N>1$

$$\begin{aligned}&\mathcal{I}_4^3(q,q')\\
&\lesssim(1+\sqrt N)\|(Q_2,\nabla Q_2)\|_{L^2_x}\|\dot\Delta_{q'}Q_2\|_{L^\infty_x}\times\|\dot S_{q'+2}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim(1+\sqrt N)\|(Q_2,\nabla Q_2)\|_{L^2_x}2^{-q'}\|\dot\Delta_{q'}\Delta Q_2\|_{L^2_x}\|\dot S_{q'+2}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim(1+\sqrt N)\|\nabla Q_2\|_{L^2_x}\|\dot\Delta_{q'}\Delta Q_2\|_{L^2_x}\|\dot S_{q'+2}(\delta u,\nabla\delta Q)\|_{L^2_x}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim(1+\sqrt N)2^{\frac{q'}{2}}\|\nabla Q_2\|_{L^2_x}\|\dot\Delta_{q'}\Delta Q_2\|_{L^2_x}\|(\delta u,\nabla\delta Q)\|_{\dot H^{-\frac12}}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x},\end{aligned}$$

which yields

$$\begin{aligned}&\sum_{q\le N}\sum_{q'\ge q-5}2^{-q}\mathcal{I}_4^3(q,q')\\
&\lesssim(1+\sqrt N)\|(Q_2,\nabla Q_2)\|_{L^2_x}\|(\delta u,\nabla\delta Q)\|_{\dot H^{-\frac12}}\\
&\qquad\times\sum_{q\le N}\sum_{q'\ge q-5}2^{\frac{q'}{2}-q}\|\dot\Delta_{q'}\Delta Q_2\|_{L^2_x}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim(1+\sqrt N)\|(Q_2,\nabla Q_2)\|_{L^2_x}\|(\delta u,\nabla\delta Q)\|_{\dot H^{-\frac12}}\\
&\qquad\times\sum_{q\in\mathbb{Z}}2^{-\frac q2}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\sum_{q'\ge q-5}2^{\frac{q'-q}{2}}\|\dot\Delta_{q'}\Delta Q_2\|_{L^2_x}\\
&\lesssim(1+\sqrt N)\|(Q_2,\nabla Q_2)\|_{L^2_x}\|(\delta u,\nabla\delta Q)\|_{\dot H^{-\frac12}}\\
&\qquad\times\Big(\sum_{q'\in\mathbb{Z}}\Big|\sum_{q\in\mathbb{Z}}2^{\frac{q-q'}{2}}1_{(-\infty,5]}(q-q')\|\dot\Delta_q\Delta Q_2\|_{L^2_x}\Big|^2\Big)^{\frac12}\|(\nabla\delta u,\Delta\delta Q)\|_{\dot H^{-\frac12}}.\end{aligned}$$

Thus by convolution,

$$
\begin{aligned}
&\sum_{q\leq N}\sum_{q'\geq q-5} 2^{-q}\mathcal{I}_4^3(q,q')\\
&\lesssim (1+\sqrt{N})\|(Q_2,\nabla Q_2)\|_{L^2_x}\|\Delta Q_2\|_{L^2_x}\|(\delta u,\nabla\delta Q)\|_{\dot H^{-\frac12}}\|(\nabla\delta u,\Delta\delta Q)\|_{\dot H^{-\frac12}}\\
&\lesssim (1+N)\|(Q_2,\nabla Q_2)\|^2_{L^2_x}\|\Delta Q_2\|^2_{L^2_x}\|(\delta u,\nabla\delta Q)\|^2_{\dot H^{-\frac12}}\\
&\qquad\qquad + C_\nu\|\nabla\delta u\|^2_{\dot H^{-\frac12}} + C_{\Gamma,L}\|\Delta\delta Q\|^2_{\dot H^{-\frac12}}.
\end{aligned}
$$

For the high frequencies $q>N$,

$$
\begin{aligned}
\sum_{q\geq N}\sum_{q'\geq q-5} 2^{-q}\mathcal{I}_2^3(q,q') &\lesssim \sum_{q\geq N}\sum_{q'\geq q-5} 2^{-q}(1+\sqrt{q-1})\|\dot S_{q-1}(Q_2,\nabla Q_2)\|_{L^2_x}\\
&\qquad\times\|\dot\Delta_{q'}Q_2\|_{L^\infty_x}\|\dot S_{q'+2}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim \|(Q_2,\nabla Q_2)\|_{L^2_x}\|(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\qquad\times\sum_{q\geq N}2^{-\frac q2}(1+\sqrt q)\sum_{q'\geq q-5}2^{\frac{q'-q}{2}}\|\dot\Delta_{q'}\nabla Q_2\|_{L^2_x}2^{-\frac{q'}{2}}\|\dot S_{q'+2}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim \|(Q_2,\nabla Q_2)\|_{L^2_x}\|(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\|(\nabla\delta u,\Delta\delta Q)\|_{\dot H^{-\frac12}}\\
&\qquad\times 2^{-\frac N2}\Big(\sum_{q\in\mathbb{Z}}\Big|\sum_{q'\geq q-5}2^{\frac{q'-q}{2}}\|\dot\Delta_{q'}\nabla Q_2\|_{L^2_x}\Big|^2\Big)^{\frac12},
\end{aligned}
$$

so that, by convolution,

$$
\begin{aligned}
&\sum_{q\geq N}\sum_{q'\geq q-5} 2^{-q}\mathcal{I}_2^3(q,q')\\
&\lesssim 2^{-N}\|(Q_2,\nabla Q_2)\|_{L^2_x}\|\nabla Q_2\|_{L^2_x}\|(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\|(\nabla\delta u,\Delta\delta Q)\|_{\dot H^{-\frac12}}.
\end{aligned}
$$

Summarising, we get

$$
\begin{aligned}
\sum_{q\in\mathbb{Z}}\sum_{q'\geq q-5} 2^{-q}\mathcal{I}_2^3(q,q') &\lesssim (1+N)\|(Q_2,\nabla Q_2)\|^2_{L^2_x}\|\Delta Q_2\|^2_{L^2_x}\|(\delta u,\nabla\delta Q)\|^2_{\dot H^{-\frac12}}\\
&\quad + C_\nu\|\nabla\delta u\|^2_{\dot H^{-\frac12}} + C_{\Gamma,L}\|\Delta\delta Q\|^2_{\dot H^{-\frac12}}\\
&\quad + 2^{-2N}\|(Q_2,\nabla Q_2)\|^2_{L^2_x}\|\nabla Q_2\|^2_{L^2_x}\|(\nabla\delta u,\Delta\delta Q)\|^2_{L^2_x},
\end{aligned}
$$

which is similar to Equation (4.34). Hence we can conclude as in Equation (4.35).
Estimate of $\mathcal{J}_q^4$. Now, we handle the last term of Equation (4.19), which is related to $i=4$, namely

$$
\begin{aligned}
&\sum_{q'\geq q-5}\int_{\mathbb{R}_2}\mathrm{tr}\big\{\dot\Delta_q\big[\dot\Delta_{q'}Q_2\mathrm{tr}\{\dot S_{q'+2}(Q_2\nabla\delta u)\}\big]\dot\Delta_q\Delta\delta Q\\
&\qquad\qquad -\dot\Delta_q\big[\dot\Delta_{q'}Q_2\mathrm{tr}\{\dot S_{q'+2}(Q_2\Delta\delta Q)\}\big]\dot\Delta_q\nabla\delta u\big\}\\
&=\sum_{q'\leq q-5}\sum_{q''\leq q'+1}\int_{\mathbb{R}_2}\mathrm{tr}\big\{\dot\Delta_q\big[\dot\Delta_{q'}Q_2\mathrm{tr}\{\dot\Delta_{q'}(Q_2\nabla\delta u)\}\big]\dot\Delta_q\Delta\delta Q\\
&\qquad\qquad -\dot\Delta_q\big[\dot\Delta_{q'}Q_2\mathrm{tr}\{\dot\Delta_{q'}(Q_2\Delta\delta Q)\}\big]\dot\Delta_q\nabla\delta u\big\}\\
&=\sum_{j=1}^4\sum_{q'\leq q-5}\sum_{q''\leq q'+1}\int_{\mathbb{R}_2}\mathrm{tr}\big\{\dot\Delta_q\big[\dot\Delta_{q'}Q_2\mathrm{tr}\{\mathcal{J}^j_{q''}(Q_2,\nabla\delta u)\}\big]\dot\Delta_q\Delta\delta Q
\end{aligned}
$$

$$-\dot{\Delta}_q\big[\dot{\Delta}_{q'}Q_2\mathrm{tr}\{\mathcal{J}^j_{q''}(Q_2,\Delta\delta Q)\}\big]\dot{\Delta}_q\nabla\delta u\big\}. \tag{4.37}$$

First, we consider the term related to $j=1$, that is

$$\begin{aligned}
\mathcal{I}^4_1(q,q',q'',q'''):=&\int_{\mathbb{R}_2}\mathrm{tr}\Big\{\dot{\Delta}_q\big[\dot{\Delta}_{q'}Q_2\mathrm{tr}\{[\dot{\Delta}_{q''},\dot{S}_{q'''-1}Q_2]\dot{\Delta}_{q'''}\nabla\delta u\}\big]\dot{\Delta}_q\Delta\delta Q\\
&\qquad-\dot{\Delta}_q\big[\dot{\Delta}_{q'}Q_2\mathrm{tr}\{[\dot{\Delta}_{q''},\dot{S}_{q'''-1}Q_2]\dot{\Delta}_{q''}\Delta\delta Q\}\big]\dot{\Delta}_q\nabla\delta u\Big\}\\
\lesssim&\|\dot{\Delta}_q\big[\dot{\Delta}_{q'}Q_2\mathrm{tr}\{[\dot{\Delta}_{q''},S_{q'''-1}Q_2]\dot{\Delta}_{q'''}(\nabla\delta u,\Delta\delta Q)\}\big]\|_{L^2_x}\|\dot{\Delta}_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
\lesssim&2^q\|\dot{\Delta}_q\big[\dot{\Delta}_{q'}Q_2\mathrm{tr}\{[\dot{\Delta}_{q''},S_{q'''-1}Q_2]\dot{\Delta}_{q'''}(\nabla\delta u,\Delta\delta Q)\}\big]\|_{L^1_x}\|\dot{\Delta}_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
\lesssim&2^{q-q''}\|\dot{\Delta}_{q'}Q_2\|_{L^\infty_x}\|\dot{S}_{q'''-1}\nabla Q_2\|_{L^2_x}\|\dot{\Delta}_{q'''}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\|\dot{\Delta}_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
\lesssim&2^{q-q'-q''}\|\dot{\Delta}_{q'}\Delta Q_2\|_{L^2_x}\|\dot{S}_{q'''-1}\nabla Q_2\|_{L^2_x}\|\dot{\Delta}_{q'''}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\|\dot{\Delta}_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
\lesssim&2^{q-q'-q''+q'''}\|\Delta Q_2\|_{L^2_x}\|\nabla Q_2\|_{L^2_x}\|\dot{\Delta}_{q'''}(\delta u,\nabla\delta Q)\|_{L^2_x}\|\dot{\Delta}_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}.
\end{aligned}\tag{4.38}$$

Hence, taking the sum in q, q', q'', and q''' (and observing that $|q''-q'''|\le 5$), we get

$$\begin{aligned}
&\sum_{q\in\mathbb{Z}}\sum_{q'\ge q-5}\sum_{q''\le q'+1}\sum_{|q'''-q''|\le 5}2^{-q}\mathcal{I}^4_1(q,q',q'',q''')\\
&\lesssim\|\nabla Q_2\|_{L^2_x}\|(\nabla\delta u,\Delta\delta Q)\|_{\dot{H}^{-\frac12}}\sum_{q,q',q''}2^{\frac{q}{2}-q'}\|\dot{\Delta}_{q'}\Delta Q_2\|_{L^2_x}\|\dot{\Delta}_{q''}(\delta u,\nabla\delta Q)\|_{L^2_x}\\
&\lesssim\|\nabla Q_2\|_{L^2_x}\|(\nabla\delta u,\Delta\delta Q)\|_{\dot{H}^{-\frac12}}\\
&\qquad\times\sum_{q\in\mathbb{Z}}2^{\frac{q}{2}}\sum_{q'\ge q-5}2^{-q'}\sum_{q''\le q'+1}2^{\frac{q''}{2}}\|\dot{\Delta}_{q'}\Delta Q_2\|_{L^2_x}2^{-\frac{q''}{2}}\|\dot{\Delta}_{q''}(\delta u,\nabla\delta Q)\|_{L^2_x}\\
&\lesssim\|\nabla Q_2\|_{L^2_x}\|(\nabla\delta u,\Delta\delta Q)\|_{\dot{H}^{-\frac12}}\\
&\qquad\times\sum_{q''\in\mathbb{Z}}2^{-\frac{q''}{2}}\|\dot{\Delta}_{q''}(\delta u,\nabla\delta Q)\|_{L^2_x}\sum_{q'\ge q''+1}2^{\frac{q''}{2}-q'}\|\dot{\Delta}_{q'}\Delta Q_2\|_{L^2_x}\sum_{q\le q'+5}2^{\frac{q}{2}}\\
&\lesssim\|\nabla Q_2\|_{L^2_x}\|(\nabla\delta u,\Delta\delta Q)\|_{\dot{H}^{-\frac12}}\\
&\qquad\times\sum_{q''\in\mathbb{Z}}2^{-\frac{q''}{2}}\|\dot{\Delta}_{q''}(\delta u,\nabla\delta Q)\|_{L^2_x}\sum_{q'\ge q''+1}2^{\frac{q''}{2}-\frac{q'}{2}}\|\dot{\Delta}_{q'}\Delta Q_2\|_{L^2_x}\\
&\lesssim\|\nabla Q_2\|_{L^2_x}\|(\nabla\delta u,\Delta\delta Q)\|_{\dot{H}^{-\frac12}}\|(\delta u,\nabla\delta Q)\|_{\dot{H}^{-\frac12}}\Big(\sum_{q''\in\mathbb{Z}}\Big|\sum_{q'\ge q''+1}2^{\frac{q''}{2}-\frac{q'}{2}}\|\dot{\Delta}_{q'}\Delta Q_2\|_{L^2_x}\Big|^2\Big)^{\frac12}\\
&\lesssim\|\nabla Q_2\|_{L^2_x}\|\Delta Q_2\|_{L^2_x}\|(\nabla\delta u,\Delta\delta Q)\|_{\dot{H}^{-\frac12}}\|(\delta u,\nabla\delta Q)\|_{\dot{H}^{-\frac12}}\\
&\lesssim\|\nabla Q_2\|^2_{L^2_x}\|\Delta Q_2\|^2_{L^2_x}\|(\delta u,\nabla\delta Q)\|^2_{\dot{H}^{-\frac12}}+C_\nu\|\nabla\delta u\|^2_{\dot{H}^{-\frac12}}+C_{\Gamma,L}\|\Delta\delta Q\|^2_{\dot{H}^{-\frac12}}.
\end{aligned}\tag{4.39}$$

When $j=2$ in Equation (4.37), we observe that

$$\begin{aligned}
&\mathcal{I}^4_2(q,q',q'',q'''):=\int_{\mathbb{R}_2}\mathrm{tr}\Big\{\dot{\Delta}_q\big[\dot{\Delta}_{q'}Q_2\mathrm{tr}\{(\dot{S}_{q'''-1}Q_2-\dot{S}_{q''-1}Q_2)\dot{\Delta}_{q''}\dot{\Delta}_{q'''}\nabla\delta u\}\big]\dot{\Delta}_q\Delta\delta Q\\
&\quad-\dot{\Delta}_q\big[\dot{\Delta}_{q'}Q_2\mathrm{tr}\{(\dot{S}_{q'''-1}Q_2-\dot{S}_{q''-1}Q_2)\dot{\Delta}_{q''}\dot{\Delta}_{q'''}\Delta\delta Q\}\big]\dot{\Delta}_q\nabla\delta u\Big\}\\
&\lesssim\|\dot{\Delta}_q\big[\dot{\Delta}_{q'}Q_2\mathrm{tr}\{(\dot{S}_{q'''-1}Q_2-\dot{S}_{q''-1}Q_2)\dot{\Delta}_{q''}\dot{\Delta}_{q'''}(\nabla\delta u,\Delta\delta Q)\}\big]\|_{L^2_x}\|\dot{\Delta}_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim 2^q\|\dot{\Delta}_q\big[\dot{\Delta}_{q'}Q_2\mathrm{tr}\{(\dot{S}_{q'''-1}Q_2-\dot{S}_{q''-1}Q_2)\dot{\Delta}_{q''}\dot{\Delta}_{q'''}(\nabla\delta u,\Delta\delta Q)\}\big]\|_{L^1_x}\\
&\qquad\times\|\dot{\Delta}_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}
\end{aligned}$$

$$\begin{aligned}
&\lesssim 2^q \|\dot\Delta_{q'} Q_2\|_{L^\infty_x} \|(\dot S_{q'''-1} Q_2 - \dot S_{q''-1} Q_2)\|_{L^2_x}\\
&\qquad\qquad \times \|\dot\Delta_{q''}\dot\Delta_{q'''}(\nabla\delta u, \Delta\delta Q)\|_{L^2_x} \|\dot\Delta_q(\nabla\delta u, \Delta\delta Q)\|_{L^2_x}\\
&\lesssim 2^{q-q'-q''} \|\dot\Delta_{q'}\Delta Q_2\|_{L^2_x} \|(\dot S_{q'''-1}\nabla Q_2 - \dot S_{q''-1}\nabla Q_2)\|_{L^2_x}\\
&\qquad\qquad \times \|\dot\Delta_{q''}(\nabla\delta u, \Delta\delta Q)\|_{L^2_x} \|\dot\Delta_q(\nabla\delta u, \Delta\delta Q)\|_{L^2_x}\\
&\lesssim 2^{q-q'} \|\dot\Delta_{q'}\Delta Q_2\|_{L^2_x} \|\nabla Q_2\|_{L^2_x} \|\dot\Delta_{q''}(\delta u, \nabla\delta Q)\|_{L^2_x} \|\dot\Delta_q(\nabla\delta u, \Delta\delta Q)\|_{L^2_x},
\end{aligned}$$

which is equivalent to the last inequality of Equation (4.38) (since $|q''-q'''|\le 5$). Hence, arguing as when proving Equation (4.39), the following estimate holds:

$$\begin{aligned}
&\sum_{q\in\mathbb{Z}}\sum_{q'\ge q-5}\sum_{q''\le q'+1}\sum_{|q'''-q''|\le 5} 2^{-q}\mathcal{I}_2^4(q,q',q'',q''')\\
&\quad\lesssim \|\nabla Q_2\|_{L^2_x}^2 \|\Delta Q_2\|_{L^2_x}^2 \|(\delta u, \nabla\delta Q)\|^2_{\dot H^{-\frac12}} + C_\nu \|\nabla\delta u\|^2_{\dot H^{-\frac12}} + C_{\Gamma,L}\|\Delta\delta Q\|^2_{\dot H^{-\frac12}}.
\end{aligned}$$

Now, let us analyze the term in Equation (4.37) related to $j=3$. Assuming $q''\le N$ for a suitable positive N, we get

$$\begin{aligned}
\mathcal{I}_3^4(q,q',q'') &:= \int_{\mathbb{R}_2} \mathrm{tr}\big\{\dot\Delta_q\big[\dot\Delta_{q'}Q_2\mathrm{tr}\{\dot S_{q''-1}Q_2\dot\Delta_{q''}\nabla\delta u\}\big]\dot\Delta_q\Delta\delta Q\\
&\qquad\qquad - \dot\Delta_q\big[\dot\Delta_{q'}Q_2\mathrm{tr}\{\dot S_{q''-1}Q_2\dot\Delta_{q''}\Delta\delta Q\}\big]\dot\Delta_q\nabla\delta u\big\}\\
&\lesssim \|\dot\Delta_q\big[\dot\Delta_{q'}Q_2\mathrm{tr}\{\dot S_{q''-1}Q_2\dot\Delta_{q''}(\nabla\delta u, \Delta\delta Q)\}\big]\|_{L^2_x}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim 2^q \|\dot\Delta_q\big[\dot\Delta_{q'}Q_2\mathrm{tr}\{\dot S_{q''-1}Q_2\dot\Delta_{q''}(\nabla\delta u, \Delta\delta Q)\}\big]\|_{L^1_x}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim 2^q \|\dot\Delta_{q'}Q_2\|_{L^2_x}\|\dot S_{q''-1}Q_2\|_{L^\infty_x}\|\dot\Delta_{q''}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim 2^q \|\dot\Delta_{q'}Q_2\|_{L^2_x}(1+\sqrt{N})\|(Q_2,\nabla Q_2)\|_{L^2_x}\|\dot\Delta_{q''}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\|\dot\Delta_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim (1+\sqrt{N})2^{\frac{3q}{2}+\frac{3q''}{2}-2q'}\|\dot\Delta_{q'}\Delta Q_2\|_{L^2_x}\|(Q_2,\nabla Q_2)\|_{L^2_x}\\
&\qquad\qquad \times 2^{-\frac{q''}{2}}\|\dot\Delta_{q''}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\|(\nabla\delta u,\Delta\delta Q)\|_{\dot H^{-\frac12}}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{q''\le N}\sum_{q'\ge q''-1}\sum_{q\le q'+5} 2^{-q}\mathcal{I}_3^4(q,q',q'')\\
&\lesssim (1+\sqrt{N})\|(Q_2,\nabla Q_2)\|_{L^2_x}\|(\nabla\delta u,\Delta\delta Q)\|_{\dot H^{-\frac12}}\\
&\qquad\times \sum_{q''\le N} 2^{-\frac{q''}{2}}\|\dot\Delta_{q''}(\delta u,\nabla\delta Q)\|_{L^2_x}\sum_{q'\ge q''-1}2^{\frac{3q''}{2}-2q'}\|\dot\Delta_{q'}\Delta Q_2\|_{L^2_x}\sum_{q\le q'+5}2^{\frac q2}\\
&\lesssim (1+\sqrt{N})\|(Q_2,\nabla Q_2)\|_{L^2_x}\|(\nabla\delta u,\Delta\delta Q)\|_{\dot H^{-\frac12}}\\
&\qquad\times \sum_{q''\le N} 2^{-\frac{q''}{2}}\|\dot\Delta_{q''}(\delta u,\nabla\delta Q)\|_{L^2_x}\sum_{q'\ge q''-1}2^{\frac{3q''}{2}-\frac{3q'}{2}}\|\dot\Delta_{q'}\Delta Q_2\|_{L^2_x}\\
&\lesssim (1+\sqrt{N})\|(Q_2,\nabla Q_2)\|_{L^2_x}\|(\nabla\delta u,\Delta\delta Q)\|_{\dot H^{-\frac12}}\\
&\qquad\times \|(\delta u,\nabla\delta Q)\|_{\dot H^{-\frac12}}\Big(\sum_{q''\in\mathbb{Z}}\Big|\sum_{q'\ge q''-1}2^{\frac32 q''-\frac32 q'}\|\dot\Delta_{q'}\Delta Q_2\|_{L^2_x}\Big|^2\Big)^{\frac12}\\
&\lesssim (1+\sqrt{N})\|(Q_2,\nabla Q_2)\|_{L^2_x}\|\Delta Q_2\|_{L^2_x}\|(\nabla\delta u,\Delta\delta Q)\|_{\dot H^{-\frac12}}\|(\delta u,\nabla\delta Q)\|_{\dot H^{-\frac12}}.
\end{aligned}$$

Considering the high frequencies $q'' > N$

$$
\begin{aligned}
&\mathcal{I}_3^4(q,q',q'')\\
&\lesssim \|\dot{\Delta}_q\big[\dot{\Delta}_{q'}Q_2\mathrm{tr}\{\dot{S}_{q''}Q_2\,\dot{\Delta}_{q''}(\nabla\delta u,\Delta\delta Q)\}\big]\|_{L^2_x}\|\dot{\Delta}_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim 2^q\|\dot{\Delta}_q\big[\dot{\Delta}_{q'}Q_2\mathrm{tr}\{\dot{S}_{q''}Q_2\,\dot{\Delta}_{q''}(\nabla\delta u,\Delta\delta Q)\}\big]\|_{L^1_x}\|\dot{\Delta}_q(\delta u,\nabla\delta Q)\|_{L^2_x}\\
&\lesssim 2^q\|\dot{\Delta}_{q'}Q_2\|_{L^2_x}\|\dot{S}_{q''}Q_2\|_{L^\infty_x}\|\dot{\Delta}_{q''}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\|\dot{\Delta}_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim 2^{\frac{3q}{2}-2q'}\|\dot{\Delta}_{q'}\Delta Q_2\|_{L^2_x}(1+\sqrt{q''})\|(Q_2,\nabla Q_2)\|_{L^2_x}\\
&\qquad\qquad\times 2^{q''}\|\dot{\Delta}_{q''}(\delta u,\nabla\delta Q)\|_{L^2_x}2^{-\frac{q}{2}}\|\dot{\Delta}_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim (1+\sqrt{q''})2^{\frac{3q}{2}+q''-2q'}\|\Delta Q_2\|_{L^2_x}\|(Q_2,\nabla Q_2)\|_{L^2_x}\|(\delta u,\nabla\delta Q)\|_{L^2_x}\|(\nabla\delta u,\Delta\delta Q)\|_{\dot{H}^{-\frac12}},
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\sum_{q''>N}\sum_{q'\geq q''-1}\sum_{q\leq q'+5}2^{-q}\mathcal{I}_3^4(q,q',q'')\\
&\lesssim \|\Delta Q_2\|_{L^2_x}\|(Q_2,\nabla Q_2)\|_{L^2_x}\|(\nabla\delta u,\Delta\delta Q)\|_{\dot{H}^{-\frac12}}\|(\delta u,\nabla\delta Q)\|_{L^2_x}\\
&\qquad\qquad\times\sum_{q''>N}(1+\sqrt{q''})2^{q''}\sum_{q'\geq q''-1}2^{-2q'}\sum_{q\leq q'+5}2^{\frac{q}{2}}\\
&\lesssim \|\Delta Q_2\|_{L^2_x}\|(Q_2,\nabla Q_2)\|_{L^2_x}\|\Delta\delta Q\|_{\dot{H}^{-\frac12}}\|(\delta u,\nabla\delta Q)\|_{L^2_x}\\
&\qquad\qquad\times\sum_{q''>N}(1+\sqrt{q''})2^{q''}\sum_{q'\geq q''-1}2^{-2q'+\frac{q'}{2}}\\
&\lesssim \|\Delta Q_2\|_{L^2_x}\|(Q_2,\nabla Q_2)\|_{L^2_x}\|(\nabla\delta u,\Delta\delta Q)\|_{\dot{H}^{-\frac12}}\|(\delta u,\nabla\delta Q)\|_{L^2_x}\times\sum_{q''>N}(1+\sqrt{q''})2^{-\frac{q''}{2}}\\
&\lesssim \|\Delta Q_2\|_{L^2_x}\|(Q_2,\nabla Q_2)\|_{L^2_x}\|(\nabla\delta u,\Delta\delta Q)\|_{\dot{H}^{-\frac12}}\|(\delta u,\nabla\delta Q)\|_{L^2_x}2^{-\frac{N}{2}}.
\end{aligned}
$$

Summarising the last inequalities, we obtain an estimate similar to Equation (4.34), so that we can conclude arguing as in Equation (4.35). Finally, it remains to examine when $j=4$, as last term. Let us define

$$
\begin{aligned}
\mathcal{I}_4^4(q,q',q'',q''') := \int_{\mathbb{R}_2}\mathrm{tr}\Big\{&\dot{\Delta}_q\big[\dot{\Delta}_{q'}Q_2\mathrm{tr}\{\dot{\Delta}_{q''}(\dot{\Delta}_{q'''}Q_2\,\dot{S}_{q'''+2}\nabla\delta u)\}\big]\dot{\Delta}_q\Delta\delta Q\\
&-\dot{\Delta}_q\big[\dot{\Delta}_{q'}Q_2\mathrm{tr}\{\dot{\Delta}_{q''}(\dot{\Delta}_{q'''}Q_2\,\dot{S}_{q'''+2}\Delta\delta Q)\}\big]\dot{\Delta}_q\nabla\delta u\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \|\dot{\Delta}_q\big[\dot{\Delta}_{q'}Q_2\mathrm{tr}\{\dot{\Delta}_{q''}[\dot{\Delta}_{q'''}Q_2\,\dot{S}_{q'''+2}(\nabla\delta u,\Delta\delta Q)]\}\big]\|_{L^2_x}\|\dot{\Delta}_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim 2^q\|\dot{\Delta}_q\big[\dot{\Delta}_{q'}Q_2\mathrm{tr}\{\dot{\Delta}_{q''}[\dot{\Delta}_{q'''}Q_2\,\dot{S}_{q'''+2}(\nabla\delta u,\Delta\delta Q)]\}\big]\|_{L^1_x}\|\dot{\Delta}_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim 2^q\|\dot{\Delta}_{q'}Q_2\|_{L^2_x}\|\dot{\Delta}_{q''}[\dot{\Delta}_{q'''}Q_2\,\dot{S}_{q'''+2}(\nabla\delta u,\Delta\delta Q)]\|_{L^2_x}\|\dot{\Delta}_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim 2^{q+q''}\|\dot{\Delta}_{q'}Q_2\|_{L^2_x}\|\dot{\Delta}_{q''}[\dot{\Delta}_{q'''}Q_2\,\dot{S}_{q'''+2}(\nabla\delta u,\Delta\delta Q)]\|_{L^1_x}\|\dot{\Delta}_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim 2^{q+q''}\|\dot{\Delta}_{q'}\Delta Q_2\|_{L^2_x}\|\dot{\Delta}_{q'''}Q_2\|_{L^2_x}\|\dot{S}_{q'''+2}(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\|\dot{\Delta}_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}\\
&\lesssim 2^{q-q'+q''-q'''}\|\dot{\Delta}_{q'}\nabla Q_2\|_{L^2_x}\|\dot{\Delta}_{q'''}\Delta Q_2\|_{L^2_x}\|\dot{S}_{q'''+2}(\delta u,\nabla\delta Q)\|_{L^2_x}\|\dot{\Delta}_q(\nabla\delta u,\Delta\delta Q)\|_{L^2_x}.
\end{aligned}
$$

Hence, taking the sum in q, q', q'' and q''', we get

$$
\sum_{q\in\mathbb{Z}}\sum_{q'\geq q-5}\sum_{q''\leq q'-1}\sum_{q'''\geq q''+5}2^{-q}\mathcal{I}_4^4(q,q',q'',q''')
$$

$$
\begin{aligned}
&\lesssim \|(\nabla\delta u,\Delta\delta Q)\|_{\dot H^{-\frac12}}\|\nabla Q_2\|_{L^2_x}\\
&\qquad\times\sum_{q\in\mathbb{Z}}\sum_{q'\geq q-5}\sum_{q''\leq q'-1}\sum_{q'''\geq q''+5}2^{\frac q2-q'+q''-q'''}\|\dot\Delta_{q'''}\Delta Q_2\|_{L^2_x}\|\dot S_{q'''+2}(\delta u,\nabla\delta Q)\|_{L^2_x}\\
&\lesssim \|(\nabla\delta u,\Delta\delta Q)\|_{\dot H^{-\frac12}}\|\nabla Q_2\|_{L^2_x}\\
&\qquad\times\sum_{q'''\in\mathbb{Z}}\sum_{q''\leq q'''-5}2^{q''-q'''}\|\dot\Delta_{q'''}\Delta Q_2\|_{L^2_x}\|\dot S_{q'''+2}(\delta u,\nabla\delta Q)\|_{L^2_x}\sum_{q'\geq q''+1}2^{-q'}\sum_{q\leq q'+5}2^{\frac q2}\\
&\lesssim \|(\nabla\delta u,\Delta\delta Q)\|_{\dot H^{-\frac12}}\|\nabla Q_2\|_{L^2_x}\\
&\qquad\times\sum_{q'''\in\mathbb{Z}}\sum_{q''\leq q'''-5}2^{q''-q'''}\|\dot\Delta_{q'''}\Delta Q_2\|_{L^2_x}\|\dot S_{q'''+2}(\delta u,\nabla\delta Q)\|_{L^2_x}\sum_{q'\geq q''+1}2^{-\frac{q'}{2}}\\
&\lesssim \|(\nabla\delta u,\Delta\delta Q)\|_{\dot H^{-\frac12}}\|\nabla Q_2\|_{L^2_x}\\
&\qquad\times\sum_{q'''\in\mathbb{Z}}\sum_{q''\leq q'''-5}2^{\frac{q''-q'''}{2}}\|\dot\Delta_{q'''}\Delta Q_2\|_{L^2_x}2^{-\frac{q'''}{2}}\|\dot S_{q'''+2}(\delta u,\nabla\delta Q)\|_{L^2_x}\\
&\lesssim \|(\nabla\delta u,\Delta\delta Q)\|_{\dot H^{-\frac12}}\|\nabla Q_2\|_{L^2_x}\sum_{q'''\in\mathbb{Z}}\|\dot\Delta_{q'''}\Delta Q_2\|_{L^2_x}2^{-\frac{q'''}{2}}\|\dot S_{q'''+2}(\delta u,\nabla\delta Q)\|_{L^2_x}\\
&\lesssim \|(\nabla\delta u,\Delta\delta Q)\|_{\dot H^{-\frac12}}\|\nabla Q_2\|_{L^2_x}\|\Delta Q_2\|_{L^2_x}\|(\delta u,\nabla\delta Q)\|_{\dot H^{-\frac12}}\\
&\lesssim \|\nabla Q_2\|^2_{L^2_x}\|\Delta Q_2\|^2_{L^2_x}\|(\delta u,\nabla\delta Q)\|^2_{\dot H^{-\frac12}}+C_\nu\|\nabla\delta u\|^2_{\dot H^{-\frac12}}+C_{\Gamma,L}\|\Delta\delta Q\|^2_{\dot H^{-\frac12}}
\end{aligned}
$$

and this concludes the estimates of the term $\mathcal{E}_1+\mathcal{E}_2$.

4.2.5. Remaining terms. For the sake of completeness, we now analyse all the remaining terms. However we point out that they are going to be estimates using simply just Theorem A.1; hence, they are not a challenging drawback. For instance, let us observe that

$$
\begin{aligned}
&L\langle(\xi\delta D+\delta\Omega)\delta Q,\Delta\delta Q\rangle_{\dot H^{-\frac12}}+L\langle(\xi D_2+\Omega_2)\delta Q,\Delta\delta Q\rangle_{\dot H^{-\frac12}}\\
&\qquad+L\langle\delta Q(\xi\delta D+\delta\Omega),\Delta\delta Q\rangle_{\dot H^{-\frac12}}+L\langle\delta Q(\xi D_2+\Omega_2),\Delta\delta Q\rangle_{\dot H^{-\frac12}}\\
&\lesssim\|\delta Q\|_{\dot H^{\frac12}}\|\nabla(u_1,u_2)\|_{L^2_x}\|\Delta\delta Q\|_{\dot H^{-\frac12}}\\
&\lesssim\|\nabla\delta Q\|_{\dot H^{-\frac12}}\times\|\nabla(u_1,u_2)\|_{L^2_x}\|\Delta\delta Q\|_{\dot H^{-\frac12}}\\
&\lesssim\|\nabla(u_1,u_2)\|^2_{L^2_x}\|\nabla\delta Q\|^2_{\dot H^{-\frac12}}+C_{\Gamma,L}\|\Delta\delta Q\|^2_{\dot H^{-\frac12}}.
\end{aligned}
$$

Moreover $La\Gamma\langle\delta Q,\Delta\delta Q\rangle_{\dot H^{-1/2}}\lesssim\|\delta Q\|^2_{\dot H^{-1/2}}+C_{\Gamma,L}\|\Delta\delta Q\|^2_{\dot H^{-1/2}}$ and

$$
\begin{aligned}
&Lb\Gamma\langle Q_1\delta Q+\delta QQ_2,\Delta\delta Q\rangle_{\dot H^{-\frac12}}\\
&\lesssim\|Q_1\delta Q+\delta QQ_2\|_{\dot H^{-\frac12}}\|\Delta Q\|_{\dot H^{-\frac12}}\\
&\lesssim\|(Q_1,Q_2)\|_{L^2_x}\|\delta Q\|_{\dot H^{\frac12}}\|\Delta\delta Q\|_{\dot H^{-\frac12}}\\
&\lesssim\|(Q_1,Q_2)\|_{L^2_x}\|\nabla\delta Q\|_{\dot H^{-\frac12}}\|\Delta\delta Q\|_{\dot H^{-\frac12}}\\
&\lesssim\|(Q_1,Q_2)\|^2_{L^2_x}\|\nabla\delta Q\|^2_{\dot H^{-\frac12}}+C_{\Gamma,L}\|\Delta\delta Q\|^2_{\dot H^{-\frac12}}.
\end{aligned}
$$

Furthermore, by a direct computation, we get

$$
\begin{aligned}
&Lc\Gamma\langle\delta Q\mathrm{tr}\{Q_1^2\},\Delta\delta Q\rangle_{\dot H^{-\frac12}}+Lc\Gamma\langle Q_2\mathrm{tr}\{Q_1\delta Q+\delta QQ_2\},\Delta\delta Q\rangle_{\dot H^{-\frac12}}\\
&\lesssim\|(Q_1^2,Q_2^2)\|_{L^2_x}\|\delta Q\|_{\dot H^{\frac12}}\|\Delta\delta Q\|_{\dot H^{-\frac12}}\lesssim\|(Q_1,Q_2)\|^2_{L^4_x}\|\nabla\delta Q\|_{\dot H^{-\frac12}}\|\Delta\delta Q\|_{\dot H^{-\frac12}}
\end{aligned}
$$

$$\lesssim \|(Q_1,Q_2)\|^2_{L^2_x}\|\nabla(Q_1,Q_2)\|^2_{L^2_x}\|\nabla\delta Q\|^2_{\dot H^{-\frac12}}+C_{\Gamma,L}\|\Delta\delta Q\|^2_{\dot H^{-\frac12}}$$

and

$$\begin{aligned}
&L\langle\delta u\cdot\nabla Q_1,\Delta\delta Q\rangle_{\dot H^{-\frac12}}+L\langle u_2\cdot\nabla\delta Q,\Delta\delta Q\rangle_{\dot H^{-\frac12}}\\
&\lesssim \|(u_2,\nabla Q_1)\|_{\dot H^{\frac34}}\|(\delta u,\nabla\delta Q)\|_{\dot H^{-\frac14}}\|\Delta\delta Q\|_{\dot H^{-\frac12}}\\
&\lesssim \|(u_2,\nabla Q_1)\|^{\frac14}_{L^2_x}\|(\nabla u_2,\Delta Q_1)\|^{\frac34}_{L^2_x}\|(\delta u,\nabla\delta Q)\|^{\frac34}_{\dot H^{-\frac12}}\|(\Delta\delta u,\Delta\delta Q)\|^{\frac14}_{\dot H^{-\frac12}}\|\Delta\delta Q\|_{\dot H^{-\frac12}}\\
&\lesssim \|(u_2,\nabla Q_1)\|^{\frac23}_{L^2_x}\|(\nabla u_2,\Delta Q_1)\|^2_{L^2_x}\|(\delta u,\nabla\delta Q)\|^2_{\dot H^{-\frac12}}+C_\nu\|\nabla\delta u\|^2_{\dot H^{-\frac12}}+C_{\Gamma,L}\|\Delta\delta Q\|^2_{\dot H^{-\frac12}}.
\end{aligned}$$

Moreover $a\xi\langle\delta QQ_1,\nabla\delta u\rangle_{\dot H^{-1/2}}\lesssim\|\delta Q\|_{\dot H^{1/2}}\|Q_1\|_{L^2}\|\nabla\delta u\|_{\dot H^{-1/2}}\lesssim\|Q_1\|^2_{L^2}\|\nabla\delta Q\|^2_{\dot H^{-1/2}}+C_\nu\|\nabla\delta u\|^2_{\dot H^{-1/2}}$,

$$\begin{aligned}
b\xi\langle\delta Q(Q_1^2-\mathrm{tr}\{Q_1^2\}\frac{\mathrm{Id}}{2}),\nabla\delta u\rangle_{\dot H^{-\frac12}}&\lesssim\|\delta Q\|_{\dot H^{\frac12}}\|Q_1^2\|_{L^2_x}\|\nabla\delta u\|_{\dot H^{-\frac12}}\\
&\lesssim\|\nabla\delta Q\|_{\dot H^{-\frac12}}\|Q_1\|^2_{L^4_x}\|\nabla\delta u\|_{\dot H^{-\frac12}}\\
&\lesssim\|\nabla\delta Q\|_{\dot H^{-\frac12}}\|Q_1\|_{L^2_x}\|\nabla Q_1\|_{L^2_x}\|\nabla\delta u\|_{\dot H^{-\frac12}}\\
&\lesssim\|Q_1\|^2_{L^2_x}\|\nabla Q_1\|^2_{L^2_x}\|\nabla\delta Q\|^2_{\dot H^{-\frac12}}+C_\nu\|\nabla\delta u\|^2_{\dot H^{-\frac12}}
\end{aligned}$$

and

$$\begin{aligned}
c\xi\langle\delta Q\mathrm{tr}(Q_1^2)Q_1,\nabla\delta u\rangle_{\dot H^{-\frac12}}&\lesssim\|\delta Q\|_{\dot H^{\frac12}}\|Q_1^2\|_{L^2_x}\|Q_1\|_{L^\infty}\|\nabla\delta u\|_{\dot H^{-\frac12}}\\
&\lesssim\|Q_1\|^2_{L^2_x}\|\nabla Q_1\|^2_{L^2_x}\|Q_1\|^2_{H^2}\|\nabla\delta Q\|^2_{\dot H^{-\frac12}}+C_\nu\|\nabla\delta u\|^2_{\dot H^{-\frac12}}.
\end{aligned}$$

Now $a\xi\langle(Q_2+\mathrm{Id}/2)\delta Q,\nabla\delta u\rangle_{\dot H^{-1/2}}\lesssim(\|Q_2\|^2_{L^2_x}+1)\|\nabla\delta Q\|^2_{\dot H^{-1/2}}+C_\nu\|\nabla\delta u\|^2_{\dot H^{-1/2}}$ and

$$\begin{aligned}
&b\xi\langle(Q_2+\frac{\mathrm{Id}}{2})(Q_1\delta Q+\delta QQ_2),\nabla\delta u\rangle_{\dot H^{-\frac12}}-b\xi\langle Q_2\mathrm{tr}\{Q_1\delta Q+\delta QQ_2\},\nabla\delta u\rangle_{\dot H^{-\frac12}}\\
&\lesssim(\|Q_2\|_{L^\infty_x}+1)\|(Q_1,Q_2)\|_{L^2_x}\|\delta Q\|_{\dot H^{\frac12}}\|\nabla\delta u\|_{\dot H^{-\frac12}}\\
&\lesssim(\|Q_2\|_{H^2}+1)\|(Q_1,Q_2)\|_{L^2_x}\times\|\nabla\delta Q\|_{\dot H^{-\frac12}}\|\nabla\delta u\|_{\dot H^{-\frac12}}\\
&\lesssim(\|Q_2\|_{H^2}+1)^2\|(Q_1,Q_2)\|^2_{L^2_x}\|\nabla\delta Q\|^2_{\dot H^{-\frac12}}+C_\nu\|\nabla\delta u\|^2_{\dot H^{-\frac12}}.
\end{aligned}$$

Equivalently, we get

$$\begin{aligned}
&c\xi\langle(Q_2+\frac{\mathrm{Id}}{2})\delta Q\mathrm{tr}\{Q_1^2\},\nabla\delta u\rangle_{\dot H^{-\frac12}}+c\xi\langle(Q_2+\frac{\mathrm{Id}}{2})Q_2\mathrm{tr}\{\delta QQ_1+Q_2\delta Q\},\nabla\delta u\rangle_{\dot H^{-\frac12}}\\
&\lesssim\|Q_2\|_{L^\infty}\|(Q_1^2,Q_2^2)\|_{L^2_x}\|\delta Q\|_{\dot H^{\frac12}}\|\nabla\delta u\|_{\dot H^{-\frac12}}\\
&\lesssim\|Q_2\|_{H^2}\|(Q_1,Q_2)\|^2_{L^4_x}\|\nabla\delta Q\|_{\dot H^{\frac12}}\|\nabla\delta u\|_{\dot H^{-\frac12}}\\
&\lesssim\|Q_2\|^2_{H^2}\|(Q_1,Q_2)\|^2_{L^2_x}\|\nabla(Q_1,Q_2)\|^2_{L^2_x}\|\nabla\delta Q\|^2_{\dot H^{-\frac12}}+C_\nu\|\nabla\delta u\|^2_{\dot H^{-\frac12}}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&L\xi\langle\delta Q\Delta\delta Q,\nabla\delta u\rangle_{\dot H^{-\frac12}}+L\xi\langle\delta Q\Delta Q_2,\nabla\delta u\rangle_{\dot H^{-\frac12}}\\
&\lesssim\|\delta Q\|_{\dot H^{\frac12}}\|\Delta(Q_1,Q_2)\|_{L^2_x}\|\nabla\delta u\|_{\dot H^{-\frac12}}\\
&\lesssim\|\Delta(Q_1,Q_2)\|^2_{L^2_x}\|\nabla\delta Q\|^2_{\dot H^{-\frac12}}+C_\nu\|\nabla\delta u\|^2_{\dot H^{-\frac12}}.
\end{aligned}$$

We can similarly control the terms from $-a\xi\langle Q_1\delta Q, \nabla\delta u\rangle_{\dot H^{-1/2}}$ to $L\xi\langle \Delta Q_2, \delta Q, \nabla\delta\rangle_{\dot H^{-1/2}}$ in (4.6), proceeding as in the previous estimates. Furthermore

$$
\begin{aligned}
&2a\xi\langle \delta Q\mathrm{tr}\{Q_1^2\}, \nabla\delta u\rangle_{\dot H^{-\frac12}} + 2a\xi\langle Q_2\mathrm{tr}\{\delta Q Q_1\}, \nabla\delta u\rangle_{\dot H^{-\frac12}} + 2a\xi\langle Q_2\mathrm{tr}\{Q_2\delta Q\}, \nabla\delta u\rangle_{\dot H^{-\frac12}}\\
\lesssim &\|\delta Q\|_{\dot H^{\frac12}}\|(Q_1^2, Q_2^2)\|_{L^2_x}\|\nabla\delta u\|_{\dot H^{-\frac12}}\\
\lesssim &\|(Q_1,Q_2)\|_{L^2_x}^2\|\nabla(Q_1,Q_2)\|_{L^2_x}^2\|\nabla\delta Q\|^2_{\dot H^{-\frac12}} + C_\nu\|\nabla\delta u\|^2_{\dot H^{-\frac12}},
\end{aligned}
$$

$$
\begin{aligned}
&2b\xi\langle \delta Q\mathrm{tr}\{Q_1^3\}, \nabla\delta u\rangle_{\dot H^{-\frac12}} + 2b\xi\langle Q_2\mathrm{tr}\{\delta Q Q_2^2\}, \nabla\delta u\rangle_{\dot H^{-\frac12}}\\
&\quad + 2b\xi\langle Q_2\mathrm{tr}\{Q_2(\delta Q Q_1 + Q_2\delta Q)\}, \nabla\delta u\rangle_{\dot H^{-\frac12}}\\
&\lesssim \|(Q_1,Q_2)\|^2_{L^2_x}\|\nabla(Q_1,Q_2)\|^2_{L^2_x}\|(Q_1,Q_2)\|^2_{H^2}\|\nabla\delta Q\|^2_{\dot H^{-\frac12}} + C_\nu\|\nabla\delta u\|^2_{\dot H^{-\frac12}},
\end{aligned}
$$

and also

$$
\begin{aligned}
&2c\xi\langle \delta Q\mathrm{tr}\{Q_1^2\}^2, \nabla\delta u\rangle_{\dot H^{-\frac12}} + 2c\xi\langle Q_2\mathrm{tr}\{\delta Q Q_1 + Q_2\delta Q\}\mathrm{tr}\{Q_1^2\}, \nabla\delta u\rangle_{\dot H^{-\frac12}}\\
&\qquad + 2c\xi\langle Q_2\mathrm{tr}\{Q_2^2\}\mathrm{tr}\{\delta Q Q_1 + Q_2\delta Q\}, \nabla\delta u\rangle_{\dot H^{-\frac12}}\\
\lesssim &\|(Q_1^4, Q_2^4)\|_{L^2_x}\|\delta Q\|_{\dot H^{\frac12}}\|\nabla\delta u\|_{\dot H^{-\frac12}}\\
\lesssim &\|(Q_1,Q_2)\|^4_{L^8_x}\|\nabla\delta Q\|_{\dot H^{-\frac12}}\|\nabla\delta u\|_{\dot H^{-\frac12}}\\
\lesssim &\|(Q_1,Q_2)\|_{L^2_x}\|\nabla(Q_1,Q_2)\|^3_{L^2_x}\|\nabla\delta Q\|_{\dot H^{-\frac12}}\|\nabla\delta u\|_{\dot H^{-\frac12}}\\
\lesssim &\|(Q_1,Q_2)\|^2_{L^2_x}\|\nabla(Q_1,Q_2)\|^6_{L^2_x}\|\nabla\delta Q\|^2_{\dot H^{-\frac12}} + C_\nu\|\nabla\delta u\|^2_{\dot H^{-\frac12}}.
\end{aligned}
$$

Furthermore, we observe that

$$
\begin{aligned}
&2L\xi\langle \delta Q\mathrm{tr}\{\delta Q\Delta\delta Q\}, \nabla\delta u\rangle_{\dot H^{-\frac12}} + 2L\xi\langle \delta Q\mathrm{tr}\{\delta Q\Delta Q_2\}, \nabla\delta u\rangle_{\dot H^{-\frac12}}\\
&\qquad + 2L\xi\langle \delta Q\mathrm{tr}\{Q_2\Delta\delta Q\}, \nabla\delta u\rangle_{\dot H^{-\frac12}} + 2L\xi\langle Q_2\mathrm{tr}\{\delta Q\Delta\delta Q\}, \nabla\delta u\rangle_{\dot H^{-\frac12}}\\
&\qquad + 2L\xi\langle \delta Q\mathrm{tr}\{Q_2\Delta Q_2\}, \nabla\delta u\rangle_{\dot H^{-\frac12}} + 2L\xi\langle Q_2\mathrm{tr}\{\delta Q\Delta Q_2\}, \nabla\delta u\rangle_{\dot H^{-\frac12}}\\
\lesssim &\|\delta Q\|_{\dot H^{\frac12}}\|\Delta(Q_1,Q_2)\|_{L^2_x}\|(Q_1,Q_2)\|_{L^\infty_x}\|\nabla\delta u\|_{\dot H^{-\frac12}}\\
\lesssim &\|\Delta(Q_1,Q_2)\|^2_{L^2_x}\|(Q_1,Q_2)\|^2_{H^2}\|\nabla\delta Q\|^2_{\dot H^{-\frac12}} + C_\nu\|\nabla\delta u\|^2_{\dot H^{-\frac12}}
\end{aligned}
$$

and

$$
\begin{aligned}
&L\langle \nabla\delta Q\odot\nabla Q_1, \nabla\delta u\rangle_{\dot H^{-\frac12}} + L\langle \nabla Q_2\odot\nabla\delta Q, \nabla\delta u\rangle_{\dot H^{-\frac12}}\\
\lesssim &\|\nabla\delta Q\|_{\dot H^{-\frac14}}\|\nabla(Q_1,Q_2)\|_{\dot H^{\frac34}}\|\nabla\delta u\|_{\dot H^{-\frac12}}\\
\lesssim &\|\nabla\delta Q\|^{\frac34}_{\dot H^{-\frac12}}\|\nabla\delta Q\|^{\frac14}_{\dot H^{\frac12}}\|\nabla(Q_1,Q_2)\|^{\frac14}_{L^2}\|\Delta(Q_1,Q_2)\|^{\frac34}_{L^2}\|\nabla\delta u\|_{\dot H^{-\frac12}}\\
\lesssim &\|\nabla\delta Q\|^{\frac34}_{\dot H^{-\frac12}}\|\Delta\delta Q\|^{\frac14}_{\dot H^{-\frac12}}\|\nabla(Q_1,Q_2)\|^{\frac14}_{L^2}\|\Delta(Q_1,Q_2)\|^{\frac34}_{L^2}\|\nabla\delta u\|_{\dot H^{-\frac12}}\\
\lesssim &\|\nabla(Q_1,Q_2)\|^{\frac23}_{L^2}\|\Delta(Q_1,Q_2)\|^2_{L^2}\|\nabla\delta Q\|^2_{\dot H^{-\frac12}} + C_{\Gamma,L}\|\Delta\delta Q\|^2_{\dot H^{-\frac12}} + C_\nu\|\nabla\delta u\|^2_{\dot H^{-\frac12}}.
\end{aligned}
$$

Moreover

$$
\begin{aligned}
&La\langle \delta Q Q_1, \nabla\delta u\rangle_{\dot H^{-\frac12}} + a\langle Q_2\delta Q, \nabla\delta u\rangle_{\dot H^{-\frac12}} - a\langle Q_1\delta Q, \nabla\delta u\rangle_{\dot H^{-\frac12}}\\
\lesssim &\|\delta Q\|_{\dot H^{\frac12}}\|(Q_1,Q_2)\|_{L^2}\|\nabla\delta u\|_{\dot H^{-\frac12}}
\end{aligned}
$$

$$\lesssim \|\nabla \delta Q\|_{\dot H^{-\frac12}} \|(Q_1,Q_2)\|_{L^2} \|\nabla \delta u\|_{\dot H^{-\frac12}}$$
$$\lesssim \|\nabla \delta Q\|^2_{\dot H^{-\frac12}} \|(Q_1,Q_2)\|^2_{L^2} + C_\nu \|\nabla \delta u\|^2_{\dot H^{-\frac12}},$$

$$-Lb\langle \delta Q(Q_1^2 - \mathrm{tr}\{Q_1^2\}\frac{\mathrm{Id}}{3}), \nabla \delta u\rangle_{\dot H^{-\frac12}} + b\langle (Q_1^2 - \mathrm{tr}\{Q_1^2\}\frac{\mathrm{Id}}{3})\delta Q, \nabla \delta u\rangle_{\dot H^{-\frac12}}$$
$$- b\langle Q_2(Q_1\delta Q + \delta Q Q_2 - \mathrm{tr}\{Q_1\delta Q + \delta Q Q_2\}\frac{\mathrm{Id}}{3}), \nabla \delta u\rangle_{\dot H^{-\frac12}}$$
$$- b\langle (Q_1\delta Q + \delta Q Q_2 - \mathrm{tr}\{Q_1\delta Q + \delta Q Q_2\}\frac{\mathrm{Id}}{3})\delta Q, \nabla \delta u\rangle_{\dot H^{-\frac12}}$$
$$\lesssim \|\delta Q\|_{\dot H^{-\frac12}} \|(Q_1^2,Q_2^2)\|_{L^2} \|\nabla \delta u\|_{\dot H^{-\frac12}}$$
$$\lesssim \|\delta Q\|_{\dot H^{\frac12}} \|(Q_1,Q_2)\|^2_{L^4} \|\nabla \delta u\|_{\dot H^{-\frac12}}$$
$$\lesssim \|\nabla \delta Q\|_{\dot H^{-\frac12}} \|(Q_1,Q_2)\|_{L^2} \|\nabla (Q_1,Q_2)\|_{L^2} \|\nabla \delta u\|_{\dot H^{-\frac12}}$$
$$\lesssim \|\nabla \delta Q\|^2_{\dot H^{-\frac12}} \|(Q_1,Q_2)\|^2_{L^2} \|\nabla (Q_1,Q_2)\|^2_{L^2} + C_\nu \|\nabla \delta u\|^2_{\dot H^{-\frac12}},$$

and

$$Lc\langle \delta Q Q_1 \mathrm{tr}\{Q_1^2\}, \nabla \delta u\rangle_{\dot H^{-\frac12}} + c\langle Q_2 \delta Q \mathrm{tr}\{Q_1^2\}, \nabla \delta u\rangle_{\dot H^{-\frac12}}$$
$$- c\langle Q_1 \delta Q \mathrm{tr}\{Q_1^2\}, \nabla \delta u\rangle_{\dot H^{-\frac12}} - c\langle \delta Q Q_2 \mathrm{tr}\{Q_1^2\}, \nabla \delta u\rangle_{\dot H^{-\frac12}}$$
$$\lesssim \|\delta Q\|_{\dot H^{\frac12}} \|(Q_1,Q_2)\|_{L^\infty_x} \|(Q_1^2,Q_2^2)\|_{L^2_x} \|\nabla \delta u\|_{\dot H^{-\frac12}}$$
$$\lesssim \|\nabla \delta Q\|_{\dot H^{-\frac12}} \|(Q_1,Q_2)\|_{H^2} \|(Q_1,Q_2)\|^2_{L^4_x} \|\nabla \delta u\|_{\dot H^{-\frac12}}$$
$$\lesssim \|\nabla \delta Q\|^2_{\dot H^{-\frac12}} \|(Q_1,Q_2)\|^2_{H^2} \|(Q_1,Q_2)\|^2_{L^2} \|\nabla (Q_1,Q_2)\|^2_{L^2} + C_\nu \|\nabla \delta u\|^2_{\dot H^{-\frac12}}.$$

Finally

$$\langle u_2\cdot \nabla \delta u, \delta u\rangle_{\dot H^{-\frac12}} = -\langle u_2 \otimes \delta u, \nabla \delta u\rangle_{\dot H^{-\frac12}} \lesssim \|u_2\|_{\dot H^{\frac12}} \|\delta u\|_{L^2} \|\nabla \delta u\|_{\dot H^{-\frac12}}$$
$$\lesssim \|u_2\|^{\frac12}_{L^2} \|\nabla u_2\|^{\frac12}_{L^2} \|\delta u\|^{\frac12}_{\dot H^{-\frac12}} \|\nabla \delta u\|^{\frac32}_{\dot H^{-\frac12}} \lesssim \|u_2\|^2_{L^2} \|\nabla u_2\|^2_{L^2} \|\delta u\|^2_{\dot H^{-\frac12}} + C_\nu \|\nabla \delta u\|^2_{\dot H^{-\frac12}}$$

and

$$\langle \delta u\cdot \nabla u_1, \delta u\rangle_{\dot H^{-\frac12}} \lesssim \|\delta u\|_{\dot H^{\frac12}} \|\nabla u_1\|_{L^2_x} \|\delta u\|_{\dot H^{-\frac12}} \lesssim C_\nu \|\nabla \delta u\|^2_{\dot H^{-\frac12}} + \|\nabla u_1\|^2_{L^2_x} \|\delta u\|^2_{\dot H^{-\frac12}}.$$

4.2.6. Conclusion. Recalling Equation (4.6) and summarising all the previous estimates, we conclude that there exists a function χ which belongs to $L^1_{loc}(\mathbb{R}_+)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) + \nu \|\nabla \delta u\|^2_{\dot H^{-\frac12}} + \Gamma L^2 \|\Delta Q\|^2_{\dot H^{-\frac12}} \lesssim \chi(t)\mu(\Phi(t)) + c_\nu \|\nabla \delta u\|^2_{\dot H^{-\frac12}} + C_{\Gamma,L} \|\Delta Q\|^2_{\dot H^{-\frac12}}$$

where μ is the Osgood modulus of continuity defined in the system (4.8). Hence, choosing $C_{\Gamma,L}$ and C_ν small enough from the beginning, we can absorb the last two terms on the right-hand side by the left-hand side, obtaining Equation (4.7). We deduce that $\Phi\equiv 0$, thanks to the Osgood Lemma and the null initial data $\Phi(0)=0$. Thus, $(\delta u, \nabla \delta Q)$ is identically zero and δQ as well, since $\delta Q(t)$ decades to 0 at infinity for almost every t. ∎

Appendix A.

THEOREM A.1. *Let s and t be two real numbers such that $|s|$ and $|t|$ belong to $[0, d/2)$. Let us assume that $s+t$ is positive. Then, for every $a \in \dot H^s(\mathbb{R}^d)$ and for every*

$b\in \dot H^t(\mathbb{R}^d)$, *the product* ab *belongs to* $\dot H^{s+t-d/2}$ *and there exists a positive constant (not dependent on* a *and* b*) such that*

$$\|ab\|_{\dot H^{s+t-d/2}} \le C\|a\|_{\dot H^s}\|b\|_{\dot H^t}.$$

Proof. At first we identify the Sobolev Spaces $\dot H^s$ and $\dot H^t$ with the Besov Spaces $\dot B^s_{2,2}$ and $\dot B^t_{2,2}$ respectively. We claim that ab belongs to $\dot B^{s+t-d/2}_{2,2}$ and

$$\|ab\|_{\dot B^{s+t-\frac{d}{2}}_{2,2}} \le C\|a\|_{\dot B^s_{2,2}}\|b\|_{\dot B^t_{2,2}},$$

for a suitable positive constant.
We decompose the product ab through the Bony decomposition, namely $ab=\dot T_a b+\dot T_b a+R(a,b)$, where

$$\dot T_a b := \sum_{q\in\mathbb{Z}} \dot\Delta_q a \dot S_{q-1} b, \qquad \dot T_b a := \sum_{q\in\mathbb{Z}} \dot S_{q-1} a \dot\Delta_q b, \qquad \dot R(a,b) := \sum_{\substack{q\in\mathbb{Z}\\ |\nu|\le 1}} \dot\Delta_q a \dot\Delta_{q+\nu} b.$$

For any $q\in\mathbb{Z}$, we have

$$\begin{aligned}&2^{q(s+t-\frac{d}{2})}\|(\dot\Delta_q \dot T_a b, \dot\Delta_q \dot T_b a)\|_{L^2}\\ &\lesssim \sum_{|q-q'|\le 5} 2^{q's}\|\dot\Delta_q a\|_{L^2_x} 2^{q'(t-\frac{d}{2})}\|\dot S_{q-1} b\|_{L^\infty} + \sum_{|q-q'|\le 5} 2^{q'(s-\frac{d}{2})}\|\dot S_{q-1} a\|_{L^\infty_x} 2^{q't}\|\dot\Delta_q b\|_{L^2}.\end{aligned}$$

Hence

$$\begin{aligned}\|(\dot T_a b, \dot T_b a)\|_{\dot B^{s+t-\frac{d}{2}}_{2,2}} &\le \|(\dot T_a b, \dot T_b a)\|_{\dot B^{s+t-\frac{d}{2}}_{2,1}}\\ &\lesssim \|a\|_{\dot B^s_{2,2}}\|b\|_{\dot B^{t-\frac{d}{2}}_{\infty,2}} + \|a\|_{\dot B^{s-\frac{d}{2}}_{\infty,2}}\|b\|_{\dot B^t_{2,2}} \lesssim \|a\|_{\dot B^s_{2,2}}\|b\|_{\dot B^t_{2,2}},\end{aligned}$$

where we have used the embedding $\dot B^\sigma_{2,2}\hookrightarrow \dot B^{\sigma-d/2}_{\infty,2}$ for any $\sigma\in\mathbb{R}$ and furthermor the following norm-equivalence

$$\|u\|_{\dot B^{\tilde\sigma}_{p,r}} \approx \|(2^{\tilde\sigma}\|S_q u\|_{L^p_x})_{q\in\mathbb{Z}}\|_{l^r(\mathbb{Z})}, \qquad u\in \dot B^{\tilde\sigma}_{p,r}$$

for any $1\le p,r\le\infty$ and $\tilde\sigma<0$.
In order to conclude the proof, we have to handle the rest $\dot R(a,b)$. By a direct computation, for any $q\in\mathbb{Z}$,

$$2^{(t+s)q}\|\dot\Delta_q \dot R(a,b)\|_{L^1_x} \le \sum_{\substack{q'\ge q-5\\ |\nu|\le 1}} 2^{(q-q')(s+t)} 2^{q's}\|\dot\Delta_{q'} a\|_{L^2_x} 2^{(q'+\nu)t}\|\dot\Delta_{q'+\nu} a\|_{L^2_x},$$

so that, thanks to the Young inequality, we deduce

$$\|\dot R(a,b)\|_{\dot B^{s+t-\frac{d}{2}}_{2,2}} \lesssim \|\dot R(a,b)\|_{\dot B^{s+t}_{1,1}} \lesssim \|a\|_{\dot B^s_{2,2}}\|b\|_{\dot B^t_{2,2}},$$

where we have used the embedding $\dot B^{s+t}_{1,1}\hookrightarrow \dot B^{s+t-d/2}_{2,2}$ and moreover $\sum_{q\le 5} 2^{q(s+t)}<\infty$, since $s+t$ is positive. ∎

THEOREM A.2. *Let N be a positive real number and f a function in H^1. Then $\dot{S}_N f$ belongs to L^∞_x and*

$$\|\dot{S}_N f\|_{L^\infty_x} \lesssim \|f\|_{L^2_x} + \sqrt{N}\|\nabla f\|_{L^2_x} \lesssim (1+\sqrt{N})\|(f,\nabla f)\|_{L^2_x}.$$

Proof. We split $\dot{S}_N f$ into two parts, namely $\dot{S}_N f = \sum_{q<0}\dot{\Delta}_q f + \sum_{0\le q<N}\dot{\Delta}_q f$. First we observe that

$$\|\sum_{q<0}\dot{\Delta}_q f\|_{L^\infty_x} \le \sum_{q<0}\|\dot{\Delta}_q f\|_{L^\infty_x} \lesssim \sum_{q<0} 2^q\|\dot{\Delta}_q f\|_{L^2_x} \lesssim \Big(\sum_{q<0}2^q\Big)\|f\|_{L^2_x}.$$

Similarly, considering the second term, we get

$$\begin{aligned}\|\sum_{0<q\le N}\dot{\Delta}_q f\|_{L^\infty_x} &\le \sum_{0<q\le N}\|\dot{\Delta}_q f\|_{L^\infty_x} \lesssim \sum_{0<q\le N} 2^q\|\dot{\Delta}_q f\|_{L^2_x}\\ &\lesssim \sum_{0<q\le N}\|\dot{\Delta}_q \nabla f\|_{L^2_x} \lesssim \Big(\sum_{0<q\le N}1\Big)^{\frac{1}{2}}\Big(\sum_{0<q\le N}\|\dot{\Delta}_q\nabla f\|^2_{L^2_x}\Big)^{\frac{1}{2}} \lesssim \sqrt{N}\|f\|_{\dot{H}^1},\end{aligned}$$

which concludes the proof of the Theorem. □

The following Lemma plays a main role in the uniqueness result of Theorem 1.1; more precisely, inequality (A.1) is the key for the double-logarithmic estimate.

LEMMA A.1. *There exists a positive constant C, such that for any $p\in[1,\infty)$, the following inequality is satisfied:*

$$\|f\|_{L^{2p}(\mathbb{R}^2)} \le C\sqrt{p}\|f\|^{\frac{1}{p}}_{L^2(\mathbb{R}^2)}\|\nabla f\|^{1-\frac{1}{p}}_{L^2(\mathbb{R}^2)}. \tag{A.1}$$

Proof. The proof of this lemma was presented in [32, Lemma 4.3] and we report it here, for the sake of simplicity. Thanks to Sobolev embeddings, we have

$$\|f\|_{L^{2p}(\mathbb{R}^2)} \le C\sqrt{p}\|f\|_{\dot{H}^{1-\frac{1}{p}}(\mathbb{R}^2)}. \tag{A.2}$$

Moreover, since $\dot{H}^{1-1/p}(\mathbb{R}^2)$ is an interpolation space between $L^2(\mathbb{R}^2)$ and $\dot{H}^1(\mathbb{R}^2)$, the following inequality is satisfied:

$$\|f\|_{\dot{H}^{1-\frac{1}{p}}(\mathbb{R}^2)} \le \|f\|^{\frac{1}{p}}_{L^2(\mathbb{R}^2)}\|\nabla f\|^{1-\frac{1}{p}}_{L^2(\mathbb{R}^2)},$$

which leads to Equation (A.1), together with Equation (A.2). □

Appendix B.

PROPOSITION B.1. *Let $(Q^{(n)},u^n)$ be a smooth solution of the system* (3.2) *in dimension $d=2$ or $d=3$, with restriction* (1.3) *and smooth initial data $(\bar{Q}(x),\bar{u}(x))$ that decays fast enough at infinity so that we can integrate by parts in space (for any $t\ge 0$) without boundary terms. We assume that $|\xi|<\xi_0$, where ξ_0 is an explicitly computable constant, scale invariant, depending on $a,b,c,d,\Gamma,\nu,\lambda$.*

For $(\bar{Q},\bar{u})\in H^1\times L^2$,we have

$$\|Q^{(n)}(t,\cdot)\|_{H^1} \le C_1 + \bar{C}_1 e^{\bar{C}_1 t}\|\bar{Q}\|_{H^1}, \forall t\ge 0, \tag{B.1}$$

with $C_1, \bar{C}_1$ *depending on* $(a,b,c,d,\Gamma,L,\nu,\bar{Q},\bar{u})$. *Moreover*

$$\|u^n(t,\cdot)\|_{L^2}^2 + \nu\int_0^t \|\nabla u^n\|_{L^2}^2 \le C_1. \tag{B.2}$$

Proof. We denote

$$X^n_{\alpha\beta} \stackrel{\text{def}}{=} L\Delta Q^{(n)}_{\alpha\beta} - cQ^{(n)}_{\alpha\beta}\mathrm{tr}((Q^{(n)})^2),\ \alpha,\beta = 1,2,3. \tag{B.3}$$

Multiplying the first equation in (3.2) by $-\lambda\bar{H}^n$ and the second one by u^n, taking the trace, and integrating over $\mathbb{R}^d$, we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\frac{1}{2}|u^n|^2 + \frac{L\lambda}{2}|\nabla Q^{(n)}|^2 + \lambda\big(\frac{a}{2}|Q^{(n)}|^2 - \frac{b}{3}\mathrm{tr}(Q^{(n)})^3 + \frac{c}{4}|Q^{(n)}|^4\big)\mathrm{d}x\\
&\quad + \nu\|\nabla u^n\|_{L^2}^2 + \Gamma\lambda L^2\|\Delta Q^{(n)}\|_{L^2}^2\\
&\quad + \Gamma\lambda c^2\|J_n(Q^{(n)}\mathrm{tr}\{Q^{(n)}\})\}\|_{L^2}^2 - 2cL\Gamma\lambda\int_{\mathbb{R}^d}\Delta Q^{(n)}_{\alpha\beta}Q^{(n)}_{\alpha\beta}\mathrm{tr}\{(Q^{(n)})^2\}\mathrm{d}x + a^2\Gamma\lambda\|Q^{(n)}\|_{L^2}^2\\
&\quad + b^2\Gamma\lambda\int_{\mathbb{R}^d}\mathrm{tr}\big\{J_n\big((Q^{(n)})^2 - \frac{\mathrm{tr}\{(Q^{(n)})^2\}}{d}\big)^2\big\}\mathrm{d}x\\
&\quad + \varepsilon\int_{\mathbb{R}^d}|R_\varepsilon u\nabla Q^{(n)}|^3\mathrm{d}x + \varepsilon\int_{\mathbb{R}^d}|R_\varepsilon\nabla u^n|^4\mathrm{d}x\\
&\le 2a\Gamma\lambda\underbrace{\int_{\mathbb{R}^d}\mathrm{tr}\{X^nQ^{(n)}\}\mathrm{d}x}_{\stackrel{\text{def}}{=}\mathcal{I}_n} - 2b\Gamma\lambda\underbrace{\int_{\mathbb{R}^d}\mathrm{tr}\{X^n(Q^{(n)})^2\}\mathrm{d}x}_{\stackrel{\text{def}}{=}\mathcal{J}_n}\\
&\quad + 2ab\Gamma\lambda\int_{\mathbb{R}^d}\mathrm{tr}\{(Q^{(n)})^3\}\mathrm{d}x + \lambda\underbrace{\int_{\mathbb{R}^d}J_n\Big(R_\varepsilon u^n\cdot\nabla Q^{(n)}_{\alpha\beta}\Big)J_n\Big(bQ^{(n)}_{\alpha\gamma}Q^{(n)}_{\gamma\beta} - cQ^{(n)}_{\alpha\beta}\big|Q^{(n)}\big|^2\Big)\mathrm{d}x}_{\stackrel{\text{def}}{=}\mathcal{II}}\\
&\quad + \lambda\int_{\mathbb{R}^d}J_n\Big(-R_\varepsilon\Omega^{(n)}_{\alpha\gamma}Q^{(n)}_{\gamma\beta} + Q^{(n)}_{\alpha\gamma}R_\varepsilon\Omega^{(n)}_{\gamma\beta}\Big)J_n\Big(bQ^{(n)}_{\alpha\delta}Q^{(n)}_{\delta\beta} - cQ^{(n)}_{\alpha\beta}\big|Q^{(n)}\big|^2\Big)\mathrm{d}x.
\end{aligned} \tag{B.4}$$

Integrating by parts we have:

$$\begin{aligned}
&-2cL\Gamma\lambda\int_{\mathbb{R}^d}\Delta Q^{(n)}_{\alpha\beta}Q^{(n)}_{\alpha\beta}\mathrm{tr}\{(Q^{(n)})^2\}dx\\
&= 2cL\Gamma\lambda\int_{\mathbb{R}^d}Q^{(n)}_{\alpha\beta,k}Q^{(n)}_{\alpha\beta,k}\mathrm{tr}\{(Q^{(n)})^2\}dx + 2cL\Gamma\lambda\int_{\mathbb{R}^d}Q^{(n)}_{\alpha\beta,k}Q^{(n)}_{\alpha\beta}\partial_k\Big(\mathrm{tr}\{(Q^{(n)})^2\}\Big)dx\\
&= 2cL\Gamma\lambda\int_{\mathbb{R}^d}|\nabla Q^{(n)}|^2\mathrm{tr}\{(Q^{(n)})^2\})\,dx + cL\Gamma\lambda\int_{\mathbb{R}^d}|\nabla\Big(\mathrm{tr}\{(Q^{(n)})^2\}\Big)|^2\,dx\\
&\ge 0
\end{aligned} \tag{B.5}$$

(where for the last inequality we used the assumption (1.3) and $L,\Gamma,\lambda > 0$). One can easily see that

$$\mathcal{I}_n = -\frac{L}{2}\|\nabla Q^{(n)}\|_{L^2}^2 - c\|Q^{(n)}\|_{L^4}^4 \tag{B.6}$$

and furthermore

$$\lambda\int_{\mathbb{R}^d}J_n\Big(-R_\varepsilon\Omega^{(n)}_{\alpha\gamma}Q^{(n)}_{\gamma\beta} + Q^{(n)}_{\alpha\gamma}R_\varepsilon\Omega^{(n)}_{\gamma\beta}\Big)J_n\Big(bQ^{(n)}_{\alpha\delta}Q^{(n)}_{\delta\beta} - cQ^{(n)}_{\alpha\beta}\big|Q^{(n)}\big|^2\Big)\,dx$$

$$\leq \frac{\varepsilon}{2}\int_{\mathbb{R}^d}|R_\varepsilon \nabla u^n|^4\,dx + C(\varepsilon)\int_{\mathbb{R}^d}|Q^{(n)}|^4\,dx + \frac{\Gamma c^2}{2}\int_{\mathbb{R}^d}|J_n(Q^{(n)}|Q^{(n)}|^2)|^2\,dx.$$

On the other hand, for any $\varepsilon > 0$ and $\tilde{C} = \tilde{C}(\varepsilon, c)$ an explicitly computable constant, we have

$$\begin{aligned}
\mathcal{J}_n =& L\int_{\mathbb{R}^d} Q^{(n)}_{\alpha\beta,kk}Q^{(n)}_{\alpha\gamma}Q^{(n)}_{\gamma\beta}\mathrm{d}x - c\int_{\mathbb{R}^d}\mathrm{tr}\{(Q^{(n)})^2\}\mathrm{tr}\{(Q)^{(n)})^3\}\mathrm{d}x\\
\leq& -L\int_{\mathbb{R}^d} Q^{(n)}_{\alpha\beta,k}Q^{(n)}_{\alpha\gamma,k}Q^{(n)}_{\gamma\beta}\mathrm{d}x - L\int_{\mathbb{R}^d} Q^{(n)}_{\alpha\beta,k}Q^{(n)}_{\alpha\gamma}Q^{(n)}_{\gamma\beta,k}\mathrm{d}x\\
&+\int_{\mathbb{R}^d}\mathrm{tr}\{(Q^{(n)})^2\}\left(\frac{\tilde{C}}{\varepsilon}\mathrm{tr}\{(Q^{(n)})^2\} + \varepsilon\,\mathrm{tr}^2\{(Q^{(n)})^2\}\right)\mathrm{d}x\\
\leq& L\varepsilon\int_{\mathbb{R}^d}|\nabla Q^{(n)}|^2\mathrm{tr}\{(Q^{(n)})^2\}\mathrm{d}x + \frac{\tilde{C}}{\varepsilon}\|\nabla Q^{(n)}\|^2_{L^2}\\
&+\int_{\mathbb{R}^d}\mathrm{tr}\{(Q^{(n)})^2\}\left(\frac{\tilde{C}}{\varepsilon}\mathrm{tr}\{(Q^{(n)})^2\} + \varepsilon\,\mathrm{tr}^2\{(Q^{(n)})^2\}\right)\mathrm{d}x.
\end{aligned}$$

Using the last four relations in Equation (B.4) and considering Equation (3.4), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\frac{1}{2}|u^n|^2 + \frac{L\lambda}{2}|\nabla Q^{(n)}|^2 + \lambda\Big(\frac{a}{2}|Q^{(n)}|^2 - \frac{b}{3}\mathrm{tr}(Q^{(n)})^3 + \frac{c}{4}|Q^{(n)}|^4\Big)\mathrm{d}x\\
&+\nu\|\nabla u^n\|^2_{L^2} + \Gamma\lambda L^2\|\Delta Q^{(n)}\|^2_{L^2} + \frac{\Gamma\lambda c^2}{2}\|J_n(Q^{(n)}\mathrm{tr}\{Q^{(n)})\})\|^2_{L^2} + a^2\Gamma\lambda\|Q^{(n)}\|^2_{L^2}\\
&+2cL\Gamma\lambda\int_{\mathbb{R}^d}|\nabla Q^{(n)}|^2\mathrm{tr}\{(Q^{(n)})^2\}\mathrm{d}x\\
&+cL\Gamma\lambda\int_{\mathbb{R}^d}|\nabla\left(\mathrm{tr}\{(Q^{(n)})^2\}\right)|^2\mathrm{d}x + \frac{\varepsilon}{2}\int_{\mathbb{R}^d}|R_\varepsilon u^n\cdot\nabla Q^{(n)}|^3\mathrm{d}x + \frac{\varepsilon}{2}\int_{\mathbb{R}^d}|\nabla R_\varepsilon u^n|^4\mathrm{d}x\\
\leq& 2|a|\Gamma\lambda\Big(\frac{L}{2}\|\nabla Q^{(n)}\|^2_{L^2} + c\|Q^{(n)}\|^4_{L^4}\Big)\\
&+2|b|\Gamma\lambda L\varepsilon\int_{\mathbb{R}^d}|\nabla Q^{(n)}|^2\mathrm{tr}\{(Q^{(n)})^2\}\mathrm{d}x + 2|b|\Gamma\lambda\frac{\tilde{C}}{\varepsilon}\|\nabla Q^{(n)}\|^2_{L^2}\\
&+2|b|\Gamma\lambda\int_{\mathbb{R}^d}\mathrm{tr}\{(Q^{(n)})^2\}\left(\frac{\tilde{C}}{\varepsilon}\mathrm{tr}\{(Q^{(n)})^2\} + \varepsilon\,\mathrm{tr}^2\{(Q^{(n)})^2\}\right)\mathrm{d}x\\
&+2|ab|\Gamma\lambda\Big(\varepsilon\|Q^{(n)}\|^2_{L^2} + \big(C(\varepsilon) + \frac{\tilde{C}}{\varepsilon}\big)\|Q^{(n)}\|^4_{L^4}\Big).
\end{aligned}$$

Taking ε small enough, we can absorb all the terms with an ϵ coefficient on the right into the left-hand side, and we are left with

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d}\frac{1}{2}|u^n|^2 + \frac{L\lambda}{2}|\nabla Q^{(n)}|^2 + \lambda\Big(\frac{a}{2}|Q^{(n)}|^2 - \frac{b}{3}\mathrm{tr}(Q^{(n)})^3 + \frac{c}{4}|Q^{(n)}|^4\Big)\mathrm{d}x\\
&\quad+\nu\|\nabla u^n\|^2_{L^2} + \Gamma\lambda L^2\|\Delta Q^{(n)}\|^2_{L^2} + \frac{\Gamma\lambda c^2}{2}\|J_n(Q^{(n)}\mathrm{tr}\{Q^{(n)})\})\|^2_{L^2} + \Gamma\lambda a^2\|Q^{(n)}\|^2_{L^2}\\
&\quad+2cL\Gamma\lambda\int_{\mathbb{R}^d}|\nabla Q^{(n)}|^2\mathrm{tr}\{(Q^{(n)})^2\}\mathrm{d}x + cL\Gamma\lambda\int_{\mathbb{R}^d}|\nabla\left(\mathrm{tr}\{(Q^{(n)})^2\}\right)|^2\mathrm{d}x\\
\leq& \bar{C}\left(\|\nabla Q^{(n)}\|^2_{L^2} + \|Q^{(n)}\|^4_{L^4}\right),
\end{aligned}$$

with $\bar{C}=\bar{C}(a,b,c)$.

The last relation is not yet enough because there are no positive terms. However, let us note that, if $a>0$ we obtain the a priori estimates by using the inequality $tr\{(Q^{(n)})^3\}\leq\frac{3}{8}tr\{(Q^{(n)})^2)\}+tr\{(Q^{(n)})^2\}^2$. If $a\leq 0$ we have to estimate separately $\|Q^{(n)}\|_{L^2}$ and this ask for a smallness condition for ξ. Indeed, proceeding as when proving Equation (2.13), we get

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big[\int_{\mathbb{R}^d}\frac{1}{2}|u^n|^2+\frac{L\lambda}{2}|\nabla Q^{(n)}|^2+\lambda\big(\frac{a}{2}|Q^{(n)}|^2-\frac{b}{3}\mathrm{tr}(Q^{(n)})^3+\frac{c}{4}|Q^{(n)}|^4\big)\mathrm{d}x+M\|Q\|_{L^2}^2\Big]\\
&\quad+\nu\|\nabla u^n\|_{L^2}^2+\Gamma\lambda L^2\|\Delta Q^{(n)}\|_{L^2}^2+\frac{\Gamma\lambda c^2}{2}\|J_n(Q^{(n)}\mathrm{tr}\{Q^{(n)})\})\|_{L^2}^2+a^2\|Q^{(n)}\|_{L^2}^2\\
&\quad+2cL\Gamma\lambda\int_{\mathbb{R}^d}|\nabla Q^{(n)}|^2\mathrm{tr}\{(Q^{(n)})^2\}\mathrm{d}x+cL\Gamma\lambda\int_{\mathbb{R}^d}|\nabla\left(\mathrm{tr}\{(Q^{(n)})^2\}\right)|^2\mathrm{d}x\\
&\quad+\frac{\varepsilon}{2}\int_{\mathbb{R}^d}|R_\varepsilon u^n\cdot\nabla Q^{(n)}|^3\mathrm{d}x+\frac{\varepsilon}{2}\int_{\mathbb{R}^d}|\nabla R_\varepsilon u^n|^4\mathrm{d}x\\
&\leq\bar{C}\left(\|\nabla Q^{(n)}\|_{L^2}^2+\|Q^{(n)}\|_{L^4}^4\right)\\
&\quad+MC(d)\varepsilon\int_{\mathbb{R}^d}|\nabla u^n|^2\mathrm{d}x+\frac{M|\xi|^2}{\varepsilon}\int_{\mathbb{R}^d}|Q^{(n)}|^2+|J_n(Q^{(n)}\mathrm{tr}\{(Q^{(n)})^2\}|^2\mathrm{d}x\\
&\quad+M\hat{C}\int_{\mathbb{R}^d}|Q^{(n)}|^2+|Q^{(n)}|^4\mathrm{d}x.
\end{aligned}
$$

We chose ε small enough so that $MC(d)\varepsilon<\nu$. Finally we make the assumption that $|\xi|$ is small enough, depending on a,b,c,d,ν such that

$$\frac{M|\xi|^2}{\varepsilon}\leq\Gamma\lambda c^2.$$

Then, taking into account that

$$\frac{M}{2}\mathrm{tr}\{(Q^{(n)})^2\}+\frac{c}{8}\mathrm{tr}^2\{(Q^{(n)})^2\}\leq(M+\frac{a}{2})\mathrm{tr}\{(Q^{(n)})^2\}-\frac{b}{3}\mathrm{tr}\{(Q^{(n)})^3\}+\frac{c}{4}\mathrm{tr}^2\{(Q^{(n)})^2\},$$

we obtain the claimed relation (B.1). ∎

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