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DIFFERENTIAL QUADRATURE-BASED NUMERICAL SOLUTIONS OF A FLUID DYNAMIC MODEL FOR SUPPLY CHAINS*

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Abstract. In this paper, we discuss a numerical approach for the simulation of a model for supply chains based on both ordinary and partial differential equations. Such a methodology foresees differential quadrature rules and a Picard-like recursion. In its former version, it was proposed for the solution of ordinary differential equations and is here extended to the case of partial differential equations. The outcome is a final non-recursive scheme, which uses matrices and vectors, with consequent advantages for the determination of the local error. A test case shows that traditional methods give worse approximations with respect to the proposed formulation.

Key words. Conservation laws, supply chains, DQ rules, simulation.

AMS subject classifications. 35L65, 65Q30, 90B30.

1. Introduction

Industrial applications, with a particular emphasis on supply systems, represent an important matter nowadays, with the aim of controlling unwanted phenomena, such as bottlenecks, dead times, and so on.

Suppliers, manufacturers, warehouses, and stores are components of supply chains and networks, where parts are produced, assembled, and distributed among various production facilities. Mathematical models are useful to capture distribution dynamics of parts and to estimate business processes. Depending on the scale, such models can be discrete, based on Discrete Event Simulation (DES), [11, 28], or continuous ([1, 2, 3, 29]), by using Ordinary and/or Partial Differential Equations (ODEs, PDEs), see [12] and [13] for exhaustive discussions. The most meaningful difference between the two classes is the description of parts as individuals or as a flow, respectively. In particular, continuous models are suitable to manage production activities where the involved quantities are large.

In this paper, we focus on the continuous approach by considering the model proposed in [15, 16] and discussed in [20] for the existence of solutions: there are suppliers on which the processing rate is constant (in order to avoid the problem of delta waves) and queues in front of each supplier. The dynamics of parts inside a processor is described by a conservation law ([5, 21]), while the evolution of the queue buffer occupancy

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is the difference of fluxes from the preceding and following suppliers. Hence, a coupled ODE-PDE model is obtained.

Different numerical approaches are possible for models with conservation laws, see [6, 7, 8, 10, 14, 23], with emphasis on various optimization problems ([9, 17, 24, 25]). In our case, the approach considers a numerical scheme in compact form to solve the system of differential equations of our supply system model. The definition of such a further numerical method is a direct consequence of many real applications, where optimization topics naturally arise, see for instance [16, 18, 19, 22].

The proposed scheme is in main part based on an extension to PDEs of the approach suggested in [31, 32] to solve Volterra integro-differential equations and some eigenvalues problems [33]. The methodology foresees a combination of Differential Quadrature (DQ) rules [4], which provide high-order finite-difference approximations, and a Picard-like recursion, whose features are suitable for serial networks. The DQ rules have been successfully used in a certain class of problems [26, 27] and even in element based approaches [34, 35].

Although the method of successive approximations is usually used as a semianalytical approach, classical quadrature, and DQ rules are introduced to handle integrals and derivatives, with the aim of improving the computational efficiency ([30]). In spite of its recursive nature, the proposed approach leads to a final non-recursive approximate solution by means of operational matrices and vectors of known quantities. So, the order of the local error is easily determined. Combining the technique of successive approximations to DQ allows to avoid slow computations (as for example when there are rational functions), without increasing the number of grid points, as instead finite difference based methods need.

In order to test the numerical approach, we consider the one supplier case that, in spite of its simplicity, has practical applications, as shown in [36] and [37]. Different initial conditions show that L_∞ norms for densities are considerably lower than the ones obtained with classical methods (described in [10]), especially when Gauss–Chebyshev–Lobatto (GCL) points are considered.

The outline of the paper is the following. In Section 2, we present the ODE-PDE model. Section 3 describes the DQ-based Picard-like numerical method. Section 4 reports a short numerical test. Conclusions ends the paper in Section 5.

2. A fluid-dynamic model for supply chains

In this section, we present an ODE-PDE model for supply chains, based on the work [1] and presented in [15, 16]. Besides the conservation laws formulation proposed in [1], such model considers time dependent queues for the description of the transition of parts among suppliers.

A supply chain is a directed graph consisting of arcs $\mathcal{J}=\{1,\dots,P\}$ and vertices $\mathcal{V}=\{1,\dots,P-1\}$. Each arc $j\in\mathcal{J}$, parameterized by an interval $[a_j,b_j]$, models a supplier. Here we consider the special case where each vertex is connected to one incoming arc and one outgoing arc and arcs are consecutively labeled, i.e. arc j is connected to arc $j+1$ and $b_j=a_{j+1}$. For the first and the last arc, we either set $a_1=-\infty$ and $b_P=+\infty$, respectively, or provide boundary data for the inflow and outflow.

For each supplier $j\in\mathcal{J}$, indicate by: $L_j>0$ the length; $T_j>0$ the processing time; $v_j:=L_j/T_j$ the processing velocity; $\mu_j>0$ the maximum processing capacity; $\rho_j(t,x)\in\left[0,\rho_j^{\max}\right]$ the density of parts at point x and time t. The evolution of parts is described by the following conservation law:

$$\partial_t\rho_j(t,x)+\partial_x f_j(\rho_j(t,x))=0,\quad \forall\, x\in[a_j,b_j],\quad t>0, \tag{2.1}$$

$$\rho_j(0,x)=\rho_{j,0}(x)\geq 0,\quad \rho_j(t,a_j)=\frac{f_{j,inc}(t)}{v_j},$$

where $f_j(\rho_j(t,x))$ is the flux function, defined as

$$f_j:[0,+\infty[\to[0,\mu_j],\quad f_j(\rho_j(t,x)):=\min\{\mu_j,v_j\rho_j(t,x)\},$$

while the initial data, $\rho_{j,0}$ and the inflow, $f_{j,inc}(t)$, have to be assigned. Equation (2.1) is interpreted as follows: parts are processed with velocity v_j and with a maximal flux μ_j.

Each supplier $j\in\mathcal{J}\setminus\{1\}$ has a queue at $x=a_j$, namely in front of itself. Such queue increases or decreases if the capacity of the supplier $j-1$ and the demand of the supplier j are not equal. In particular, it is a time dependent function $t\to q_j(t)$, whose evolution is described by:

$$\frac{d}{dt}q_j(t)=f_{j-1}(\rho_{j-1}(t,b_{j-1}))-f_{j,inc}(t),\quad j\in\mathcal{J}\setminus\{1\},\tag{2.2}$$

$$q_j(0)=q_{j,0}\geq 0,$$

where $f_{j-1}(\rho_{j-1}(t,b_{j-1}))$ is given by the density of parts on supplier $j-1$.

Notice that the flux on the outgoing arc j is defined as:

$$f_{j,inc}(t):=\begin{cases}\varphi(t), & j=1,\\ \min\{f_{j-1}(\rho_{j-1}(t,b_{j-1})),\mu_j\}, & q_j(t)=0,\ j\in\mathcal{J}\setminus\{1\},\\ \mu_j, & q_j(t)>0,\ j\in\mathcal{J}\setminus\{1\},\end{cases}\tag{2.3}$$

with the following interpretation: if $j=1$, namely we consider the first arc of the supply chain, it represents an assigned input profile $\varphi(t)$ on the left boundary $\{(t,a_1):t\in\mathbb{R}\}$. If $j\in\mathcal{J}\setminus\{1\}$, Equation (2.3) is strictly dependent on the capacity of the queue: If the queue buffer is empty, the inflow to supplier j and the outflow from supplier $j-1$ are equal; otherwise, the inflow is maximal. Hence, when $q_j(t)>0$, the processing occurs at μ_j, the maximal possible rate, so as to empty the queue as fast as possible.

Finally, the complete system of equations is:

$$\partial_t\rho_j(t,x)+\partial_x\min\{\mu_j,v_j\rho_j(t,x)\}=0,\ \forall\ x\in[a_j,b_j],\quad t>0,\quad j\in\mathcal{J},\tag{2.4}$$

$$\rho_j(0,x)=\rho_{j,0}(x)\geq 0,\quad \forall\ x\in[a_j,b_j],\quad \rho_j(t,a_j)=\frac{f_{j,inc}(t)}{v_j},\quad t>0,\tag{2.5}$$

$$\frac{d}{dt}q_j(t)=f_{j-1}(\rho_{j-1}(t,b_{j-1}))-f_{j,inc}(t),\ j\in\mathcal{J}\setminus\{1\},\tag{2.6}$$

$$q_j(0)=q_{j,0}\geq 0,\tag{2.7}$$

$$f_{j,inc}(t):=\begin{cases}\varphi(t), & j=1,\\ \min\{f_{j-1}(\rho_{j-1}(t,b_{j-1})),\mu_j\}, & q_j(t)=0,\ j\in\mathcal{J}\setminus\{1\},\\ \mu_j, & q_j(t)>0,\ j\in\mathcal{J}\setminus\{1\}.\end{cases}\tag{2.8}$$

The definition of a solution to the supply chain model (2.4)–(2.8) is provided in an exhaustive way in [12] and [15], following the wave- or front-tracking method ([5, 21]).

Precisely: first the existence results for the Cauchy problem on a single arc without coupling condition is discussed; then, the definition of the solution at a single node is defined; finally, the obtained results are combined to prove the existence for the whole system.
Existence and uniqueness of the solution on the whole network is also discussed in [20], where the authors prove the Lipschitz continuous dependence on the initial data and construct approximate solutions using the front tracking method.

REMARK 2.1. Considering the supply-demand framework suggested by J. P. Lebacque, the discussed model has analogies with the road traffic theory. (For details, see [13], Section 5.2.3, page 106.)

3. The proposed numerical scheme

We introduce a numerical scheme aimed at computing the solution to the model described in the previous section. Numerical results for dynamics of parts are obtained by finding suitable approximations of $\rho_j(t,x)$, $j\in\mathcal{J}$, and $q_j(t)$, $j\in\mathcal{J}\setminus\{1\}$.

The numerical scheme is defined by combining DQ rules [4] and a Picard-like recursion. Indeed, such a fusion leads to a final non-recursive approximate solution through expressions having matrices, with consequent advantages in terms either of computation of the local error or simulation of supply system with a high number of arcs.

In what follows, we omit the subscript j from $\rho_j(t,x)$ and $q_j(t)$, and discuss the situation for which $f=v\rho$ and $q>0$, as the other cases are straightforward. First, we consider numerics for density of parts; then, in order to consider coupling conditions at nodes, a suitable analysis of queues is made to obtain the final approximation scheme.

3.1. Density of parts.

Consider the operator $L_t=\partial_t$ and its inverse

$$L_t^{-1}(\cdot)=\int_0^t(\cdot)dt.$$

By applying L_t^{-1} on both sides of (2.1), we get

$$\rho(t,x)=\rho_0(t,x)-L_t^{-1}(L_x(f(\rho(t,x))))$$

where $L_x=\partial_x$.

Following the idea of the successive approximations, the solution $\overline{\rho}(t,x)$ is written as a series of unknown functions $\rho_i(t,x)$, which are found recursively, namely

$$\overline{\rho}(t,x)=\sum_{k=0}^{+\infty}\rho_k(t,x), \tag{3.1}$$

$$\rho(0,x)=\rho_0(x),$$

$$\rho_{k+1}(t,x)=-vL_t^{-1}(L_x(\rho_k(t,x))),\ k\in\mathbb{N},$$

where the term v appears from the assumption $f(\rho_k(t,x))=v\rho_k(t,x)$.

Let $l_i(t)$ and $l_j(x)$ be the Lagrange polynomials for time and space variables, N and M the number of grid points at abscissae $t_1<t_2<\cdots<t_{N-1}<t_N$ and $x_1<x_2<\cdots<x_{M-1}<x_M$, respectively.

Using numerical integration and DQ rules, i.e. by writing integrals and derivatives as weighted sums, we obtain for a single time interval in the following compact form:

$$\rho_{k+1}(t,x) = -v\sum_{i=1}^{N} C_i(t)\sum_{j=1}^{M} A_j^{(1)}(x)\rho_k(t_i,x_j),$$

where

$$\rho_{k+1}(t,x) = -v\mathbf{C}(t)\overline{\mathbf{A}}(x)\rho_k, \tag{3.2}$$

where ρ_k is the vector

$$\rho_k^T = (\rho_{k,11},\ldots,\rho_{k,N1},\ldots,\rho_{k,1M},\ldots,\rho_{k,NM}),$$

with $\rho_{k,ij} = \rho_k(t_i,x_j)$; $\mathbf{C}$ is the row vector of the weights $C_i(t) = \int_0^t l_i(z)dz$, namely:

$$\mathbf{C}(t) = (C_1(t),\ldots,C_N(t));$$

$\overline{\mathbf{A}}(x)$ is a matrix obtained by row sub-vectors $\mathbf{A}_i(x)$ with non-null value just on the diagonal, i.e.

$$\mathbf{A}_i(x) = (A_1^{(1)}(x),\ldots,A_M^{(1)}(x)),$$

with $i = 1,\ldots,N$, while functions $A_j^{(1)}(x) = \dfrac{dl_j(x)}{dx}$ are the weights of the quadrature rules. As

$$\rho_k = \overline{v}\overline{\mathbf{B}}\rho_{k-1} = (\overline{v}\overline{\mathbf{B}})^k\rho_0,$$

where $\overline{\mathbf{B}}$ is a square matrix of size MN obtained as:

$$\overline{\mathbf{B}} = \begin{pmatrix} A_{11}^{(1)}C_{11} & \ldots & A_{1M}^{(1)}C_{1N} \\ \vdots & \ddots & \vdots \\ A_{M1}^{(1)}C_{N1} & \ldots & A_{MM}^{(1)}C_{NN} \end{pmatrix},$$

with $A_{kj}^{(1)} = A_k^{(1)}(x_j)$, $C_{ki} = C_k(t_i)$; $\overline{v} = -v$ and

$$\rho_0^T = (\rho_{0,11},\ldots,\rho_{0,N1},\ldots,\rho_{0,1M},\ldots,\rho_{0,NM}),$$

equation (3.2) is written as:

$$\rho_{k+1} = \overline{v}\mathbf{C}(t)\overline{\mathbf{A}}(x)(\overline{v}\overline{\mathbf{B}})^k\rho_0.$$

Via division of the time domain into sub-domains with length Δt, a recurrence relation is obtained, considering that

$$\rho^{[p,r]}(t_N,x) = \rho^{[p,r+1]}(0,x),$$

where the apex r indicates the rth interval and p is the number of terms to truncate the sum in Equation (3.1). The solution is given by the first p series terms (besides ρ_0) in the time interval r,

$$\rho^{[p,r]}(t,x) = \sum_{i=0}^{p}\rho_i(t,x) = \rho_0(x) + \overline{v}\mathbf{C}(t)\overline{\mathbf{A}}(x)\sum_{i=0}^{p-1}(\overline{v}\overline{\mathbf{B}})^i\rho_0.$$

As for $\overline{\rho}(t,x)$, we have the following.

THEOREM 3.1. *Let $\overline{\lambda}(\overline{\mathbf{B}})$ be the spectral radius of the matrix $\overline{\mathbf{B}}$. If $\overline{\lambda}(\overline{v}\overline{\mathbf{B}})\leq 1$, then the solution $\overline{\rho}(t,x)$ in equation (3.1) is*

$$\overline{\rho}(t,x)=\rho_0(x)+\overline{v}\ \mathbf{C}(t)\overline{\mathbf{A}}(x)(\ \mathbf{I}-\overline{v}\overline{\mathbf{B}})^{-1}\rho_0. \tag{3.3}$$

Proof. Since the proposed approximation has the same form deduced in [31] for ODEs, Equation (3.3) is simply obtained following Theorem 1 in [31]. ∎

Now, consider the error $e(t,x):=\rho(t,x)-\overline{\rho}(t,x)$. From Theorem 3.1, we get:

LEMMA 3.2. *If Δt is the length of the time interval, then the error $e(\Delta t,x)$, evaluated at the end of the single time interval and at $\overline{x}$, assumes the finite value:*

$$e(\Delta t,\overline{x})=-\overline{v}\ \mathbf{C}(\Delta t)\overline{\mathbf{A}}(\overline{x})(\ \mathbf{I}-\overline{v}\overline{\mathbf{B}})^{-1}\rho_0+O(\Delta t).$$

Proof. Consider the Taylor expansion of $\rho(t,x)$ at the abscissa $\overline{x}$ around $t=0$, truncated at the linear term. We obtain:

$$\rho(t,\overline{x})=\rho_0(x)+O(t),$$

and the conclusion easily follows. ∎

3.2. Queues and final scheme. Now, we obtain the final compact scheme by finding an approximation for q. Under the assumption $f=v\rho$, the flux on the previous arc is $v\rho(x_M)$, where x_M represents the last space point of the previous arc. Instead, assuming $q>0$, the flux on the current arc is simply μ. Hence, Equation (2.2) reads as:

$$\frac{d}{dt}q(t)=v\rho(x_M)-\mu. \tag{3.4}$$

By applying the inverse operator L_t^{-1} to equation (3.4), we obtain:

$$q(t)=q_0+v\mathbf{C}(t)\left(\rho(x_M)-\mu\right), \tag{3.5}$$

where μ is a vector of which N entries are all equal to μ, while the vector $\rho(x_M)$ is given by

$$\rho(x_M)=\rho_0(x_M)+\overline{v}\mathbf{D}(\ \mathbf{I}-\overline{v}\overline{\mathbf{B}})^{-1}\rho_0,$$

and the $N\times NM$ matrix $\mathbf{D}=\overline{\mathbf{C}\mathbf{A}}(x_M)$, being $\overline{\mathbf{C}}$ the matrix with elements $C_{ij}=C(t_j)$.

Writing equation (3.5) for each time t_i, $i=1,\ldots,N$, we get the vector

$$\mathbf{q}=\overline{\mathbf{q}}_0+\mathbf{H}\rho_0,$$

where $\mathbf{H}=\overline{v}\overline{\mathbf{C}}\mathbf{D}(\ \mathbf{I}-\overline{v}\overline{\mathbf{B}})^{-1}$, $\overline{\mathbf{q}}_0=q(0)\mathbf{I}_N+\overline{\mathbf{C}}(\rho_0(x_M)-\mu)$ and $\mathbf{I}_N$ is the identity matrix of order N.

From equation (3.3), for the time interval r we obtain

$$\rho^{[r]}=\mathbf{P}\rho^{[r-1]},$$

with

$$\mathbf{P}=\mathbf{I}+\overline{v}\overline{\mathbf{B}}(\mathbf{I}-\overline{v}\overline{\mathbf{B}})^{-1}.$$

As

$$q(0)^{[r]} = q(x_M)^{[r-1]},$$

the final computational scheme in compact form becomes

$$\mathbf{w}^{[k]} = \mathbf{Q}\overline{\mathbf{w}}^{[k-1]}, \tag{3.6}$$

with $\mathbf{w}^{[k]T} = (\mathbf{q}, \rho)^T$, $\overline{\mathbf{w}}^{[k]T} = (\overline{\mathbf{q}}, \rho)^T$ and $\mathbf{Q}$ defined as

$$\mathbf{Q} = \begin{bmatrix} \mathbf{I}_N & \mathbf{H} \\ \mathbf{0} & \mathbf{P} \end{bmatrix},$$

where $\mathbf{0}$ indicates an $NM \times N$ null matrix.

REMARK 3.1. The computational cost of the proposed approach is due to the inversion of matrix $\mathbf{B}$, i.e. it is $O(N^3M^3)$. A useful comparison in made considering standard numerical schemes, such as upwind for densities and explicit Euler for queues (see [10]). In this last case, the update of densities for each arc depends on space grid size while, for boundary data, there is a dependence on either time grid size or the whole number of arcs for the system. Notice that the proposed numerical scheme considers, inside the matrix inversion, the computation either of densities or queues. Such cost can be decreased by suitable algorithms (see [30]).

REMARK 3.2. Stability of (3.6) is achieved if $\overline{\lambda}(\mathbf{Q}) \leq 1$, where $\overline{\lambda}(\mathbf{Q})$ is the spectral radius of $\mathbf{Q}$.

4. A numerical test

In this section, we consider the simulation of a supply chain with one supplier having $L_1 = T_1 = 1$ (hence $v_1 = 1$) and $\mu_1 = 50$. In what follows, we indicate simply by $\rho(t,x)$ the density of the unique arc of the supply chain and, according to the model described in Section 2, we have no queues. All the results were obtained under the Theorem 1 (see [31]) and by satisfying the stability condition $\overline{\lambda}(\mathbf{Q}) \leq 1$.

Consider the initial and boundary conditions

$$\rho(0,x) = e^{-\left(x-\frac{1}{100}\right)^2}, \quad \rho(t,0) = e^{-\left(-v_1 t-\frac{1}{100}\right)^2}. \tag{4.1}$$

Table 2 shows the L_∞ norm for the ρ function, by using several values of M and Δ, with $N = 6$ either for uniform grid points or GCL ones. Notice that GCL points allow more accurate results.

M	Δ	$L_\infty^{(1)}$	$L_\infty^{(2)}$
9	1/20	7.94×10^{-8}	7.24×10^{-7}
13	1/20	8.71×10^{-10}	4.43×10^{-9}
13	1/50	6.87×10^{-10}	7.72×10^{-11}
13	1/100	2.43×10^{-10}	2.00×10^{-11}

TABLE 2: L_∞ norm for $\rho(t,x)$ for uniform grid points ($L_\infty^{(1)}$); GCL points ($L_\infty^{(2)}$).

The approximation $\rho(t,x)$ for $M = 13$, $N = 6$, and $\Delta = 1/100$ is in Figure 4.1, left. Now, we assume the following initial and boundary conditions, respectively,

$$\rho(0,x) = \sin(\pi x)^a, \quad \rho(t,0) = \sin(-v_1 t)^a, \tag{4.2}$$

in two different cases, $a=10$ and $a=100$. Table 3 shows the L_∞ norm for the ρ function and different values of M and Δ, with $N=6$, $a=10$, by using GCL points. Results obtained by using uniform grid points are considerably worst and are not listed in the table for the sake of brevity. For instance, for $a=10$, $M=19$, $\Delta=1/500$, uniform grid points provide a result such as $L_\infty=1.3\times10^{-2}$, while by means of GCL we get $L_\infty=4.7\times10^{-5}$, as shown in Table 3. Similar numerical results for $a=100$ are in Table 4.

M	Δ	L_∞
13	1/200	5.143×10^{-3}
19	1/200	1.97×10^{-4}
16	1/500	2.5×10^{-4}
19	1/500	4.7×10^{-5}

TABLE 3: L_∞ norm for $\rho(t,x)$ in case of GCL points for $\rho_0=\sin(\pi x)^{10}$.

M	Δ	L_∞
13	1/200	1.17×10^{-2}
19	1/200	1.3×10^{-2}
16	1/500	9.89×10^{-3}
19	1/500	8.21×10^{-7}

TABLE 4: L_∞ norm for $\rho(t,x)$ in case of GCL points for $\rho_0=\sin(\pi x)^{100}$.

For $a=100$ a noticeable improvement in numerical results is achieved using a smaller time interval Δ, with few grid points.

The approximation of $\rho(t,x)$ for $a=10$, $N=6$, $M=19$, $\Delta=1/500$ is in Figure 4.1, right.

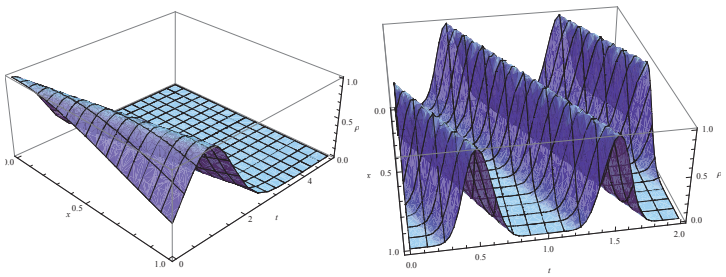

FIG. 4.1. *Approximation of $\rho(t,x)$ for conditions 4.1 (left) and 4.2 (right).*

A further comparison is made analyzing the simulation of the same supply chain using classical methods (upwind scheme for Equation (2.1) and the explicit Euler scheme for Equation (2.2), see [10]) for cases in which $\rho_0=\sin(\pi x)^{10}$ and $\rho_0=\sin(\pi x)^{100}$. Consider the results in Table 5, obtained using the space grid mesh $\Delta x=\dfrac{L_1}{M}$ and the time

grid mesh $\Delta t = \Delta$ so as to satisfy the CFL condition $\Delta x \geq v_1 \Delta t$.

M	Δ	$L_\infty^{(\mathrm{I})}$	$L_\infty^{(\mathrm{II})}$
13	1/200	4.34×10^{-2}	2.06×10^{-1}
19	1/200	1.57×10^{-3}	1.03×10^{-2}
16	1/500	1.6×10^{-3}	5.78×10^{-2}
19	1/500	3.6×10^{-4}	6.41×10^{-5}

TABLE 5: L_∞ norm for $\rho(t,x)$ using classical methods for cases in which $\rho_0 = \sin(\pi x)^{10}$ ($L_\infty^{(\mathrm{I})}$) and $\rho_0 = \sin(\pi x)^{100}$ ($L_\infty^{(\mathrm{II})}$).

The results obtained via upwind and Euler schemes are worst, in terms of L_∞ norm, than the ones obtained using the DQ rules.

5. Conclusions

In this paper, we proposed a numerical scheme to find approximate solutions to a continuous model for supply systems.

A scheme in compact and matricial form, combining DQ rules and a Picard-like recursion, has been analyzed.

A simple test case has been useful to test the obtained numerical scheme. It has been showed that upwind and Euler methods provide worst approximations, in terms of L_∞ norm, than the proposed approach.

Future research activities aim to study deeply some properties of the DQ-based Picard-like scheme, considering road traffic and telecommunication networks, as well as simulation results of supply systems with more arcs.

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