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THE SYMMETRIC STRUCTURE OF GREEN–NAGHDI TYPE EQUATIONS*

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Abstract. The notion of symmetry classically defined for hyperbolic systems of conservation laws is extended to the case of evolution equations of conservative form for which the flux function can be an operator. We explain how such a symmetrization can work from a general point of view using an extension of the classical Godunov structure. We then apply it to the Green–Naghdi type equations which are a dispersive extension of the hyperbolic shallow-water equations. In fact, in the case of these equations, the general Godunov structure of the system is obtained from its Hamiltonian structure.

Key words. Symmetric systems, conservation law, strict convexity, variational derivative, Green–Naghdi equations.

AMS subject classifications. 35Q35, 35L65, 37L50, 70S10.

1. Introduction

Incompressible Euler equations and water waves problem model free surface incompressible fluids under the influence of the gravity. The complexity of these systems leads to consider averaged geophysical models to describe coastal oceanic flows. We focus on a particular type of these reduced models called the Green–Naghdi type model [13], which writes

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x(hu^2) + \partial_x(gh^2/2 + \alpha h^2 \ddot{h}) = 0. \end{cases} \tag{1.1}$$

The unknown h represents the fluid height and is assumed to be positive, while u is the averaged horizontal velocity. Moreover, the material derivative $\dot{()}$ is defined by $\dot{()} = \partial_t() + u\partial_x()$, α is a positive real number and g is the gravity constant.

If $\alpha = 0$, system (1.1) is hyperbolic and equivalent to the Saint-Venant equations (and to the barotropic Euler equations). System (1.1) with $\alpha \neq 0$ is different from the Saint-Venant system by the dispersive term $\partial_x(\alpha h^2 \ddot{h})$. It has been rigorously derived for $\alpha = \frac{1}{3}$ from the water wave problem for irrotational flows by Li [19] and by Alvarez and Lannes [1]. In [15], Ionescu derived the same system by a variational method considering the Lagrangian formulation of the irrotational incompressible Euler equations. In [4], the authors obtain system (1.1) for $\alpha = \frac{1}{4}$ by a different but a formal method without any hypothesis on the irrotationality of the fluid.

It is worth remarking that system (1.1) admits the following conservation law (see for instance [9, 10]),

$$\partial_t E + \partial_x (u(E+p)) = 0, \tag{1.2}$$

where the energy E is defined by

$$E = gh^2/2 + hu^2/2 + \alpha h^3 (\partial_x u)^2/2, \tag{1.3}$$

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and p by

$$p = gh^2/2 + \alpha h^2 \ddot{h}. \tag{1.4}$$

Contrary to the case of hyperbolic systems, the energy E and the pressure p are not functions of the unknown but smooth operators acting on the space of functions the unknown belongs to.

The aim of this paper is to extend the notion of symmetry classically defined for hyperbolic systems, to more general type of equations, including the Green–Naghdi model (1.1). We first recall the definition of symmetrizability for hyperbolic systems and its relation with the existence of a convex entropy.

1.1. Symmetric structure of hyperbolic systems of conservation laws. Let us provide a brief review on the symmetrization of hyperbolic systems of conservation laws. We consider the system

$$\partial_t U + \partial_x F(U) = 0 \tag{1.5}$$

where the flux $F:\mathbb{R}^N \to \mathbb{R}^N$, $N \geq 1$, is a smooth function. We only consider in the sequel smooth solutions $U:\mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^N$.

The hyperbolic system (1.5) is called *symmetrizable* if there exists a change of variable $U \mapsto Q$ such that Equation (1.5) is equivalent to

$$A_0(Q)\partial_t Q + A_1(Q)\partial_x Q = 0, \tag{1.6}$$

where $A_0(Q)$ is a symmetric positive definite matrix and $A_1(Q)$ is a symmetric one.

Moreover, a pair of smooth functions (E,P) from $\mathbb{R}^N$ to $\mathbb{R}$ such that $\nabla_U^2 E(U)$ positive definite is an *entropy pair* for system (1.5) if any solution U to Equation (1.5) satisfies

$$\partial_t E(U) + \partial_x P(U) = 0, \tag{1.7}$$

or equivalently if

$$(\nabla_U F(U))^T \nabla_U E(U) = \nabla_U P(U). \tag{1.8}$$

Using Poincaré's theorem [5], the latter condition, which is nothing but an integrability condition, is equivalent to the symmetry condition

$$\nabla^2 E(U) \nabla F(U) = \left(\nabla^2 E(U) \nabla F(U)\right)^T.$$

The following classical proposition illustrates how the notions of entropy and symmetry are related. We also provide the associated proof in order to compare it to the generalized case of the next section.

PROPOSITION 1.1 ([3, 6, 12, 20, 21]). *Let us assume that the hyperbolic system* (1.5) *admits an entropy pair* (E,P). *Then, it is symmetrizable under any change of variable* $U \mapsto V$ *under the form*

$$A_0(V)\partial_t V + A_1(V)\partial_x V = 0,$$

where

$$A_0(V) = (\nabla_V U)^T \nabla_U^2 E(U)\ \nabla_V U, \text{ and } A_1(V) = (\nabla_V U)^T \nabla_U^2 E(U)\ \nabla_U F(U) \nabla_V U. \tag{1.9}$$

Proof. Considering a change of variable $U \mapsto V$, system (1.5) becomes

$$\nabla_V U \partial_t V + \nabla_U F(U) \nabla_V U \partial_x V = 0. \tag{1.10}$$

We now apply $(\nabla_V U)^T \nabla_U^2 E(U)$ to the left-hand side and obtain

$$(\nabla_V U)^T \nabla_U^2 E(U)\ \nabla_V U\ \partial_t V + (\nabla_V U)^T \nabla_U^2 E(U)\ \nabla_U F(U)\ \nabla_V U\ \partial_x V = 0. \tag{1.11}$$

The symmetric matrix $A_0(V) = (\nabla_V U)^T \nabla_U^2 E(U)\ \nabla_V U$ is positive definite due to the strict convexity of the entropy. Therefore, we just need to prove the symmetry of $\nabla_U^2 E(U)\ \nabla_U F(U)$. To do so, we consider the change of variable $U \mapsto Q$ where Q is the entropy variable, i.e.

$$Q = \nabla_U E(U). \tag{1.12}$$

This change of variable is valid since E is strictly convex. As a consequence, the Legendre transform $E^\star$ of E defined by

$$E^\star(Q) = Q \cdot (\nabla_U E)^{-1}(Q) - E((\nabla_U E)^{-1}(Q)), \tag{1.13}$$

satisfies

$$U = \nabla_Q E^\star(Q). \tag{1.14}$$

Let us now define the scalar function $\hat{P}$ by

$$\hat{P}(Q) = Q \cdot F(U(Q)) - P(U(Q)). \tag{1.15}$$

Then, we use relation (1.8) to get

$$\nabla_Q \hat{P}(Q) = F(U). \tag{1.16}$$

Hence,

$$\nabla_U^2 E(U)\ \nabla_U F(U) = \nabla_U^2 E(U)\ \nabla_Q^2 \hat{P}(Q)\ \nabla_U Q = \nabla_U^2 E(U)\ \nabla_Q^2 \hat{P}(Q)\ \nabla_U^2 E(U)$$

is symmetric. □

Gathering Equations (1.14) and (1.16), we remark that system (1.5) is equivalent to

$$\partial_t (\nabla_Q E^\star(Q)) + \partial_x \left(\nabla_Q \hat{P}(Q)\right) = 0.$$

In other words, system (1.5) admits a so-called *Godunov structure* [12]. Note that such a structure can be used to deduce the existence of an entropy pair since it implies the symmetry of $\nabla_U^2 E(U)\ \nabla_U F(U)$, and thus the integrability of $(\nabla_U F(U))^T \nabla_U E(U)$.

Remark 1.1. Let us consider a system of the form (1.5) which admits an entropy pair (E,P). Assume that there exists a decomposition of the unknown $U = (U_1, U_2)$ such that the application $\phi \mapsto \nabla_{U_2} E(U_1, \phi)$ is invertible. Then, the change of variable

$$U \mapsto V = (U_1, \nabla_{U_2} E(U_1, U_2))$$

is particularly interesting since $A_0(V)$ is block diagonal (this is a direct consequence of the expression (1.9) of $A_0(V)$). Indeed, this can be useful to deduce equivalent normal forms of system (1.5) when studying for instance parabolic regularizations [17].

In the case of the Saint-Venant equations, with $U=(h,hu)$ and $E=gh^2/2+hu^2/2$, let us compare two symmetric forms. If we consider the entropy variable $Q=\nabla_U E(U)=(gh-u^2/2,u)$, one has

$$A_0(Q)\partial_t Q+A_1(Q)\partial_x Q=0,$$

where

$$A_0(Q)=\frac{1}{g}\begin{pmatrix}1 & u\\ u & gh+u^2\end{pmatrix} \quad \text{and} \quad A_1(Q)=\frac{1}{g}\begin{pmatrix}u & gh+u^2\\ gh+u^2 & 3ghu+u^3\end{pmatrix}.$$

On the other hand, using the change of variable $U\mapsto V=(h,\nabla_{hu}E(U))=(h,u)$, the Saint-Venant equations become

$$A_0(V)\partial_t V+A_1(V)\partial_x V=0,$$

with

$$A_0(V)=\begin{pmatrix}g & 0\\ 0 & h\end{pmatrix}, \quad \text{and} \quad A_1(V)=\begin{pmatrix}gu & gh\\ gh & hu\end{pmatrix}.$$

The notion of symmetrizability is crucial to be useful to prove the local well-posedness of hyperbolic systems (see [3] for instance) as well as the stability of constant solutions of hyperbolic systems with dissipative terms [14, 17, 23, 24]. Let us now recall some properties of the Green–Naghdi equations.

1.2. Hamiltonian structure of the Green–Naghdi equations. Following Li [19], let us consider the unknown $U=(h,m)$ defined by

$$m=\mathcal{L}_h(u)=hu-\alpha\left(h^3u_x\right)_x. \tag{1.17}$$

The change of variable $(h,u)\mapsto(h,m)$ is valid since the Sturm–Liouville operator $\mathcal{L}_h$ is an isomorphism from $\mathbb{H}^s(\mathbb{R})$ to $\mathbb{H}^{s-2}(\mathbb{R})$, for $s\geq 2$, due to the fact that h is positively bounded by below[1]. Let us also mention that the variable m has been used in [10] to define the generalized velocity $k=\frac{m}{h}$.

We illustrate in the following proposition the Hamiltonian structure of the Green–Naghdi equations inherited from the structure of incompressible Euler equations with a free surface. To state this result, we adopt classical notations of variational derivatives and second variations (see for instance [11, 22]).

PROPOSITION 1.2 ([19]). *Let $\bar{h}>0$ be a real constant. System* (1.1) *is equivalent to*

$$\partial_t U=\mathcal{J}(U)\delta\mathcal{H}_{\bar{h}}(U), \tag{1.18}$$

where

$$U=(h,m)=(h,\mathcal{L}_h(u)),$$

[1] Operator $\mathcal{L}_h$ is a diffeomorphism from $\mathbb{H}^{s+2}(\mathbb{R})$ to $\mathbb{H}^s(\mathbb{R})$ if h is close enough to a constant state $\bar{h}$ for the norm $\mathbb{H}^n$ with $n\geq 2$. This assumption is considered in Section 3.1 while symmetrizing the Green–Naghdi equations.

$$\mathcal{H}_{\bar{h}}(h,u)=\int_{\mathbb{R}} gh(h-\bar{h})/2+hu^2/2+\alpha h^3(u_x)^2/2, \tag{1.19}$$

and

$$\mathcal{J}(U)=-\begin{pmatrix} 0 & \partial_x(h()) \\ h\partial_x & \partial_x(m())+m\partial_x \end{pmatrix}. \tag{1.20}$$

More precisely, we have for all test functions (ϕ,ψ)

$$\mathcal{J}(U)\begin{pmatrix}\phi\\ \psi\end{pmatrix}=-\begin{pmatrix} \partial_x(h\psi) \\ h\partial_x\phi+\partial_x(m\psi)+m\partial_x\psi \end{pmatrix}. \tag{1.21}$$

By classical calculations, we have

$$\delta\mathcal{H}_{\bar{h}}(U)=(\sigma,u),$$

with

$$\sigma=gh-g\bar{h}/2-u^2/2-\frac{3}{2}\alpha h^2(u_x)^2. \tag{1.22}$$

The variable σ has been used in [9] for the canonical representation of the Green–Naghdi equations.

The function $\mathcal{H}_{\bar{h}}$ is the integral of the relative energy

$$E_{\bar{h}}=gh(h-\bar{h})/2+hu^2/2+\alpha h^3(u_x)^2/2, \tag{1.23}$$

which, following the same calculations as those which lead to Equation (1.2), satisfies the conservation law

$$\partial_t E_{\bar{h}}+\partial_x(u(E_{\bar{h}}+p))=0, \tag{1.24}$$

where p is given by Equation (1.4). The first consequence is the conservation of the Hamiltonian $\mathcal{H}_{\bar{h}}$ over time by integration in space[2]. This important property can also be obtained using the Hamiltonian structure (1.18) of the system and the fact that $\mathcal{J}(U)$ is a skew-symmetric operator acting on the space of vector-valued functions whose second component converges to 0 at infinity. Hence,

$$\frac{d}{dt}\mathcal{H}(U(t))=\int_{\mathbb{R}}\delta\mathcal{H}(U)\cdot\partial_t U=\int_{\mathbb{R}}\delta\mathcal{H}(U)\cdot\mathcal{J}(U)\delta\mathcal{H}(U)=0.$$

1.3. General idea. The generalization of the notion of symmetrizability to dispersive perturbations of hyperbolic systems has been studied by several authors. For instance, Gavrilyuk and Gouin in [8] (see also [2]) investigate the symmetric structure of Euler–Korteweg models and some p-systems. Similar ideas can be partially adapted to some generalized p-systems like bubbly fluid equations and to modified Lagrangian Green–Naghdi [7].

These generalizations are investigated with the hope of extending the results on hyperbolic systems to their dispersive perturbations. In the very recent work [18], we use the generalized symmetric structure presented in this work (more precisely in Section 3)

[2]It has been shown in [16,19] that the Green–Naghdi equations endowed with the unknown $(h-\bar{h},u)$ are well-posed in $C\left([0,T);\mathbb{H}^s(\mathbb{R})\times\mathbb{H}^{s+1}(\mathbb{R})\right)$ for some $T>0$ and $s\geq 2$. Hence, u is a continuous function vanishing at infinity using the Sobolev embedding theorem.

to prove the asymptotic stability of constant solutions of the Green–Naghdi equations with viscosity. Let us note that the symmetric structure presented here for Green–Naghdi equations holds only in a small enough neighborhood of constant solutions, this is to say that we consider the symmetrizability as a local notion. As we can see in [18], this is not an obstacle to prove the stability of equilibriums since the solution of the viscous Green–Naghdi equations remains close to equilibriums for initial data close enough to these solutions.

In this paper, we consider general systems written under the following conservative form

$$\partial_t U + \partial_x F(U) = 0. \tag{1.25}$$

The unknown U is supposed to belong to $C([0,T);\mathcal{A})$ for some $T>0$ where $\mathcal{A}$ is a Banach subspace of continuous functions of $\mathbb{L}^2(\mathbb{R},\mathbb{R}^N)$ converging to 0 at infinity. We also assume that the derivative of all elements of $\mathcal{A}$ belongs to $\mathcal{A}$. Let us note that F is not anymore a function of $\mathbb{R}^N$ but a smooth application defined from $\mathcal{A}$ to $\mathcal{A}$. This is actually the case for the Green–Naghdi equations. As we will see in Subsection 3.1, the Green–Naghdi equations under the Hamiltonian variable (h,m) fits the abstract form (1.25) with no loss of derivatives through F.

For sake of simplicity, we mainly consider the one-dimensional problem (1.25). We provide some generalizations of the previous notions used in the hyperbolic case, symmetrizability, Godunov structure, and relate it, in the case of Green–Naghdi equations, to the existence of a Hamiltonian structure. The extension of the results of the next section to the multi-dimensional case will be addressed at the end of the next section. Section 3 is devoted to the particular case of the Green–Naghdi equations.

2. Weak symmetric structure

The aim of this part is to provide a sufficient condition for the symmetrizability of system (1.25) under any variable. First, we provide an adapted notion of symmetrizability and define the Legendre transform of a variational function. Then, we will see how a convenient strictly convex function can lead to the symmetrizability.

The notion of symmetry we consider here is based on the $\mathbb{L}^2$ scalar product, and not on the scalar product of $\mathbb{R}^N$. More precisely, an operator $\mathcal{F}:\mathcal{A}\subset\mathbb{L}^2(\mathbb{R},\mathbb{R}^N)\to\mathbb{L}^2(\mathbb{R},\mathbb{R}^N)$ is said to be *symmetric* if

$$\int_{\mathbb{R}} \phi\cdot\mathcal{F}(\psi) = \int_{\mathbb{R}} \mathcal{F}(\phi)\cdot\psi \quad \forall\phi,\psi\in\mathcal{A},$$

and *positive definite* if, for all $\phi\in\mathcal{A}\setminus\{0\}$, $\int_{\mathbb{R}}\phi\cdot\mathcal{F}\phi>0$.

Definition 2.1 (**Weak symmetrizability**). *System* (1.25) *is called weakly symmetrizable if there exists a change of variable $U\mapsto V$ such that Equation* (1.25) *is equivalent to*

$$A_0(V)\partial_t V + A_1(V)\partial_x(V) = 0, \tag{2.1}$$

where $A_0(V)$ is a symmetric positive definite operator and $A_1(V)$ is a symmetric one.

Definition 2.2 (**Legendre transform**). *Let Ω be an open convex subset of a Banach space $\mathcal{A}\subset\mathbb{L}^2(\mathbb{R},\mathbb{R}^N)$ and consider a smooth application $E:\Omega\to\mathbb{L}^1(\mathbb{R})$ together with the variational function $\mathcal{H}:\Omega\to\mathbb{R}$ defined by*

$$\mathcal{H}(U) = \int_{\mathbb{R}} E(U).$$

Assume that there exists an open set $\Omega^\star$ of a Banach space $\mathcal{B}\subset\mathbb{L}^2(\mathbb{R},\mathbb{R}^N)$ such that the application

$$\delta_U\mathcal{H}:\begin{cases}\Omega\to\Omega^\star\\ U\mapsto\delta_U\mathcal{H}(U)\end{cases}$$

is a diffeomorphism. The Legendre transform $\mathcal{H}^\star$ of $\mathcal{H}$ is defined on $\Omega^\star$ by

$$\mathcal{H}^\star(Q)=\int_{\mathbb{R}}Q\cdot(\delta_U\mathcal{H})^{-1}(Q)-E\big((\delta_U\mathcal{H})^{-1}(Q)\big). \tag{2.2}$$

Let us note that the Legendre transform $\mathcal{H}^\star$ of a function $\mathcal{H}$ satisfying the assumptions of Definition 2.1, also satisfies the assumptions of the definition. Moreover, basic computations show that the Legendre transform of $\mathcal{H}^\star$ is nothing but $\mathcal{H}$. In other words,

$$\mathcal{H}^{\star\star}=\mathcal{H}.$$

We now state one of the fundamental properties of the Legendre transform of a strictly convex variational function (i.e. a function with a definite positive second variation). Let us remark here that contrary to the finite dimensional case, the variational derivative of a smooth strictly convex function is not necessarily a diffeomorphism. Therefore, we still need to assume in the sequel that its variational derivative defines a diffeomorphism as in Definition 2.2.

PROPOSITION 2.1. *The Legendre transform $\mathcal{H}^\star$ of a strictly convex function $\mathcal{H}$ which satisfies the assumptions of Definition 2.2 is strictly convex.*

Proof. Considering the expression (2.2) of the Legendre transform, we remark that

$$\delta_Q\mathcal{H}^\star(Q)=(\delta_U\mathcal{H})^{-1}(Q).$$

In other words,

$$Q=\delta_U\mathcal{H}(U)\iff U=\delta_Q\mathcal{H}^\star(Q).$$

Hence, the definite positivity of the second variation of $\mathcal{H}$ implies the definite positivity of the second variation of $\mathcal{H}^\star$. More precisely, we have

$$\delta_U^2\mathcal{H}(U)=D_UQ(U),$$

and

$$\delta_Q^2\mathcal{H}^\star(Q)=D_QU(Q).$$

Therefore,

$$\delta_U^2\mathcal{H}(U)=\big(\delta_Q^2\mathcal{H}^\star(Q)\big)^{-1}.$$

□

The following theorem provides the connection between the convexity of $\mathcal{H}$ and the existence of a general Godunov structure (this notion has been introduced in [9] and is recalled in the following statement).

THEOREM 2.1. *We use the same notations and assumptions as in Definition 2.2. Assume that $\mathcal{H}$ is strictly convex on Ω. If*

$$\delta_U^2\mathcal{H}(U)D_UF(U) \text{ is symmetric,} \tag{2.3}$$

then system (1.25) *admits a* general Godunov structure*: there exists a change of variable $U\mapsto Q$ defined on Ω and a function R, together with $\mathcal{R}(Q)=\int_{\mathbb{R}}R(Q)$, such that system* (1.25) *is equivalent to*

$$\partial_t(\delta_Q\mathcal{H}^\star(Q))+\partial_x(\delta_Q\mathcal{R}(Q))=0, \tag{2.4}$$

as long as the solution U remains in Ω.

Proof. Let us first consider the change of variable $U\mapsto Q$ defined by

$$Q=\delta_U\mathcal{H}(U). \tag{2.5}$$

or equivalently by

$$U=\delta_Q\mathcal{H}^\star(Q). \tag{2.6}$$

Considering the fact that $\delta_U^2\mathcal{H}(U)D_UF(U)$ is symmetric on the open convex set Ω, there exists, by Poincaré's theorem [5], a differentiable application $\mathcal{N}:\Omega\to\mathbb{R}$ such that

$$D_U\mathcal{N}(U)\phi=\int_{\mathbb{R}}\delta_U\mathcal{H}(U)\cdot D_UF(U)\phi \quad \forall\phi\in\mathcal{A}. \tag{2.7}$$

We now define the function $\mathcal{R}$ by

$$\mathcal{R}(Q)=\int_{\mathbb{R}}Q\cdot F(U(Q))-\mathcal{N}(U(Q)). \tag{2.8}$$

We differentiate Equation (2.8) and take the action on a test function ψ. This leads to

$$D_Q\mathcal{R}(Q)\psi=\int_{\mathbb{R}}F(U(Q))\cdot\psi+Q\cdot D_UF(U)D_QU(\psi)-D_U\mathcal{N}(U)D_QU(\psi).$$

Then, we have by Equation (2.5),

$$D_Q\mathcal{R}(Q)\psi=\int_{\mathbb{R}}F(U(Q))\cdot\psi+\delta_U\mathcal{H}(U)\cdot D_UF(U)D_QU(\psi)-D_U\mathcal{N}(U)D_QU(\psi).$$

Finally, using Equation (2.7), we find

$$D_Q\mathcal{R}(Q)\psi=\int_{\mathbb{R}}F(U(Q))\cdot\psi,$$

or equivalently

$$\delta_Q\mathcal{R}(Q)=F(U(Q)). \tag{2.9}$$

Considering system (1.25) together with Equations (2.6) and (2.9), we obtain Equation (2.4). ∎

The general Godunov structure (2.4) directly implies the weak symmetrizability of system (1.25) with respect to the unknown Q, since it lets us write the system under

$$\delta_Q^2\mathcal{H}^\star(Q)\partial_t Q+\delta_Q^2\mathcal{R}(Q)\partial_x Q=0.$$

Let us now state in the following theorem, other consequences of a general Godunov structure for system (1.25).

THEOREM 2.2. *We use the same notations and assumptions as in Definition 2.2. Assume that $\mathcal{H}^\star$ is strictly convex on $\Omega^\star$. Then, the general Godunov system (2.4) is weakly symmetrizable for any change of variable $Q\mapsto V$. More precisely, it is written under the form*

$$A_0(V)\partial_t V+A_1(V)\partial_x V=0,$$

where the symmetric operators are given by

$$A_0(V)=(D_V U)^T\delta_U^2\mathcal{H}(U)D_V U, \tag{2.10}$$

$$A_1(V)=(D_V U)^T\delta_U^2\mathcal{H}(U)D_U F(U)D_V U, \tag{2.11}$$

with $U=\delta_Q\mathcal{H}^\star(Q)$, $F(U)=\delta_Q\mathcal{R}(Q)$, and $\mathcal{H}$ the Legendre transform of $\mathcal{H}^\star$.

Proof. Setting $U=\delta_Q\mathcal{H}^\star(Q)$ and $F(U)=\delta_Q\mathcal{R}(Q(U))$, system (2.4) writes

$$\partial_t U+\partial_x F(U)=0. \tag{2.12}$$

We now consider the change of variable $U\mapsto V$ and write Equation (2.12) under

$$D_V U\partial_t V+D_U F(U)D_V U\partial_x V=0. \tag{2.13}$$

Then, we denote by $\mathcal{H}$ the Legendre transform of $\mathcal{H}^\star$ and take the left action of $(D_V U)^T\ \delta_U^2\mathcal{H}(U)$ on Equation (2.13). This leads to

$$(D_V U)^T\delta_U^2\mathcal{H}(U)D_V U\partial_t V+(D_V U)^T\delta_U^2\mathcal{H}(U)D_U F(U)D_V U\partial_x V=0. \tag{2.14}$$

Hence, the theorem is proved if we show that $\delta_U^2\mathcal{H}(U)D_U F(U)$ is symmetric. To do so, let us differentiate the following application

$$\mathcal{N}(U):=\int_{\mathbb{R}} Q(U)\cdot F(U)-\mathcal{R}(Q(U)),$$

and find

$$D_U\mathcal{N}(U)\phi=\int_{\mathbb{R}}(F(U)-\delta_Q\mathcal{R}(Q))\cdot D_U Q\phi+Q\cdot D_U F(U)\phi\quad\forall\phi\in\mathcal{A}.$$

On the other hand, $\delta_Q\mathcal{R}(Q)=F(U)$ and $Q=\delta_U\mathcal{H}(U)$. Therefore,

$$D_U\mathcal{N}(U)\phi=\int_{\mathbb{R}}\delta_U\mathcal{H}(U)\cdot D_U F(U)\phi\quad\forall\phi\in\mathcal{A}.$$

The symmetry of the operator $\delta_U^2\mathcal{H}(U)D_U F(U)$ is just a consequence of the integrability of $\phi\mapsto\int_{\mathbb{R}}\delta_U\mathcal{H}(U)\cdot D_U F(U)\phi$. ∎

Let us gather the two previous results in the following corollary.

COROLLARY 2.1. *We use the same notations and assumptions as in Definition 2.2. Assume that $\mathcal{H}$ is strictly convex on Ω. The three following statements are equivalent:*

(1) System (1.25) *owns a general Godunov structure using the Legendre transform $\mathcal{H}^\star$ of $\mathcal{H}$.*

(2) The operator $\delta_U^2\mathcal{H}(U)D_UF(U)$ is symmetric.

(3) System (1.25) *is weakly symmetrizable under any change of variable $U\mapsto V$ with the expressions* (2.10) *and* (2.11) *for symmetric operators.*

One can see that these relations are very similar to the case of hyperbolic systems. It remains to check whether or not one can add to these statements the existence of a conservation law.

PROPOSITION 2.2. *Assume any of the three statements of Corollary 2.1. Assume also that there exists a pair of functions (E,R) which defines $\mathcal{H}(U)=\int_{\mathbb{R}}E(U)$ and $\mathcal{R}(Q)=\int_{\mathbb{R}}R(Q)$ describing the general Godunov form* (2.4) *of system* (1.25)*. Then, the solution U to system* (1.25) *satisfies*

$$\int_{\mathbb{R}}\big(\partial_t E(U)+\partial_x N(U)\big)=0, \tag{2.15}$$

where

$$N(U)=Q(U)\cdot F(U)-R(Q(U)).$$

Proof. We take the left side action of $D_UE(U)$ on Equation (1.25) and find

$$D_UE(U)\partial_tU+D_UE(U)D_UF(U)\partial_xU=0. \tag{2.16}$$

We then take the integral on $\mathbb{R}$ and use the definition of the variational derivative to get

$$\int_{\mathbb{R}}D_UE(U)\partial_tU+\delta_U\mathcal{H}(U)\cdot D_UF(U)\partial_xU=0. \tag{2.17}$$

On the other hand, as done in the proof of Theorem 2.2, we have

$$\int_{\mathbb{R}}D_UN(U)\phi=\int_{\mathbb{R}}\delta_U\mathcal{H}(U)\cdot D_UF(U)\phi \quad \forall\phi\in\mathcal{A}. \tag{2.18}$$

Therefore,

$$\int_{\mathbb{R}}D_UN(U)\partial_xU=\int_{\mathbb{R}}\delta_U\mathcal{H}(U)\cdot D_UF(U)\partial_xU. \tag{2.19}$$

Hence, we can write Equation (2.17) as

$$\int_{\mathbb{R}}D_UE(U)\partial_tU+D_UN(U)\partial_xU=0, \tag{2.20}$$

which provides Equation (2.15). ∎

Let us remark that contrary to the case of hyperbolic systems, the reciprocal of Proposition 2.2 is false since Equations (2.18) and (2.19) are no longer equivalent. Indeed, $\delta_U\mathcal{H}(U)$ as well as the components of $D_UN(U)$ depend not only on U but also on its derivatives.

Let us also remark that the notion of symmetry introduced for Equation (1.25) corresponds to the symmetry for the $\mathbb{L}^2$ scalar product and is a weak notion while the symmetry of hyperbolic system is a strong one. This is due to the fact that the assertion

$$\int_{\mathbb{R}} \phi.\mathcal{F}\psi = \int_{\mathbb{R}} \mathcal{F}\psi\cdot\phi \quad \forall \phi,\psi \text{ test functions}, \tag{2.21}$$

does not imply

$$\phi.\mathcal{F}\psi = \mathcal{F}\psi\cdot\phi \quad \forall \phi,\psi \text{ test functions}. \tag{2.22}$$

Therefore, the weak symmetry of the system does not lead to a conservation law but to an equality of the form (2.15). However, as we can see in [18], this definition is strong enough to allow us to generalize the hyperbolic techniques to the Green–Naghdi equations. In fact, if we considered a stronger definition like the one deduced in Equation (2.22) for the symmetric operator and a stronger condition such as the symmetry of $D_U^2 E(U) D_U F(U)$ for Theorem 2.1, we would obtain a conservation law in addition to similar theorems. However, less equations would be covered (i.e. the result would be less general). Moreover, the strong symmetry of $D_U^2 E(U) D_U F(U)$ is more tedious to be checked than the weak symmetry of $\delta_U^2 \mathcal{H}(U) D_U F(U)$. We end this section by two remarks. The first one is about an interesting change of variable (similarly to Remark 1.1) while the second deals with the multi-dimensional case.

Remark 2.1. Let us consider system (1.25) with a variational function $\mathcal{H}$ such that $\delta_U^2 \mathcal{H}(U) D_U F(U)$ is a symmetric operator. Assume that there exists a decomposition of the unknown $U = (U_1, U_2)$ such that the application $\phi \mapsto \delta_{U_2}\mathcal{H}(U_1,\phi)$ is invertible. Then, the change of variable

$$U \mapsto (V_1, V_2) = (U_1, \delta_{U_2}\mathcal{H}(U_1, U_2)) \tag{2.23}$$

is very interesting since it leads to a block diagonal structure of the matrix operator $A_0(V)$ defined by Equation (2.10). Using this expression, we have

$$A_0(V) = \begin{pmatrix} A_0^{11} & A_0^{12} \\ A_0^{21} & A_0^{22} \end{pmatrix},$$

where

$$\begin{aligned}
A_0^{11} &= \delta_{U_1}^2 \mathcal{H}(U) + \delta_{U_2 U_1}^2 \mathcal{H}(U)\ D_{V_1} U_2 + (D_{V_1} U_2)^T \delta_{U_1 U_2}^2 \mathcal{H}(U) + (D_{V_1} U_2)^T \delta_{U_2}^2 \mathcal{H}(U)\ D_{V_1} U_2 \\
A_0^{12} &= \delta_{U_2 U_1}^2 \mathcal{H}(U)\ D_{V_2} U_2 + (D_{V_1} U_2)^T \delta_{U_2}^2 \mathcal{H}(U) D_{V_2} U_2, \\
A_0^{21} &= (A_0^{12})^T = (D_{V_2} U_2)^T\ \delta_{U_1 U_2}^2 \mathcal{H}(U) + (D_{V_2} U_2)^T \delta_{U_2}^2 \mathcal{H}(U) D_{V_1} U_2, \\
A_0^{22} &= (D_{V_2} U_2)^T\ \delta_{U_2}^2 \mathcal{H}(U)\ D_{V_2} U_2.
\end{aligned}$$

Therefore, $A_0(V)$ is block diagonal since

$$A_0^{21} = (A_0^{12})^T = (D_{V_2} U_2)^T\ \delta_{U_1 U_2}^2 \mathcal{H}(U) + (D_{V_2} U_2)^T \delta_{U_2}^2 \mathcal{H}(U)\ D_{V_1} U_2 = 0.$$

This is due to the fact that Equation (2.23) implies that

$$\begin{aligned}
&(D_{V_2} U_2)^T\ \delta_{U_1 U_2}^2 \mathcal{H}(U) + (D_{V_2} U_2)^T \delta_{U_2}^2 \mathcal{H}(U)\ D_{V_1} U_2 \\
=&(D_{V_2} U_2)^T\ D_{U_1} V_2 + (D_{V_2} U_2)^T\ D_{U_2} V_2\ D_{V_1} U_2
\end{aligned}$$

$$
\begin{aligned}
&=(D_{V_2}U_2)^T\ D_{U_1}V_2\ D_{V_1}U_1+(D_{V_2}U_2)^T\ D_{U_2}V_2\ D_{V_1}U_2\\
&=(D_{V_2}U_2)^T\ (D_{U_1}V_2\ D_{V_1}U_1+D_{U_2}V_2\ D_{V_1}U_2)\\
&=(D_{V_2}U_2)^T\ D_{V_1}V_2=0.
\end{aligned}
$$

Remark 2.2. Let us consider the multi-dimensional version of system (1.25)

$$\partial_t U+\sum_{i=1}^{n}\partial_{x_i}F_i(U)=0. \tag{2.24}$$

One can easily extend the previous results. Consider a variational function $\mathcal{H}(U)=\int_{\mathbb{R}}E(U)$ which admits a Legendre transform, as in Definition 2.2. Then the following three statements are equivalent:

(1) The operators $\delta_U^2\mathcal{H}(U)D_U F_i(U)$ are symmetric for all $i\in\{1,\dots,n\}$.

(2) System (2.24) admits a general Godunov structure, i.e. there exist functions R_i and the associated $\mathcal{R}_i(Q)=\int_{\mathbb{R}}R_i(Q)$ such that system (2.24) is equivalent to

$$\partial_t(\delta_Q\mathcal{H}^\star(Q))+\sum_{i=1}^{n}\partial_{x_i}(\delta_Q\mathcal{R}_i(Q))=0.$$

(3) System (2.24) is symmetrizable under any change of variable $U\mapsto V$ i.e. it is equivalent to

$$A_0(V)\partial_t V+\sum_{i=1}^{n}A_i(V)\partial_{x_i}V=0,$$

where the symmetric positive definite operator $A_0(V)$ is given by

$$A_0(V)=(D_V U)^T\delta_U^2\mathcal{H}(U)D_V U, \tag{2.25}$$

and the symmetric operators $A_i(V)$ by

$$A_i(V)=(D_V U)^T\delta_U^2\mathcal{H}(U)D_U F_i(U)D_V U. \tag{2.26}$$

Moreover, if one of these statements is satisfied, the solution to system (1.25) satisfies

$$\int_{\mathbb{R}}\partial_t E(U)+\sum_{i=1}^{n}\partial_{x_i}\left(Q\cdot F_i(U)-R_i(Q)\right)=0.$$

3. Application to Green–Naghdi type equations

3.1. Symmetrization of the Green–Naghdi system. In this part, we are going to apply the result of the previous section to the Green–Naghdi type system (1.1) around constant solutions $(\bar h,\bar h\bar u)$, with $\bar h>0$ and $\bar u\in\mathbb{R}$. First, we show that system (1.1) is of the form (1.25) under convenient variables.

Proposition 3.1. *Let $s\geq 2$ be an integer and set $\mathcal{A}=\mathbb{H}^s(\mathbb{R})\times\mathbb{H}^{s-1}(\mathbb{R})$. Then, using the variable*

$$U=(\eta,w)$$

with $\eta=h-\bar{h}$ and $w=\mathcal{L}_h(u)-\bar{h}\bar{u}$, system (1.1) *is of the form* (1.25) *where $F:\mathcal{A}\to\mathcal{A}$ is differentiable.*

Proof. We denote $\mathcal{L}_h(u)$ by m and $\bar{h}\bar{u}$ by $\bar{m}$. Let us first prove that system (1.1) can be written as

$$\begin{cases} \frac{\partial \eta}{\partial t}+\frac{\partial}{\partial x}(hu)=0, \\ \frac{\partial w}{\partial t}+\frac{\partial}{\partial x}(mu)+\frac{\partial}{\partial x}\left(-2\alpha h^3(\partial_x u)^2+\frac{g}{2}h^2\right)=0 \end{cases} \tag{3.1}$$

This is a consequence of the Hamiltonian structure (1.18) of the system. Indeed, developing Equation $(1.18)_1$, we easily find Equation $(3.1)_1$. Then, we develop Equation $(1.18)_2$ to get

$$\partial_t m+h\partial_x\sigma+\partial_x(mu)+m\partial_x u=0,$$

where σ is given by Equation (1.22). Now, using the expression (1.22) of σ together with the fact that $m=\mathcal{L}_h(u)$, we find Equation $(3.1)_2$. One can deduce

$$F(U)=\begin{pmatrix} (\eta+\bar{h})\mathcal{L}_h^{-1}(w+\bar{m})-\bar{h}\bar{u} \\ (w+\bar{m})\mathcal{L}_h^{-1}(w+\bar{m})-2\alpha(\eta+\bar{h})^3(\partial_x\mathcal{L}_h^{-1}(w+\bar{m}))^2+\frac{g}{2}(\eta+\bar{h})^2-\frac{g}{2}\bar{h}^2-\bar{m}\bar{u} \end{pmatrix}.$$

Let us now check the properties of F. Assuming that $h\in\mathbb{H}^s(\mathbb{R})+\bar{h}$ is positively bounded by below, $\mathcal{L}_h$ is a diffeomorphism from $\mathbb{H}^{s+1}(\mathbb{R})+\bar{u}$ to $\mathbb{H}^{s-1}(\mathbb{R})+\bar{h}\bar{u}$. This together with the fact that $\mathbb{H}^{s-1}(\mathbb{R})$ is an algebra for $s\geq 2$ ensures that F is an application from $\mathcal{A}$ to $\mathcal{A}$. For instance, let us consider the first component of $F(U)$. Since $w\in\mathbb{H}^{s-1}(\mathbb{R})$, we obtain that $\mathcal{L}_h^{-1}(w+\bar{m})$ belongs to $\mathbb{H}^{s+1}(\mathbb{R})+\bar{u}$ and thus, to $\mathbb{H}^s(\mathbb{R})+\bar{u}$. On the other hand, $\eta+\bar{h}\in\mathbb{H}^s(\mathbb{R})+\bar{h}$. Hence, the product is in $\mathbb{H}^s(\mathbb{R})+\bar{h}\bar{u}$. A similar logic can be applied to get a similar result on the second component of F.
The differentiability of F is due to the fact that it is a composition of differentiable applications. ∎

We are now going to see that system (1.1) satisfies the assumptions of theorems 2.1 and 2.2 and Corollary 2.1 presented in Section 2.

PROPOSITION 3.2. *Let us consider a constant solution $\bar{V}=(\bar{h},\bar{u})$ with $\bar{h}>0$. Then, there exists a neighborhood in $\mathbb{H}^s(\mathbb{R})\times\mathbb{H}^{s+1}(\mathbb{R})$ of $\bar{V}$, such that as long as the solution $V=(h,u)$ remains in this neighborhood, System* (1.1) *is symmetrizable under any change of variable defined on this neighborhood. In other words, system* (1.1) *is locally weakly symmetrizable around constant solutions.*

Proof. Let us prove that system (1.1) admits a general Godunov structure of the form (2.4) using the function

$$\mathcal{H}_{\bar{h},\bar{u}}(U)=\int_{\mathbb{R}}\frac{gh(h-\bar{h})}{2}+\frac{h(u-\bar{u})^2}{2}+\frac{\alpha h^3(u_x)^2}{2}.$$

Let us first remark that $\mathcal{H}_{\bar{h},\bar{u}}(U)$ is strictly convex in a small neighborhood[3] of $\bar{U}=U(\bar{V})$. The explicit representation formula of the second variation of $\mathcal{H}_{\bar{h},\bar{u}}$ is provided in Appendix A. For all test functions ϕ_1 and ϕ_2, one has[4]

[3] for the classical norm of $\mathbb{H}^s(\mathbb{R})\times\mathbb{H}^{s-1}(\mathbb{R})$.

[4] $\mathcal{L}_h^{-\frac{1}{2}}$ is the symmetric operator such that $\mathcal{L}_h^{-\frac{1}{2}}\circ\mathcal{L}_h^{-\frac{1}{2}}=\mathcal{L}_h^{-1}$. The existence of this operator is guaranteed by the symmetry definite positivity of $\mathcal{L}_h^{-1}$.

$$\int_{\mathbb{R}} \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \cdot \delta^2 \mathcal{H}_{\bar{h},\bar{u}}(U) \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$$
$$= \int_{\mathbb{R}} (g - 3\alpha h(u_x)^2)(\phi_1)^2 + \left(\mathcal{L}_h^{-\frac{1}{2}}(-u\phi_1 + 3\alpha\partial_x(h^2 u_x \phi_1)) + \mathcal{L}_h^{-\frac{1}{2}}(\phi_2)\right)^2.$$

Now, considering the fact that $g - 3\alpha h(u_x)^2$ is bounded positively by below for (h,u) close enough to $\bar{V}$[5] (therefore, for U close enough to $\bar{U}$), the strict convexity of $\mathcal{H}_{\bar{h},\bar{u}}$ on the small neighborhood of $\bar{U}$ is concluded. We can formulate this conclusion as following:

There exists a neighborhood in $\mathbb{H}^s(\mathbb{R}) \times \mathbb{H}^{s-1}(\mathbb{R})$ of $\bar{U} = (0,0)$ such that as long as the solution $U = (\eta, w)$ is in this neighborhood, $\delta_U^2 \mathcal{H}_{\bar{h},\bar{u}}$ is positive definite. In particular, we have on $\bar{U}$,

$$\delta^2 \mathcal{H}_{\bar{h},\bar{u}}(\bar{U}) = \begin{pmatrix} g & 0 \\ 0 & \mathcal{L}_h^{-1}\end{pmatrix}. \tag{3.2}$$

Let us also remark that $\delta^2 \mathcal{H}_{\bar{h},\bar{u}}(U)$ is an isomorphism from $\mathbb{H}^s(\mathbb{R}) \times \mathbb{H}^{s-1}(\mathbb{R})$ to $\mathbb{H}^s(\mathbb{R}) \times \mathbb{H}^{s+1}(\mathbb{R})$ if U is close enough to $\bar{U}$. Hence, the variational derivative $\delta\mathcal{H}_{\bar{h},\bar{u}}$ defines a diffeomorphism on a small enough neighborhood of the equilibrium $\bar{U}$. This is a consequence of the inverse function theorem considering the injectivity of $\delta\mathcal{H}_{\bar{h},\bar{u}}(U)$ for U close to $\bar{U}$.

We now consider the Legendre transform $\mathcal{H}^\star_{\bar{h},\bar{u}}$ which is defined by

$$\mathcal{H}^\star_{\bar{h},\bar{u}}(Q) = \int_{\mathbb{R}} Q \cdot U - E_{\bar{h},\bar{u}}, \tag{3.3}$$

where

$$E_{\bar{h},\bar{u}} = \frac{gh(h-\bar{h})}{2} + \frac{h(u-\bar{u})^2}{2} + \alpha h^3 (u_x)^2.$$

and

$$Q = \delta_U \mathcal{H}_{\bar{h},\bar{u}}(U). \tag{3.4}$$

One can check that $Q = (\sigma, u-\bar{u})$, with $\sigma = gh - g\bar{h}/2 - u^2/2 + \bar{u}^2/2 - \frac{3}{2}\alpha h^2 (u_x)^2$. This leads to the following expression for $\mathcal{H}^\star_{\bar{h},\bar{u}}$

$$\mathcal{H}^\star_{\bar{h},\bar{u}}(Q) = \int_{\mathbb{R}} \frac{g(h-\bar{h})^2}{2} + \frac{\bar{h}(u-\bar{u})^2}{2} - \alpha h^3 (u_x)^2 + \frac{3}{2}\alpha h^2 \bar{h}(u_x)^2.$$

We just now need to remark that there exists a function $\mathcal{R}$ of Q such that

$$F(U) = \delta_Q \mathcal{R}(Q).$$

We can get to this equality setting

$$\mathcal{R}(Q) = \int_{\mathbb{R}} gu\left(\frac{h^2 - \bar{h}^2}{2}\right) - \alpha h^3 u (u_x)^2 - \bar{h}\bar{u}\sigma - \bar{h}\bar{u}^2(u-\bar{u}) + g\bar{h}^2\bar{u}/2. \tag{3.5}$$

[5] for the classical norm of $\mathbb{H}^s(\mathbb{R}) \times \mathbb{H}^{s+1}(\mathbb{R})$.

Hence, the system is equivalent on a small enough neighborhood of $\bar{V}$ to

$$\partial_t\left(\delta_Q \mathcal{H}^{\star}_{\bar{h},\bar{u}}(Q)\right) + \partial_x\left(\delta_Q \mathcal{R}(Q)\right) = 0. \tag{3.6}$$

Now, using Theorem 2.2, we can conclude the weak symmetrizability of the system under any change of variable around constant solutions. □

Let us remark that the quantity $E_{\bar{h},\bar{u}}$ introduced in the proof of Proposition 3.2, is actually an energy for the system. Indeed, we can check that the solution of system (1.1) satisfies

$$\partial_t E_{\bar{h},\bar{u}} + \partial_x\left(u E_{\bar{h},\bar{u}} + (u-\bar{u})p\right) = 0. \tag{3.7}$$

where p is defined by Equation (1.4).

Proposition 3.2 together with Theorem 2.2 implies the symmetrizability of the system under any variable around constant solutions. We now provide in the two following propositions some explicit symmetric forms of system (1.1).

PROPOSITION 3.3. *The Green–Naghdi type system* (1.1) *can be written under the symmetric form*

$$A_0(Q)\partial_t Q + A_1(Q)\partial_x(Q) = 0, \tag{3.8}$$

where $Q = (\sigma, u-\bar{u})$ *is defined by Equation* (3.4) *and*

$$A_0(Q) = \tag{3.9}$$
$$\begin{pmatrix} \frac{1}{g-3\alpha h(u_x)^2} & \frac{u+3\alpha h^2 u_x \partial_x}{g-3\alpha h(u_x)^2} \\ \frac{u}{g-3\alpha h(u_x)^2} - 3\alpha\partial_x\left(\frac{h^2 u_x}{g-3\alpha h(u_x)^2}()\right) & \mathcal{L}_h + \frac{u\left(u+3\alpha h^2 (u_x)\partial_x\right)}{g-3\alpha h(u_x)^2} - 3\alpha\partial_x\left(h^2 u_x \frac{u()+3\alpha h^2 u_x \partial_x()}{g-3\alpha h(u_x)^2}\right) \end{pmatrix}$$

and

$$A_1(Q) = \tag{3.10}$$
$$\begin{pmatrix} \frac{u}{g-3\alpha h(u_x)^2} & h + \frac{u^2+3\alpha h^2 u u_x \partial_x}{g-3\alpha h(u_x)^2} \\ h + \frac{u^2}{g-3\alpha h(u_x)^2} - 3\alpha\partial_x\left(\frac{h^2 u(u_x)}{g-3\alpha h(u_x)^2}()\right) & 3hu + \frac{u^3+3\alpha h^2 u^2 u_x \partial_x}{g-3\alpha h(u_x)^2} - \alpha\partial_x\left(h^3 u_x()\right) - \alpha u \partial_x\left(h^3\partial_x()\right) \\ & -3\alpha\partial_x\left(\frac{h^2 u^2 u_x + 3\alpha h^4 u (u_x)^2 \partial_x()}{g-3\alpha h(u_x)^2}\right) \end{pmatrix}.$$

Proof. This is a consequence of the general Godunov structure (3.6) of the system. We just need to set $A_0(Q) = \delta^2_Q \mathcal{H}^{\star}_{\bar{h},\bar{u}}(Q)$ and $A_1(Q) = \delta^2_Q \mathcal{R}(Q)$ to get the result. □

Let us remark that the operators $A_0(Q)$ and $A_1(Q)$ defined by Equations (3.9) and (3.10) are second order differential operators.

PROPOSITION 3.4. *The Green–Naghdi type system* (1.1) *is symmetric under the unknown* $V = (h, u)$ *of the form*

$$A_0(V)\partial_t V + A_1(V)\partial_x(V) = 0, \tag{3.11}$$

with

$$A_0(V) = \begin{pmatrix} g-3\alpha h(u_x)^2 & 0 \\ 0 & \mathcal{L}_h \end{pmatrix}, \tag{3.12}$$

and

$$A_1(V) = \begin{pmatrix} gu - 3\alpha h u (u_x)^2 & gh - 3\alpha h^2 (u_x)^2 \\ gh - 3\alpha h^2 (u_x)^2 & hu + 2\alpha\partial_x(h^3 u_x) - \alpha h^3 u_x \partial_x - \alpha u \partial_x(h^3\partial_x()) \end{pmatrix}. \tag{3.13}$$

Proof. This proposition is just a consequence of Theorem 2.2 and Proposition 3.2. In fact, we check that the change of variable $U \mapsto \tilde{V}$ such that

$$\begin{cases} U = (\eta, w), \\ \tilde{V} = (\eta, \delta_w \mathcal{H}_{\bar{h},\bar{u}}(U)), \end{cases}$$

leads to $\tilde{V} = (h - \bar{h}, u - \bar{u})$ which is nothing but V within a constant. This fact is true since $\delta_w \mathcal{H}_{\bar{h},\bar{u}}(U) = \mathcal{L}_h^{-1}(m) - \bar{u}$. This change of variable is valid by the properties of the Sturm–Liouville operator $\mathcal{L}_h$ while h is positively bounded by below. Hence, the system is symmetric with

$$A_0(V) = (D_V U)^T \delta_U^2 \mathcal{H}_{\bar{h},\bar{u}}(U) D_V U,$$

and

$$A_1(V) = (D_V U)^T \delta_U^2 \mathcal{H}_{\bar{h},\bar{u}}(U) \nabla_U F(U) D_V U.$$

Basic computations (similar to those presented in Appendix A) show that their analytic expressions are given by Equations (3.12) and (3.13). ∎

Let us remark that similarly to Proposition 3.3, the operators $A_0(V)$ and $A_1(V)$ are second order differential operators. However, the analytic expressions of these operators are much simpler than the expressions of $A_0(Q)$ and $A_1(Q)$ in Proposition 3.3. In fact, as explained in Remark 2.1, the symmetric positive definite operator of Proposition 3.4 is diagonal.

Remark 3.1. A similar structure to Equation (3.11) (but non symmetric) is used in [16] to study the linearized Green–Naghdi system in order to prove the local well-posedness.

Let us now apply Proposition 2.2 to the Green–Naghdi type equations to get a conserved quantity. According to this proposition, as long as the solution U remains close $\bar{U}$, it satisfies

$$\int_{\mathbb{R}} \partial_t E_{\bar{h},\bar{u}}(U) + \partial_x N(U) = 0, \tag{3.14}$$

where

$$N(U) = Q \cdot F(U) - R(U),$$

with

$$R(U) = gu\left(\frac{h^2 - \bar{h}^2}{2}\right) - \alpha h^3 u (u_x)^2 - \bar{h}\bar{u}\sigma - \bar{h}\bar{u}^2(u - \bar{u}) + g\bar{h}^2\bar{u}/2$$

given by Equation(3.5). Now, we use the expressions of Q, $F(U)$, and $R(U)$ and we find

$$N(U) = \frac{ghu(h - \bar{h})}{2} + \left(\frac{gh^2 + hu^2}{2} + 3\alpha h^3 (u_x)^2\right)(u - \bar{u}) + \frac{\alpha}{2} h^3 u (u_x)^2.$$

Since $(h - \bar{h}, u - \bar{u}) \in \mathbb{H}^s(\mathbb{R}) \times \mathbb{H}^{s+1}(\mathbb{R})$ and s large, we remark that

$$\lim_{x \to \pm\infty} N(U) = 0,$$

which gives

$$\frac{d}{dt}\int_{\mathbb{R}} E_{\bar{h},\bar{u}}(U)=0.$$

Hence, we conclude the conservation of the energy integral $\mathcal{H}_{\bar{h},\bar{u}}(U)$ from the general Godunov structure of the system. Let us note that we could get the conservation of the energy integral simply by integrating the energy conservation law (3.7).

3.2. Two-dimensional extension. Let us fix $\bar{V}=(\bar{h},\bar{u},\bar{v})\in\mathbb{R}^3$ with $\bar{h}>0$ and consider the 2D Green–Naghdi model

$$\partial_t h+\partial_x hu+\partial_y hv=0, \tag{3.15a}$$

$$\partial_t hu+\partial_x hu^2+\partial_y huv+\partial_x(gh^2/2+\alpha h^2\ddot{h})=0, \tag{3.15b}$$

$$\partial_t hv+\partial_x huv+\partial_y hv^2+\partial_y(gh^2/2+\alpha h^2\ddot{h})=0, \tag{3.15c}$$

where $\dot{h}=\partial_t h+u\partial_x h+v\partial_y h$.

This system is equivalent to

$$\partial_t U+\partial_x F_1(U)+\partial_y F_2(U)=0, \tag{3.16}$$

where $U=(h-\bar{h},m-\bar{h}\bar{u},n-\bar{h}\bar{v})$, with $(m,n)=\mathcal{L}_h(u,v)$ and $\mathcal{L}_h(u,v)=h(u,v)-\alpha\nabla\left(h^3\mathrm{div}(u,v)\right)$. The transformation $(m,n)\mapsto(u,v)$ is well-defined if h is strictly positively bounded by below. Indeed in this case, $\mathcal{L}_h$ is an isomorphism acting on the space

$$\mathbb{H}^{s+1}(\mathrm{div})=\{(u,v)\in(\mathbb{H}^s(\mathbb{R}^2)+\bar{u})\times(\mathbb{H}^s(\mathbb{R}^2)+\bar{v}) \text{ such that } \mathrm{div}(u,v)\in\mathbb{H}^s(\mathbb{R}^2)\}.$$

The fluxes are defined by

$$F_1(U)=\begin{pmatrix} hu \\ gh^2/2+hu^2-2\alpha h^3(\mathrm{div}(u,v))^2-\alpha u\partial_x\left(h^3\mathrm{div}(u,v)\right)+\alpha h^3\mathrm{div}(u,v)v_y \\ huv-\alpha\partial_y\left(h^3u\mathrm{div}(u,v)\right)\end{pmatrix},$$

and

$$F_2(U)=\begin{pmatrix} hv \\ huv-\alpha\partial_x\left(h^3v\mathrm{div}(u,v)\right) \\ gh^2/2+hv^2-2\alpha h^3(\mathrm{div}(u,v))^2-\alpha v\partial_y\left(h^3\mathrm{div}(u,v)\right)+\alpha h^3\mathrm{div}(u,v)u_x\end{pmatrix}.$$

PROPOSITION 3.5. *The solution of system* (3.15) *satisfies the following conservation law*

$$\partial_t E_{\bar{V}}+\partial_x\left(uE_{\bar{V}}+(u-\bar{u})p\right)+\partial_y\left(vE_{\bar{V}}+(v-\bar{v})p\right)=0. \tag{3.18}$$

where

$$E_{\bar{V}}=gh(h-\bar{h})/2+h(u-\bar{u})^2/2+h(v-\bar{v})^2/2+\alpha h^3(u_x+v_y)^2/2, \tag{3.19}$$

with p given by Equation (1.4).

Let us consider the space integral $\mathcal{H}_{\bar{V}}$ of the energy $E_{\bar{V}}$,

$$\mathcal{H}_{\bar{V}}(U)=\int_{\mathbb{R}^2}E_{\bar{V}}(U). \tag{3.20}$$

Similarly to the one dimensional case, this function is strictly convex as an application of U while $V=(h,u,v)$ is close enough to the equilibrium $\bar{V}=(\bar{h},\bar{u},\bar{v})$, i.e. $\delta_U^2\mathcal{H}_{\bar{V}}(U)$ is positive definite for U close to $\bar{U}=U(\bar{V})=(0,0,0)$. Let us now consider the change of variable

$$U\mapsto Q=\delta_U\mathcal{H}_{\bar{V}}(U),$$

defined around $\bar{U}$. Similarly to the 2-dimensional case, this is a diffeomorphism since $\delta_U\mathcal{H}_{\bar{V}}$ is injective on a small enough neighborhood of $\bar{U}$. Moreover, $\delta_U^2\mathcal{H}_{\bar{V}}(U)$ is an isomorphism for all U close to $\bar{U}$. The invertibility of $\delta_U\mathcal{H}_{\bar{V}}$ is then just a consequence of the inverse function theorem. One can check that

$$Q=\begin{pmatrix} gh-g\bar{h}/2-(u^2-\bar{u}^2)/2-(v^2-\bar{v}^2)/2-3\alpha h^2(\mathrm{div}(u,v))^2/2 \\ u-\bar{u} \\ v-\bar{v}\end{pmatrix}. \tag{3.21}$$

We are going to see in the following proposition that the 2-dimensional Green–Naghdi Equation (3.15) admits a general Godunov structure using the variable Q.

PROPOSITION 3.6. *Let $s>4$. There exists a neighborhood for the norm $\mathbb{H}^s\times\mathbb{H}^{s+1}(\mathrm{div})$ of $\bar{V}$ such that as long as the solution V of* (3.15) *remains in this neighborhood, the system is equivalent to*

$$\partial_t(\delta_Q\mathcal{H}_{\bar{V}}^\star(Q))+\partial_x(\delta_Q\mathcal{R}_1(Q))+\partial_y(\delta_Q\mathcal{R}_2(Q))=0, \tag{3.22}$$

where Q is defined by (3.21) *and $\mathcal{R}_1$ and $\mathcal{R}_2$ are two functions defined on a neighborhood of $\bar{Q}=Q(\bar{V})=(g\bar{h}/2,0,0)$ (see system* (3.24) *for some explicit representation formulas).*

Proof. Let us first remark that the Legendre transform $\mathcal{H}_{\bar{V}}^\star$ of the energy integral $\mathcal{H}_{\bar{V}}$ is defined by

$$\begin{aligned}\mathcal{H}_{\bar{V}}^\star(Q)&=\int_{\mathbb{R}^2}Q\cdot U-E_{\bar{V}}(U)\\ &=\int_{\mathbb{R}^2}g(h-\bar{h})^2/2+\bar{h}(u-\bar{u})^2/2+\bar{h}(v-\bar{v})^2/2-\alpha h^3(\mathrm{div}(u,v))^2+\frac{3}{2}\alpha h^2\bar{h}(\mathrm{div}(u,v))^2.\end{aligned}$$

We know by Definition 2.2 of the Legendre transform that we have

$$U=\delta_Q\mathcal{H}_{\bar{V}}^\star(Q). \tag{3.23}$$

Let us now consider the variational functions $\mathcal{R}_1$ and $\mathcal{R}_2$ defined by

$$\mathcal{R}_1(Q)=\int_{\mathbb{R}^2}g\left(\frac{uh^2-\bar{u}\bar{h}^2}{2}\right)+\bar{h}\bar{u}(u^2-\bar{u}^2)-\alpha h^3u(\mathrm{div}(u,v))^2, \tag{3.24a}$$

and

$$\mathcal{R}_2(Q)=\int_{\mathbb{R}^2}g\left(\frac{vh^2-\bar{v}\bar{h}^2}{2}\right)+\bar{h}\bar{v}(v^2-\bar{v}^2)-\alpha h^3v(\mathrm{div}(u,v))^2. \tag{3.24b}$$

We can easily check that

$$F_1(U)=\delta_Q\mathcal{R}_1(Q), \tag{3.25}$$

$$F_2(U) = \delta_Q \mathcal{R}_2(Q). \tag{3.26}$$

Considering Equation (3.23) together with Equations (3.25) and (3.26) we get the result. □

Now, according to Remark 2.2, the 2-dimensional Green–Naghdi system (3.15) is symmetrizable under any change of variable around any constant solution $\bar{V}$. Especially, the general Godunov structure of the system leads directly to the following symmetric structure under the unknown Q:

$$A_0(Q)\partial_t Q + A_1(Q)\partial_x Q + A_2(Q)\partial_y Q = 0, \tag{3.27}$$

where

$$A_0(Q) = \delta_Q^2 \mathcal{H}_{\bar{V}}^\star(Q), \tag{3.28a}$$
$$A_1(Q) = \delta_Q^2 \mathcal{R}_1(Q), \tag{3.28b}$$

and

$$A_2(Q) = \delta_Q^2 \mathcal{R}_2(Q). \tag{3.28c}$$

Considering the fact that we can recover the physical variable $V = (h,u,v)$ using the partial variational derivative of the energy integral, we have the following corollary.

Corollary 3.1. *The two-dimensional Green–Naghdi Equation* (3.15) *is symmetric under the physical variable* $V = (h,u,v)$ *of the form*

$$A_0(V)\partial_t V + A_1(V)\partial_x V + A_2(V)\partial_y V = 0. \tag{3.29}$$

where

$$A_0(V) = \begin{pmatrix} g - 3\alpha h(\mathrm{div}(u,v))^2 & 0 & 0 \\ 0 & h - \alpha\partial_x(h^3\partial_x) & -\alpha\partial_x(h^3\partial_y) \\ 0 & -\alpha\partial_y(h^3\partial_x) & h - \alpha\partial_y(h^3\partial_y) \end{pmatrix}$$

is block diagonal.

Proof. We first consider the change of variable $U \mapsto \tilde{V}$ where

$$U = (h,m,n),$$

and

$$\tilde{V} = (h, \delta_{(m,n)}\mathcal{H}_{\bar{V}}(U)) = (h, u - \bar{u}, v - \bar{v})$$

is nothing but V within a constant. This change of variable is valid by the invertibility of $\mathcal{L}_h$ on $\mathbb{H}^{s+1}(\mathrm{div})$ since h is positively bounded by below and the physical speed (u,v) belongs to $\mathbb{H}^{s+1}(\mathrm{div})$. We then use Remark 2.2 to find the following expression for the operators

$$A_0(V) = (D_V U)^T \ \delta_U^2 \mathcal{H}_{\bar{V}}(U)\ D_V U,$$

$$A_1(V) = (D_V U)^T \ \delta_U^2 \mathcal{H}_{\bar{V}}(U)\ D_U F_1(U)\ D_V U,$$

and

$$A_2(V) = (D_V U)^T \ \delta_U^2 \mathcal{H}_{\bar{V}}(U)\ D_U F_2(U)\ D_V U.$$

Using Remark 2.1, we could predict the block diagonal structure of $A_0(V)$. □

Let us mention that similarly to the first dimensional case, the conservation over time of the energy integral $\mathcal{H}_{\bar{V}}$ can be concluded.

4. Conclusion

A generalization of the notion of symmetry classically defined for hyperbolic systems has been presented. This generalization is mainly based on the generalization of Godunov systems introduced in [9]. We prove that all general Godunov systems are symmetrizable under any change of variable. We also see that this structure leads to a conserved quantity. Then, we check that the one and two dimensional Green–Naghdi equations are general Godunov systems as long as the solution remains close enough to equilibriums. Therefore, there are symmetrizable under any change of variable defined on a small neighborhood of constant solutions. Moreover, the conserved quantity deduced by the general Godunov structure of the system is nothing but the energy integral which represents the total physical energy of the system.

Let us also mention that we write the Green–Naghdi equation on a quite simple structure under the physical variable. This is due to the fact that the physical variable can be obtained from the Hamiltonian variable by a partial change of variables. In fact, this leads to a bloc diagonal operator for the symmetric structure. The symmetric structure of the Green–Naghi equations under the variable (h,u) is also used in [18] to prove the non linear stability of constant solutions of the system with viscosity.

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Appendix A. Computation of the second variation. In this part, we compute the second variation with respect to $U=(h,m)$ of

$$\mathcal{H}_{\bar{h},\bar{u}}(U)=\int_{\mathbb{R}}\frac{gh(h-\bar{h})}{2}+\frac{h(u-\bar{u})^2}{2}+\frac{\alpha h^3(u_x)^2}{2}.$$

Let us first compute the variational derivative with respect to h of $\mathcal{H}_{\bar{h},\bar{u}}(U)$. In fact, fixing the function m, we have for all test functions ϕ,

$$\begin{aligned}\mathcal{H}_{\bar{h},\bar{u}}(h+\phi,m)&=\int_{\mathbb{R}}E_{\bar{h},\bar{u}}(h+\phi,u)\\&=\int_{\mathbb{R}}E_{\bar{h},\bar{u}}(h,u)+D_hE_{\bar{h},\bar{u}}(h,u)(\phi)+o(\|\phi\|),\\&=\mathcal{H}_{\bar{h},\bar{u}}(h,u)+\int_{\mathbb{R}}D_hE_{\bar{h},\bar{u}}(h,u)(\phi)+o(\|\phi\|)\end{aligned}$$

where $\lim_{\phi\to 0}\frac{o(\|\phi\|)}{\|\phi\|}=0$. Using the definition of $E_{\bar{h},\bar{u}}$, we have

$$\begin{aligned}\mathcal{H}_{\bar{h},\bar{u}}(h+\phi,m)=\mathcal{H}_{\bar{h},\bar{u}}(h,u)&+\int_{\mathbb{R}}gh\phi-\frac{g\bar{h}}{2}\phi+\frac{(u-\bar{u})^2}{2}\phi\\&+\int_{\mathbb{R}}h(u-\bar{u})D_hu(\phi)+\frac{3}{2}\alpha h^2(u_x)^2\phi+\alpha h^3u_x\partial_xD_hu(\phi)+o(\|\phi\|).\end{aligned}\tag{A.1}$$

In order to compute $D_hu(\phi)$, we consider Equation (1.17) where m is defined. We differentiate this relation with respect to h and take the action on ϕ. We find

$$0=u\phi+hD_hu(\phi)-\alpha\partial_x(3h^2\phi u_x)-\alpha\partial_x(h^3\partial_x(D_hu(\phi))).$$

This leads to

$$D_h u(\phi) = \mathcal{L}_h^{-1}\left(3\alpha\partial_x(h^2 u_x \phi) - u\phi\right).$$

Injecting this into Equation (A.1), we get

$$\begin{aligned}\mathcal{H}_{\bar{h},\bar{u}}(h+\phi,m) = \mathcal{H}_{\bar{h},\bar{u}}(h,u) + \int_{\mathbb{R}} gh\phi - \frac{g\bar{h}}{2}\phi + \frac{(u-\bar{u})^2}{2}\phi \\ + \int_{\mathbb{R}} \frac{3}{2}\alpha h^2 (u_x)^2\phi + \left(h(u-\bar{u}) + \alpha h^3 u_x \partial_x\right)\mathcal{L}_h^{-1}\left(3\alpha\partial_x(h^2 u_x\phi) - u\phi\right) + o(\|\phi\|).\end{aligned}$$

Hence, we have after an integration by part

$$\begin{aligned}\mathcal{H}_{\bar{h},\bar{u}}(h+\phi,m) = &\mathcal{H}_{\bar{h},\bar{u}}(h,u) + \int_{\mathbb{R}} gh\phi - \frac{g\bar{h}}{2}\phi + \frac{(u-\bar{u})^2}{2}\phi + \frac{3}{2}\alpha h^2(u_x)^2\phi \\ &+ \int_{\mathbb{R}} \left(h(u-\bar{u}) - \partial_x(\alpha h^3 u_x)\right)\mathcal{L}_h^{-1}\left(3\alpha\partial_x(h^2 u_x\phi) - u\phi\right) + o(\|\phi\|),\end{aligned}$$

or equivalently

$$\begin{aligned}\mathcal{H}_{\bar{h},\bar{u}}(h+\phi,m) = &\mathcal{H}_{\bar{h},\bar{u}}(h,u) + \int_{\mathbb{R}} gh\phi - \frac{g\bar{h}}{2}\phi + \frac{(u-\bar{u})^2}{2}\phi + \frac{3}{2}\alpha h^2(u_x)^2\phi \\ &\int_{\mathbb{R}} + \mathcal{L}_h(u-\bar{u})\cdot\mathcal{L}_h^{-1}\left(3\alpha\partial_x(h^2 u_x\phi) - u\phi\right) + o(\|\phi\|).\end{aligned}$$

Now considering the fact that $\mathcal{L}_h$ is symmetric and using another integration by part, we get

$$\mathcal{H}_{\bar{h},\bar{u}}(h+\phi,m) = \mathcal{H}_{\bar{h},\bar{u}}(h,u) + \int_{\mathbb{R}}\left(gh - g\bar{h}/2 - \frac{u^2}{2} + \frac{\bar{u}^2}{2} - \frac{3}{2}\alpha h^2(u_x)^2\right)\phi + o(\|\phi\|).$$

Then, we have

$$\delta_h\mathcal{H}_{\bar{h},\bar{u}}(U) = gh - g\bar{h}/2 - \frac{u^2}{2} + \frac{\bar{u}^2}{2} - \frac{3}{2}\alpha h^2(u_x)^2,$$

which is nothing but the quantity called σ in Section 1.2.
Using exactly the same type of computations, we find

$$\delta_m\mathcal{H}_{\bar{h},\bar{u}}(U) = u - \bar{u}.$$

On the other hand, we know by the definition of the second variation that

$$\delta_U^2\mathcal{H}_{\bar{h},\bar{u}}(U) = D_U\delta_U\mathcal{H}_{\bar{h},\bar{u}}(U) = \begin{pmatrix} D_h\sigma(U) & D_m\sigma(U) \\ D_h u(U) & D_m u(U)\end{pmatrix}.$$

Then, similar computations lead to

$$\delta^2\mathcal{H}_{\bar{h},\bar{u}}(U) = \\ \begin{pmatrix} g - 3\alpha h(u_x)^2 - \left(u + 3\alpha h^2 u_x\partial_x\right)\mathcal{L}_h^{-1}\left(-u() + 3\alpha\partial_x(h^2 u_x())\right) & -\left(u + 3\alpha h^2 u_x \partial_x\right)\mathcal{L}_h^{-1} \\ \mathcal{L}_h^{-1}\left(-u() + 3\alpha\partial_x(h^2 u_x())\right) & \mathcal{L}_h^{-1}\end{pmatrix}.$$

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