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GLOBAL EXISTENCE AND BOUNDEDNESS IN A 2D KELLER–SEGEL–STOKES SYSTEM WITH NONLINEAR DIFFUSION AND ROTATIONAL FLUX*

XIE LI[†], YULAN WANG[‡], AND ZHAOYIN XIANG[§]

Abstract. In this paper, we investigate the degenerate Keller–Segel–Stokes system (KSS) in a bounded convex domain $\Omega\subset\mathbb{R}^2$ with smooth boundary. A particular feature is that the chemotactic sensitivity S is a given parameter matrix on $\Omega\times[0,\infty)^2$ whose Frobenius norm satisfies $|S(x,n,c)|\leq C_S$ with some $C_S>0$. It is shown that for any porous medium diffusion $m>1$, the system (KSS) with nonnegative and smooth initial data possesses at least a global-in-time weak solution, which is uniformly bounded.

Key words. Global existence, boundedness, Keller–Segel–Stokes system, tensor-valued sensitivity.

AMS subject classifications. 35K55, 35Q92, 35Q35, 92C17.

1. Introduction

In this paper, we investigate the global existence and boundedness of weak solutions to the 2D Keller–Segel–Stokes system with porous medium diffusion and rotational flux:

$$\begin{cases} n_t+\mathbf{u}\cdot\nabla n=\Delta n^m-\nabla\cdot\big(nS(x,n,c)\nabla c\big), & x\in\Omega, t>0,\\ c_t+\mathbf{u}\cdot\nabla c=\Delta c-c+n, & x\in\Omega, t>0,\\ \mathbf{u}_t+\nabla P=\Delta\mathbf{u}+n\nabla\phi(x), & x\in\Omega, t>0,\\ \nabla\cdot\mathbf{u}=0, & x\in\Omega, t>0, \end{cases} \tag{1.1}$$

where n, c, $\mathbf{u}$, and P denote, respectively, the density of cells, chemical concentration, velocity field and pressure of the fluid. The prescribed functions S and ϕ stand for the chemotactic sensitivity and the potential of the gravitational field within which the cells are driven through buoyant forces, respectively. In this system, chemotaxis and fluid are coupled through both the transport of the cells and the chemical defined by the terms $\mathbf{u}\cdot\nabla n$ and $\mathbf{u}\cdot\nabla c$, and the external force $n\nabla\phi$ exerted on the fluid by the cells.

To motivate our study, we recall some related progresses on system (1.1). We will start by the classical Keller–Segel model.

Keller–Segel model. In their pioneering work [19], Keller and Segel heuristically derived the celebrated model

$$\begin{cases} n_t=\Delta n-\nabla\cdot(n\nabla c), & x\in\Omega, t>0,\\ c_t=\Delta c-c+n, & x\in\Omega, t>0, \end{cases} \tag{1.2}$$

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[†]School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu 611731, P.R. China; College of Mathematic and Information, China West Normal University, Nanchong 637002, P.R. China (xieli-520@163.com).
[‡]School of Science, Xihua University, Chengdu 610039, China (wangyulan-math@163.com).
[§]School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu 611731, P.R. China (zxiang@uestc.edu.cn).

which has been widely investigated (see [1, 13–15] for a survey). One of the most important features studied over the last few years is related to the blow-up of solutions to system (1.2) whose connection to real process behavior still needs to be explained, as blow-up phenomena are not observed in reality. To avoid the blow-up, one can use a nonlinear porous-medium-like diffusion instead of a linear one. Precisely, for any $m>1$, the 2D Keller–Segel model

$$\begin{cases} n_t = \Delta n^m - \nabla\cdot(n\nabla c), & x\in\Omega, t>0, \\ c_t = \Delta c - c + n, & x\in\Omega, t>0 \end{cases} \tag{1.3}$$

with the homogeneous Neumann boundary condition possesses global bounded solutions for arbitrary large initial data, while for $m\le 1$, system (1.3) admits the blow-up solutions under some technical assumptions (see [7, 18, 20, 22, 27, 28] and references therein).

Chemotaxis-(Navier–)Stokes model. In some cases of chemotactic movement in flowing environments the mutual influence between the cells and the fluids may be significant. Considering that the motion of the fluids is determined by the incompressible (Navier–)Stokes equations, Tuval et al. [31] proposed the following chemotaxis-(Navier–)Stokes system to describe such coupled biological phenomena in the context of *signal consumption* by cells:

$$\begin{cases} n_t + \mathbf{u}\cdot\nabla n = \Delta n - \nabla\cdot(\chi n\nabla c), \\ c_t + \mathbf{u}\cdot\nabla c = \Delta c - nf(c), \\ \mathbf{u}_t + \kappa(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla P = \Delta\mathbf{u} + n\nabla\phi, \\ \nabla\cdot\mathbf{u} = 0. \end{cases} \tag{1.4}$$

Here the coefficient κ is related to the strength of nonlinear fluid convection and the oxygen consumption rate $f(c)$ are supposed to be given functions. This inevitably couples the known obstacles from the theory of fluid equations to the typical difficulties arising in the study of chemotaxis systems. Despite this challenge, there are numerous analytical approaches, which addressed issues of well-posedness for corresponding initial-value problems in either bounded or unbounded domains, with various assumptions on the scalar functions χ, f, and ϕ (see e.g. [3–6, 9, 25, 26, 32, 37, 38, 42, 45] and references therein) in the past several years. We refer to [8, 10, 24, 29, 30, 46] and references therein for the nonlinear diffusion models of a porous medium type Δn^m, instead of Δn.

Chemotaxis-(Navier–)Stokes model with rotational flux. Recent modeling approaches suggested that an adequate description of bacterial motion near surfaces of their surrounding fluid should involve rotational components in the cross-diffusive flux (see [43, 44]). Natural generalizations of chemotaxis and chemotaxis-fluid systems thereby obtained should thus model the evolution of the cell density by equations of the form

$$\begin{cases} n_t + \mathbf{u}\cdot\nabla n = \Delta n^m - \nabla\cdot\big(nS(x,n,c)\nabla c\big), \\ c_t + \mathbf{u}\cdot\nabla c = \Delta c - nf(c), \\ \mathbf{u}_t + \kappa(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla P = \Delta\mathbf{u} + n\nabla\phi, \\ \nabla\cdot\mathbf{u} = 0, \end{cases} \tag{1.5}$$

where S is a function with values in $\mathbb{R}^{N\times N}$. Models with rotational chemotaxis which additionally account for interaction with the surrounding fluid have only been studied very rudimentarily so far. For any $m>1$, Ishida [17] showed the global existence and boundedness of weak solutions to system (1.5). In his recent works [40, 41], Winkler further gave a complete analysis for the chemotaxis-Stokes system (1.5) with $m=1$ in the 2D and the 3D case. Very recently, Wang and Cao [33] obtained the existence of global solutions to system (1.5) with $m=1$ and some decay sensitivity S in the three dimension. We also refer to [2, 21] for the more related works on the fluid-free versions (i.e., $\mathbf{u}\equiv 0$) of system (1.5).

Keller–Segel–Stokes model. It is worth noticing that the results obtained so far indicate that in contrast to the standard Keller–Segel model (1.2), phenomena of finite-time blow-up, which represents maybe the most extreme facet of bacterial aggregation, do not occur for system (1.4) or system (1.5) involving *chemical signal consumption* even though the Stokes-fluid is included. Very recently, the second and third [35] authors established the global existence and boundedness of solutions to the following Keller–Segel–Stokes system in two or three dimensional bounded domains

$$\begin{cases} n_t+\mathbf{u}\cdot\nabla n=\Delta n-\nabla\cdot\big(nS(x,n,c)\nabla c\big), & x\in\Omega, t>0,\\ c_t+\mathbf{u}\cdot\nabla c=\Delta c-c+n, & x\in\Omega, t>0,\\ \mathbf{u}_t+\nabla P=\Delta\mathbf{u}+n\nabla\phi, & x\in\Omega, t>0,\\ \nabla\cdot\mathbf{u}=0, & x\in\Omega, t>0, \end{cases} \tag{1.6}$$

where the chemotactic sensitivity $S(x,n,c)$ is tensor-valued and satisfies

$$|S(x,n,c)|\le C_S(1+n)^{-\alpha} \tag{1.7}$$

with some constants $\alpha>0$ and $C_S>0$. In two dimensional case, the existence of global bounded solutions to the corresponding Keller–Segel-Navier–Stokes system has been established in [36].

Main results. Due to the possible blow-up in the classical Keller–Segel model, the assumption $\alpha>0$ in Equation (1.7) is crucial to ensure the global existence of solutions to system (1.6). The early literature have contained some evidence confirming the intuitive idea that the tendency toward blow-up can be weakened if the diffusion is enhanced. Thus an understanding of the competitive interaction among the Keller–Segel chemotaxis mechanism, the Stokes-fluid, the rotational sensitivity, and the nonlinear diffusion is an interesting topic. Motivated by the above works, we will establish the global existence and boundedness of weak solutions to the 2D Keller–Segel–Stokes system (1.1) in this paper. In order to specify the framework of our analysis, we specify the precise problem context by considering system (1.1) along with boundary conditions

$$(\nabla n^m-nS(x,n,c)\nabla c)\cdot\nu=\nabla c\cdot\nu=0,\quad \mathbf{u}=0,\qquad x\in\partial\Omega, t>0 \tag{1.8}$$

and the initial conditions

$$n(x,0)=n_0(x),\quad c(x,0)=c_0(x),\quad \mathbf{u}(x,0)=\mathbf{u}_0(x),\qquad x\in\partial\Omega. \tag{1.9}$$

We shall assume throughout that the initial data satisfy

$$\begin{cases} n_0\in C^0(\bar\Omega),\quad n_0\ge 0\quad \textit{and}\quad n_0\not\equiv 0\quad \text{in}\quad \bar\Omega,\\ c_0\in W^{1,\infty}(\Omega),\quad c_0\ge 0\quad \textit{and}\quad c_0\not\equiv 0\quad \text{in}\quad \bar\Omega,\\ \mathbf{u}_0\in D(A_r^\beta)\quad \text{for some } \beta\in(\frac{1}{2},1) \text{ and for all } r\in(1,\infty), \end{cases} \tag{1.10}$$

where A_r denotes the Stokes operator with domain $D(A_r):=W^{2,r}(\Omega)\cap W_0^{1,r}(\Omega)\cap L_\sigma^r(\Omega)$ and $L_\sigma^r(\Omega):=\{\mathbf{v}\in L^r(\Omega)|\nabla\cdot\mathbf{v}=0\}$ for all $r\in(1,\infty)$. As for the gravitational potential

ϕ in (1.1), we require that

$$\phi \in W^{1,\infty}(\Omega). \tag{1.11}$$

Moreover, we assume that the chemotactic sensitivity $S=(S_{ij})_{i,j\in\{1,2\}}$ satisfies

$$S_{ij}(x,n,c)\in C^2(\bar{\Omega}\times[0,\infty)\times[0,\infty)), \tag{1.12}$$

and that the Frobenius norm of S satisfies

$$|S(x,n,c)|\le C_S \tag{1.13}$$

for some positive constant C_S.

Under these assumptions, our main result on global existence and boundedness of weak solutions to system (1.1) is as follows.

THEOREM 1.1. *Let $m>1$ and $\Omega\subset\mathbb{R}^2$ be a bounded convex domain with smooth boundary $\partial\Omega$, and assume that Equations* (1.8)–(1.13) *hold. Then system* (1.1) *has a global-in-time weak solution $(n,c,\mathbf{u},P)$ which is uniformly bounded in the sense that*

$$\|n(\cdot,t)\|_{L^\infty(\Omega)}+\|c(\cdot,t)\|_{W^{1,\infty}(\Omega)}+\|\mathbf{u}(\cdot,t)\|_{W^{1,\infty}(\Omega)}\le C \qquad \text{for all} \quad t\in(0,\infty)$$

with some positive constant C.

Main difficulties. System (1.1) incorporates degenerate diffusion, fluid and rotational flux, which involves more complex cross-diffusion mechanisms and brings about many considerable mathematical difficulties. Firstly, due to $m>1$, Equation $(1.1)_1$ may be degenerate at $n=0$ and, in general, system (1.1) does not allow for classical solvability as the well-known porous medium equations. Thus we need to introduce the following definition of weak solutions to system (1.1).

DEFINITION 1.1. *Assume that $(n_0,c_0,\mathbf{u}_0)$ satisfy conditions* (1.10). *Then a triple of functions $(n,c,\mathbf{u})$ defined in $\Omega\times(0,\infty)$ is called a global weak solution of the initial-boundary value problem* (1.1), *if*

$$\begin{aligned}
&n\in L^\infty((0,\infty);L^\infty(\Omega)), \quad \nabla n^m\in L^2_{loc}((0,\infty);L^2(\Omega)),\\
&c\in L^\infty((0,\infty);W^{1,\infty}(\Omega)), \quad \text{and}\\
&\mathbf{u}\in L^\infty((0,\infty);W^{1,\infty}(\Omega)) \quad \text{such that } \nabla\cdot\mathbf{u}=0 \text{ in the distributional sense in } \Omega\times(0,\infty),
\end{aligned}$$

and for all $\varphi\in C_0^\infty\left(\bar{\Omega}\times[0,\infty)\right)$ and $\zeta\in C_0^\infty\left(\bar{\Omega}\times[0,\infty),\mathbb{R}^2\right)$ with $\nabla\cdot\zeta=0$, the following integral equalities hold:

$$\begin{aligned}
&\int_0^\infty\int_\Omega n\varphi_t dxdt+\int_\Omega n_0\varphi(\cdot,0)dx\\
&=\int_0^\infty\int_\Omega \nabla n^m\cdot\nabla\varphi dxdt-\int_0^\infty\int_\Omega n(S(x,n,c)\nabla c)\cdot\nabla\varphi dxdt-\int_0^\infty\int_\Omega n\mathbf{u}\cdot\nabla\varphi dxdt,
\end{aligned}$$

$$\begin{aligned}
&\int_0^\infty\int_\Omega c\varphi_t dxdt+\int_\Omega c_0\varphi(\cdot,0)dx\\
&=\int_0^\infty\int_\Omega \nabla c\cdot\nabla\varphi dxdt+\int_0^\infty\int_\Omega c\varphi dxdt-\int_0^\infty\int_\Omega n\varphi dxdt-\int_0^\infty\int_\Omega c\mathbf{u}\cdot\nabla\varphi dxdt,
\end{aligned}$$

and

$$\int_0^\infty \int_\Omega \mathbf{u}\cdot\zeta_t dxdt + \int_\Omega \mathbf{u}_0\cdot\zeta(\cdot,0)dx = \int_0^\infty \int_\Omega \nabla\mathbf{u}\cdot\nabla\zeta dxdt - \int_0^\infty \int_\Omega n\nabla\phi\cdot\zeta dxdt.$$

Secondly, the tensor-valued sensitivity functions result in new mathematical difficulties, mainly linked to the fact that a chemotaxis system with such rotational fluxes thereby loses an energy-like structure. Thirdly, unlike the *signal consumption* system (1.4) and system (1.5), we cannot gain the L^∞ estimates of c via the maximum principle directly. To overcome these difficulties, we will establish the existence of global bounded weak solutions by presenting several new *a priori* estimates. Our method is motivated by [34, 41].

The rest of this paper is organized as follows. In Section 2, we first establish the global existence and boundedness for the regularized system of (1.1). Then we will deal with the general case by an approximation procedure in Section 3.

Notation: Sometimes, we will use C,C_i to denote some uniform constants which may be different on different lines.

2. Non-degenerate problems

As mentioned in the introduction, system (1.1) is degenerate at $n=0$, which results in the failure of the classical parabolic regularity theory. On the other hand, the nonlinear boundary condition on n also brings about a great challenge to the study of system (1.1). To overcome these difficulties, we shall first consider the regularized version of system (1.1) in this section.

Let $\{\rho_\epsilon\}_{\epsilon\in(0,1)} \subset C_0^\infty(\Omega)$ be a family of standard cut-off functions satisfying $0\le\rho_\epsilon\le 1$ in Ω for all $\epsilon\in(0,1)$ and $\rho_\epsilon\to 1$ in Ω pointwisely as $\epsilon\to 0$. Then we can construct the following approximate sequence for S:

$$S_\epsilon(x,n,c)=\rho_\epsilon S(x,n,c), \qquad (x,n,c)\in\bar\Omega\times[0,\infty)\times[0,\infty).$$

For each $\epsilon\in(0,1)$, we introduce the following regularized problem:

$$\begin{cases} n_{\epsilon t}+\mathbf{u}_\epsilon\cdot\nabla n_\epsilon=\Delta(n_\epsilon+\epsilon)^m-\nabla\cdot\big(n_\epsilon S_\epsilon(x,n_\epsilon,c_\epsilon)\nabla c_\epsilon\big), & x\in\Omega,\ t>0,\\ c_{\epsilon t}+\mathbf{u}_\epsilon\cdot\nabla c_\epsilon=\Delta c_\epsilon-c_\epsilon+n_\epsilon, \quad x\in\Omega,\ t>0,\\ \mathbf{u}_{\epsilon t}+\nabla P_\epsilon=\Delta\mathbf{u}_\epsilon+n_\varepsilon\nabla\phi, \quad x\in\Omega,\ t>0,\\ \nabla\cdot\mathbf{u}_\epsilon=0, \quad x\in\Omega, t>0,\\ \dfrac{\partial n_\epsilon}{\partial\nu}=\dfrac{\partial c_\epsilon}{\partial\nu}=0, \quad \mathbf{u}_\epsilon=0, \quad x\in\partial\Omega,\ t>0,\\ n_\epsilon(x,0)=n_0(x), \quad c_\epsilon(x,0)=c_0(x), \quad \mathbf{u}_\epsilon(x,0)=\mathbf{u}_0(x), \quad x\in\Omega. \end{cases} \tag{2.1}$$

2.1. Local existence and mass conservation. In this subsection, we give the local existence of solutions to the regularized problem (2.1) and the mass conservation of cells.

Lemma 2.1 (Local existence for the regularized system). *Let $m>1$ and suppose that Equations (1.10)–(1.13) hold. Then there exist a maximal existence time T^* and a unique classical solution $(n_\epsilon,c_\epsilon,\mathbf{u}_\epsilon,P_\epsilon)$ to system (2.1) in $\bar\Omega\times(0,T^*)$ such that*

$$\begin{aligned} &n_\epsilon\in C^0\big(\bar\Omega\times[0,T^*)\big)\cap C^{2,1}\big(\bar\Omega\times(0,T^*)\big),\\ &c_\epsilon\in C^0\big(\bar\Omega\times[0,T^*)\big)\cap C^{2,1}\big(\bar\Omega\times(0,T^*)\big),\\ &\mathbf{u}_\epsilon\in C^0\big(\bar\Omega\times[0,T^*)\big)\cap C^{2,1}\big(\bar\Omega\times(0,T^*)\big),\\ &P_\epsilon\in C^{1,0}\big(\bar\Omega\times(0,T^*)\big). \end{aligned}$$

Moreover, we have $n_\epsilon > 0$ *and* $c_\epsilon > 0$ *in* $\bar{\Omega} \times (0, T^*)$*, and if* $T^* < \infty$*, then*

$$\lim_{t \to T^*} \|n_\epsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|c_\epsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|\mathbf{u}_\epsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} = \infty. \tag{2.2}$$

Proof. The proof of the local-in-time existence of the classical solution is based on the Schauder fixed point theorem (see e.g. [41]). Also, the proof of uniqueness is standard. We omit these details here. ∎

We next state the mass conservation property of n_ϵ. In the following, we use T^* to denote the maximal existence time of solution $(n_\epsilon, c_\epsilon, \mathbf{u}_\epsilon)$.

Lemma 2.2. *Let* $(n_\epsilon, c_\epsilon, \mathbf{u}_\epsilon)$ *be a classical solution to system* (2.1) *in* $\bar{\Omega} \times (0, T^*)$*. Then we have*

$$\|n_\epsilon(\cdot,t)\|_{L^1(\Omega)} = \|n_0(\cdot,t)\|_{L^1(\Omega)} \qquad \text{for any } t \in (0, T^*). \tag{2.3}$$

Proof. Indeed, Equation (2.3) can be obtained by taking an integration of Equation $(2.1)_1$ over Ω and using the nonnegativity of n_ϵ. ∎

2.2. Regularity of $\mathbf{u}_\epsilon$. In this subsection, we shall establish the $W^{1,r}$ regularity of $\mathbf{u}_\epsilon$. Before starting our analysis, we first briefly collect some known facts concerning the Stokes operator from [11] and [12].

For each $r \in (1, \infty)$, the Helmholtz projection $\mathcal{P}_r$ acts as an orthogonal projector from $L^r(\Omega)$ onto its subspace $L^r_\sigma(\Omega) := \{\mathbf{v} \in L^r(\Omega) \,|\, \nabla \cdot \mathbf{v} = 0\}$ of all solenoidal vector fields. The realization A_r of the Stokes operator A in $L^r_\sigma(\Omega)$ with domain $D(A_r) := W^{2,r}(\Omega) \cap W^{1,r}_0(\Omega) \cap L^r_\sigma(\Omega)$ is sectorial in $L^r_\sigma(\Omega)$ and possesses closed fractional powers A^β_r with dense domain for any $\beta \in \mathbb{R}$. Moreover, $(e^{-tA_r})_{t \geq 0}$ is an analytic semigroup in $L^r_\sigma(\Omega)$ generated by A_r. Notice that $\mathcal{P}_r$, A^β_r, and $(e^{-tA_r})_{t \geq 0}$ are actually independent of $r \in (1, \infty)$ whenever they are applied to smooth functions. Thus we will omit the subscript r in $\mathcal{P}_r$, A^β_r and $(e^{-tA_r})_{t \geq 0}$ whenever there is no danger of confusion.

The following basic conclusion plays an important role in our estimates due to the appearance of Stokes-fluids, which can be obtained by a direct modification of its three-dimensional version in Winkler [41, Lemma 3.3].

Lemma 2.3. *Suppose that* $1 \leq p < p_0 < \infty$*, and that* $\delta \in (0,1)$ *satisfying* $\delta > \frac{1}{p} - \frac{1}{p_0}$*. Then there exists a positive constant* C *such that*

$$\|A^{-\delta} \mathcal{P} \psi\|_{L^{p_0}(\Omega)} \leq C \|\psi\|_{L^p(\Omega)} \qquad \text{for all } \psi \in C^\infty_0(\Omega). \tag{2.4}$$

Consequently, the operator $A^{-\delta}\mathcal{P}$ *possesses a unique extension to all of* $L^{p_0}(\Omega)$ *with norm controlled according to estimate* (2.4).

As the first application of the estimate (2.4), we have the following L^p estimate for $\mathbf{u}_\epsilon$ (see [39, Lemma 3.1(i)] or [34, Lemma 2.4]).

Lemma 2.4. *Let* $(n_\epsilon, c_\epsilon, \mathbf{u}_\epsilon)$ *be a classical solution to system* (2.1) *in* $\bar{\Omega} \times (0, T^*)$ *such that assumptions* (1.11)–(1.13) *hold. Then, for any* $p \in (1, +\infty)$*, there exists a positive constant* $C = C(p, \mathbf{u}_0, n_0, \phi)$ *such that*

$$\|\mathbf{u}_\epsilon(\cdot,t)\|_{L^p(\Omega)} \leq C \qquad \text{for all } t \in (0, T^*).$$

Another application of estimate (2.4) is the following $W^{1,r}$ estimate of $\mathbf{u}_\epsilon$, whose proof is similar to that of its three-dimensional version (see [41, Corollary 3.4] or [34, Lemma 2.5]).

LEMMA 2.5. *Suppose that* $(n_\epsilon, c_\epsilon, \mathbf{u}_\epsilon)$ *is a classical solution to system* (2.1) *in* $\bar{\Omega}\times(0,T^*)$ *and that assumptions* (1.11)–(1.13) *hold. Let* $p\in[1,\infty)$ *and* $r\in[1,\infty]$ *be such that*

$$\begin{cases} r<\dfrac{2p}{2-p}, & p\le 2,\\ r\le\infty, & p>2. \end{cases}$$

Then for all $K>0$ *there exists* $C=C(p,r,K,\mathbf{u}_0,\phi)$ *such that if for* $T>0$ *we have*

$$\|n_\epsilon(\cdot,t)\|_{L^p(\Omega)}\le K \qquad \text{for all } t\in(0,T),$$

then

$$\|D\mathbf{u}_\epsilon(\cdot,t)\|_{L^r(\Omega)}\le C \qquad \text{for all } t\in(0,T).$$

2.3. Estimates for c_ϵ. In this subsection, we will give some estimates for c_ϵ. The first one is an L^p-estimate, which can be proved by using the regularity information on n_ϵ and $\mathbf{u}_\epsilon$ obtained so far (see Lemma 2.6 in [34]).

LEMMA 2.6. *Let* $(n_\epsilon, c_\epsilon, \mathbf{u}_\epsilon)$ *be a classical solution to system* (2.1) *in* $\bar{\Omega}\times(0,T^*)$ *and the assumptions* (1.11)–(1.13) *hold. Then, for any* $p\in[1,\infty)$, *there exists a positive constant* C *depending only on* $p, n_0, c_0, \mathbf{u}_0$ *such that*

$$\|c_\epsilon(\cdot,t)\|_{L^p(\Omega)}\le C \qquad \text{for all } t\in(0,T^*).$$

The following conclusion is an interpolation inequality for C^2 functions on $\bar{\Omega}$, which has been proved by [34] (see Lemma 2.7 there).

LEMMA 2.7. *Suppose that* $\Omega\subset\mathbb{R}^2$ *is a bounded domain with smooth boundary. Let* $q>1$, $\gamma>1$ *and* $\rho\ge\frac{2\gamma}{2\gamma-1}(q+1)$. *Then there exists a constant* $C=C(q,\gamma,\rho)>0$ *such that the inequality*

$$\|\nabla c\|_{L^\rho(\Omega)}^\rho\le C\left\||\nabla c|^{q-1}|D^2c|\right\|_{L^2(\Omega)}^{\frac{(2\gamma-1)\rho-2\gamma}{q(2\gamma-1)}}\left(1+\|c\|_{L^{\frac{2\gamma}{\gamma-1}}(\Omega)}^{\frac{\rho}{q+1}}\right)+C\|c\|_{L^{\frac{2\gamma}{\gamma-1}}(\Omega)}^{\frac{\rho}{q+1}}$$

holds for any $c\in C^2(\bar{\Omega})$ *satisfying* $c\frac{\partial c}{\partial\nu}=0$ *on* $\partial\Omega$, *where* D^2c *denotes the Hessian of* c.

As a direct application of Lemma 2.6 and Lemma 2.7, we have the following corollary, which will be used in the sequel to obtain the estimates of ∇c_ϵ.

COROLLARY 2.1. *Suppose that* $(n_\epsilon, c_\epsilon, \mathbf{u}_\epsilon)$ *is a classical solution to system* (2.1) *in* $\bar{\Omega}\times(0,T^*)$ *and that assumptions* (1.11)–(1.13) *hold. Let* $q>1$, $\gamma>1$, *and* $\rho\ge\frac{2\gamma}{2\gamma-1}(q+1)$. *Then there exists a positive constant* $C=C(q,\gamma,\rho)$ *such that*

$$\|\nabla c_\epsilon\|_{L^\rho(\Omega)}^\rho\le C\left(\int_\Omega|\nabla c_\epsilon|^{q-1}|D^2c_\epsilon|dx\right)^{\frac{(2\gamma-1)\rho-2\gamma}{q(2\gamma-1)}}+C \qquad \text{for all } t\in(0,T^*). \tag{2.5}$$

2.4. A coupled estimate for $\|n_\epsilon(\cdot,t)\|_{L^k(\Omega)}$ and $\|\nabla c_\epsilon(\cdot,t)\|_{L^{2q}(\Omega)}$. The goal of this subsection is to establish a coupled estimate for $\|n_\epsilon(\cdot,t)\|_{L^k(\Omega)}$ and $\|\nabla c_\epsilon(\cdot,t)\|_{L^{2q}(\Omega)}$. To this end, we first need to choose some parameters appropriately.

LEMMA 2.8. *Let $m>1$. Then, for any k and q satisfying $\max\{m,\frac{k+2}{m}\}<q<k$, there exist some $\gamma>1$, $\zeta>1$, and $\mu>1$ such that*

$$\frac{k-m+\frac{1}{\theta}}{m+k-1}+\frac{(2\gamma-1)\theta-\gamma}{(2\gamma-1)q\theta}<1, \tag{2.6}$$

where $\theta:=\frac{\gamma}{2\gamma-1}(q+1)$,

$$(q-1)\mu>\frac{\zeta(q+1)}{2\zeta-1} \tag{2.7}$$

and

$$\frac{1+\frac{1}{\mu}}{m+k-1}+\frac{(2\zeta-1)(q-1)\mu-\zeta}{(2\zeta-1)q\mu}<1. \tag{2.8}$$

Proof. We first fix $\zeta>1$ and then take μ large enough such that Equation (2.7) holds. For Equations (2.6) and (2.8), we notice that they are equivalent to

$$k+2-\frac{1}{\gamma}<(2m-1)q+m \quad \text{and} \quad \frac{1}{m+k-1}+\frac{1}{(m+k-1)\mu}<\frac{\zeta}{(2\zeta-1)q\mu}+\frac{1}{q}, \tag{2.9}$$

respectively. Due to $q>\frac{k+2}{m}$, it is clear that the first inequality of (2.9) is true for any $\gamma>1$. On the other hand, to show the second inequality of (2.9), it is enough to further take μ large enough by $q<k$. This completes the proof of Lemma 2.8. □

LEMMA 2.9. *Let $\Omega\subset\mathbb{R}^2$ be a bounded convex domain with smooth boundary and $m>1$. Suppose that $(n_\epsilon,c_\epsilon,\mathbf{u}_\epsilon)$ is a classical solution to system (2.1) in $\bar\Omega\times(0,T^*)$ and that assumptions (1.11)–(1.13) hold. Then, for any k and q satisfying $\max\{m,\frac{k+2}{m}\}<q<k$, there exists a positive constant C independent of ϵ such that*

$$\|n_\epsilon(\cdot,t)\|_{L^k(\Omega)}\le C \qquad \text{for all} \quad t\in(0,T^*) \tag{2.10}$$

and

$$\|\nabla c_\epsilon(\cdot,t)\|_{L^{2q}(\Omega)}\le C \qquad \text{for all} \quad t\in(0,T^*). \tag{2.11}$$

Proof. Due to the boundedness of Ω and Hölder's inequality, we only need to pay attention to the case that k and q are large enough. We will divide the proof into several steps.

Step 1. Estimates for n_ϵ. Multiplying Equation $(2.1)_1$ by $k(n_\epsilon+\epsilon)^{k-1}$ and integrating over Ω, we have

$$\begin{aligned}&\frac{d}{dt}\int_\Omega(n_\epsilon+\epsilon)^k dx+\int_\Omega \mathbf{u}_\epsilon\cdot\nabla(n_\epsilon+\epsilon)^k dx-k\int(n_\epsilon+\epsilon)^{k-1}\Delta(n_\epsilon+\epsilon)^m dx\\ =&-k\int_\Omega(n_\epsilon+\epsilon)^{k-1}\nabla\cdot\big(n_\epsilon S_\epsilon(x,n_\epsilon,c_\epsilon)\nabla c_\epsilon\big)dx.\end{aligned}$$

It then follows from the integration by parts and the equation enforcing that $\mathbf{u}_\epsilon$ is divergence free that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega (n_\epsilon+\epsilon)^k dx + mk(k-1)\int_\Omega (n_\epsilon+\epsilon)^{m+k-3}|\nabla n_\epsilon|^2 dx\\ =&k(k-1)\int_\Omega (n_\epsilon+\epsilon)^{k-2} n_\epsilon \nabla n_\epsilon\cdot\big(S_\epsilon(x,n_\epsilon,c_\epsilon)\nabla c_\epsilon\big)dx\end{aligned}\tag{2.12}$$

for all $t\in(0,T^*)$. Notice that Equation (1.13) and Young's inequality yield

$$\begin{aligned}&\int_\Omega (n_\epsilon+\epsilon)^{k-2} n_\epsilon \nabla n_\epsilon\cdot\big(S_\epsilon(x,n_\epsilon,c_\epsilon)\nabla c_\epsilon\big)dx\\ \le& C_S\int_\Omega (n_\epsilon+\epsilon)^{k-1}|\nabla n_\epsilon||\nabla c_\epsilon|dx\\ \le&\frac{m}{2}\int_\Omega (n_\epsilon+\epsilon)^{k+m-3}|\nabla n_\epsilon|^2 dx+\frac{C_S^2}{2m}\int_\Omega (n_\epsilon+\epsilon)^{k+1-m}|\nabla c_\epsilon|^2 dx,\end{aligned}$$

which together with Equation (2.12) gives

$$\begin{aligned}&\frac{d}{dt}\int_\Omega (n_\epsilon+\epsilon)^k dx + \frac{mk(k-1)}{2}\int_\Omega (n_\epsilon+\epsilon)^{m+k-3}|\nabla n_\epsilon|^2 dx\\ \le&\frac{k(k-1)C_S^2}{2m}\int_\Omega (n_\epsilon+\epsilon)^{k+1-m}|\nabla c_\epsilon|^2 dx\end{aligned}\tag{2.13}$$

for all $t\in(0,T^*)$.

Step 2. Estimates for ∇c_ϵ. Applying ∇ to Equation $(2.1)_2$ and then multiplying the resulting equation by $2q|\nabla c_\epsilon|^{2q-2}\nabla c_\epsilon$, we have

$$\begin{aligned}&\frac{d}{dt}\int_\Omega |\nabla c_\epsilon|^{2q}dx-2q\int_\Omega|\nabla c_\epsilon|^{2(q-1)}\nabla c_\epsilon\cdot\Delta\nabla c_\epsilon dx+2q\int_\Omega|\nabla c_\epsilon|^{2q}dx\\ =&2q\int_\Omega|\nabla c_\epsilon|^{2(q-1)}\nabla c_\epsilon\cdot\nabla n_\epsilon dx-2q\int_\Omega|\nabla c_\epsilon|^{2(q-1)}\nabla c_\epsilon\cdot\nabla\big(\mathbf{u}_\epsilon\cdot\nabla c_\epsilon\big)dx\end{aligned}$$

for all $t\in(0,T^*)$. Noticing the point-wise identity $2\nabla c\cdot\nabla\Delta c=\Delta|\nabla c|^2-2|D^2c|^2$ and using the integration by parts, we deduce

$$\begin{aligned}&\frac{d}{dt}\int_\Omega |\nabla c_\epsilon|^{2q}dx+q(q-1)\int_\Omega|\nabla c_\epsilon|^{2(q-2)}\big|\nabla|\nabla c_\epsilon|^2\big|^2dx\\ &+2q\int_\Omega|\nabla c_\epsilon|^{2(q-1)}|D^2c_\epsilon|^2dx+2q\int_\Omega|\nabla c_\epsilon|^{2q}dx\\ =&2q\int_\Omega|\nabla c_\epsilon|^{2(q-1)}\nabla n_\epsilon\cdot\nabla c_\epsilon dx+2q(q-1)\int_\Omega(\mathbf{u}_\epsilon\cdot\nabla c_\epsilon)|\nabla c_\epsilon|^{2(q-2)}\nabla c_\epsilon\cdot\nabla|\nabla c_\epsilon|^2dx\\ &+2q\int_\Omega(\mathbf{u}_\epsilon\cdot\nabla c_\epsilon)|\nabla c_\epsilon|^{2(q-1)}\Delta c_\epsilon dx+q\int_{\partial\Omega}|\nabla c_\epsilon|^{2(q-1)}\frac{\partial|\nabla c_\epsilon|^2}{\partial\nu}dx.\end{aligned}\tag{2.14}$$

For the first term on the right-hand side of Equation (2.14), due to $|\Delta c_\epsilon|^2\le 2|D^2c_\epsilon|^2$, we can use the integration by parts and Young's inequality to obtain

$$\begin{aligned}
&2q\int_\Omega |\nabla c_\epsilon|^{2(q-1)}\nabla n_\epsilon\cdot\nabla c_\epsilon dx\\
=&-2q\int_\Omega |\nabla c_\epsilon|^{2(q-1)}n_\epsilon\Delta c_\epsilon dx-2q(q-1)\int_\Omega|\nabla c_\epsilon|^{2(q-2)}n_\epsilon\nabla c_\epsilon\cdot\nabla|\nabla c_\epsilon|^2dx\\
\le&2\sqrt{2}q\int_\Omega|\nabla c_\epsilon|^{2(q-1)}n_\epsilon|D^2c_\epsilon|dx+2q(q-1)\int_\Omega|\nabla c_\epsilon|^{2q-3}n_\epsilon\left|\nabla|\nabla c_\epsilon|^2\right|dx\\
\le&q\int_\Omega|\nabla c_\epsilon|^{2(q-1)}|D^2c_\epsilon|^2dx+\frac{q(q-1)}{2}\int_\Omega|\nabla c_\epsilon|^{2(q-2)}\left|\nabla|\nabla c_\epsilon|^2\right|^2dx\\
&\quad+2q^2\int_\Omega|\nabla c_\epsilon|^{2(q-1)}n_\epsilon^2dx.
\end{aligned}\tag{2.15}$$

Similarly, we have

$$\begin{aligned}
&2q(q-1)\int_\Omega(\mathbf{u}_\epsilon\cdot\nabla c_\epsilon)\cdot|\nabla c_\epsilon|^{2(q-2)}\nabla c_\epsilon\cdot\nabla|\nabla c_\epsilon|^2dx\\
\le&\frac{q(q-1)}{4}\int_\Omega|\nabla c_\epsilon|^{2(q-2)}\left|\nabla|\nabla c_\epsilon|^2\right|^2dx+4q(q-1)\int_\Omega|\mathbf{u}_\epsilon\cdot\nabla c_\epsilon|^2|\nabla c_\epsilon|^{2(q-1)}\\
\le&\frac{q(q-1)}{4}\int_\Omega|\nabla c_\epsilon|^{2(q-2)}\left|\nabla|\nabla c_\epsilon|^2\right|^2dx+4q(q-1)\int_\Omega|\mathbf{u}_\epsilon|^2|\nabla c_\epsilon|^{2q}dx
\end{aligned}\tag{2.16}$$

and

$$\begin{aligned}
&2q\int_\Omega(\mathbf{u}_\epsilon\cdot\nabla c_\epsilon)|\nabla c_\epsilon|^{2(q-1)}\Delta c_\epsilon dx\\
\le&\frac{q}{2}\int_\Omega|\nabla c_\epsilon|^{2(q-1)}|D^2c_\epsilon|^2dx+4q\int_\Omega|\mathbf{u}_\epsilon|^2|\nabla c_\epsilon|^{2q}dx.
\end{aligned}\tag{2.17}$$

Since Ω is convex and $\frac{\partial c_\epsilon}{\partial\nu}=0$ on $\partial\Omega$, it follows from [28, Lemma 3.2] that $\frac{\partial|\nabla c_\epsilon|^2}{\partial\nu}\le 0$ on $\partial\Omega$, which implies that

$$q\int_{\partial\Omega}|\nabla c_\epsilon|^{2(q-1)}\frac{\partial|\nabla c_\epsilon|^2}{\partial\nu}dx\le 0.\tag{2.18}$$

We remark that this is the only place where the convexity is needed.

Substituting Equations (2.15)–(2.18) into Equation (2.14), we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega|\nabla c_\epsilon|^{2q}dx+\frac{q-1}{q}\int_\Omega\left|\nabla|\nabla c_\epsilon|^q\right|^2dx+\frac{q}{2}\int_\Omega|\nabla c_\epsilon|^{2(q-1)}|D^2c_\epsilon|^2dx+2q\int_\Omega|\nabla c_\epsilon|^{2q}dx\\
\le&2q^2\int_\Omega|\nabla c_\epsilon|^{2(q-1)}n_\epsilon^2dx+4q^2\int_\Omega|\nabla c_\epsilon|^{2q}|\mathbf{u}_\epsilon|^2dx
\end{aligned}\tag{2.19}$$

for all $t\in(0,T^*)$. Here we used the identity $\frac{1}{4}|\nabla c_\epsilon|^{2q-4}\left|\nabla|\nabla c_\epsilon|^2\right|^2=\frac{1}{q^2}\left|\nabla|\nabla c_\epsilon|^q\right|^2$.

Step 3. Coupled estimates for n_ϵ and ∇c_ϵ. Combining Equations (2.13) and (2.19), we have

$$\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega(n_\epsilon+\epsilon)^kdx+\int_\Omega|\nabla c_\epsilon|^{2q}dx\right)+\frac{2mk(k-1)}{(m+k-1)^2}\int_\Omega|\nabla(n_\epsilon+\epsilon)^{\frac{m+k-1}{2}}|^2dx\\
&\quad+\frac{q-1}{q}\int_\Omega\left|\nabla|\nabla c_\epsilon|^q\right|^2dx+\frac{q}{2}\int_\Omega|\nabla c_\epsilon|^{2(q-1)}|D^2c_\epsilon|^2dx+2q\int_\Omega|\nabla c_\epsilon|^{2q}dx
\end{aligned}$$

$$\leq \frac{k(k-1)C_S^2}{2m}\int_\Omega (n_\epsilon+\epsilon)^{k+1-m}|\nabla c_\epsilon|^2dx+2q^2\int_\Omega|\nabla c_\epsilon|^{2(q-1)}n_\epsilon^2dx+4q^2\int_\Omega|\nabla c_\epsilon|^{2q}|\mathbf{u}_\epsilon|^2dx$$
$$:=I_1+I_2+I_3. \tag{2.20}$$

We shall show that I_1, I_2, and I_3 can be bounded by the LHS of (2.20) one by one. For this purpose, we first let $\gamma>1$, $\zeta>1$, and $\mu>1$ be determined by Lemma 2.8.

For I_1, it follows from Hölder's inequality that

$$I_1\leq \frac{k(k-1)C_S^2}{2m}\left\|(n_\epsilon+\epsilon)^{\frac{m+k-1}{2}}\right\|^{\frac{2(k+1-m)}{m+k-1}}_{L^{\frac{2(k+1-m)\theta'}{m+k-1}}(\Omega)}\|\nabla c_\epsilon\|^2_{L^{2\theta}(\Omega)}, \tag{2.21}$$

where $\theta=\frac{\gamma}{2\gamma-1}(q+1)$ and $\frac{1}{\theta'}+\frac{1}{\theta}=1$. By $k>m$, we see

$$\frac{2(k+1-m)\theta'}{m+k-1}>\frac{2}{m+k-1},$$

which together with the Gagliardo–Nirenberg inequality yields

$$\begin{aligned}
&\left\|(n_\epsilon+\epsilon)^{\frac{m+k-1}{2}}\right\|^{\frac{2(k+1-m)}{m+k-1}}_{L^{\frac{2(k+1-m)\theta'}{m+k-1}}(\Omega)}\\
\leq &C_1\left\|\nabla(n_\epsilon+\epsilon)^{\frac{m+k-1}{2}}\right\|^{\frac{2(k+1-m)\kappa_1}{m+k-1}}_{L^2(\Omega)}\left\|(n_\epsilon+\epsilon)^{\frac{m+k-1}{2}}\right\|^{\frac{2(k+1-m)(1-\kappa_1)}{m+k-1}}_{L^{\frac{2}{m+k-1}}(\Omega)}\\
&+C_1\left\|(n_\epsilon+\epsilon)^{\frac{m+k-1}{2}}\right\|^{\frac{2(k+1-m)}{m+k-1}}_{L^{\frac{2}{m+k-1}}(\Omega)}\\
=&C_1\left\|\nabla(n_\epsilon+\epsilon)^{\frac{m+k-1}{2}}\right\|^{\frac{2(k+1-m)\kappa_1}{m+k-1}}_{L^2(\Omega)}\|(n_\epsilon+\epsilon)\|^{(k+1-m)(1-\kappa_1)}_{L^1(\Omega)}+C_1\|(n_\epsilon+\epsilon)\|^{k+1-m}_{L^1(\Omega)},
\end{aligned}$$

where $\kappa_1=1-\frac{1}{(k+1-m)\theta'}$ and C_1 is a positive constant independent of ϵ. Thus by the mass conservation property (2.3) and the fact that $\epsilon<1$, we obtain

$$\left\|(n_\epsilon+\epsilon)^{\frac{m+k-1}{2}}\right\|^{\frac{2(k+1-m)}{m+k-1}}_{L^{\frac{2(k+1-m)\theta'}{m+k-1}}(\Omega)}\leq C_2\left\|\nabla(n_\epsilon+\epsilon)^{\frac{m+k-1}{2}}\right\|^{\frac{2(k+1-m)\kappa_1}{m+k-1}}_{L^2(\Omega)}+C_2, \tag{2.22}$$

where C_2 is a positive constant independent of ϵ. To estimate the factor related to ∇c_ϵ in Equation (2.21), we use Equation (2.5) with $\rho=2\theta$ to obtain

$$\|\nabla c_\epsilon\|^2_{L^{2\theta}(\Omega)}\leq C_3\left\||\nabla c_\epsilon|^{q-1}D^2n_\epsilon\right\|^{\frac{2\theta(2\gamma-1)-2\gamma}{q\theta(2\gamma-1)}}_{L^2(\Omega)}+C_3 \tag{2.23}$$

for some positive constant C_3 independent of ϵ. Set

$$a_1:=\frac{2(k+1-m)\kappa_1}{m+k-1}=\frac{2(k-m+\frac{1}{\theta})}{m+k-1}\qquad\text{and}\qquad b_1:=\frac{2\theta(2\gamma-1)-2\gamma}{q\theta(2\gamma-1)}.$$

It then follows from Equation (2.6) that $a_1+b_1<2$. Thus, by Equations (2.21)–(2.23) and Young's inequality, we can deduce that

$$\begin{aligned}
I_1\leq&\frac{k(k-1)C_S^2}{2m}C_2C_3\left(\left\|\nabla(n_\epsilon+\epsilon)^{\frac{m+k-1}{2}}\right\|^{a_1}_{L^2(\Omega)}+1\right)\left(\left\||\nabla c_\epsilon|^{q-1}D^2c\right\|^{b_1}_{L^2(\Omega)}+1\right)\\
\leq&\frac{mk(k-1)}{2(m+k-1)^2}\int_\Omega\left|\nabla(n_\epsilon+\epsilon)^{\frac{m+k-1}{2}}\right|^2dx+\frac{q}{8}\int_\Omega|\nabla c_\epsilon|^{2(q-1)}|D^2c_\epsilon|^2dx+C_4
\end{aligned} \tag{2.24}$$

for all $t\in(0,T^*)$, where C_4 is a positive constant independent of ϵ.

The term I_2 can be similarly dealt with. We give a sketch for completeness. Let $\frac{1}{\mu'}+\frac{1}{\mu}=1$. Then, by Hölder's inequality, we have

$$I_2\le 2q^2\left\|(n_\epsilon+\epsilon)^{\frac{m+k-1}{2}}\right\|_{L^{\frac{4\mu'}{m+k-1}}(\Omega)}^{\frac{4}{m+k-1}}\|\nabla c_\epsilon\|_{L^{2(q-1)\mu}(\Omega)}^{2(q-1)}. \tag{2.25}$$

Similarly to Equation (2.22), we can deduce that

$$\left\|(n_\epsilon+\epsilon)^{\frac{m+k-1}{2}}\right\|_{L^{\frac{4\mu'}{m+k-1}}(\Omega)}^{\frac{4}{m+k-1}}\le C_5\left\|\nabla(n_\epsilon+\epsilon)^{\frac{m+k-1}{2}}\right\|_{L^2(\Omega)}^{\frac{4\kappa_2}{m+k-1}}+C_5 \tag{2.26}$$

for some positive constant C_5 independent of ϵ, where $\kappa_2=1-\frac{1}{2\mu'}$.

Due to Equation (2.7), we can use Equation (2.5) with $\rho=2(q-1)\mu$ and $\gamma=\zeta$ to find a positive constant C_6 independent of ϵ such that

$$\|\nabla c_\epsilon\|_{L^{2(q-1)\mu}(\Omega)}^{2(q-1)}\le C_6\left\||\nabla c_\epsilon|^{q-1}D^2 n_\epsilon\right\|_{L^2(\Omega)}^{\frac{2\mu(2\zeta-1)(q-1)-2\zeta}{q\mu(2\zeta-1)}}+C_6. \tag{2.27}$$

Denote

$$a_2:=\frac{4\kappa_2}{m+k-1}=\frac{4-\frac{2}{\mu'}}{m+k-1},\qquad \text{and}\qquad b_2:=\frac{2\mu(2\zeta-1)(q-1)-2\zeta}{q\mu(2\zeta-1)}.$$

Then Equation (2.8) implies that $a_2+b_2<2$, which enables us to use Equations (2.25)–(2.27) and Young's inequality to obtain

$$\begin{aligned}I_2\le&2q^2\left(C_5\left\|\nabla(n_\epsilon+\epsilon)^{\frac{m+k-1}{2}}\right\|_{L^2(\Omega)}^{a_2}+C_5\right)\left(C_6\left\||\nabla c_\epsilon|^{q-1}D^2c\right\|_{L^2(\Omega)}^{b_2}+C_6\right)\\ \le&\frac{mk(k-1)}{2(m+k-1)^2}\int_\Omega|\nabla(n_\epsilon+\epsilon)^{\frac{m+k-1}{2}}|^2dx+\frac{q}{8}\int_\Omega|\nabla c_\epsilon|^{2(q-1)}|D^2c_\epsilon|^2dx+C_7\end{aligned} \tag{2.28}$$

for all $t\in(0,T^*)$, where C_7 is a positive constant independent of ϵ.

We now turn to the estimate of I_3. We first fix an $\alpha\in(1,\infty)$. By using Hölder's inequality and Lemma 2.4, we have

$$\begin{aligned}\int_\Omega|\nabla c_\epsilon|^{2q}|\mathbf{u}_\epsilon|^2dx\le&\left\||\nabla c_\epsilon|^{2q}\right\|_{L^\alpha(\Omega)}\left\||\mathbf{u}_\epsilon|^2\right\|_{L^{\frac{\alpha}{\alpha-1}}(\Omega)}\\ \le&C_8\left\||\nabla c_\epsilon|\right\|_{L^{2q\alpha}(\Omega)}^{2q}\left\||\mathbf{u}_\epsilon|\right\|_{L^{\frac{2\alpha}{\alpha-1}}(\Omega)}^{2}\\ \le&C_9\left\||\nabla c_\epsilon|\right\|_{L^{2q\alpha}(\Omega)}^{2q},\end{aligned} \tag{2.29}$$

where C_8 and C_9 are two positive constants independent of ϵ. If we choose ξ such that $\xi\ge\frac{q\alpha}{2q\alpha-q-1}$, then we have $2q\alpha\ge\frac{2\xi}{2\xi-1}(q+1)$. Thus, we can use Equation (2.5) with $\rho=2q\alpha$ and $\gamma=\xi$ to find two positive constants C_{10} and C_{11} independent of ϵ such that

$$\begin{aligned}C_9\left\||\nabla c_\epsilon|\right\|_{L^{2q\alpha}(\Omega)}^{2q}\le&C_{10}\left(\int_\Omega|\nabla c_\epsilon|^{q-1}|D^2c_\epsilon|dx\right)^{\frac{2(2\xi-1)q\alpha-2\xi}{(2\xi-1)q\alpha}}+C_{10}\\ \le&C_{11}\left(\int_\Omega|\nabla c_\epsilon|^{2(q-1)}|D^2c_\epsilon|^2dx\right)^{\frac{(2\xi-1)q\alpha-\xi}{(2\xi-1)q\alpha}}+C_{10},\end{aligned}$$

which together with Equation (2.29) and Young's inequality gives

$$I_3 \le \frac{q}{4}\int_\Omega |\nabla c_\epsilon|^{2(q-1)}|D^2 c_\epsilon|^2 dx + C_{12} \tag{2.30}$$

for all $t\in(0,T^*)$, where C_{12} is a positive constant independent of ϵ.

Substituting Equations (2.24), (2.28), and (2.30) into (2.20), we can deduce

$$\begin{aligned}&\frac{d}{dt}\left(\int_\Omega (n_\epsilon+\epsilon)^k dx + \int_\Omega |\nabla c_\epsilon|^{2q}dx\right) + \frac{mk(k-1)}{(m+k-1)^2}\int_\Omega |\nabla (n_\epsilon+\epsilon)^{\frac{m+k-1}{2}}|^2 dx\\ &\quad + \frac{q-1}{q}\int_\Omega |\nabla|\nabla c_\epsilon|^q|^2 dx + 2q\int_\Omega |\nabla c_\epsilon|^{2q}dx\\ &\le C_{13}\end{aligned} \tag{2.31}$$

for all $t\in(0,T^*)$, where C_{13} is a positive constant independent of ϵ.

Step 4. Conclusion of the estimates for n_ϵ and ∇c_ϵ. It is clear that we can obtain a bound for n_ϵ and ∇c_ϵ by taking a time integral on both sides of Equation (2.31). However, such a bound will increase with the time t. Indeed, we can employ the ODE comparison argument to improve this bound. Firstly, it follows from the interpolation inequality and the mass conservation that

$$\begin{aligned}\int_\Omega (n_\epsilon+\epsilon)^k dx =& \|(n_\epsilon+\epsilon)^{\frac{m+k-1}{2}}\|^{\frac{2k}{m+k-1}}_{L^{\frac{2k}{m+k-1}}(\Omega)}\\ \le& C_{14}\|\nabla (n_\epsilon+\epsilon)^{\frac{m+k-1}{2}}\|^{\frac{2k\lambda_1}{m+k-1}}_{L^2(\Omega)}\|(n_\epsilon+\epsilon)^{\frac{m+k-1}{2}}\|^{\frac{2k(1-\lambda_1)}{m+k-1}}_{L^{\frac{2}{m+k-1}}(\Omega)}\\ &+C_{14}\|(n_\epsilon+\epsilon)^{\frac{m+k-1}{2}}\|^{\frac{2k}{m+k-1}}_{L^{\frac{2}{m+k-1}}(\Omega)}\\ \le& C_{15}\|\nabla (n_\epsilon+\epsilon)^{\frac{m+k-1}{2}}\|^{\frac{2k\lambda_1}{m+k-1}}_{L^2(\Omega)} + C_{15}\end{aligned}$$

for all $t\in(0,T^*)$, where $\lambda_1 = 1-\frac{1}{k}\in(0,1)$, and where C_{14} and C_{15} are two positive constants independent of ϵ. Then we can use Young's inequality to obtain

$$\int_\Omega (n_\epsilon+\epsilon)^k dx \le \frac{mk(k-1)}{(m+k-1)^2}\int_\Omega |\nabla (n_\epsilon+\epsilon)^{\frac{m+k-1}{2}}|^2 dx + C_{16} \tag{2.32}$$

for all $t\in(0,T^*)$, where C_{16} is a positive constant independent of ϵ. Next, we substitute Equation (2.32) into Equation (2.31) to conclude that

$$\frac{d}{dt}\left(\int_\Omega (n_\epsilon+\epsilon)^k dx + \int_\Omega |\nabla c_\epsilon|^{2q}dx\right) + \left(\int_\Omega (n_\epsilon+\epsilon)^k dx + \int_\Omega |\nabla c_\epsilon|^{2q}dx\right) \le C_{17}$$

for all $t\in(0,T^*)$, where C_{17} is a positive constant independent of ϵ. Thus a standard ODE comparison argument shows that

$$\int_\Omega (n_\epsilon+\epsilon)^k dx + \int_\Omega |\nabla c_\epsilon|^{2q}dx \le \max\left\{\int_\Omega (n_0+1)^k dx + \int_\Omega |\nabla c_0|^{2q}dx, C_{17}\right\}$$

for all $t\in(0,T^*)$. This gives the desired estimates (2.10) and (2.11) and thus we have completed the proof of Lemma 2.9. □

2.5. Global existence and boundedness for the regularized problem. In this subsection, we shall establish the global existence and boundedness of solutions to the regularized problem (2.1). For this purpose, we first increase the integrability in lemmas 2.4, 2.5, 2.6 and 2.9, which can be done by means of a Moser-type iteration in conjunction with standard parabolic regularity arguments.

LEMMA 2.10. *Let $m>1$, $\epsilon\in(0,1)$, and $\Omega\subset\mathbb{R}^2$ be a bounded convex domain with smooth boundary. Suppose that $(n_\epsilon,c_\epsilon,\mathbf{u}_\epsilon)$ is a classical solution to system (2.1) in $\bar\Omega\times(0,T^*)$ and that the assumptions (1.11)–(1.13) hold. Then there exists a positive constant C such that for all $\epsilon\in(0,1)$, it holds that*

$$\|n_\epsilon(\cdot,t)\|_{L^\infty(\Omega)}\le C \qquad \text{for all} \quad t\in(0,T^*) \tag{2.33}$$

and

$$\|c_\epsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)}\le C \qquad \text{for all} \quad t\in(0,T^*), \tag{2.34}$$

as well as

$$\|\mathbf{u}_\epsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)}\le C \qquad \text{for all} \quad t\in(0,T^*). \tag{2.35}$$

Proof. Firstly, by taking $k>2$ in Equation (2.10), we can allow for an application of Lemma 2.5 with $r:=\infty$ to assert that $D\mathbf{u}_\epsilon$ is bounded in $L^\infty\big(\Omega\times(0,T^*)\big)$, which together with Lemma 2.4 and the interpolation inequality also yields the boundedness of $\mathbf{u}_\epsilon$. This gives Equation (2.35).

Secondly, we rewrite Equation $(2.1)_1$ as

$$n_{\epsilon t}=\Delta(n_\epsilon+\epsilon)^m-\nabla\cdot\big(n_\epsilon\mathbf{u}_\epsilon+n_\epsilon S_\epsilon(x,n_\epsilon,c_\epsilon)\nabla c_\epsilon\big)$$

by using the solenoidality of $\mathbf{u}_\epsilon$. Then the boundedness of n_ϵ can be obtained by [28, Lemma A.1]. Indeed, Hölder's inequality implies that the assumptions of [28, Lemma A.1] are valid provided that we take the parameters k and q in Lemma 2.9 appropriately large. This establishes Equation (2.33).

Finally, the boundedness of c_ϵ and ∇c_ϵ can be derived from Equations (2.33) and (2.35) by applying the standard parabolic regularity theory to Equation $(2.1)_2$ (see [16, Lemma 4.1] for instance). This yields Equation (2.34). ∎

THEOREM 2.1. *Let $m>1$ and $\Omega\subset\mathbb{R}^2$ be a bounded convex domain with smooth boundary. Then, for any $\epsilon\in(0,1)$, system (2.1) admits a unique global in time classical solution $(n_\epsilon,c_\epsilon,\mathbf{u}_\epsilon)$ which is uniformly bounded in $\Omega\times(0,\infty)$ with respect to ϵ.*

Proof. According to Lemma 2.10 and the extension criterion (2.2), we can deduce that $T^*=\infty$. By Lemma 2.10, we also see that $(n_\epsilon,c_\epsilon,\mathbf{u}_\epsilon)$ is uniformly bounded in $\Omega\times(0,\infty)$ with respect to ϵ. This completes the proof of Theorem 2.1. ∎

3. Degenerate problems

In this section, we shall construct a global weak solution to system (1.1), (1.8), (1.9) with general tensor-valued sensitivity S, which satisfies Equations (1.12) and (1.13). We shall invoke the global-in-time classical solutions to system (2.1) to approximate the weak solution of system (1.1), (1.8), (1.9).

3.1. Further regularity properties of approximate solutions. In order to investigate the convergence of solutions to system (2.1) as $\epsilon \to 0$, we need some further regularity properties for them.

LEMMA 3.1. *Suppose that $(n_\epsilon, c_\epsilon, \mathbf{u}_\epsilon)$ is a classical solution to system (2.1) in $\bar{\Omega}\times(0,+\infty)$. Then there exists a positive constant C independent of ϵ such that*

$$\|n_\epsilon\|_{L^\infty(\Omega\times(0,\infty))} \le C, \qquad \|c_\epsilon\|_{L^\infty(0,\infty;W^{1,\infty}(\Omega))} \le C, \tag{3.1}$$

and

$$\|\mathbf{u}_\epsilon(\cdot,t)\|_{L^\infty(0,\infty;W^{1,\infty}(\Omega))} \le C. \tag{3.2}$$

Moreover, for any $T>0$ and $\gamma > m-1$, we have

$$\int_0^T \int_\Omega |\nabla(n_\epsilon+\epsilon)^\gamma|^2 dxdt \le C(m,\gamma,T). \tag{3.3}$$

Proof. The uniform estimates (3.1) and (3.2) are the direct results of Lemma 2.10 and Theorem 2.1.

It remains to show Equation (3.3). For this purpose, we integrate Equation (2.13) over $(0,T)$ to obtain

$$\begin{aligned}&\int_\Omega (n_\epsilon+\epsilon)^k(\cdot,T)dx + \frac{mk(k-1)}{2}\int_0^T\int_\Omega (n_\epsilon+\epsilon)^{m+k-3}|\nabla n_\epsilon|^2 dxdt\\ &\le \frac{k(k-1)C_S^2}{2m}\int_0^T\int_\Omega (n_\epsilon+\epsilon)^{k+1-m}|\nabla c_\epsilon|^2 dxdt + \int_\Omega (n_0+\epsilon)^k dx\end{aligned}$$

for any $k>1$. In particular, by taking $k = 2\gamma+1-m$, we deduce that

$$\begin{aligned}&\int_0^T\int_\Omega \left|\nabla (n_\epsilon+\epsilon)^{\frac{m+k-1}{2}}\right|^2 dxdt\\ &\le C(\gamma,m)\int_0^T\int_\Omega (n_\epsilon+\epsilon)^{2(\gamma+1-m)}|\nabla c_\epsilon|^2 dxdt + C(\gamma,m)\\ &\le C(\gamma,m)\big(\|n_\epsilon\|_{L^\infty(\Omega\times(0,\infty))}+\epsilon\big)^{2(\gamma+1-m)}\|\nabla c_\epsilon\|^2_{L^\infty(\Omega\times(0,\infty))}|\Omega|T + C(\gamma,m)\\ &\le C(\gamma,m,T).\end{aligned}$$

Here we used the uniform estimate (3.1) in the last inequality. This gives (3.3). ∎

With the help of Lemma 3.1, we can obtain the uniform Hölder continuity of c_ϵ, ∇c_ϵ, and $\mathbf{u}_\epsilon$ by the standard parabolic regularity theory.

LEMMA 3.2. *Suppose that $(n_\epsilon, c_\epsilon, \mathbf{u}_\epsilon)$ is a classical solution to system (2.1) in $\bar{\Omega}\times(0,+\infty)$. Then there exist $\sigma\in(0,1)$ and $C>0$ independent of ϵ such that*

$$\|c_\epsilon\|_{C^{\sigma,\frac{\sigma}{2}}\left(\bar{\Omega}\times[t,t+1]\right)} \le C \qquad \text{for all } t>0$$

and

$$\|\mathbf{u}_\epsilon\|_{C^{\sigma,\frac{\sigma}{2}}\left(\overline{\Omega}\times[t,t+1]\right)} \le C \qquad \text{for all } t>0.$$

Moreover, for each $t_0 > 0$, we can also find $C(t_0) > 0$ such that

$$\|\nabla c_\epsilon\|_{C^{\sigma,\frac{\sigma}{2}}\left(\bar{\Omega}\times[t,t+1]\right)} \le C(t_0) \qquad \text{for all } t > t_0.$$

Proof. This can be done by taking a similar procedure as the proof of Lemma 3.18 and Lemma 3.19 in Winkler [41]. Indeed, the regularity of c_ϵ and ∇c_ϵ can be derived by applying the parabolic regularity theory to the equation

$$c_{\epsilon t} - \Delta c_\epsilon = -\mathbf{u}_\epsilon \cdot \nabla c_\epsilon - c_\epsilon + n_\epsilon$$

due to Lemma 2.10. On the other hand, the regularity of $\mathbf{u}_\epsilon$ can be obtained by applying the semigroup estimation techniques to the variation-of-constants representation

$$\mathbf{u}_\epsilon(t) = e^{-t\mathcal{A}}\mathbf{u}_0 + \int_0^t e^{-(t-s)\mathcal{A}}\mathcal{P}(n_\epsilon(s)\nabla\phi)ds.$$

This completes the proof of Lemma 3.2. ∎

We now deduce some regularity properties of time derivatives. The first one is related to the time derivatives of certain powers of n_ϵ.

LEMMA 3.3. *Let $m > 1$ and $\gamma > \max\{1, m-1\}$. Assume that $S(x,n,c)$ satisfies Equations* (1.12) *and* (1.13). *Suppose that $(n_\epsilon, c_\epsilon, \mathbf{u}_\epsilon)$ is a classical solution to system* (2.1) *in $\bar{\Omega}\times(0,+\infty)$. Then, for all $T > 0$, there exists a positive constant $C(T)$ such that*

$$\int_0^T \left\|\frac{\partial}{\partial t}(n_\epsilon+\epsilon)^\gamma\right\|_{(W_0^{2,2}(\Omega))^*} dt \le C(T) \qquad \text{for all} \quad \epsilon \in (0,1). \tag{3.4}$$

Proof. We take a similar procedure as the proof of Lemma 3.22 in Winkler [41]. For any $\psi \in W_0^{2,2}(\Omega)$, we take $\gamma(n_\epsilon+\epsilon)^{\gamma-1}\psi$ as a test function on Equation $(2.1)_1$ to obtain

$$\begin{aligned}
&\int_\Omega \frac{\partial}{\partial t}(n_\epsilon+\epsilon)^\gamma \psi dx \\
=&\gamma\int_\Omega \Big(\Delta(n_\epsilon+\epsilon)^m - \nabla\cdot\big(n_\epsilon S_\epsilon(x,n_\epsilon,c_\epsilon)\nabla c_\epsilon\big) - \mathbf{u}_\epsilon\cdot\nabla n_\epsilon\Big)(n_\epsilon+\epsilon)^{\gamma-1}\psi dx.
\end{aligned}$$

It then follows from the integration by parts that

$$\begin{aligned}
&\int_\Omega \frac{\partial}{\partial t}(n_\epsilon+\epsilon)^\gamma \psi dx \\
=& -m\gamma(\gamma-1)\int_\Omega (n_\epsilon+\epsilon)^{m+\gamma-3}|\nabla n_\epsilon|^2\psi dx - m\gamma\int_\Omega (n_\epsilon+\epsilon)^{m+\gamma-2}\nabla n_\epsilon\cdot\nabla\psi dx \\
&+\gamma(\gamma-1)\int_\Omega n_\epsilon(n_\epsilon+\epsilon)^{\gamma-2}\nabla n_\epsilon\cdot\big(S_\epsilon(x,n_\epsilon,c_\epsilon)\nabla c_\epsilon\big)\psi dx \\
&+\gamma\int_\Omega n_\epsilon(n_\epsilon+\epsilon)^{\gamma-1}\big(S_\epsilon(x,n_\epsilon,c_\epsilon)\nabla c_\epsilon\big)\cdot\nabla\psi dx + \int_\Omega (n_\epsilon+\epsilon)^\gamma \mathbf{u}_\epsilon\cdot\nabla\psi dx \\
=:& J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned} \tag{3.5}$$

We now estimate the terms $J_1, J_2, \ldots, J_5$ one by one. To this end, we first invoke Lemma 3.1 to pick a positive constant C_{18} such that

$$n_\epsilon \le C_{18}, \quad |\nabla c_\epsilon| \le C_{18}, \quad \text{and} \quad |\mathbf{u}_\epsilon| \le C_{18} \qquad \text{in} \quad \Omega\times(0,\infty) \tag{3.6}$$

for all $\epsilon\in(0,1)$. Then, for J_1, we have

$$\begin{aligned}|J_1|=&\frac{4m\gamma(\gamma-1)}{(m+\gamma-1)^2}\int_\Omega\left|\nabla(n_\epsilon+\epsilon)^{\frac{\gamma+m-1}{2}}\right|^2\psi dx\\ \le&\frac{4m\gamma(\gamma-1)}{(m+\gamma-1)^2}\|\psi\|_{L^\infty(\Omega)}\int_\Omega\left|\nabla(n_\epsilon+\epsilon)^{\frac{\gamma+m-1}{2}}\right|^2dx.\end{aligned}\tag{3.7}$$

For J_2, it follows from Hölder's inequality that

$$\begin{aligned}|J_2|=&\frac{m\gamma}{m+\gamma-1}\left|\int_\Omega\nabla(n_\epsilon+\epsilon)^{m+\gamma-1}\cdot\nabla\psi dx\right|\\ \le&\frac{m\gamma}{m+\gamma-1}\|\nabla(n_\epsilon+\epsilon)^{m+\gamma-1}\|_{L^2(\Omega)}\|\nabla\psi\|_{L^2(\Omega)}\\ \le&\frac{m\gamma}{2(m+\gamma-1)}\left(\int_\Omega\left|\nabla(n_\epsilon+\epsilon)^{\gamma+m-1}\right|^2dx+1\right)\|\nabla\psi\|_{L^2(\Omega)}.\end{aligned}\tag{3.8}$$

Similarly, for J_3, we have

$$\begin{aligned}|J_3|\le&\gamma(\gamma-1)\int_\Omega(n_\epsilon+\epsilon)^{\gamma-1}|\nabla n_\epsilon|dxC_SC_{18}\|\psi\|_{L^\infty(\Omega)}\\ \le&(\gamma-1)C_SC_{18}\int_\Omega|\nabla(n_\epsilon+\epsilon)^\gamma|dx\|\psi\|_{L^\infty(\Omega)}\\ \le&(\gamma-1)C_SC_{18}\int_\Omega\left(|\nabla(n_\epsilon+\epsilon)^\gamma|^2+1\right)dx\|\psi\|_{L^\infty(\Omega)}.\end{aligned}\tag{3.9}$$

For J_4 and J_5, we also have

$$\begin{aligned}|J_4|\le&\gamma\int_\Omega(n_\epsilon+\epsilon)^\gamma\left|S_\epsilon(x,n_\epsilon,c_\epsilon)\right||\nabla c_\epsilon||\nabla\psi|dx\\ \le&\gamma(C_{18}+1)^\gamma C_SC_{18}|\Omega|^{\frac12}\|\nabla\psi\|_{L^2(\Omega)}\end{aligned}\tag{3.10}$$

and

$$|J_5|\le\int_\Omega(n_\epsilon+\epsilon)^\gamma|\mathbf{u}_\epsilon||\nabla\psi|dx\le(C_{18}+1)^\gamma C_{18}|\Omega|^{\frac12}\|\nabla\psi\|_{L^2(\Omega)}.\tag{3.11}$$

Due to $W_0^{2,2}(\Omega)\hookrightarrow L^\infty(\Omega)$, substituting Equations (3.7)–(3.11) into Equation (3.5), we conclude that there exists a positive constant C_{19} independent of ϵ such that

$$\begin{aligned}&\left|\int_\Omega\frac{\partial}{\partial t}(n_\epsilon+\epsilon)^\gamma\psi dx\right|\\ \le&C_{19}\Big(\int_\Omega\left|\nabla(n_\epsilon+\epsilon)^{\frac{\gamma+m-1}{2}}\right|^2dx+\int_\Omega\left|\nabla(n_\epsilon+\epsilon)^{\gamma+m-1}\right|^2dx\\ &+\int_\Omega\left|\nabla(n_\epsilon+\epsilon)^\gamma\right|^2dx+1\Big)\|\psi\|_{W_0^{2,2}(\Omega)}.\end{aligned}$$

Integrating with respect to time, we have

$$\begin{aligned}&\int_0^T\left\|\frac{\partial}{\partial t}(n_\epsilon+\epsilon)^\gamma\right\|_{(W_0^{2,2}(\Omega))^*}dt\\ \le&C_{19}\Big(\int_0^T\int_\Omega\left|\nabla(n_\epsilon+\epsilon)^{\frac{\gamma+m-1}{2}}\right|^2dx+\int_0^T\int_\Omega\left|\nabla(n_\epsilon+\epsilon)^{\gamma+m-1}\right|^2dx\\ &+\int_0^T\int_\Omega\left|\nabla(n_\epsilon+\epsilon)^\gamma\right|^2dx+T\Big).\end{aligned}$$

By $\gamma>m-1$, we may use Equation (3.3) to derive the desired bound (3.4). $\square$

3.2. Convergence of a subsequence. With the help of *a priori* estimates, in this subsection, we shall extract a suitable subsequence from $(n_\epsilon, c_\epsilon, \mathbf{u}_\epsilon)$ such that it is convergent.

LEMMA 3.4. *Let $m>1$. Suppose that $(n_\epsilon, c_\epsilon, \mathbf{u}_\epsilon)$ is a classical solution to system* (2.1) *in $\bar{\Omega}\times(0,+\infty)$. Then there exist a triple of functions $(n,c,\mathbf{u})$ satisfying*

$$\begin{aligned}
&n\in L^\infty((0,\infty);L^\infty(\Omega)), \quad \nabla n^m\in L^2_{loc}((0,\infty);L^2(\Omega)),\\
&c\in L^\infty((0,\infty);W^{1,\infty}(\Omega)), \quad \textit{and}\\
&\mathbf{u}\in L^\infty((0,\infty);W^{1,\infty}(\Omega)) \quad \textit{such that } \nabla\cdot\mathbf{u}=0 \textit{ in the distributional sense in } \Omega\times(0,\infty),
\end{aligned}$$

and a subsequence $\{\epsilon_j\}_{j=1}^\infty$ converging to zero as $j\to\infty$ such that

$$n_{\epsilon_j} \rightharpoonup n \qquad \text{weakly}* \quad \text{in} \quad L^\infty\big((0,\infty);L^\infty(\Omega)\big), \tag{3.12}$$

$$\nabla n^m_{\epsilon_j} \rightharpoonup \nabla n^m \qquad \text{in} \quad L^2_{loc}\big((0,\infty);W^{1,2}(\Omega)\big), \tag{3.13}$$

$$c_{\epsilon_j} \rightharpoonup c \qquad \text{and} \qquad \nabla c_{\epsilon_j} \rightharpoonup \nabla c \qquad \text{weakly}* \quad \text{in} \quad L^\infty\big((0,\infty);L^\infty(\Omega)\big), \tag{3.14}$$

$$n_{\epsilon_j} S_{\epsilon_j}(x,n_{\epsilon_j},c_{\epsilon_j}) \to nS(x,n,c) \qquad \text{strongly in} \quad L^2_{loc}\big([0,\infty);L^2(\Omega)\big), \tag{3.15}$$

$$\mathbf{u}_{\epsilon_j} \rightharpoonup \mathbf{u} \qquad \text{and} \qquad \nabla\mathbf{u}_{\epsilon_j} \rightharpoonup \nabla\mathbf{u} \qquad \text{weakly}* \quad \text{in} \quad L^\infty\big((0,\infty);L^\infty(\Omega)\big) \tag{3.16}$$

and

$$c_{\epsilon_j} \to c, \qquad \nabla c_{\epsilon_j} \to \nabla c \qquad \text{and} \qquad \mathbf{u}_{\epsilon_j} \to \mathbf{u} \qquad \text{in} \quad C^0_{loc}\big(\overline{\Omega}\times[0,\infty)\big) \tag{3.17}$$

as $j\to\infty$.

Proof. Due to Lemma 3.1, we can find a subsequence $\{\epsilon_j\}_{j=1}^\infty$ converging to zero as $j\to\infty$ and a triple $(n,c,\mathbf{u})$ such that conditions (3.12), (3.14), and (3.16) hold.

By using the uniform estimate (3.3) with $\gamma=m$, we can also have a subsequence (still denoted by $\{\epsilon_j\}_{j=1}^\infty$) such that

$$\nabla n^m_{\epsilon_j} \rightharpoonup \nabla n^m \qquad \text{in} \quad L^2\big((0,T);W^{1,2}(\Omega)\big)$$

for any $T>0$, which yields condition (3.13).

Due to Lemma 3.2, we can use the Arzelà-Ascoli theorem and a standard extraction procedure to find a sequence (still denoted by $\{\epsilon_j\}_{j=1}^\infty$) such that condition (3.17) holds. Due to $\nabla\cdot\mathbf{u}_\epsilon=0$, we also have $\nabla\cdot\mathbf{u}=0$ in the distributional sense in $\Omega\times(0,\infty)$.

On the other hand, we fix a $\gamma>m-1$. Then Lemma 3.1 and Lemma 3.3 assert that for any $T>0$,

$$(n_\epsilon^\gamma)_{\epsilon\in(0,1)} \qquad \text{is bounded in}\, L^2\big((0,T);W^{1,2}(\Omega)\big)$$

and

$$(\frac{\partial}{\partial t}(n_\epsilon+\epsilon)^\gamma)_{\epsilon\in(0,1)} \qquad \text{is bounded in}\, L^1\big((0,T);(W_0^{2,2}(\Omega))^*\big).$$

Since

$$W^{1,2}(\Omega)\hookrightarrow_{\text{compact}} L^2(\Omega)\hookrightarrow_{\text{continuous}} \big(W_0^{2,2}(\Omega)\big)^*,$$

an application of Aubin-Lions lemma (see Chapter IV, [23]) yields the strong precompactness of $(n_{\epsilon_j}+\epsilon_j)^\gamma$ in $L^2([0,T];L^2(\Omega))$. By using the Egorov's theorem, we can find a subsequence (still denoted by $\{\epsilon_j\}_{j=1}^\infty$) fulfilling

$$(n_{\epsilon_j}+\epsilon_j)^\gamma \to n^\gamma \quad \text{strongly in} \quad L^2\big([0,T];L^2(\Omega)\big)$$

and

$$(n_{\epsilon_j}+\epsilon_j)^\gamma \to n^\gamma \quad \text{a.e. in}\, \Omega\times(0,\infty)$$

as $j\to\infty$, which implies that

$$(n_{\epsilon_j}+\epsilon_j)\to n \quad \text{a.e. in}\, \Omega\times(0,\infty) \tag{3.18}$$

as $j\to\infty$. By conditions (3.17) and (3.18) and the definition of S_ϵ, we may further infer that

$$S_{\epsilon_j}(x,n_{\epsilon_j},c_{\epsilon_j})\to S(x,n,c) \qquad \text{a.e. in}\, \Omega\times(0,\infty)$$

and thus

$$n_{\epsilon_j}S_{\epsilon_j}(x,n_{\epsilon_j},c_{\epsilon_j})\to nS(x,n,c) \qquad \text{a.e. in}\, \Omega\times(0,\infty)$$

as $j\to\infty$. Then we may use the Lebesgue's dominated theorem, along with a subsequence (still denoted by $\{\epsilon_j\}_{j=1}^\infty$), to infer that

$$n_{\epsilon_j}S_{\epsilon_j}(x,n_{\epsilon_j},c_{\epsilon_j})\to nS(x,n,c) \quad \text{strongly in}\, L^2_{loc}\big([0,\infty);L^2(\Omega)\big)$$

as $j\to\infty$. This completes the proof of condition (3.15). ◻

3.3. Solution properties of $(n,c,\mathbf{u})$. In this subsection, we shall show that the triple $(n,c,\mathbf{u})$ obtained in Lemma 3.4 is a global weak solution to system (1.1)–(1.9).

Proof. (Proof of Theorem 1.1.) In Equations $(2.1)_1$, $(2.1)_2$, and $(2.1)_3$, we take $\varphi\in C_0^\infty\big(\bar\Omega\times[0,\infty)\big)$ and $\zeta\in C_0^\infty\big(\bar\Omega\times[0,\infty),\mathbb{R}^2\big)$ with $\nabla\cdot\zeta=0$ as test functions and then obtain

$$\begin{aligned}&\int_0^\infty\int_\Omega n_{\epsilon_j}\varphi_t dxdt+\int_\Omega n_0\varphi(\cdot,0)dx\\ &=\int_0^\infty\int_\Omega \nabla(n_{\epsilon_j}+\epsilon_j)^m\cdot\nabla\varphi dxdt-\int_0^\infty\int_\Omega\big(n_{\epsilon_j}S_{\epsilon_j}(x,n_{\epsilon_j},c_{\epsilon_j})\nabla c_{\epsilon_j}\big)\cdot\nabla\varphi dxdt\\ &\quad-\int_0^\infty\int_\Omega n_{\epsilon_j}\mathbf{u}_{\epsilon_j}\cdot\nabla\varphi dxdt\end{aligned}$$

and

$$\begin{aligned}&\int_0^\infty\int_\Omega c_{\epsilon_j}\varphi_t dxdt+\int_\Omega c_0\varphi(\cdot,0)dx\\ &=\int_0^\infty\int_\Omega\nabla c_{\epsilon_j}\cdot\nabla\varphi dxdt+\int_0^\infty\int_\Omega c_{\epsilon_j}\varphi dxdt-\int_0^\infty\int_\Omega n_{\epsilon_j}\varphi dxdt-\int_0^\infty\int_\Omega c_{\epsilon_j}\mathbf{u}_{\epsilon_j}\cdot\nabla\varphi dxdt\end{aligned}$$

as well as

$$\int_0^\infty\int_\Omega \mathbf{u}_{\epsilon_j}\cdot\zeta_t dxdt+\int_\Omega\mathbf{u}_0\cdot\zeta(\cdot,0)dx=\int_0^\infty\int_\Omega\nabla\mathbf{u}_{\epsilon_j}\cdot\nabla\zeta dxdt-\int_0^\infty\int_\Omega n_{\epsilon_j}\nabla\phi\cdot\zeta dxdt.$$

By Lemma 3.4, we can deduce that

$$\int_0^\infty \int_\Omega n_{\epsilon_j} \varphi_t dxdt \to \int_0^\infty \int_\Omega n\varphi_t dxdt,$$

$$\int_0^\infty \int_\Omega \big(n_{\epsilon_j} S_{\epsilon_j}(x, n_{\epsilon_j}, c_{\epsilon_j}) \nabla c_{\epsilon_j}\big) \cdot \nabla\varphi dxdt \to \int_0^\infty \int_\Omega \big(nS(x,n,c)\nabla c\big) \cdot \nabla\varphi dxdt,$$

$$\int_0^\infty \int_\Omega \nabla(n_{\epsilon_j} + \epsilon_j)^m \cdot \nabla\varphi dxdt \to \int_0^\infty \int_\Omega \nabla n^m \cdot \nabla\varphi dxdt,$$

and

$$\int_0^\infty \int_\Omega n_{\epsilon_j} \mathbf{u}_{\epsilon_j} \cdot \nabla\varphi dxdt \to \int_0^\infty \int_\Omega n\mathbf{u} \cdot \nabla\varphi dxdt$$

as $j \to \infty$. Then we obtain

$$\begin{aligned}
&\int_0^\infty \int_\Omega n\varphi_t dxdt + \int_\Omega n_0 \varphi(0) dx \\
=& \int_0^\infty \int_\Omega \nabla n^m \cdot \nabla\varphi dxdt - \int_0^\infty \int_\Omega n(S(x,n,c)\nabla c) \cdot \nabla\varphi dxdt - \int_0^\infty \int_\Omega n\mathbf{u} \cdot \nabla\varphi dxdt.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_0^\infty \int_\Omega c\varphi_t dxdt + \int_\Omega c_0 \varphi(\cdot,0) dx \\
=& \int_0^\infty \int_\Omega \nabla c \cdot \nabla\varphi dxdt + \int_0^\infty \int_\Omega c\varphi dxdt - \int_0^\infty \int_\Omega n\varphi dxdt - \int_0^\infty \int_\Omega c\mathbf{u} \cdot \nabla\varphi dxdt
\end{aligned}$$

and

$$\int_0^\infty \int_\Omega \mathbf{u} \cdot \zeta_t dxdt + \int_\Omega \mathbf{u}_0 \cdot \zeta(\cdot,0) dx = \int_0^\infty \int_\Omega \nabla\mathbf{u} \cdot \nabla\zeta dxdt - \int_0^\infty \int_\Omega n\nabla\phi \cdot \zeta dxdt.$$

This means that $(n, c, \mathbf{u})$ is a global-in-time weak solution of system (1.1)–(1.9).

Finally, the boundedness of $(n, c, \mathbf{u})$ may result from the boundedness of $(n_\epsilon, c_\epsilon, \mathbf{u}_\epsilon)$ and the Banach-Alaoglu theorem. This completes the proof of Theorem 1.1. □

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