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NON-RELATIVISTIC AND LOW MACH NUMBER LIMITS OF TWO P1 APPROXIMATION MODEL ARISING IN RADIATION HYDRODYNAMICS*

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Abstract. In this paper we study the non-relativistic and low Mach number limits of two $P1$ approximation model arising in radiation hydrodynamics in $\mathbb{T}^3$, i.e. the barotropic model and the Navier–Stokes–Fourier model. For the barotropic model, we consider the case that the initial data is a small perturbation of stable equilbria while for the Navier–Stokes–Fourier model, we consider the case that the initial data is large. For both models, we prove the convergence to the solution of the incompressible Navier–Stokes equations with/without stationary transport equations.

Keywords. Radiation hydrodynamics, low Mach number limit, non-relativistic limit.

AMS subject classifications. 35Q30, 35Q70, 35B25.

1. Introduction

Radiation hydrodynamics studies the propagation of thermal radiation through a fluid, and the effect of this radiation on the hydrodynamics describing the fluid motion [3, 16, 18]. Usually, the state of the radiation can described by a kinetic or transport equation. However, from the computation of view, the total system to the Radiation hydrodynamics are too complicated to compute effective and some simplified models are introduced [2, 10, 11, 20]. The simplified models can also be derived by taking singular limits, for example, the diffusion limit [6, 7, 10, 12], low Mach number limit [5, 9] and non-relativistic limit [14, 17, 19].

In this paper we study the non-relativistic and low Mach number limits of two $P1$ approximation model arising in radiation hydrodynamics in $\mathbb{T}^3$, i.e. the barotropic model and the Navier–Stokes–Fourier model. We first consider the following barotropic model of radiative flow:

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \frac{1}{\epsilon_1^2}\nabla p(\rho) - \mu \Delta u - (\lambda + \mu)\nabla \operatorname{div} u = \epsilon_2 I_1, \tag{1.2}$$

$$\epsilon_2 \partial_t I_0 + \operatorname{div} I_1 = B(\rho) - I_0, \tag{1.3}$$

$$\epsilon_2 \partial_t I_1 + \nabla I_0 = -I_1 \quad \text{in} \quad \mathbb{T}^3 \times (0, \infty). \tag{1.4}$$

Recently, Danchin and Ducomet [5] studied the low Mach number limit ($\epsilon_1 \to 0$ and $\epsilon_2 > 0$ is fixed) for the system (1.1)–(1.4) in the critical regularity framework. Here we study the limit of $\epsilon := (\epsilon_1, \epsilon_2) \to 0$. It is convenient to consider the flow with small density variation, i.e.,

$$\rho = 1 + \epsilon_1 \sigma,$$

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and we will take $p'(1)=1$, $B(1)=1$, and setting $b:=B(\rho)-B(1), j_0:=I_0-B(1)$ and $j_1:=I_1$. Then we can rewrite the system (1.1)–(1.4) as follows:

$$\partial_t\sigma+\operatorname{div}(\sigma u)+\frac{1}{\epsilon_1}\operatorname{div}u=0, \tag{1.5}$$

$$\rho\partial_t u+\rho u\cdot\nabla u+\frac{1}{\epsilon_1}p'(\rho)\nabla\sigma-\mu\Delta u-(\lambda+\mu)\nabla\operatorname{div}u=\epsilon_2 j_1, \tag{1.6}$$

$$\epsilon_2\partial_t j_0+\operatorname{div}j_1=b-j_0, \tag{1.7}$$

$$\epsilon_2\partial_t j_1+\nabla j_0=-j_1, \tag{1.8}$$

with the initial condition

$$(\sigma,u,j_0,j_1)(\cdot,0)=(\sigma_0,u_0,j_{00},j_{10}) \text{ in } \mathbb{T}^3. \tag{1.9}$$

Here, we impose the following regularity conditions on the initial data:

$$\begin{cases}\sigma_0,u_0\in H^2,\ j_{00},j_{10}\in H^1,\ (\partial_t\sigma,\partial_t u,\partial_t j_0,\partial_t j_1)(\cdot,0)\in L^2,\\ \displaystyle\int\sigma_0dx=0,\ \int\rho_0u_0dx=0,\ \int j_{10}dx=0,\ \frac12\le\rho_0\le\frac32.\end{cases} \tag{1.10}$$

We will prove the following theorem.

THEOREM 1.1. *Let $0<\epsilon_1,\ \epsilon_2<1$, and assume that condition* (1.10) *holds. Then, there exists a positive constant α such that if*

$$\|(\sigma_0^\epsilon,u_0^\epsilon)\|_{H^2}+\|(j_{00}^\epsilon,j_{10}^\epsilon)\|_{H^1}+\|\partial_t(\sigma^\epsilon,u^\epsilon,j_0^\epsilon,j_1^\epsilon)(0)\|_{L^2}\le\alpha, \tag{1.11}$$

then for any $\epsilon_1,\epsilon_2\in(0,\epsilon_0]$ where $0<\epsilon_0<1$ is some constant, the problem (1.5)–(1.9) *has a unique strong solution $(\sigma^\epsilon,u^\epsilon,j_0^\epsilon,j_1^\epsilon)$ satisfying*

$$\begin{cases}\displaystyle\int\sigma^\epsilon dx=0,\ \int\rho^\epsilon u^\epsilon dx=0,\ \int j_1^\epsilon dx=0,\ \frac12\le\rho^\epsilon\le\frac32,\\ \sigma^\epsilon\in L^\infty(0,\infty;H^2),\ \partial_t\sigma^\epsilon\in L^\infty(0,\infty;L^2)\cap L^2(0,\infty;H^1),\\ u^\epsilon\in L^\infty(0,\infty;H^2)\cap L^2(0,\infty;H^3),\ \partial_t u^\epsilon\in L^\infty(0,\infty;L^2)\cap L^2(0,\infty;H^1),\\ j_0^\epsilon,j_1^\epsilon\in L^\infty(0,\infty;L^2)\cap L^2(0,\infty;H^1),\ \partial_t(j_0^\epsilon,j_1^\epsilon)\in L^2(0,\infty;L^2),\end{cases} \tag{1.12}$$

with the corresponding norms that are uniformly bounded with respect to ϵ. Furthermore, $(\sigma^\epsilon,u^\epsilon,j_0^\epsilon,j_1^\epsilon)$ converge to $(\sigma,u,j_0=0,j_1=0)$ in certain Sobolev space as $\epsilon\to0$, and there exists a function $\pi(x,t)$ such that (u,π) in $C(0,\infty;H^2)$ solves the following problem of the incompressible Navier–Stokes equations:

$$\begin{cases}\partial_t u+u\cdot\nabla u+\nabla\pi-\mu\Delta u=0,\ \operatorname{div}u=0,\\ u(\cdot,0)=u_0 \ \text{ in } \ \mathbb{T}^3,\end{cases} \tag{1.13}$$

where u_0 is the weak limit of u_0^ϵ in H^2 with $\operatorname{div}u_0=0$ in $\mathbb{T}^3$.

REMARK 1.1. In the assumption (1.11), $\partial_t\sigma^\epsilon(0)$ is indeed defined by $-\operatorname{div}(\sigma_0^\epsilon,u_0^\epsilon)+\frac{1}{\epsilon_1}\operatorname{div}u_0^\epsilon$ through the density equation. Further, $\partial_t u^\epsilon(0)$, $\partial_t j_0^\epsilon(0)$, and $\partial_t j_1^\epsilon(0)$ are defined by an analogous way.

Next we consider the singular limits of the following compressible Navier–Stokes–Fourier-P1 approximate model arising in radiation hydrodynamics [3, 5]:

$$\partial_t\rho+\operatorname{div}(\rho u)=0, \tag{1.14}$$

$$\partial_t(\rho u)+\operatorname{div}(\rho u\otimes u)+\frac{1}{\epsilon_1^2}\nabla p-\mu\Delta u-(\lambda+\mu)\nabla\operatorname{div}u=\epsilon_2 I_1, \tag{1.15}$$

$$\partial_t(\rho e)+\operatorname{div}(\rho u e)+p\operatorname{div}u-\operatorname{div}(\kappa\nabla\mathcal{T})=\epsilon_1^2\left(\frac{\mu}{2}|\nabla u+\nabla u^t|^2+\lambda(\operatorname{div}u)^2\right)+I_0-\mathcal{T}^4, \tag{1.16}$$

$$\epsilon_2\partial_t I_0+\operatorname{div}I_1=\mathcal{T}^4-I_0, \tag{1.17}$$

$$\epsilon_2\partial_t I_1+\nabla I_0=-I_1 \quad \text{in} \quad \mathbb{T}^3\times(0,\infty). \tag{1.18}$$

For simplicity, we will consider the case that the fluid is a polytropic ideal gas, that is

$$e:=C_V\mathcal{T},\ p:=R\rho\mathcal{T}. \tag{1.19}$$

In the following, we introduce the new unknowns σ and θ with

$$\rho:=1+\epsilon_1\sigma,\ \mathcal{T}:=1+\epsilon_1\theta. \tag{1.20}$$

Then the non-dimensional system (1.14)–(1.18) can be rewritten as

$$\partial_t\sigma+\operatorname{div}(\sigma u)+\frac{1}{\epsilon_1}\operatorname{div}u=0, \tag{1.21}$$

$$\rho\partial_t u+\rho u\cdot\nabla u+\frac{R}{\epsilon_1}(\nabla\sigma+\nabla\theta)+R\nabla(\sigma\theta)-\mu\Delta u-(\lambda+\mu)\nabla\operatorname{div}u=\epsilon_2 I_1, \tag{1.22}$$

$$\begin{aligned}&C_V\rho(\partial_t\theta+u\cdot\nabla\theta)+R(\rho\theta+\sigma)\operatorname{div}u+\frac{R}{\epsilon_1}\operatorname{div}u-\kappa\Delta\theta\\ =&\epsilon_1\left(\frac{\mu}{2}|\nabla u+\nabla u^t|^2+\lambda(\operatorname{div}u)^2\right)+I_0-(1+\epsilon_1\theta)^4,\end{aligned} \tag{1.23}$$

$$\epsilon_2\partial_t I_0+\operatorname{div}I_1=(1+\epsilon_1\theta)^4-I_0, \tag{1.24}$$

$$\epsilon_2\partial_t I_1+\nabla I_0=-I_1 \quad \text{in} \quad \mathbb{T}^3\times(0,\infty). \tag{1.25}$$

We impose the initial conditions

$$(\sigma,u,\theta,I_0,I_1)(\cdot,0)=(\sigma_0,u_0,\theta_0,I_{00},I_{10}) \quad \text{in} \quad \mathbb{T}^3. \tag{1.26}$$

Very recently, Jiang–Li–Xie [14] studied the singular limit $\epsilon_2\to0$, when $\epsilon_1=1$. The aim of this paper is to study the singular limit $\epsilon:=(\epsilon_1,\epsilon_2)\to0$.

Definition 1.1.

$$|||u|||_{k,j}:=\sum_{i=0}^{j}\|\partial_t^i u\|_{H^{k-i}(\mathbb{T}^3)},$$

$$|||u|||_{k,j}(0):=\sum_{i=0}^{j}\|\partial_t^i u(0)\|_{H^{k-i}(\mathbb{T}^3)}.$$

Remark 1.2. To simplify the statement, similar to Remark 1.1, we have used $\partial_t u(0)$ to signify the quantity $\partial_t u|_{t=0}$ obtained through Equation (1.22), and $\partial_t^2 u(0)$ is given recursively by $\partial_t(1.22)$ in the same manner. Similarly, we can define $\partial_t\sigma(0)$, $\partial_t\theta(0)$, $\partial_t I_0(0)$, $\partial_t I_1(0)$, $\partial_t^2\sigma(0)$, $\partial_t^2\theta(0)$, $\partial_t^2 I(0)$, and $\partial_t^2 I_1(0)$.

A local existence result for the problem (1.21)–(1.26) in the following sense can be shown in a similar way as in [22]. Thus we omit the details of the proof.

Proposition 1.1 (Local existence). *Let $0<\epsilon_1,\epsilon_2<1$. Suppose that the initial data $(\sigma_0^\epsilon,u_0^\epsilon,\theta_0^\epsilon,I_{00}^\epsilon,I_{10}^\epsilon)$ satisfies that $1+\epsilon_1\sigma_0^\epsilon\geq m>0$ for some positive constant m, and*

$$\partial_t^k\sigma^\epsilon(0),\partial_t^k u^\epsilon(0),\partial_t^k\theta^\epsilon(0),\partial_t^k I_0^\epsilon(0),\partial_t^k I_1^\epsilon(0)\in H^{2-k}(\mathbb{T}^3),\quad k=0,1,2.$$

Then there exists a positive constant $T^\epsilon>0$ such that the problem (1.21)–(1.26) has a unique solution $(\sigma^\epsilon,u^\epsilon,\theta^\epsilon,I_0^\epsilon,I_1^\epsilon)$, satisfying that $1+\epsilon_1\sigma^\epsilon>0$ in $\mathbb{T}^3\times(0,T^\epsilon)$, and for $k=0,1,2$,

$$\begin{aligned}&\partial_t^k\sigma^\epsilon,\partial_t^k I_0^\epsilon,\partial_t^k I_1^\epsilon\in C([0,T^\epsilon];H^{2-k}),\\&\partial_t^k u^\epsilon,\partial_t^k\theta^\epsilon\in C([0,T^\epsilon];H^{2-k})\cap L^2(0,T^\epsilon,H^{3-k}).\end{aligned}$$

Theorem 1.2. *Assume that $(\sigma^\epsilon,u^\epsilon,\theta^\epsilon,I_0^\epsilon,I_1^\epsilon)$ is the unique solution obtained in Proposition 1.1, where the initial data $(\sigma_0^\epsilon,u_0^\epsilon,\theta_0^\epsilon,I_{00}^\epsilon,I_{10}^\epsilon)$ satisfies*

$$|||(\sigma^\epsilon,u^\epsilon,\theta^\epsilon,I_{00}^\epsilon,I_{10}^\epsilon)|||_{2,2}(0)+\|(1+\epsilon_1\sigma_0^\epsilon)^{-1}\|_{L^\infty}\leq D_0. \tag{1.27}$$

Then there exist positive constants T_0 and D such that $(\sigma^\epsilon,u^\epsilon,\theta^\epsilon,I_0^\epsilon,I_1^\epsilon)$ satisfies the following uniform estimates:

$$\begin{aligned}&\sup_{0\leq t\leq T_0}(|||(\sigma^\epsilon,u^\epsilon,\theta^\epsilon)|||_{2,2}+\|(1+\epsilon_1\sigma^\epsilon)^{-1}\|_{L^\infty})(t)+\left(\int_0^{T_0}|||(u^\epsilon,\theta^\epsilon)|||_{3,2}^2dt\right)^{\frac12}\\&\quad+\sup_{0\leq t\leq T_0}(\sqrt{\epsilon_2}\|(I_0^\epsilon,I_1^\epsilon)\|_{H^2}+\|(I_0^\epsilon,I_1^\epsilon)\|_{H^1}+\|\partial_t(I_0^\epsilon,I_1^\epsilon)\|_{L^2})(t)\\&\quad+\left(\int_0^{T_0}(\|(I_0^\epsilon,I_1^\epsilon)\|_{H^2}^2+\|\partial_t(I_0^\epsilon,I_1^\epsilon)\|_{H^1}^2+\|\partial_t^2(I_0^\epsilon,I_1^\epsilon)\|_{L^2}^2)dt\right)^{\frac12}\leq D,\end{aligned} \tag{1.28}$$

with D_0,T_0, and D independent of $\epsilon>0$. Furthermore, $(\sigma^\epsilon,u^\epsilon,\theta^\epsilon,I_0^\epsilon,I_1^\epsilon)$ converges to $(\sigma,u,\theta,I_0=1,I_1=0)$ in certain Sobolev space as $\epsilon\to0$, and there exists a function $\pi(x,t)$ such that (u,π) in $C([0,T_0];H^2)$ solves the following problem of the incompressible Navier–Stokes equations

$$\begin{cases}\partial_t u+u\cdot\nabla u+\nabla\pi-\mu\Delta u=0,\ \operatorname{div}u=0,\\ u(\cdot,0)=u_0\quad in\quad \mathbb{T}^3,\end{cases} \tag{1.29}$$

where u_0 is the weak limit of u_0^ϵ in H^2 with $\operatorname{div}u_0=0$ in $\mathbb{T}^3$.

Definition 1.2. *We denote*

$$\begin{aligned}M^\epsilon(t):=&\sup_{0\leq s\leq t}(|||(\sigma^\epsilon,u^\epsilon,\theta^\epsilon)|||_{2,2}(s)+\|(1+\epsilon_1\sigma^\epsilon)^{-1}\|_{L^\infty}(s))\\&+\sup_{0\leq s\leq t}(\sqrt{\epsilon_2}\|(I_0^\epsilon,I_1^\epsilon)\|_{H^2}+\|(I_0^\epsilon,I_1^\epsilon)\|_{H^1}+\|\partial_t(I_0^\epsilon,I_1^\epsilon)\|_{L^2})(s)\\&+\left(\int_0^t|||(u^\epsilon,\theta^\epsilon)|||_{3,2}^2ds\right)^{\frac12}+\left(\int_0^t|||(I_0^\epsilon,I_1^\epsilon)|||_{2,2}^2ds\right)^{\frac12},\end{aligned}$$

$$M_0^\epsilon := M^\epsilon(t=0).$$

THEOREM 1.3. *Let T^ϵ be the maximal time of existence for the problem (1.21)–(1.26) in the sense of Proposition 1.1. Then for any $t \in [0, T^\epsilon)$, we have that*

$$M^\epsilon(t) \leq C_0(M_0^\epsilon)\exp\left[t^{\frac{1}{4}} C(M^\epsilon(t))\right], \tag{1.30}$$

for some given nondecreasing continuous function $C_0(\cdot)$ and $C(\cdot)$.

Below we shall use C to denote the generic positive constant which may change from line to line and is independent of ϵ_1 and ϵ_2.

The remainder of this paper is devoted to the proofs of theorems 1.1, 1.2, and 1.3.

2. Proof of Theorem 1.1

Because the local existence for the problem (1.5)–(1.9) with fixed ϵ is standard [21], we only need to prove Equation (1.12). To this end, we only need to prove that there exists a constant $\eta \ll 1$ such that if

$$\sup_{0 \leq t \leq T} (\|(\sigma, u)(\cdot, t)\|_{H^2} + \|\partial_t(\sigma, u)(\cdot, t)(\cdot, t)\|) \leq \eta, \tag{2.1}$$

then for any $t \in [0, T]$, it holds

$$\begin{aligned}
&\|(\sigma, u)(\cdot, t)\|_{H^2}^2 + \|\partial_t(\sigma, u)(\cdot, t)\|_{L^2}^2 + \epsilon_2 \|(j_0, j_1)(\cdot, t)\|_{H^1}^2 \\
&\quad + \int_0^t (\|u\|_{H^3}^2 + \|\partial_t(\sigma, u)\|_{H^1}^2 + \|(j_0, j_1)\|_{H^1}^2 + \|\partial_t(j_0, j_1)\|_{L^2}^2) ds \\
&\leq C\|(\sigma_0, u_0)\|_{H^2}^2 + C\epsilon_2 \|(j_0, j_1)\|_{H^1}^2 + C\|\partial_t(\sigma, u, j_0, j_1)(\cdot, 0)\|_{L^2}^2.
\end{aligned} \tag{2.2}$$

In fact, once Equation (2.2) is obtained, Equation (1.12) can be obtained directly and the limit process is just an easy application of the uniform estimates and Arzelá–Ascoli theorem.

First, we have that

$$\int \sigma dx = 0, \ \int j_1 dx = 0, \ \int \rho u dx = 0, \ \int \rho \partial_t u dx = 0, \tag{2.3}$$

and

$$\frac{1}{2} \leq \rho \leq \frac{3}{2}. \tag{2.4}$$

Testing Equation (1.5) with σ, we see that

$$\frac{1}{2}\frac{d}{dt}\int \sigma^2 dx + \frac{1}{\epsilon_1}\int \sigma \operatorname{div} u dx = -\frac{1}{2}\int \sigma^2 \operatorname{div} u dx \leq C\|\sigma\|_{L^4}^2 \|\operatorname{div} u\|_{L^2} \leq C\eta \|\nabla\sigma\|_{L^2}^2. \tag{2.5}$$

Testing Equation (1.6) with u and using Equations (1.1) and (2.1), we find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho |u|^2 dx + \int (\mu |\nabla u|^2 + (\lambda + \mu)(\operatorname{div} u)^2) dx + \frac{1}{\epsilon_1}\int u \nabla \sigma dx \\
&= \int \frac{p'(1) - p'(\rho)}{\epsilon_1} u \nabla \sigma dx + \epsilon_2 \int j_1 u dx \\
&\leq C\|u\|_{L^3}\|\sigma\|_{L^6}\|\nabla\sigma\|_{L^2} + \epsilon_2 \|j_1\|_{L^2}\|u\|_{L^2}
\end{aligned}$$

$$\leq C\eta\|\nabla\sigma\|_{L^2}^2+\epsilon_2\|j_1\|_{L^2}\|u\|_{L^2}. \tag{2.6}$$

Summing up Equations (2.5) and (2.6), we infer that

$$\frac{1}{2}\frac{d}{dt}\int(\sigma^2+\rho|u|^2)dx+\int(\mu|\nabla u|^2+(\lambda+\mu)(\operatorname{div}u)^2)dx\leq C\eta\|\nabla\sigma\|_{L^2}^2+\epsilon_2\|j_1\|_{L^2}\|u\|_{L^2}. \tag{2.7}$$

Applying ∇ to Equation (1.5), testing by $\nabla\sigma$, and using Equation (2.1), we derive that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int|\nabla\sigma|^2+\frac{1}{\epsilon_1}\int\nabla\operatorname{div}u\nabla\sigma dx=&-\int\nabla\operatorname{div}(\sigma u)\nabla\sigma dx\\ \leq& C\|u\|_{H^2}\|\sigma\|_{H^2}\|\nabla\sigma\|_{L^2}\leq C\eta\|\sigma\|_{H^2}^2.\end{aligned} \tag{2.8}$$

Testing Equation (1.6) with $\nabla\operatorname{div}u$ and using Equation (2.1), we get

$$\begin{aligned}&(\lambda+2\mu)\|\nabla\operatorname{div}u\|_{L^2}^2-\frac{1}{\epsilon_1}\int\nabla\operatorname{div}u\nabla\sigma dx\\ =&\int(\rho u_t+\rho u\cdot\nabla u)\nabla\operatorname{div}udx+\int\frac{p'(\rho)-p'(1)}{\epsilon_1}\nabla\sigma\nabla\operatorname{div}udx+\epsilon_2\int j_1\nabla\operatorname{div}udx\\ =&-\frac{1}{2}\frac{d}{dt}\int\rho(\operatorname{div}u)^2dx+\frac{1}{2}\int\rho u\nabla(\operatorname{div}u)^2dx+\int\rho u\cdot\nabla u\cdot\nabla\operatorname{div}udx\\ &+\int\frac{p'(\rho)-p'(1)}{\epsilon_1}\nabla\sigma\nabla\operatorname{div}udx+\epsilon_2\int j_1\nabla\operatorname{div}udx\\ \leq&-\frac{1}{2}\frac{d}{dt}\int\rho(\operatorname{div}u)^2dx+C\|\rho\|_{L^\infty}\|u\|_{L^\infty}\|\operatorname{div}u\|_{L^2}\|\nabla\operatorname{div}u\|_{L^2}\\ &+C\|\rho\|_{L^\infty}\|u\|_{L^\infty}\|\nabla u\|_{L^2}\|\nabla\operatorname{div}u\|_{L^2}+C\|\sigma\|_{L^6}\|\nabla\sigma\|_{L^3}\|\nabla\operatorname{div}u\|_{L^2}\\ &+\epsilon_2\|j_1\|_{L^2}\|\nabla\operatorname{div}u\|_{L^2}\\ \leq&-\frac{1}{2}\frac{d}{dt}\int\rho(\operatorname{div}u)^2dx+C\eta\|u\|_{H^2}^2+C\eta\|\sigma\|_{H^2}^2+\epsilon_2\|j_1\|_{L^2}\|\nabla\operatorname{div}u\|_{L^2}.\end{aligned} \tag{2.9}$$

Summing up Equations (2.8) and (2.9), we have that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int(|\nabla\sigma|^2+\rho(\operatorname{div}u)^2)dx+(\lambda+2\mu)\int|\nabla\operatorname{div}u|^2dx\\ \leq& C\eta\|\sigma\|_{H^2}^2+C\eta\|u\|_{H^2}^2+\epsilon_2\|j_1\|_{L^2}\|\nabla\operatorname{div}u\|_{L^2}.\end{aligned} \tag{2.10}$$

Denote $\omega:=\operatorname{rot}u$. Applying rot to Equation (1.6), we observe that

$$\rho\partial_t\omega+\rho u\cdot\nabla\omega-\mu\Delta\omega=f+\epsilon_2\operatorname{rot}j_1, \tag{2.11}$$

with

$$f=\nabla\rho\times\partial_t u+\sum_i\nabla(\rho u_i)\times\partial_i u.$$

Testing Equation (2.11) with $\partial_t\omega-\Delta\omega$, we have that

$$\begin{aligned}&\frac{\mu}{2}\frac{d}{dt}\int|\operatorname{rot}\omega|^2dx+\int(\mu|\Delta\omega|^2+\rho|\partial_t\omega|^2)dx\\ =&\int[\rho\partial_t\omega\Delta\omega+(f-\rho u\cdot\nabla\omega)(\partial_t\omega-\Delta\omega)]dx+\epsilon_2\int\operatorname{rot}j_1(\partial_t\omega-\Delta\omega)dx\end{aligned}$$

$$=:\ell_1+\ell_2. \tag{2.12}$$

It has been proved in [13] that

$$\ell_1\le -\frac{1}{2}\frac{d}{dt}\int \rho|\mathrm{rot}\omega|^2dx + C\eta\|u\|_{H^3}^2 + C\eta\|\partial_t u\|_{H^1}^2.$$

We bound ℓ_2 as follows.

$$\ell_2\le \epsilon_2\|\mathrm{rot}\, j_1\|_{L^2}(\|\partial_t\omega\|_{L^2}+\|\Delta\omega\|_{L^2}).$$

Substituting the above estimates into Equation (2.12), we have that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int(\mu|\mathrm{rot}\omega|^2+\rho|\mathrm{rot}\omega|^2)dx+\int(\mu|\Delta\omega|^2+\rho|\partial_t\omega|^2)dx\\ \le& C\eta\|u\|_{H^3}^2+C\eta\|\partial_t u\|_{H^1}^2+\epsilon_2\|\mathrm{rot}\, j_1\|_{L^2}(\|\partial_t\omega\|_{L^2}+\|\Delta\omega\|_{L^2}).\end{aligned} \tag{2.13}$$

Applying ∂_t to Equation (1.6), testing by $\nabla\mathrm{div}\, u$, and using Equation (2.1), we obtain

$$\begin{aligned}&\frac{\lambda+2\mu}{2}\frac{d}{dt}\int|\nabla\mathrm{div}\, u|^2dx-\frac{d}{dt}\int\rho\partial_t u\nabla\mathrm{div}\, u dx-\frac{1}{\epsilon_1}\int\nabla\partial_t\sigma\nabla\mathrm{div}\, u dx\\ =&\int\{\left[\partial_t\left(\frac{p'(\rho)-p'(1)}{\epsilon_1}\nabla\sigma\right)+\epsilon_1\partial_t\sigma u\cdot\nabla u+\rho(\partial_t u\cdot\nabla u+u\cdot\nabla\partial_t u)\right]\nabla\mathrm{div}\, u\\ &+\rho(\mathrm{div}\,\partial_t u)^2+\nabla\rho\partial_t u\mathrm{div}\,\partial_t u\}dx+\epsilon_2\int\partial_t j_1\nabla\mathrm{div}\, u dx\\ =:&\ell_3+\ell_4.\end{aligned} \tag{2.14}$$

It has been proved in [13] that

$$\ell_3\le C\eta\|\nabla\partial_t\sigma\|_{L^2}^2+C\eta\|\nabla\mathrm{div}\, u\|_{L^2}^2+C\|\partial_t u\|_{H^1}^2.$$

We bound ℓ_4 as follows.

$$\ell_4\le\epsilon_2\|\partial_t j_1\|_{L^2}\|\nabla\mathrm{div}\, u\|_{L^2}.$$

Substituting the above estimates into Equation (2.14), we have that

$$\begin{aligned}&\frac{\lambda+2\mu}{2}\frac{d}{dt}\int|\nabla\mathrm{div}\, u|^2dx-\frac{d}{dt}\int\rho\partial_t u\nabla\mathrm{div}\, u dx-\frac{1}{\epsilon_1}\int\nabla\partial_t\sigma\nabla\mathrm{div}\, u dx\\ \le& C\eta\|\nabla\partial_t\sigma\|_{L^2}^2+C\eta\|\nabla\mathrm{div}\, u\|_{L^2}^2+C\|\partial_t u\|_{H^1}^2+\epsilon_2\|\partial_t j_1\|_{L^2}\|\nabla\mathrm{div}\, u\|_{L^2}.\end{aligned} \tag{2.15}$$

Applying ∇ to Equation (1.5), testing by $\nabla\partial_t\sigma$, and using Equation (2.1), we have that

$$\begin{aligned}&\|\nabla\partial_t\sigma\|_{L^2}^2+\frac{1}{\epsilon_1}\int\nabla\partial_t\sigma\nabla\mathrm{div}\, u dx=-\int\nabla\mathrm{div}\,(\sigma u)\nabla\partial_t\sigma dx\\ \le& C\|\sigma\|_{H^2}\|u\|_{H^2}\|\nabla\partial_t\sigma\|_{L^2}\\ \le& C\eta\|u\|_{H^2}^2+C\eta\|\nabla\partial_t\sigma\|_{L^2}^2.\end{aligned} \tag{2.16}$$

Summing up Equations (2.15) and (2.16), we have that

$$\frac{\lambda+2\mu}{2}\frac{d}{dt}\int|\nabla\mathrm{div}\, u|^2dx-\frac{d}{dt}\int\rho\partial_t u\nabla\mathrm{div}\, u dx+\int|\nabla\partial_t\sigma|^2dx$$

$$\leq C\eta\|\nabla\partial_t\sigma\|_{L^2}^2 + C\eta\|u\|_{H^2}^2 + C\|\partial_t u\|_{H^1}^2 + \epsilon_2\|\partial_t j_1\|_{L^2}\|\nabla\mathrm{div}\, u\|_{L^2}. \tag{2.17}$$

Applying ∇^2 to Equation (1.5), testing by $\nabla^2\sigma$ and using Equation (2.1), we have that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int|\nabla^2\sigma|^2dx + \frac{1}{\epsilon_1}\int\nabla^2\mathrm{div}\, u\nabla^2\sigma dx\\
=&-\int[(u\cdot\nabla)\nabla^2\sigma + 2\nabla u\cdot\nabla(\nabla\sigma) + \nabla^2 u\cdot\nabla\sigma + \nabla^2(\sigma\mathrm{div}\, u)]\nabla^2\sigma dx\\
\leq& C\|u\|_{H^3}\|\sigma\|_{H^2}^2 \leq C\eta\|u\|_{H^3}^2 + C\eta\|\sigma\|_{H^2}^2.
\end{aligned} \tag{2.18}$$

Applying ∇ to Equation (1.6), testing by $\nabla^2\mathrm{div}\, u$ and using Equation (2.1), calculating as that in [13], we have that

$$\begin{aligned}
&(\lambda+2\mu)\|\nabla^2\mathrm{div}\, u\|_{L^2}^2 - \frac{1}{\epsilon_1}\int\nabla^2\mathrm{div}\, u\nabla^2\sigma dx\\
\leq& C\delta\|\nabla^2\mathrm{div}\, u\|_{L^2}^2 + C\|\nabla\partial_t u\|_{L^2}^2 + C\eta\|u\|_{H^2}^2\\
&+C\|\nabla\mathrm{rot}\,\omega\|_{L^2}^2 + \epsilon_2\|\nabla j_1\|_{L^2}\|\nabla^2\mathrm{div}\, u\|_{L^2}.
\end{aligned} \tag{2.19}$$

for any $0<\delta<1$.

Summing up Equations (2.18) and (2.19) and taking δ small enough, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int|\nabla^2\sigma|^2dx + (\lambda+\mu)\int|\nabla^2\mathrm{div}\, u|^2dx\\
\leq& C\eta\|u\|_{H^3}^2 + C\eta\|\sigma\|_{H^2}^2 + C\|\nabla\partial_t u\|_{H^2}^2 + C\|\nabla\mathrm{rot}\,\omega\|_{L^2}^2 + \epsilon_2\|\nabla j_1\|_{L^2}\|\nabla^2\mathrm{div}\, u\|_{L^2}.
\end{aligned} \tag{2.20}$$

Applying ∂_t to Equation (1.5), testing by $\partial_t\sigma$ and using Equation (2.1), we have that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int(\partial_t\sigma)^2dx + \frac{1}{\epsilon_1}\int\partial_t\sigma\mathrm{div}\,\partial_t u dx = \int(\partial_t\sigma u + \sigma\partial_t u)\nabla\partial_t\sigma dx\\
\leq& (\|\partial_t\sigma\|_{L^3}\|u\|_{L^6} + \|\sigma\|_{L^6}\|\partial_t u\|_{L^3})\|\nabla\partial_t\sigma\|_{L^2}\\
\leq& C\eta\|\partial_t\sigma\|_{H^1}^2 + C\eta\|\partial_t u\|_{H^1}^2.
\end{aligned} \tag{2.21}$$

Applying ∂_t to Equation (1.6), testing by $\partial_t u$, using Equation (2.1), and calculating as that in [13], we reach

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\rho|\partial_t u|^2dx + \int(\mu|\nabla\partial_t u|^2 + (\lambda+\mu)(\mathrm{div}\,\partial_t u)^2)dx - \frac{1}{\epsilon_1}\int\partial_t\sigma\mathrm{div}\,\partial_t u dx\\
\leq& C\eta\|\partial_t\sigma\|_{H^1}^2 + C\eta\|\partial_t u\|_{H^1}^2 + C\eta\|u\|_{H^2}^2 + \epsilon_2\|\partial_t j_1\|_{L^2}\|\partial_t u\|_{L^2}.
\end{aligned} \tag{2.22}$$

Summing up Equations (2.21) and (2.22), we conclude that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int((\partial_t\sigma)^2 + \rho|\partial_t u|^2)dx + \int(\mu|\nabla\partial_t u|^2 + (\lambda+\mu)(\mathrm{div}\,\partial_t u)^2)dx\\
\leq& C\eta\|\partial_t\sigma\|_{H^1}^2 + C\eta\|\partial_t u\|_{H^1}^2 + C\eta\|u\|_{H^2}^2 + \epsilon_2\|\partial_t j_1\|_{L^2}\|\partial_t u\|_{L^2}.
\end{aligned} \tag{2.23}$$

It follows from Equations (1.6) and (2.1) that

$$\frac{1}{\epsilon_1}\|\nabla\sigma\|_{L^2} \leq C\|\partial_t u\|_{L^2} + C\|u\|_{H^2} + C\epsilon_2\|j_1\|_{L^2}, \tag{2.24}$$

$$\frac{1}{\epsilon_1}\|\nabla^2\sigma\|_{L^2} \le C\|u\|_{H^3} + C\|\partial_t u\|_{H^1} + C\epsilon_2\|\nabla j_1\|_{L^2}. \tag{2.25}$$

The inequalities (2.3), (2.24), and (2.25) imply that

$$\frac{1}{\epsilon_1}\|\sigma\|_{H^2} \le C\|u\|_{H^3} + C\|\partial_t u\|_{H^1} + C\epsilon_2\|j_1\|_{H^1}. \tag{2.26}$$

In addition, we use the following fact to control $\|u\|_{H^3}$,

$$\|u\|_{H^3} \le C(\|\mathrm{div}\, u\|_{H^2} + \|\mathrm{rot}\, u\|_{H^2} + \|u\|_{H^1}). \tag{2.27}$$

Testing Equations (1.7) and (1.8) with j_0 and j_1, respectively, and summing up the result, we get

$$\begin{aligned}&\frac{\epsilon_2}{2}\frac{d}{dt}\int (j_0^2 + |j_1|^2)dx + \int (j_0^2 + |j_1|^2)dx = \int b j_0 dx\\ \le&\|b\|_{L^2}\|j_0\|_{L^2} \le C\epsilon_1\|\sigma\|_{L^2}\|j_0\|_{L^2}.\end{aligned} \tag{2.28}$$

Applying ∇ to Equations (1.7) and (1.8), testing by ∇j_0 and ∇j_1, respectively, summing up the result, we have that

$$\begin{aligned}&\frac{\epsilon_2}{2}\frac{d}{dt}\int (|\nabla j_0|^2 + |\nabla j_1|^2)dx + \int (|\nabla j_0|^2 + |\nabla j_1|^2)dx\\ =&\int \nabla b \nabla j_0 dx \le \|\nabla b\|_{L^2}\|\nabla j_0\|_{L^2} \le C\epsilon_1\|\nabla\sigma\|_{L^2}\|\nabla j_0\|_{L^2}.\end{aligned} \tag{2.29}$$

Applying ∂_t to Equations (1.7) and (1.8), testing by $\partial_t j_0$ and $\partial_t j_1$, respectively, and summing up the result, we arrive at

$$\begin{aligned}&\frac{\epsilon_2}{2}\frac{d}{dt}\int ((\partial_t j_0)^2 + |\partial_t j_1|^2)dx + \int ((\partial_t j_0)^2 + |\partial_t j_1|^2)dx\\ =&\int \partial_t b \partial_t j_0 dx \le \|\partial_t b\|_{L^2}\|\partial_t j_0\|_{L^2}\\ \le& C\epsilon_1\|\partial_t\sigma\|_{L^2}\|\partial_t j_0\|_{L^2}.\end{aligned} \tag{2.30}$$

Denote

$$\begin{aligned}\Phi(t) :=& \frac{1}{2}\int (\sigma^2 + \rho|u|^2 + |\nabla\sigma|^2 + \rho(\mathrm{div}\, u)^2 + \mu|\mathrm{rot}\,\omega|^2 + \rho|\mathrm{rot}\,\omega|^2\\ &+ k_1(\lambda+2\mu)|\nabla\mathrm{div}\, u|^2 - k_1\rho\partial_t u\nabla\mathrm{div}\, u + 2k_2|\nabla^2\sigma|^2\\ &+ (\partial_t\sigma)^2 + \rho|\partial_t u|^2 + \epsilon_2(j_0^2 + j_1^2 + |\nabla j_0|^2 + |\nabla j_1|^2 + |\partial_t j_0|^2 + |\partial_t j_1|^2))dx,\\ \Psi(t) :=& \int (\mu|\mathrm{rot}\, u|^2 + (\lambda+2\mu)(\mathrm{div}\, u)^2 + (\lambda+2\mu)|\nabla\mathrm{div}\, u|^2 + \mu|\Delta\omega|^2\\ &+ \rho|\partial_t\omega|^2 + k_1|\nabla\partial_t\sigma|^2 + k_2(\lambda+\mu)|\nabla^2\mathrm{div}\, u|^2 + \mu|\mathrm{rot}\,\partial_t u|^2\\ &+ (\lambda+2\mu)(\mathrm{div}\,\partial_t u)^2 + j_0^2 + j_1^2 + |\nabla j_0|^2 + |\nabla j_1|^2 + (\partial_t j_0)^2 + |\partial_t j_1|^2)dx.\end{aligned}$$

Applying (2.7)+(2.10)+(2.13)+k_1(2.17)+k_2(2.20)+(2.23)+(2.28)+(2.29)+(2.30) with suitable small k_1 and k_2, using Equations eqref20.26 and (2.27), taking ϵ and η small enough, we conclude that

$$\frac{d}{dt}\Phi + C_0\Psi \le 0.$$

Integrating the above inequality, we obtain Equation (1.12). Here we used the well-known Poincaré inequality

$$\|u\|_{L^2} = \left\| u - \frac{\int \rho u dx}{\int \rho dx} \right\|_{L^2} \leq C\|\nabla u\|_{L^2}.$$

Thus we complete the proof of Theorem 1.1.

3. Proof of theorems 1.2 and 1.3

This section is devoted to the proof of theorems 1.2 and 1.3. In this section we shall prove theorems 1.2 and 1.3 by combining the ideas developed in [1,4,8,15]. First, by taking the similar arguments to those in [1,8,15], we know that in order to prove Equation (1.28), we only need to show Equation (1.30).

Below we shall drop the superscript "ϵ" of $\rho^\epsilon, \sigma^\epsilon, u^\epsilon, \theta^\epsilon$, etc. for the sake of simplicity; moreover, we write $M^\epsilon(t)$ and $M^\epsilon(0)$ as M and M_0, respectively. Since the physical constants κ, C_V, and R do not bring any essential difficulties in our arguments, we shall take $\kappa = C_V = R = 1$.

First, by the same calculations as that in [4], we get

$$(|||\rho|||_{2,2} + \|\rho^{-1}\|_{L^\infty})(t) \leq C_0(M_0)\exp(\sqrt{t}C(M)), \tag{3.1}$$

$$\|(\sigma,u,\theta)(t)\|_{L^2}^2 + \|(u,\theta)\|_{L^2(0,t;H^1)}^2 \leq C_0(M_0)\exp(\sqrt{t}C(M)), \tag{3.2}$$

$$\|(\nabla\sigma, \operatorname{div} u, \nabla\theta)(t)\|_{L^2}^2 + \|(\nabla \operatorname{div} u, \Delta\theta)\|_{L^2(0,t;L^2)}^2 \leq C_0(M_0)\exp(\sqrt{t}C(M)). \tag{3.3}$$

Lemma 3.1. *For any* $0 \leq t \leq \min(T^\epsilon, 1)$*, we have that*

$$\|\operatorname{rot} u(t)\|_{L^2}^2 + \|\operatorname{rot}^2 u\|_{L^2(0,t;L^2)}^2 \leq C_0(M_0)\exp(\sqrt{t}C(M)).$$

Proof. Let $\omega := \operatorname{rot} u$. From Equation (1.22) we easily derive

$$\rho(\partial_t \omega + u\cdot\nabla\omega) - \mu\Delta\omega = K + \epsilon_2 \operatorname{rot} I_1, \tag{3.4}$$

where $K := -(\partial_j\rho\partial_t u_i - \partial_i\rho\partial_t u_j) - [\partial_j(\rho u_k)\partial_k u_i - \partial_i(\rho u_k)\partial_k u_j]$.

Testing Equation (3.4) with ω and using Equation (1.14), we see that

$$\begin{aligned}
&\frac{1}{2}\|\sqrt{\rho}\omega(t)\|_{L^2}^2 + \mu\|\operatorname{rot}\omega\|_{L^2(0,t;L^2)}^2 \\
=& C_0(M_0) + \int_0^t \int K\omega dx ds + \epsilon_2 \int_0^t \int \operatorname{rot} I_1 \omega dx ds \\
\leq& C_0(M_0)\exp(\sqrt{t}C(M)) + \epsilon_2 \int_0^t \|I_1\|_{H^1}\|\omega\|_{L^2} ds \\
\leq& C_0(M_0)\exp(\sqrt{t}C(M)),
\end{aligned}$$

which leads to the lemma, where we used the estimate in [4]:

$$\int_0^t \int K\omega dx ds \leq C_0(M_0)\exp(\sqrt{t}C(M)).$$

□

Lemma 3.2. *For any* $0 \leq t \leq \min(T^\epsilon, 1)$*, we have that*

$$\|\partial_t(\sigma,u,\theta)(t)\|_{L^2}^2 + \|(\operatorname{rot}\partial_t u, \operatorname{div}\partial_t u, \nabla\partial_t\theta)\|_{L^2(0,t;L^2)}^2 \leq C_0(M_0)\exp(\sqrt{t}C(M)).$$

Proof. Applying the operator ∂_t to Equations (1.21)–(1.23), we find that

$$\partial_t^2\sigma+\frac{1}{\epsilon_1}\operatorname{div}\partial_t u=-\operatorname{div}\partial_t(\sigma u), \tag{3.5}$$

$$\rho(\partial_t^2 u+u\cdot\nabla\partial_t u)+\frac{R}{\epsilon_1}(\nabla\partial_t\sigma+\nabla\partial_t\theta)-\mu\Delta\partial_t u-(\lambda+\mu)\nabla\operatorname{div}\partial_t u$$
$$=-\partial_t\rho\partial_t u-\partial_t(\rho u)\cdot\nabla u-R\nabla\partial_t(\sigma\theta)+\epsilon_2\partial_t I_1, \tag{3.6}$$

$$C_V\rho(\partial_t^2\theta+u\cdot\nabla\partial_t\theta)+\frac{R}{\epsilon_1}\operatorname{div}\partial_t u-\kappa\Delta\partial_t\theta$$
$$=\epsilon_1\partial_t\left(\frac{\mu}{2}|\nabla u+\nabla u^t|^2+\lambda(\operatorname{div}u)^2\right)-C_V\partial_t\rho\partial_t\theta$$
$$-C_V\partial_t(\rho u)\cdot\nabla\theta-R\partial_t((\rho\theta+\sigma)\operatorname{div}u)+\partial_t(I_0-(1+\epsilon_1\theta)^4). \tag{3.7}$$

Testing Equations (3.5), (3.6), and (3.7) with $R\partial_t\sigma$, $\partial_t u$, and $\partial_t\theta$, respectively, then doing as that in [4], we reach the lemma. □

By the very similar calculations as in [4], we get

$$\|(\nabla\operatorname{div}u,\Delta\theta)(t)\|_{L^2}^2+\|\partial_t\nabla(\sigma,\theta)\|_{L^2(0,t;L^2)}^2\le C_0(M_0)\exp(\sqrt{t}C(M)). \tag{3.8}$$

LEMMA 3.3. *For any $0\le t\le\min(T^\epsilon,1)$, we have that*

$$\|\operatorname{rot}^2 u(t)\|_{L^2}^2+\|\Delta\operatorname{rot}u\|_{L^2(0,t;L^2)}^2\le C_0(M_0)\exp(\sqrt{t}C(M)).$$

Proof. Testing Equation (3.4) with $-\Delta\omega$, we obtain

$$\frac{1}{2}\|\sqrt{\rho}\operatorname{rot}\omega\|_{L^2}^2(t)+\mu\|\Delta\omega\|_{L^2(0,t;L^2)}^2$$
$$\le C_0(M_0)-\int_0^t\int K\Delta\omega\,dxds+\int_0^t\int\rho u\cdot(\nabla|\operatorname{rot}\omega|^2+\Delta\omega\nabla\omega)dxds$$
$$-\epsilon_2\int_0^t\int\operatorname{rot}I_1\Delta\omega\,dxds=:K_1+K_2+K_3+K_4. \tag{3.9}$$

It has been proved in [4] that

$$K_1+K_2+K_3\le C_0(M_0)\exp(\sqrt{t}C(M)).$$

We bound K_4 as follows.

$$K_4\le\epsilon_2\int_0^t\|\operatorname{rot}I_1\|_{L^2}\|\Delta\omega\|_{L^2}ds\le\epsilon_2\sqrt{t}C(M).$$

Substituting the above estimates into Equation (3.9) gives the lemma. □

LEMMA 3.4. *For any $0\le t\le\min(T^\epsilon,1)$, we have that*

$$\int_0^t\|\operatorname{rot}\partial_t\omega\|_{H^1}ds\le C_0(M_0)\exp(\sqrt{t}C(M)).$$

Proof. Applying the operator ∂_t to Equation (3.4), we deduce that

$$\rho(\partial_t^2\omega+u\cdot\nabla\partial_t\omega)-\mu\Delta\partial_t\omega=Q+\epsilon_2\partial_t\operatorname{rot}I_1, \tag{3.10}$$

where

$$Q := \partial_t K - \partial_t(\rho u)\cdot\nabla\omega - \partial_t\rho\partial_t\omega.$$

Then by the very similar calculations as that in [4], we reach the lemma. □

LEMMA 3.5. *For any $0 \le t \le \min(T^\epsilon, 1)$, we have that*

$$\|\partial_t\omega(t)\|_{L^2}^2 + \|\partial_t\nabla\omega\|_{L^2(0,t;L^2)}^2 \le C_0(M_0)\exp(\sqrt{t}C(M)).$$

Proof. Testing Equation (3.10) with $\partial_t\omega$ and using Equation (1.14), we observe that

$$\begin{aligned}&\frac{1}{2}\|\sqrt{\rho}\partial_t\omega\|_{L^2}^2(t) + \mu\|\nabla\partial_t\omega\|_{L^2(0,t;L^2)}^2\\ =&C_0(M_0) + \int_0^t\int Q\partial_t\omega dxds + \epsilon_2\int_0^t\int \partial_t \mathrm{rot}\, I_1\partial_t\omega dxds.\end{aligned} \tag{3.11}$$

It has been proved in [4] that

$$\int_0^t\int Q\partial_t\omega dxds \le C_0(M_0)\exp(\sqrt{t}C(M)).$$

We bound the third term of right-hand side of Equation (3.11) as

$$\begin{aligned}&\epsilon_2\int_0^t\int \partial_t \mathrm{rot}\, I_1\partial_t\omega dxds\\ \le&\epsilon_2\int_0^t \|\partial_t \mathrm{rot}\, I_1\|_{L^2}\|\partial_t\omega\|_{L^2}ds\\ \le&\epsilon_2 C(M)\int_0^t \|\partial_t \mathrm{rot}\, I_1\|_{L^2}ds\\ \le&\sqrt{t}C(M).\end{aligned}$$

Substituting the above estimates into Equation (3.11) yields the lemma. □

Now, by the very similar calculations as that in [4], we conclude that

$$\|\partial_t(\nabla\sigma, \mathrm{div}\, u, \nabla\theta)(t)\|_{L^2}^2 + \|\partial_t(\nabla\mathrm{div}\, u, \Delta\theta)\|_{L^2(0,t;L^2)}^2 \le C_0(M_0)\exp\left(t^{\frac{1}{4}}C(M)\right), \tag{3.12}$$

$$\|\nabla^2\sigma(t)\|_{L^2}^2 + \|\nabla^2\mathrm{div}\, u\|_{L^2(0,t;L^2)}^2 \le C_0(M_0)\exp(\sqrt{t}C(M)), \tag{3.13}$$

$$\|\Delta\theta\|_{L^2(0,t;H^1)} \le C_0(M_0)\exp\left(t^{\frac{1}{4}}C(M)\right), \tag{3.14}$$

$$\|\partial_t^2(\sigma,u,\theta)(t)\|_{L^2}^2 + \|\partial_t^2(u,\theta)\|_{L^2(0,t;H^1)}^2 \le C_0(M_0)\exp\left(t^{\frac{1}{4}}C(M)\right). \tag{3.15}$$

Finally, we estimate I_0 and I_1 in order to close the energy estimate.

LEMMA 3.6. *For any $0 \le t \le \min(T^\epsilon, 1)$, we have that*

$$\begin{aligned}&\sqrt{\epsilon_2}\|(I_0,I_1)(t)\|_{H^2}^2 + \|(I_0,I_1)(t)\|_{H^1}^2 + \|\partial_t(I_0,I_1)(t)\|_{L^2}^2 + \int_0^t |||(I_0,I_1)|||_{2,2}^2 ds\\ \le&C_0(M_0)\exp(\sqrt{t}C(M)).\end{aligned}$$

Proof. Testing Equations (1.24) and (1.25) with I_0 and I_1, respectively, summing up the resulting inequality, we infer that

$$\begin{aligned}
&\frac{\epsilon_2}{2}\int(I_0^2+|I_1|^2)(t)dx+\int_0^t\int(I_0^2+|I_1|^2)dxds\\
\le&C_0(M_0)+\int_0^t\int(1+\epsilon_1\theta)^4I_0dxds\\
\le&C_0(M_0)\exp(\sqrt{t}C(M)).
\end{aligned}\tag{3.16}$$

Applying Δ to Equations (1.24) and (1.25), testing by ΔI_0 and ΔI_1, respectively, summing up the resulting equalities, we have that

$$\begin{aligned}
&\frac{\epsilon_2}{2}\int((\Delta I_0)^2+|\Delta I_1|^2)(t)dx+\int_0^t\int((\Delta I_0)^2+|\Delta I_1|^2)dxds\\
\le&C_0(M_0)+\int_0^t\int\Delta(1+\epsilon_1\theta)^4\cdot\Delta I_0dxds\\
\le&C_0(M_0)+\int_0^t\|\Delta(1+\epsilon_1\theta)^4\|_{L^2}\|\Delta I_0\|_{L^2}ds\\
\le&C_0(M_0)\exp(\sqrt{t}C(M)).
\end{aligned}\tag{3.17}$$

Applying $\partial_t\nabla$ to Equations (1.24) and (1.25), testing by $\nabla\partial_tI_0$ and $\nabla\partial_tI_1$, respectively, summing up the resulting equalities, we have that

$$\begin{aligned}
&\frac{\epsilon_2}{2}\int(|\nabla\partial_tI_0|^2+|\nabla\partial_tI_1|^2)(t)dx+\int_0^t\int(|\nabla\partial_tI_0|^2+|\nabla\partial_tI_1|^2)dxds\\
\le&C_0(M_0)+\int_0^t\int\partial_t\nabla(1+\epsilon_1\theta)^4\partial_t\nabla I_0dxds\\
\le&C_0(M_0)\exp(\sqrt{t}C(M)).
\end{aligned}\tag{3.18}$$

Applying ∂_t^2 to Equations (1.24) and (1.25), testing by $\partial_t^2I_0$ and $\partial_t^2I_1$, respectively, summing up the resulting equalities, we have that

$$\begin{aligned}
&\frac{\epsilon_2}{2}\int((\partial_t^2I_0)^2+|\partial_t^2I_1|^2)(t)dx+\int_0^t\int((\partial_t^2I_0)^2+|\partial_t^2I_1|^2)dxds\\
\le&C_0(M_0)+\int_0^t\int\partial_t^2(1+\epsilon_1\theta)^4\partial_t^2I_0dxds\\
\le&C_0(M_0)\exp(\sqrt{t}C(M)).
\end{aligned}\tag{3.19}$$

It follows from Equations (3.16) and (3.18) that

$$\begin{aligned}
\|(I_0,I_1)(t)\|_{H^1}^2\le&C_0(M_0)+2\int_0^t\int\nabla I_0\cdot\partial_t\nabla I_0dxds+2\int_0^t\int I_0\partial_tI_0dxds\\
&+2\int_0^t\int\nabla I_1\cdot\partial_t\nabla I_1dxds+2\int_0^t\int I_1\partial_tI_1dxds\\
\le&C_0(M_0)\exp(\sqrt{t}C(M)).
\end{aligned}\tag{3.20}$$

$$\|\partial_t(I_0,I_1)(t)\|_{L^2}^2\le C_0(M_0)+2\int_0^t\int\partial_tI_0\partial_t^2I_0dxds+2\int_0^t\int\partial_tI_1\partial_t^2I_1dxds$$

$$\leq C_0(M_0)\exp(\sqrt{t}C(M)). \tag{3.21}$$

Combining Equations (3.16)–(3.21), we prove the lemma. □

By collecting all the above results together, we completes the proof of Equation (1.30).

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