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ON A LIMIT OF PERTURBED CONSERVATION LAWS WITH DIFFUSION AND NON-POSITIVE DISPERSION*

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Abstract. We consider a conservation law perturbed by a linear diffusion and a general form of non-positive dispersion. We prove the convergence of the corresponding solution to the entropy weak solution of the hyperbolic conservation law.

Key words. Diffusion, dispersion, KdV equation, Burgers' equation, hyperbolic conservation laws, entropy measure-valued solutions.

AMS subject classifications. 35G25, 35L65, 35Q53, 76B15.

1. Introduction

We consider the initial value problem

$$u_t + f(u)_x = \varepsilon u_{xx} + \delta g(u_{xx})_x \tag{1.1}$$
$$u(x,0) = u_0(x), \tag{1.2}$$

where ε and δ are small parameters, and g is a non-positive function, and we focus on the specific form

$$g(v) = -|v|^n,$$

where $n \geq 1$. Note that if $n = 1$, then g is Lipschitz; if $1 < n < 2$, then g is C^1; and for $n \geq 2$, g is C^2.

When $\delta = 0$ we reduce to the viscous (generalized) Burgers equation and the approximate solutions $u^{\varepsilon,0}$ converge to the entropy solution of the hyperbolic equation (called the *vanishing viscosity method*, see, e.g., Whitham [21] or Kružkov [9])

$$u_t + f(u)_x = 0 \tag{1.3}$$
$$u(x,0) = u_0(x). \tag{1.4}$$

On the other hand, when $\varepsilon = 0$, if we consider the flux function $f(u) = u^2$ and the linear dispersion δu_{xxx} we obtain the Korteweg–de Vries equation. The approximate solutions $u^{0,\delta}$ do not converge in a strong topology (see Lax and Levermore [11]). So, as parameters ε and δ vanish, we are concerned with singular limits and to ensure convergence it is necessary to be in the dominant dissipation regime.

The pioneer study of these singular limits was given by Schonbek [16] about the (generalized) Korteweg–de Vries–Burgers equation

$$u_t + f(u)_x = \varepsilon u_{xx} - \delta u_{xxx}.$$

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In the case of a convex flux function $f(u)$, she proved the convergence of the solutions of this perturbed equation to the entropy solution of (1.3), when both ε and δ tend to zero, *at least* under the condition $\delta = O(\varepsilon^3)$ (depending on the behavior of the flux f). Also, according to Perthame and Ryzhic [15], the sharp condition should be $\delta = o(\varepsilon)$. LeFloch and Natalini [13] proved the convergence in the case of a nonlinear viscosity function β and linear capillarity

$$u_t + f(u)_x = \varepsilon \beta(u_x)_x - \delta u_{xxx}.$$

Then, Correia and LeFloch [5] improved the estimates in Schonbek [16] and LeFloch and Natalini [13] and for the first time treated the multidimensional equation, but still in the case of a nonlinear viscosity function and linear capillarity. In fact, the dominant dissipation regime is also assured by the nonlinear viscosity. In our case, we consider the reverse situation.

In general for $\varepsilon = 0$, like for the Korteweg–de Vries equation, the divergent behaviour is expected, as we are considering "pure-dispersive equations". But, Brenier and Levy [3] considered the fully nonlinear equation

$$u_t + f(u)_x = -\delta(u_{xx}^2)_x$$

as a nonlinear generalization of the Korteweg–de Vries equation. Such nonlinear dispersion significantly affects the dispersive behaviour of the solutions that differs completely from the linear case. In particular, Brenier and Levy [3] conjectured that for strictly convex flux functions f and for the following perturbed problem

$$u_t + f(u)_x = -\delta g(u_{xx})_x - \varepsilon u_{xxxx},$$

we have convergence when ε and δ tend to zero under the condition $\varepsilon = o(\delta)$.

The paper is organized as follows. In Section 2, we present the main results of convergence. Section 3 deals with the uniform estimates needed for convergence. Finally, Section 4 is devoted to proving the convergence to the entropy solution of the hyperbolic equation, when both ε and δ go to zero.

2. Main results

Two main convergence results are presented. The first one is concerned with $g(v) = -|v|$ (i.e. $n=1$) while the second one is devoted to the case $g(v) = -v^2$ (i.e. $n=2$).

2.1. Case f convex, $\delta > 0$ and $g(u_{xx}) = -|u_{xx}|$.

In this case, we prove the following result.

THEOREM 2.1. *Let $\varepsilon > 0$, $\delta = o(\varepsilon^2)$, and $f : R \to R$ be a convex flux function satisfying*

$$f''(u) \le C(1 + |u|^\beta), \ \textit{where } 0 \le \beta < 3.$$

Then, setting $u = u_{\varepsilon,\delta}$ the solution of (1.1)-(1.2), the family solutions $(u_{\varepsilon,\delta})$ converges to the entropy solution of (1.3)-(1.4).

In the case of the linear dispersion, i.e. $g(u_{xx}) = u_{xx}$ treated in [16], Schonbek gets for a general flux satisfying $f''(u) \le C$, a convergence with rate $\delta = O(\varepsilon^3)$. Also, when $f''(u) \le C(1 + |u|)$, the author obtain the convergence with the rate $\delta = O(\varepsilon^4)$. The case of $g(u_{xx}) = -|u_{xx}|$ seems giving a weakly dispersive effects than a classical linear dispersion.

2.2. Case f convex, $\delta>0$ and $g(u_{xx})=-u_{xx}^2$. Here function g is regular, and we obtain:

THEOREM 2.2. *Let $\varepsilon>0$, $\delta=o(\varepsilon^{5/2})$, and $f:R\to R$ be a convex flux function satisfying*

$$f''(u)\leq C(1+|u|^{\beta}),\ \textit{where}\ 0\leq\beta<1/2.$$

Then, setting $u=u_{\varepsilon,\delta}$ the solution of (1.1)-(1.2), the family solutions $(u_{\varepsilon,\delta})$ converges to the entropy solution of (1.3)-(1.4).

The dispersion here is strongly nonlinear but regular, which provides the well-posedness [1, 2]. We can see that comparing to the results in [16], the rate are quite similar when the flux is convex and satisfies $f''(u)\leq C$ ($\delta=o(\varepsilon^{5/2})$, whereas $\delta=O(\varepsilon^3)$ in [16]).

3. A priori estimates

Assume that η is a regular function and φ a function defined by $\varphi'=\eta' f'$, and let us multiply (1.1) by $\eta'(u)$. We obtain

$$\eta(u)_t+\varphi(u)_x=\varepsilon\left(\eta'(u)u_x\right)_x-\varepsilon\eta''(u)u_x^2+\delta\left(\eta'(u)g(u_{xx})\right)_x-\delta\eta''(u)u_x g(u_{xx}). \tag{3.1}$$

Integrating over $(0,t)\times R$ with $\eta(u)=|u|^{\alpha+1}$, the conservative terms vanish and we obtain the following lemma.

LEMMA 3.1. *Let $\alpha\geq 1$ and $g:R\to R$ be any dispersion function. Each solution of (1.1) satisfies for $t\in[0,T]$*

$$\int_R |u(t)|^{\alpha+1}dx+(\alpha+1)\alpha\varepsilon\int_0^t\int_R |u|^{\alpha-1}u_x^2\,dxds+(\alpha+1)\alpha\delta\int_0^t\int_R |u|^{\alpha-1}u_x g(u_{xx})\,dxds$$
$$=\int_R |u_0|^{\alpha+1}dx. \tag{3.2}$$

Usually, taking $\alpha=1$ in (3.2), we deduce the a priori L^2 first energy estimates.

Let us introduce the functions $\mathcal{G}$, and G defined by $\mathcal{G}''=G'=g$, i.e,

$$G(u)=-\frac{1}{n+1}|u|^n u \quad\text{and}\quad \mathcal{G}=-\frac{1}{(n+1)(n+2)}|u|^{n+2}.$$

Using the multiplier $(q+2)(|u_x|^q u_x)_x$ to (1.1), we have

$$\begin{aligned}&\left((q+2)u_t|u_x|^q u_x\right)_x-(|u_x|^{q+2})_t\\ &=-(q+2)(q+1)|u_x|^q u_{xx}f'(u)u_x+\varepsilon(q+2)(q+1)|u_x|^q u_{xx}^2\\ &\quad+\delta(q+2)(q+1)|u_x|^q u_{xx}g'(u_{xx})u_{xxx}\\ &=-(q+1)(|u_x|^{q+2})_x f'(u)+\varepsilon(q+2)(q+1)|u_x|^q u_{xx}^2\\ &\quad+\delta(q+2)(q+1)n|u_x|^q G(u_{xx})_x,\end{aligned}$$

and we get the estimate

$$\begin{aligned}&\left((q+2)u_t|u_x|^q u_x\right)_x-(|u_x|^{q+2})_t\\ &=-\left((q+1)|u_x|^{q+2}f'(u)\right)_x+(q+1)|u_x|^{q+2}u_x f''(u)+\varepsilon(q+2)(q+1)|u_x|^q u_{xx}^2\\ &\quad+\left(\delta(q+2)(q+1)n|u_x|^q G(u_{xx})\right)_x\end{aligned}$$

$$-\delta(q+2)(q+1)q(n+2)n|u_x|^{q-2}u_x\mathcal{G}(u_{xx}). \tag{3.3}$$

Similarly, using the multiplier $(q+2)(u_x^{q+1})_x$ to (1.1), we can write

$$\begin{aligned}
&\big((q+2)u_t u_x^{q+1}\big)_x-(u_x^{q+2})_t\\
&=-(q+2)(q+1)u_x^{q+1}f'(u)u_{xx}+\varepsilon(q+2)(q+1)u_x^q u_{xx}^2\\
&\quad+\big(\delta(q+2)(q+1)nu_x^q G(u_{xx})\big)_x-\delta(q+2)(q+1)q(n+2)nu_x^{q-1}\mathcal{G}(u_{xx}),
\end{aligned}$$

thus

$$\begin{aligned}
&\big((q+2)u_t u_x^{q+1}\big)_x-(u_x^{q+2})_t\\
&=-\big((q+1)u_x^{q+2}f'(u)\big)_x+(q+1)u_x^{q+3}f''(u)+\varepsilon(q+2)(q+1)u_x^q u_{xx}^2\\
&\quad+\big(\delta(q+2)(q+1)nu_x^q G(u_{xx})\big)_x-\delta(q+2)(q+1)q(n+2)nu_x^{q-1}\mathcal{G}(u_{xx}).
\end{aligned} \tag{3.4}$$

Integrating (3.3) and (3.4) over $R\times[0,t]$ provides

$$\begin{aligned}
&\int_R |u_x(t)|^{q+2}dx+\varepsilon(q+2)(q+1)\int_0^t\int_R |u_x|^q u_{xx}^2\,dxds\\
&=\int_R |u_0'|^{q+2}dx-(q+1)\int_0^t\int_R u_x|u_x|^{q+2}f''(u)\,dxds\\
&\qquad+\delta(q+2)(q+1)q(n+2)n\int_0^t\int_R u_x|u_x|^{q-2}\mathcal{G}(u_{xx})\,dxds,
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
&\int_R u_x(t)^{q+2}dx+\varepsilon(q+2)(q+1)\int_0^t\int_R u_x^q u_{xx}^2\,dxds\\
&=\int_R (u_0')^{q+2}dx-(q+1)\int_0^t\int_R u_x^{q+3}f''(u)\,dxds\\
&\qquad+\delta(q+2)(q+1)q(n+2)n\int_0^t\int_R u_x^{q-1}\mathcal{G}(u_{xx})\,dxds.
\end{aligned} \tag{3.6}$$

We now define the sets, for $t\in[0,T]$

$$\mathcal{U}_t^+=\{x\in R; u_x(x,t)>0\},$$

and

$$\mathcal{U}_t^-=\{x\in R; u_x(x,t)<0\}.$$

Adding (3.6) into (3.5) for q odd, we obtain the following.

LEMMA 3.2. *Let q be a odd number. Then, each solution of (1.1) satisfies for $t\in[0,T]$*

$$\begin{aligned}
&\int_{\mathcal{U}_t^+}|u_x(t)|^{q+2}dx+\varepsilon(q+2)(q+1)\int_0^t\int_{\mathcal{U}_s^+}|u_x|^q u_{xx}^2\,dxds\\
&\qquad+\delta(q+2)(q+1)q(n+2)n\int_0^t\int_{\mathcal{U}_s^+}|u_x|^{q-1}|\mathcal{G}(u_{xx})|\,dxds\\
&\qquad+(q+1)\int_0^t\int_{\mathcal{U}_s^+}|u_x|^{q+3}f''(u)\,dxds
\end{aligned}$$

$$= \int_{\mathcal{U}_0^+} |u_0'|^{q+2}\, dx, \tag{3.7}$$

where the last left-hand side term can be replaced by

$$-(q+2)(q+1)\int_0^t \int_{\mathcal{U}_s{}^+} |u_x|^{q+1} f'(u)\, u_{xx}\, dxds. \tag{3.8}$$

Now, the combination of lemmas 3.1 and 3.2 gives the following estimate

PROPOSITION 3.3. *Let $\varepsilon,\delta>0$, and $f:R\to R$ be a convex flux function. The solution $u=u_{\varepsilon,\delta}$ of (1.1)-(1.2) satisfies the uniform estimate*

$$\int_R |u(t)|^{\alpha+1}\, dx + \varepsilon \int_0^t \int_R |u|^{\alpha-1} u_x^2\, dxds + \delta \int_0^t \int_R |u|^{\alpha-1} |u_x|\, |u_{xx}|^n\, dxds \le C, \tag{3.9}$$

for all $\frac{5+n}{2n+1} \le \alpha < \frac{4+n}{n}$.

Proof. When $g(u)=-|u|^n$, equality (3.2) writes

$$\begin{aligned} &\int_R |u(t)|^{\alpha+1}\, dx + \alpha(\alpha+1)\varepsilon \int_0^t \int_R |u|^{\alpha-1} u_x^2\, dxds \\ &= \|u_0\|_{\alpha+1}^{\alpha+1} + \alpha(\alpha+1)\,\delta \int_0^t \int_R |u|^{\alpha-1} u_x\, |u_{xx}|^n\, dxds. \end{aligned} \tag{3.10}$$

Also, when f is convex, we can rewrite (3.7) for $q\ge 1$ odd as

$$\begin{aligned} &\int_{\mathcal{U}_t^+} |u_x(t)|^{q+2}\, dx + \varepsilon \int_0^t \int_{\mathcal{U}_s^+} |u_x|^q\, u_{xx}^2\, dxds \\ &\quad + \delta \int_0^t \int_{\mathcal{U}_s^+} |u_x|^{q-1} |u_{xx}|^{n+2}\, dxds + \int_0^t \int_{\mathcal{U}_s^+} |u_x|^{q+3} f''(u)\, dxds \le C. \end{aligned} \tag{3.11}$$

However, using Young inequality, we get

$$\begin{aligned} &\delta \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha-1} u_x\, |u_{xx}|^n\, dxds \\ &= \int_0^t \int_{\mathcal{U}_s^+} \left(\frac{1}{ct^{\frac{\alpha-1}{\alpha+1}}} |u|^{\alpha-1} \right) \left(ct^{\frac{\alpha-1}{\alpha+1}}\, |u_x| \right) (\delta |u_{xx}|^n)\, dxds \\ &\le \left(\frac{1}{tc^{\frac{\alpha+1}{\alpha-1}}} \right) \left(\frac{\alpha-1}{\alpha+1} \right) \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha+1}\, dxds + \frac{c^k\, t^{k\frac{\alpha-1}{\alpha+1}}}{k} \int_0^t \int_{\mathcal{U}_s^+} |u_x|^k\, dxds \\ &\quad + \frac{n}{n+2} \delta^{1+\frac{2}{n}} \int_0^t \int_{\mathcal{U}_s^+} |u_{xx}|^{n+2}\, dxds, \end{aligned} \tag{3.12}$$

where c and k are two constants such that

$$c^{\frac{\alpha+1}{\alpha-1}} = 4\alpha(\alpha-1),$$

and

$$\frac{1}{k} + \frac{n}{n+2} + \frac{\alpha-1}{\alpha+1} = 1.$$

Thus,

$$k=\frac{(n+2)(\alpha+1)}{(4+n)-n\alpha},$$

and if $\frac{5+n}{2n+1}\leq\alpha<\frac{4+n}{n}$, we get $k\geq 3$.

Now, q is chosen odd such that $2+q\geq k$ to obtain

$$|u_x|^k\leq|u_x|^3+|u_x|^{q+2}.$$

Using (3.11) with $q=1$ and $q\geq k-2$ odd, we obtain

$$\int_0^t\int_{\mathcal{U}_s^+}|u_x|^k\,dxds+\delta\int_0^t\int_{\mathcal{U}_s^+}|u_{xx}|^{n+2}\,dxds\leq C. \tag{3.13}$$

Now, integrating (3.10) over $[0,t]$, we get

$$\begin{aligned}\int_0^t\int_{\mathcal{U}_s^+}|u|^{\alpha+1}\,dxds&\leq\int_0^t\int_R|u|^{\alpha+1}\,dxds\\&\leq tC+\alpha(\alpha+1)t\delta\int_0^t\int_{\mathcal{U}_s^+}|u|^{\alpha-1}u_x\,|u_{xx}|^n\,dxds.\end{aligned} \tag{3.14}$$

Thus, substituting (3.13) and (3.14) into (3.12), it becomes

$$\delta\int_0^t\int_{\mathcal{U}_s^+}|u|^{\alpha-1}u_x\,|u_{xx}|^n\,dxds\leq C+\frac{1}{4}\delta\int_0^t\int_{\mathcal{U}_s^+}|u|^{\alpha-1}u_x\,|u_{xx}|^n\,dxds \tag{3.15}$$

and we obtain

$$\delta\int_0^t\int_{\mathcal{U}_s^+}|u|^{\alpha-1}u_x\,|u_{xx}|^n\,dxds\leq C. \tag{3.16}$$

Finally, substituting (3.16) into (3.10) we obtain (3.9). □

3.1. Case f convex, $\delta>0$ and $g(u_{xx})=-|u_{xx}|$. We are concerned here with the equation

$$u_t+f(u)_x=\varepsilon u_{xx}-\delta|u_{xx}|_x. \tag{3.17}$$

PROPOSITION 3.4. *Let $\varepsilon>0$, and $f:R\to R$ be a convex flux function, such that*

$$f''(u)\leq C(1+|u|^\beta),\ \textit{where}\ 0\leq\beta<3.$$

Then, the solution $u=u_{\varepsilon,\delta}$ of (3.17) satisfies the estimate

$$\int_R u_x(t)^2\,dx+\varepsilon\int_0^t\int_R u_{xx}^2\,dxds\leq C+\frac{C}{\delta}, \tag{3.18}$$

where $C>0$ is a constant independent of ε and δ.

In addition, if $\delta=O(\varepsilon^2)$, the estimate (3.9) with $\alpha=1$ is

$$\int_R u(t)^2\,dx+\varepsilon\int_0^t\int_R u_x^2\,dxds+\delta\int_0^t\int_R|u_x||u_{xx}|\,dxds\leq C. \tag{3.19}$$

Proof. On the one hand, (3.5) is rewritten with $q=0$ as

$$\int_R u_x(t)^2\,dx+2\varepsilon\int_0^t\int_R u_{xx}^2\,dxds$$
$$=\int_R (u_0')^2\,dx+2\int_0^t\int_R f'(u)\,u_x\,u_{xx}\,dxds. \tag{3.20}$$

Since f satisfies $f''(u)\le C(1+|u|^\beta)$, with $0\le\beta<3$. Then

$$|f'(u)-f'(0)|\le C(|u|+|u|^{\beta+1}),$$

where C is a generic constant. Thus,

$$\left|\int_0^t\int_R f'(u)\,u_x\,u_{xx}\,dxds\right|\le C\int_0^t\int_R |u||u_x||u_{xx}|\,dxds$$
$$+C\int_0^t\int_R |u|^{\beta+1}|u_x||u_{xx}|\,dxds. \tag{3.21}$$

Applying (3.9) with $n=1$, $\alpha=2$ and $\alpha=\beta+2<5$, we get

$$\left|\int_0^t\int_R f'(u)\,u_x\,u_{xx}\,dxds\right|\le\frac{C}{\delta}. \tag{3.22}$$

Finally, substituting (3.22) into (3.20) provides (3.18).

On the other hand, estimate (3.2), with $\alpha=1$ is written as

$$\int_R u(t)^2\,dx+2\varepsilon\int_0^t\int_R u_x^2\,dxds=\|u_0\|_2^2+2\delta\int_0^t\int_R u_x\,|u_{xx}|\,dxds. \tag{3.23}$$

Then

$$\int_R u(t)^2\,dx+2\varepsilon\int_0^t\int_R u_x^2\,dxds\le C+2\delta\int_0^t\int_R |u_x||u_{xx}|\,dxds. \tag{3.24}$$

Now, if $\delta\le K\varepsilon^2$

$$\delta\int_0^t\int_R |u_x||u_{xx}|\,dxds\le\frac{\sqrt{\delta}}{\sqrt{K\varepsilon}}\int_0^t\int_R \left(\sqrt{\varepsilon}|u_x|\right)\left(\sqrt{K\delta\varepsilon}|u_{xx}|\right)dxds.$$
$$\le\frac{\sqrt{\delta}}{2\sqrt{K\varepsilon}}\left(\varepsilon\int_0^t\int_R u_x^2\,dxds+K\delta\varepsilon\int_0^t\int_R u_{xx}^2\,dxds\right)$$
$$\le\frac{1}{2}\left(\varepsilon\int_0^t\int_R u_x^2\,dxds+K\delta\varepsilon\int_0^t\int_R u_{xx}^2\,dxds\right). \tag{3.25}$$

Using (3.18) and substituting (3.25) into (3.24) gives (3.19). □

PROPOSITION 3.5. *Let $\varepsilon>0$, $\delta=o(\varepsilon^2)$, and $f:R\to R$ be a convex flux function satisfying*

$$f''(u)\le C(1+|u|^\beta),\ \text{where } 0\le\beta<3.$$

Then, the solution $u=u_{\varepsilon,\delta}$ satisfies *(a)* $\{\varepsilon u_x^2\}$ *is bounded in* $L^1((0,t)\times R)$.

(b) $\{\varepsilon u_x\}\to 0$ *as* $\varepsilon\to 0$, *in* $L^2((0,t)\times R)$.

(c) $\{\delta u_x|u_{xx}|\}\to 0$ *as* $\varepsilon\to 0$, *in* $L^1((0,t)\times R)$.

(d) $\{\delta|u_{xx}|\}\to 0$ *as* $\varepsilon\to 0$, *in* $L^2((0,t)\times R)$.

Proof. The statements (a) and (b) are obtained thanks to (3.19).

Now, in the same manner we obtained (3.25), we have

$$\delta\int_0^t\int_R |u_x||u_{xx}|\,dxds \le \frac{\sqrt{\delta}}{2\varepsilon}\left(\varepsilon\int_0^t\int_R u_x^2\,dxds+\delta\varepsilon\int_0^t\int_R u_{xx}^2\,dxds\right),$$

and using (3.18) and (3.19) we get

$$\delta\int_0^t\int_R |u_x||u_{xx}|\,dxds \le C\frac{\sqrt{\delta}}{\varepsilon}, \tag{3.26}$$

which gives (c) as soon as $\delta=o(\varepsilon^2)$.

Finally, (d) is obtained thanks to (3.18) since,

$$\delta^2\int_0^t\int_R u_{xx}^2\,dxds \le \frac{\delta}{\varepsilon}\left(\delta\varepsilon\int_0^t\int_R u_{xx}^2\,dxds\right)\le C\frac{\delta}{\varepsilon}. \tag{3.27}$$

□

3.2. Case f convex, $\delta>0$, and $g(u_{xx})=-u_{xx}^2$. We are concerned here with the equation

$$u_t+f(u)_x=\varepsilon u_{xx}-\delta(u_{xx}^2)_x. \tag{3.28}$$

It is noteworthy that the initial value problem associated with (3.28) is well-posed [1, 2].

PROPOSITION 3.6. *Let $\varepsilon>0$, $\delta=O(\varepsilon)$ and $f:R\to R$ be a convex flux function. Then, the solution $u=u_{\varepsilon,\delta}$ of (3.28) satisfies the estimate (3.9) with $\alpha=1$, i.e.,*

$$\int_R u(t)^2\,dx+\varepsilon\int_0^t\int_R u_x^2\,dxds+\delta\int_0^t\int_R |u_x|\,u_{xx}^2\,dxds\le C. \tag{3.29}$$

If in addition

$$f''(u)\le C(1+|u|^\beta),\ \text{where } 0\le\beta<1/2,$$

then the solution $u=u_{\varepsilon,\delta}$ checks

$$\int_R u_x(t)^2\,dx+\varepsilon\int_0^t\int_R u_{xx}^2\,dxds\le C+C\delta^{-1/2}\varepsilon^{-1/4}. \tag{3.30}$$

Proof. On the one hand, (3.2) with $n=2$, $\alpha=1$ is written

$$\int_R u(t)^2\,dx+2\varepsilon\int_0^t\int_R u_x^2\,dxds=\|u_0\|_2^2+2\delta\int_0^t\int_R u_x u_{xx}^2\,dxds, \tag{3.31}$$

and, from (3.7) with $q=1$, we get

$$\varepsilon\int_0^t\int_{\mathcal{U}_s^+} u_x u_{xx}^2\,dxds\le C. \tag{3.32}$$

Thus, if $\delta \le k\varepsilon$, (3.32) and (3.31) give (3.29).

On the other hand, assuming $f''(u) \le C(1+|u|^{\beta})$, with $0 \le \beta < 1/2$, we have

$$|f'(u) - f'(0)| \le C(|u| + |u|^{\beta+1}).$$

Then, to estimate the last term in (3.20), we proceed as in the case $n=1$:

$$\begin{aligned}
&\left|\int_0^t \int_R f'(u)\, u_x u_{xx}\, dxds\right| \\
&\le C\int_0^t \int_R |u||u_x||u_{xx}|\, dxds + C\int_0^t \int_R |u|^{\beta+1}|u_x||u_{xx}|\, dxds \\
&\le C\delta^{-1/2}\varepsilon^{-1/4}\int_0^t \int_R \left(\delta^{1/2}|u|^{1/2}|u_x|^{1/2}|u_{xx}|\right)\left(|u|^{1/2}\right)\left(\varepsilon^{1/4}|u_x|^{1/2}\right) dxds \\
&\qquad + C\delta^{-1/2}\varepsilon^{-1/4}\int_0^t \int_R \left(\delta^{1/2}|u|^{\beta+1/2}|u_x|^{1/2}|u_{xx}|\right)\left(|u|^{1/2}\right)\left(\varepsilon^{1/4}|u_x|^{1/2}\right) dxds.
\end{aligned}$$

Using Young's inequality, it becomes

$$\begin{aligned}
&\left|\int_0^t \int_R f'(u)\, u_x u_{xx}\, dxds\right| \\
&\le C\delta^{-1/2}\varepsilon^{-1/4}\Bigg(\frac{\delta}{2}\int_0^t \int_R |u|^{2\beta+1}|u_x|u_{xx}^2\, dxds \\
&\quad + \frac{\delta}{2}\int_0^t \int_R |u||u_x|u_{xx}^2\, dxds + \frac{1}{2}\int_0^t \int_R u^2\, dxds + \frac{\varepsilon}{2}\int_0^t \int_R u_x^2\, dxds\Bigg). \qquad (3.33)
\end{aligned}$$

Inequality (3.9) with $n=2$, $\alpha=2$ and $\alpha=2\beta+2<3$ is written as

$$\delta\int_0^t \int_R |u||u_x|u_{xx}^2\, dxds + \delta\int_0^t \int_R |u|^{2\beta+1}|u_x|u_{xx}^2\, dxds \le C. \qquad (3.34)$$

Now, substituting (3.34) and (3.29) into (3.33), we obtain

$$\left|\int_0^t \int_R f'(u)\, u_x u_{xx}\, dxds\right| \le C\delta^{-1/2}\varepsilon^{-1/4}. \qquad (3.35)$$

Finally, substituting (3.35) into (3.20) we get the required estimate (3.30). □

PROPOSITION 3.7. *Let $\varepsilon > 0$, $\delta = o(\varepsilon^{5/2})$, and $f : R \to R$ be a convex flux function satisfying*

$$f''(u) \le C(1+|u|^{\beta}),\ \text{where } 0 \le \beta < 1/2.$$

Then, the solution $u = u_{\varepsilon,\delta}$ of (3.28) satisfies

(a) $\{\varepsilon u_x^2\}$ is bounded in $L^1((0,t)\times R)$.
(b) $\{\varepsilon u_x\} \to 0$ as $\varepsilon \to 0$, in $L^2((0,t)\times R)$.
(c) $\{\delta u_x^- u_{xx}^2\}$, as $u_x^- = \max(0, -u_x)$, is bounded in $L^1((0,t)\times R)$.
(d) $\{\delta u_x^+ u_{xx}^2\} \to 0$, as $u_x^+ = \max(0, u_x)$ when $\varepsilon \to 0$ in $L^1((0,t)\times R)$.
(e) $\{\delta u_{xx}^2\} \to 0$ as $\varepsilon \to 0$, in $L^1((0,t)\times R)$.

Proof. The statements (a), (b), and (c) are obtained from (3.29).

Now, (d) is obtained from (3.7) with $q=1$ since

$$\delta\int_0^t\int_{\mathcal{U}_s{}^+} u_x u_{xx}^2\,dxds \le \frac{\delta}{\varepsilon}\left(\varepsilon\int_0^t\int_{\mathcal{U}_s{}^+} u_x u_{xx}^2\,dxds\right)\le C\,\frac{\delta}{\varepsilon}. \tag{3.36}$$

Finally, (3.30) provides (e) since

$$\delta\int_0^t\int_R u_{xx}^2\,dxds \le \delta^{\frac12}\varepsilon^{-5/4}(\delta^{\frac12}\varepsilon^{5/4}\int_0^t\int_R u_{xx}^2\,dxds)\le C\sqrt{\delta\varepsilon^{-5/2}}. \tag{3.37}$$

□

4. Convergence proof

We now define the measure-valued solutions to the first order Cauchy problem (1.3)-(1.4) as DiPerna [7].

Definition 4.1. *Assume that $u_0\in L^1(R)\cap L^q(R)$ and $f\in\mathcal{C}(R)$ satisfies the growth condition*

$$|f(u)|\le\mathcal{O}(|u|^m) \text{ as } |u|\to\infty, \qquad \textit{for some } m\in[0,q). \tag{4.1}$$

A Young's measure ν is called an entropy measure-valued (e.m.-v.) solution to (1.3)-(1.4) if

$$\langle\nu,|u-k|\rangle_t+\langle\nu,\mathrm{sgn}(u-k)(f(u)-f(k))\rangle_x\le 0, \qquad \textit{for all } k\in R, \tag{4.2}$$

in the sense of distributions on $(0,T)\times R$, and

$$\lim_{t\to 0^+}\frac{1}{t}\int_0^t\int_K\langle\nu_{(x,s)},|u-u_0(x)|\rangle\,dxds=0, \qquad \textit{for all compact set } K\subseteq R. \tag{4.3}$$

A representation theorem of Young's measures associated with a sequence of uniformly bounded functions of L^q is used to link the structure of measure and the strong convergence [16].

Lemma 4.2. *Let $\{u_n\}_{n\in N}$ be a bounded sequence in $L^\infty((0,T);L^q(R))$. Then there exists a subsequence denoted by $\{\tilde{u}_n\}_{n\in N}$ and a weakly-$\star$ measurable mapping $\nu:R\times(0,T)\to Prob(R)$ such that, for all functions $h\in\mathcal{C}(R)$ satisfying (4.1), $\langle\nu_{(x,t)},h\rangle$ is $L^\infty((0,T);L^{q/m}_{loc}(R))$ and the following limit representation holds:*

$$\iint_{R\times(0,T)}\langle\nu_{(x,t)},h\rangle\,\phi(x,t)\,dxdt=\lim_{n\to\infty}\iint_{R\times(0,T)}h(\tilde{u}_n(x,t))\,\phi(x,t)\,dxdt, \tag{4.4}$$

for all $\phi\in L^1(R\times(0,T))\cap L^\infty(R\times(0,T))$.

Conversely, given ν, there exists a sequence $\{u_n\}$ satisfying the same conditions as above and such that (4.4) holds for any h satisfying (4.1).

Proof of the main results. We begin proving (4.2) by using Proposition 3.5 (resp. Proposition 3.7), for $n=1$ (resp. $n=2$), and we apply the Young measure representation theorem in the suitable L^q space (4.4) to show that ν satisfies (4.2). Also, we use a standard regularization of $\mathrm{sgn}(u-k)(f(u)-f(k))$ and $|u-k|$ $(k\in R)$, since it is sufficient to show that there exists a bounded measure $\mu\le 0$ such that

$$\eta(u)_t+q(u)_x\longrightarrow\mu \quad \textit{in } \mathcal{D}'(R\times(0,T)) \tag{4.5}$$

for an arbitrary convex function η (we assume that η' and η'' are bounded on R).

Now, to prove (4.5), we rewrite formula (3.1) in the form

$$\eta(u)_t + q(u)_x = \mu_1 + \mu_2 + \mu_3 + \mu_4, \tag{4.6}$$

where,

$$\begin{aligned}
\mu_1 &:= \varepsilon\left(\eta'(u)\,u_x\right)_x;\\
\mu_2 &:= -\varepsilon\eta''(u)\,u_x^2;\\
\mu_3 &:= \delta\left(\eta'(u)\,g(u_{xx})\right)_x;\\
\mu_4 &:= -\delta\,\eta''(u)\,u_x\,g(u_{xx}).
\end{aligned}$$

We distinguish the case $n=1$ from $n=2$.

Case n = 1: $g(u_{xx}) = -|u_{xx}|$, f convex and $\delta = o(\varepsilon^2)$. We have

$$\begin{aligned}
|<\mu_1,\theta>| &\leq \varepsilon\int_0^T\int_R |\theta_x\,\eta'(u)\,u_x|\,dxds\\
&\leq \varepsilon\int_0^T\int_R |\theta_x\,u_x|\,dxds \leq C||\theta_x||_{L^2}||\varepsilon\,u_x||_{L^2},
\end{aligned} \tag{4.7}$$

$$|<\mu_2,\theta>| \leq \varepsilon\int_0^T\int_R |\theta\,\eta''(u)\,u_x^2|\,dxds \leq C||\theta||_{L^\infty}||\varepsilon\,u_x^2||_{L^1}. \tag{4.8}$$

Since η is a convex function, we notice for a non-negative function θ

$$<\mu_2,\theta> = -\varepsilon\int_0^T\int_R \theta\,\eta''(u)\,u_x^2\,dxds \leq 0. \tag{4.9}$$

In the same way, we have

$$\begin{aligned}
|<\mu_3,\theta>| &\leq \delta\int_0^T\int_R |\theta_x\,\eta'(u)\,|u_{xx}||\,dxds \leq C\,\delta\int_0^T\int_R |\theta_x\,|u_{xx}||\,dxds\\
&\leq C||\theta_x||_{L^2}||\delta\,|u_{xx}|||_{L^2},
\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}
|<\mu_4,\theta>| &\leq \delta\int_0^T\int_R |\theta\,\eta''(u)\,u_x|u_{xx}||\,dxds \leq C\,\delta\int_0^T\int_R |\theta\,u_x|u_{xx}||\,dxds\\
&\leq C||\theta||_{L^\infty}||\delta\,u_x|u_{xx}|||_{L^1}.
\end{aligned} \tag{4.11}$$

Combining (4.7), (4.8), (4.9), (4.10), and (4.11), with (a), (b), (c), and (d) in Proposition 3.5, gives (4.5) where μ is non-positive bounded measure.

Case n = 2: $g(u_{xx}) = -u_{xx}^2$, f convex and $\delta = o(\varepsilon^{5/2})$. Estimates (4.7), (4.8), and (4.9) remain true. Concerning μ_3, we have

$$\begin{aligned}
|<\mu_3,\theta>| &\leq \delta\int_0^T\int_R |\theta_x\,\eta'(u)\,u_{xx}^2|\,dxds \leq C\,\delta\int_0^T\int_R |\theta_x\,u_{xx}^2|\,dxds\\
&\leq C||\theta_x||_{L^\infty}||\delta\,u_{xx}^2||_{L^1}.
\end{aligned} \tag{4.12}$$

Now, μ_4 is split up as

$$\mu_4 = \mu_{41} + \mu_{42},$$

with

$$\mu_{41} := \delta\,\eta''(u)\,u_x^+\,u_{xx}^2;$$
$$\mu_{42} := -\delta\,\eta''(u)\,u_x^-\,u_{xx}^2,$$

where $u_x^+ = \max(0,u_x)$ and $u_x^- = \max(0,-u_x)$. Then we have

$$\begin{aligned}|<\mu_{41},\theta>| &\le \delta\int_0^T\int_R |\theta\eta''(u)\,u_x^+u_{xx}^2|\,dxds \le C\delta\int_0^T\int_R |\theta\,u_x^+\,u_{xx}^2|\,dxds\\ &\le C||\theta||_{L^\infty}||\delta\,u_x^+\,u_{xx}^2||_{L^1},\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}|<\mu_{42},\theta>| &\le \delta\int_0^T\int_R |\theta\eta''(u)\,u_x^-u_{xx}^2|\,dxds \le C\delta\int_0^T\int_R |\theta\,u_x^-\,u_{xx}^2|\,dxds\\ &\le C||\theta||_{L^\infty}||\delta\,u_x^-\,u_{xx}^2||_{L^1}.\end{aligned} \tag{4.14}$$

Again, since $\eta'' \ge 0$, we get for a non-negative function θ

$$<\mu_{42},\theta> = -\delta\int_0^T\int_R \theta\,\eta''(u)\,u_x^-u_{xx}^2\,dxds \le 0. \tag{4.15}$$

Finally, from inequalities (4.7), (4.8), (4.9), (4.12), (4.13), (4.14), and (4.15), combined with (a), (b), (c), and (d) in Proposition 3.7, we obtain (4.5) where μ is non-positive bounded measure.

Now we will prove (4.3). We follow arguments of DiPerna [7] and Szepessy [18]: we have to check that, for each compact set K of R,

$$\lim_{t\to 0+}\frac{1}{t}\int_0^t\int_K \langle \nu_{(x,s)}, |u-u_0(x)|\rangle\,dxds = \lim_{t\to 0+}\lim_{\varepsilon\to 0+}\frac{1}{t}\int_0^t\int_K \left|u^{\varepsilon,\delta}(x,s)-u_0(x)\right|dxds = 0.$$

By Jensen's inequality

$$\frac{1}{t}\int_0^t\int_K \left|u^{\varepsilon,\delta}(x,s)-u_0(x)\right|dxds \le m(K)^{1/2}\left(\frac{1}{t}\int_0^t\int_K \left(u^{\varepsilon,\delta}(x,s)-u_0(x)\right)^2 dxds\right)^{1/2},$$

where $m(K)$ denotes the Lebesgue measure of K. Then we will establish that

$$\lim_{t\to 0+}\lim_{\varepsilon\to 0+}\frac{1}{t}\int_0^t\int_K \left(u^{\varepsilon,\delta}(x,s)-u_0(x)\right)^2 dxds = 0.$$

Let $K_i \subset K_{i+1}$ ($i = 0,1,\dots$) be an increasing sequence of compact sets such that $K_0 = K$ and $\cup_{i\ge 0}K_i = R$. Using the identity $u^2 - u_0^2 - 2u_0(u-u_0) = (u-u_0)^2$, we get for all $i = 0,1,\dots$

$$\begin{aligned}&\frac{1}{t}\int_0^t\int_K \left(u^{\varepsilon,\delta}(\cdot,s)-u_0\right)^2 dxds\\ &\le \frac{1}{t}\int_0^t\left(\int_{K_i}|u^{\varepsilon,\delta}(\cdot,s)|^2\,dx - \int_{K_i}u_0^2\,dx - 2\int_{K_i}u_0\left(u^{\varepsilon,\delta}(\cdot,s)-u_0\right)dx\right)ds\\ &\le \int_{R\setminus K_i}u_0^2\,dx + \frac{1}{t}\int_0^t\left(\int_R |u^{\varepsilon,\delta}(\cdot,s)|^2\,dx - \int_R u_0^2\,dx\right)\end{aligned}$$

$$+\frac{2}{t}\int_0^t \left|\int_{K_i} u_0\left(u^{\varepsilon,\delta}(\cdot,s)-u_0\right)dx\right| ds. \tag{4.16}$$

For the first term of the right-hand side, we clearly have

$$\lim_{i\to\infty}\int_{R\setminus K_i} u_0^2\,dx=0.$$

Now, substituting (3.26) into (3.23), in the case where $n=1$, and respectively (3.36) into (3.31), in the case $n=2$, we obtain

$$\int_R |u^{\varepsilon,\delta}(\cdot,s)|^2\,dx-\int_R u_0^2\,dx\le C\frac{\sqrt{\delta}}{\varepsilon},$$

and respectively,

$$\int_R |u^{\varepsilon,\delta}(\cdot,s)|^2\,dx-\int_R u_0^2\,dx\le C\frac{\delta}{\varepsilon}.$$

In both cases, the right-hand side of these inequalities tends to zero when $\varepsilon\to 0$.

To estimate the last term in the inequality (4.16), we choose $\{\theta_n\}_{n\in N}\subset\mathcal{C}_0^\infty(R)$ such that

$$\lim_{n\to\infty}\theta_n=u_0\quad in\ L^2(R).$$

Then, the Cauchy–Schwarz inequality gives

$$\begin{aligned}\left|\int_{K_i} u_0\left(u^{\varepsilon,\delta}(\cdot,s)-u_0\right)dx\right|\le&\int_{K_i}|u_0-\theta_n|\left|u^{\varepsilon,\delta}(\cdot,s)-u_0\right|dx\\&+\left|\int_{K_i}\theta_n\left(u_0^{\varepsilon,\delta}-u_0\right)+\int_{K_i}\theta_n\left(u^{\varepsilon,\delta}(\cdot,s)-u_0^{\varepsilon,\delta}\right)dx\right|\\\le&\|u_0-\theta_n\|_{L^2(R)}\left(\|u^{\varepsilon,\delta}(\cdot,s)\|_{L^2(R)}+\|u_0\|_{L^2(R)}\right)\\&+\|\theta_n\|_{L^2(R)}\|u_0^{\varepsilon,\delta}-u_0\|_{L^2(R)}+\left|\int_0^s\int_{K_i}\theta_n\,\partial_s u^{\varepsilon,\delta}\,dxd\tau\right|.\end{aligned}$$

In view of (3.19) for $n=1$ and respectively (3.29) for $n=2$, we have

$$\|u_0-\theta_n\|_{L^2(R)}\left(\|u^{\varepsilon,\delta}(\cdot,s)\|_{L^2(R)}+\|u_0\|_{L^2(R)}\right)\le\left(\|u_0\|_{L^2(R)}+C\right)\|u_0-\theta_n\|_{L^2(R)},$$

which tends to zero when $n\to\infty$, since $\lim_{\varepsilon\to 0+}\|u_0^{\varepsilon,\delta}-u_0\|_{L^2(R)}=0$. Finally, it remains to prove that

$$\lim_{t\to 0+}\lim_{\varepsilon\to 0+}\frac{1}{t}\int_0^t\left|\int_0^s\int_{K_i}\theta_n\,\partial_s u^{\varepsilon,\delta}\,dxd\tau\right|ds=0.$$

We have, by (1.1),

$$\begin{aligned}\left|\int_0^s\int_{K_i}\theta_n\,\partial_s u^{\varepsilon,\delta}\,dxd\tau\right|=&\left|\int_0^s\int_{K_i}\theta_n\left(-f(u^{\varepsilon,\delta})_x+\varepsilon u_{xx}^{\varepsilon,\delta}+\delta g\left(u_{xx}^{\varepsilon,\delta}\right)_x\right)dxd\tau\right|\\\le&\int_0^s\int_{K_i}\left|(\theta_n)_x\,f\left(u^{\varepsilon,\delta}\right)\right|dxd\tau+\varepsilon\int_0^s\int_{K_i}\left|(\theta_n)_x\,u_x^{\varepsilon,\delta}\right|dxd\tau\end{aligned}$$

$$+\delta\int_0^s\int_{K_i}\left|(\theta_n)_x\,g\left(u_{xx}^{\varepsilon,\delta}\right)\right|dxd\tau$$
$$:=I_1+I_2+I_3.$$

To compute each quantity I_1, I_2, and I_3, we distinguish the cases $n=1$ from $n=2$.

Case n = 1: $g(u_{xx})=-|u_{xx}|$, f convex and $\delta=o(\varepsilon^2)$. Since f is such that

$$0\le f''(u)\le C(1+|u|^\beta),$$

where $\beta<3$. Thus,

$$|f(u)|\le C(1+|u|^m),$$

where $m<5$. Then, Proposition 3.3 implies

$$\int_0^s\int_{K_i}|u^{\varepsilon,\delta}|^m\,dx\,d\tau\le\int_0^s\int_R|u^{\varepsilon,\delta}|^m\,dx\,d\tau\le C\,s, \tag{4.17}$$

and

$$\begin{aligned}I_1=&\int_0^s\int_{K_i}|(\theta_n)_x|\left|f\left(u^{\varepsilon,\delta}\right)\right|dx\,d\tau\\ \le& C\int_0^s\int_{K_i}|(\theta_n)_x|\,dx\,d\tau+C\int_0^s\int_{K_i}|(\theta_n)_x|\,|u^{\varepsilon,\delta}|^m\,dx\,d\tau\\ \le& C\int_0^s\int_{K_i}|(\theta_n)_x|\,dx\,d\tau+C\|(\theta_n)_x\|_{L^\infty(R)}\int_0^s\int_{K_i}|u^{\varepsilon,\delta}|^m\,dx\,d\tau\\ \le& C\,s\|(\theta_n)_x\|_{L^1(R)}+C\,s\|(\theta_n)_x\|_{L^\infty(R)}.\end{aligned} \tag{4.18}$$

Thanks to (3.19), we can write

$$\begin{aligned}I_2=&\varepsilon\int_0^s\int_{K_i}|(\theta_n)_x|\left|u_x^{\varepsilon,\delta}\right|dx\,d\tau\\ \le&\left(\varepsilon\int_0^s\int_{K_i}|(\theta_n)_x|^2\,dxd\tau\right)^{\frac12}\left(\varepsilon\int_0^s\int_{K_i}\left|u_x^{\varepsilon,\delta}\right|^2dxd\tau\right)^{\frac12}\\ \le& C\,\varepsilon^{\frac12}\,s^{\frac12}\,\|(\theta_n)_x\|_{L^2(R)}.\end{aligned} \tag{4.19}$$

Finally for I_3, using (3.27), we get

$$\begin{aligned}I_3=&\delta\int_0^s\int_{K_i}\left|(\theta_n)_x g\left(u_{xx}^{\varepsilon,\delta}\right)\right|dxd\tau=\int_0^s\int_{K_i}\delta\,|(\theta_n)_x|\left|u_{xx}^{\varepsilon,\delta}\right|dxd\tau\\ \le&\left(\int_0^s\int_{K_i}|(\theta_n)_x|^2\,dxd\tau\right)^{\frac12}\left(\delta^2\int_0^s\int_{K_i}\left|u_{xx}^{\varepsilon,\delta}\right|^2dxd\tau\right)^{\frac12}\\ \le& s^{\frac12}\,\|(\theta_n)_x\|_{L^2(R)}\left(\delta^2\int_0^s\int_{K_i}\left|u_{xx}^{\varepsilon,\delta}\right|^2dxd\tau\right)^{\frac12}\\ \le& s^{\frac12}\left(\frac{\delta}{\varepsilon}\right)^{\frac12}\|(\theta_n)_x\|_{L^2(R)}.\end{aligned} \tag{4.20}$$

Now, from (4.18), (4.19), and (4.20), we deduce

$$\lim_{t\to0+}\lim_{\varepsilon\to0+}\frac1t\int_0^t\left|\int_0^s\int_{K_i}\theta_n\,u_s^{\varepsilon,\delta}\,dxd\tau\right|ds$$

$$\leq \lim_{t\to 0+}\lim_{\varepsilon\to 0+}\frac{C}{t}\left(t^2(\|(\theta_n)_x\|_{L^1(R)}+\|(\theta_n)_x\|_{L^\infty(R)})+\varepsilon^{\frac{1}{2}}t^{\frac{3}{2}}\|(\theta_n)_x\|_{L^2(R)}\right.$$
$$\left.+t^{\frac{3}{2}}\left(\frac{\delta}{\varepsilon}\right)^{\frac{1}{2}}\|(\theta_n)_x\|_{L^2(R)}.\right)$$
$$\leq \lim_{t\to 0+}\lim_{\varepsilon\to 0+} C\left(t+\varepsilon^{\frac{1}{2}}t^{\frac{1}{2}}+t^{\frac{1}{2}}\left(\frac{\delta}{\varepsilon}\right)^{\frac{1}{2}}\right),$$

and since $\delta=o(\varepsilon^2)$, we obtain the desired conclusion, and Theorem 2.1 is proved.

Case n = 2: $g(u_{xx})=-u_{xx}^2$, f convex and $\delta=o(\varepsilon^{5/2})$. Here, f is such that

$$0\leq f''(u)\leq C(1+|u|^\beta),$$

where $\beta<1/2$, thus,

$$|f(u)|\leq C(1+|u|^m),$$

where $m<5/2$.

Estimates I_1 and I_2 are obtained in the same manner as $n=1$ using (3.29) instead of (3.18). From (3.37) we obtain

$$I_3=\delta\int_0^s\int_{K_i}\left|(\theta_n)_x g\left(u_{xx}^{\varepsilon,\delta}\right)\right| dxd\tau=\int_0^s\int_{K_i}\delta\,|(\theta_n)_x|\left|u_{xx}^{\varepsilon,\delta}\right|^2 dxd\tau$$
$$\leq \|(\theta_n)_x\|_{L^\infty(R)}\left(\delta\int_0^s\int_{K_i}\left|u_{xx}^{\varepsilon,\delta}\right|^2 dxd\tau\right)$$
$$\leq \sqrt{\delta\varepsilon^{-5/2}}\,\|(\theta_n)_x\|_{L^\infty(R)}. \tag{4.21}$$

Finally, using (4.18), (4.19), and (4.21), we obtain

$$\lim_{t\to 0+}\lim_{\varepsilon\to 0+}\frac{1}{t}\int_0^t\left|\int_0^s\int_{K_i}\theta_n u_s^{\varepsilon,\delta}\,dxd\tau\right|ds$$
$$\leq \lim_{t\to 0+}\lim_{\varepsilon\to 0+}\frac{C}{t}\left(t^2(\|(\theta_n)_x\|_{L^1(R)}+\|(\theta_n)_x\|_{L^\infty(R)})+\varepsilon^{\frac{1}{2}}t^{\frac{3}{2}}\|(\theta_n)_x\|_{L^2(R)}\right.$$
$$\left.+t\sqrt{\delta\varepsilon^{-5/2}}\,\|(\theta_n)_x\|_{L^\infty(R)}.\right)$$
$$\leq \lim_{t\to 0+}\lim_{\varepsilon\to 0+} C\left(t+\varepsilon^{\frac{1}{2}}t^{\frac{1}{2}}+\sqrt{\delta\varepsilon^{-5/2}}\right),$$

and since $\delta=o(\varepsilon^{5/2})$, we obtain the desired conclusion, and Theorem 2.2 is proved.

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