

AN IMPROVED RESULT ON RAYLEIGH–TAYLOR INSTABILITY OF NONHOMOGENEOUS INCOMPRESSIBLE VISCOUS FLOWS*

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Abstract. In [F. Jiang and S. Jiang, Adv. Math., 264, 831–863, 2014], the author and Jiang investigated the instability of Rayleigh–Taylor steady-state of a three-dimensional nonhomogeneous incompressible viscous flow driven by gravity in a bounded domain Ω of class C^2 . In particular, we proved the steady-state is nonlinearly unstable under a restrictive condition of that the derivative function of steady density possesses a positive lower bound. In this article, by exploiting a standard energy functional and more-refined analysis of error estimates in the bootstrap argument, we further show the nonlinear instability result without the restrictive condition.

Key words. Navier–Stokes equations, steady state solutions, Rayleigh–Taylor instability.

AMS subject classifications. 76E09, 76D05.

1. Introduction

The motion of a three-dimensional (3D) nonhomogeneous incompressible viscous fluid in the presence of a uniform gravitational field in a bounded domain $\Omega \subset \mathbb{R}^3$ of C^2 -class is governed by the following Navier–Stokes equations

$$\begin{cases} \rho_t + v \cdot \nabla \rho = 0, \\ \rho v_t + \rho v \cdot \nabla v + \nabla p = \mu \Delta v - g \rho e_3, \\ \operatorname{div} v = 0, \end{cases} \quad (1.1)$$

where the unknowns $\rho := \rho(t, x)$, $v := v(t, x)$, and $p := p(t, x)$ denote the density, velocity, and pressure of the fluid, respectively; $\mu > 0$ stands for the coefficient of shear viscosity, $g > 0$ for the gravitational constant, $e_3 = (0, 0, 1)$ for the vertical unit vector, and $-g e_3$ for the gravitational force. In the system (1.1) the equation (1.1)₁ is the continuity equation, while (1.1)₂ describes the balance law of momentum.

We studied the instability of the following Rayleigh–Taylor (RT) steady-state to the system (1.1) as in [16]:

$$v(t, x) \equiv 0 \text{ and } \nabla \bar{p} = -g \bar{\rho} e_3 \quad \text{in } \Omega, \quad (1.2)$$

where the steady density $\bar{\rho}$ satisfies:

$$\bar{\rho} \in C^2(\bar{\Omega}), \quad \inf_{x \in \Omega} \{\bar{\rho}(x)\} > 0, \text{ and } \partial_{x_3} \bar{\rho}(x_0) > 0 \text{ for some } x_0 \in \Omega. \quad (1.3)$$

It is easy to show that the steady density $\bar{\rho}$ only depends on x_3 , the third component of x . Hence we can denote $\bar{\rho}' := \partial_{x_3} \bar{\rho}$ for simplicity. Moreover, we can give explicitly the associated steady pressure \bar{p} determined by $\bar{\rho}$. The third condition posed on $\bar{\rho}$ in (1.3) means that there is a region in which the RT density profile has larger density with increasing x_3 (height), thus leading to the nonlinear RT instability as shown in Theorem 1.1 below. RT instability is well known as gravity-driven instability in fluids when a heavy fluid is on top of a light one.

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To investigate the RT instability of the system (1.1) around the steady-state (1.2), we denote the perturbation by

$$\varrho = \rho - \bar{\rho}, \quad u = v - 0, \quad q = p - \bar{p},$$

then, (ϱ, u, q) satisfies the perturbed equations:

$$\begin{cases} \varrho_t + u \cdot \nabla(\varrho + \bar{\rho}) = 0, \\ (\varrho + \bar{\rho})u_t + (\varrho + \bar{\rho})u \cdot \nabla u + \nabla q = \mu \Delta u - g \varrho e_3, \\ \operatorname{div} u = 0. \end{cases} \tag{1.4}$$

To complete the statement of the perturbed problem, we specify the initial and boundary conditions

$$(\varrho, u)|_{t=0} = (\varrho_0, u_0) \quad \text{in } \Omega \tag{1.5}$$

and

$$u|_{\partial\Omega} = 0 \quad \text{for any } t > 0. \tag{1.6}$$

Moreover, the initial data should satisfy the compatibility conditions $u_0|_{\partial\Omega} = 0$ and $\operatorname{div} u_0 = 0$. If we linearize the equations (1.4) around the steady-state $(\bar{\rho}, 0)$, then the resulting linearized equations read as

$$\begin{cases} \varrho_t + \bar{\rho}' u_3 = 0, \\ \bar{\rho} u_t + \nabla q = \mu \Delta u - g \varrho e_3, \\ \operatorname{div} u = 0, \end{cases} \tag{1.7}$$

where u_3 denotes the third component of u .

Here we briefly introduce the research progress for RT instability of continuous flows, please refer to [12, 13, 22, 24] for incompressible and compressible stratified fluids, and [3, 14, 19, 20] for stratified MHD fluids. Instability of the linearized problem (i.e. linear instability) for an incompressible fluid was first introduced by Rayleigh in 1883 [23]. In 2003, Hwang and Guo [15] proved the nonlinear RT instability of $\|(\varrho, u)\|_{L^2(\Omega)}$ in the sense of Hadamard for a 2D nonhomogeneous incompressible inviscid fluid (i.e. $\mu = 0$ in the equation (1.4)) with boundary condition $u \cdot n|_{\partial\Omega} = 0$, where $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \mid -l < x_2 < m\}$ and n denotes the outer normal vector to $\partial\Omega$. Jiang et al. [17] showed the nonlinear RT instability of $\|u_3\|_{L^2(\mathbb{R}^3)}$ for the Cauchy problem of (1.4) in the sense of Lipschitz structure, and further gave the nonlinear RT instability of $\|u_3\|_{L^2(\Omega)}$ in [18] in the sense of Hadamard in an unbounded horizontal period domain Ω .

Recently, for a general bounded domain Ω , the author and Jiang showed that the steady-state (1.2) to the linearized problem (1.4)–(1.6) is linear unstable (i.e., the linear solution grows in time in $H^2(\Omega)$) by constructing a standard energy functional for the time-independent system of (1.7) and exploiting a modified variational method. Based on the linear instability result, they further showed the nonlinear instability of the perturbed problem (1.4)–(1.6) by a bootstrap technique under the following restrictive condition (i.e., the derivative function of steady-density enjoys a positive lower bound):

$$\inf_{x \in \Omega} \{\bar{\rho}'(x)\} > 0. \tag{1.8}$$

The bootstrap technique has its origins in the paper of Guo and Strauss [10, 11]. It was developed by Friedlander et al. [5], and widely quoted in the nonlinear instability

literature, see [1, 4, 6–9, 21] for example. However the Duhamel’s principle in the standard bootstrap argument can not be directly applied to show the nonlinear instability of the problem (1.4)–(1.6), see [16] for the details. To circumvent this obstacle, The author and Jiang used some specific energy error estimates to replace Duhamel’s principle, in which the key step is to deduce an error estimate for (ϱ^d, u^d) in $L^2(\Omega)$ (i.e the $L^2(\Omega)$ -norm of difference between a nonlinear solution $(\varrho^\delta, u^\delta)$ to the problem (1.4)–(1.6) and a linear solution (ϱ^a, u^a) to the problem (1.5)–(1.7)) in the bootstrap technique. To this purpose, they introduced a new energy functional under the condition (1.8) to avoid the integrand term

$$\int_0^t \langle (\varrho^\delta + \bar{\rho})u_\tau^d, u_\tau^d \rangle d\tau, \tag{1.9}$$

since the energy estimate of Gronwall-type (see (2.3)) does not directly offer any estimate for the term $\langle (\varrho^\delta + \bar{\rho})u_\tau^d, u_\tau^d \rangle$. Here $\langle \cdot, \cdot \rangle$ denotes the corresponding dual product between the two spaces $H_\sigma^{-1}(\Omega)$ and $H_\sigma^1(\Omega)$, and $H_\sigma^{-1}(\Omega)$ represents the dual space of $H_\sigma^1(\Omega) := \{u \in H_0^1(\Omega) \mid \operatorname{div} u = 0\}$. Using the new energy functional, they can get a sharp growth rate Λ of any linear solution (ϱ, u) in the norm “ $\sqrt{\|\varrho\|_{L^2}^2 + \|u\|_{L^2(\Omega)}^2}$ ”. Thus, applying this property to the process of specific energy error estimates, they easily obtained the desired error estimate, and thus showed the nonlinear instability.

This article is devoted to canceling the condition (1.8) in the proof of nonlinear instability in [16]. More precisely, we establish the following improved result by using a standard energy functional and more-refined analysis techniques to deduce the error estimate for $\|(\varrho^d, u^d)\|_{L^2(\Omega)}$ in the bootstrap argument, which will be showed in Section 3.

THEOREM 1.1. *Assume that the steady density $\bar{\rho}$ satisfies (1.3). Then, the steady-state (1.2) of the system (1.4)–(1.6) is unstable in the Hadamard sense, that is, there are positive constants $\Lambda, m_0, \varepsilon$, and δ_0 , and functions $(\bar{\varrho}_0, \bar{u}_0) \in H^2(\Omega) \times H^2(\Omega)$, such that for any $\delta \in (0, \delta_0)$ and initial data $(\varrho_0, u_0) := (\delta \bar{\varrho}_0, \delta \bar{u}_0)$ there is a unique strong solution $(\varrho, u) \in C^0([0, T^{\max}), H^2(\Omega) \times H^2(\Omega))$ of (1.4)–(1.6) with an associated pressure $q \in C^0([0, T^{\max}), H^1(\Omega))$, such that*

$$\|\varrho(T^\delta)\|_{L^2(\Omega)}, \|(u_1, u_2)(T^\delta)\|_{L^2(\Omega)}, \|u_3(T^\delta)\|_{L^2(\Omega)} \geq \varepsilon$$

for some escape time $T^\delta := \frac{1}{\Lambda} \ln \frac{2\varepsilon}{m_0 \delta} \in (0, T^{\max})$, where T^{\max} denotes the maximal time of existence of the solution (ϱ, u) .

By virtue of [16], the key step in the proof of Theorem 1.1 is to establish a error estimate

$$\|(\varrho^d, u^d)\|_{L^2(\Omega)} \leq C \delta^3 e^{3\Lambda t} \text{ for some constant } C \tag{1.10}$$

without the restrictive condition (1.8) (i.e., Lemma 3.1). Here we sketch the main idea in the proof of (1.10) without the presence of (1.8). In view of the property of standard energy functional (see (2.1)), Λ is also a sharp growth rate of any linear solution (ϱ, u) in the norm “ $\sqrt{\|\varrho\|_{L^2}^2 + \|u\|_{H^2(\Omega)}^2}$ ”, see [16, Proposition 3.3]. When applying the sharp growth rate of the standard energy functional to the process of specific energy error estimates, we need to deal with the difficulty arising from the term (1.9). However, by

a classical regularization method, we can show that

$$\begin{aligned} & 2 \int_0^t \langle (\varrho^\delta + \bar{\rho})u_\tau^d, u_\tau^d \rangle d\tau \\ &= \int_\Omega (\varrho^\delta + \bar{\rho})|u_t^d(t)|^2 dx - \int_\Omega (\varrho^\delta(0) + \bar{\rho})|u_t^d(0)|^2 dx + \int_0^t \int_\Omega \varrho_\tau^\delta |u_\tau^d|^2 dx d\tau, \end{aligned}$$

Then, we can deduce from the error equations (see (3.9)) that

$$\|\sqrt{\varrho^\delta + \bar{\rho}}u_t^d(t)\|_{L^2}^2 + 2\mu \int_0^t \|\nabla u_\tau^d\|_{L^2}^2 d\tau = \int g\bar{\rho}'|u_3^d(t)|^2 dx + R_1 + R_2(t),$$

where the two higher-order terms R_1 and $R_2(t)$ (see (3.14) and (3.15) for their definitions) can be controlled by $\delta^3 e^{3\Lambda t}$. Using the definition of sharp growth rate, we can further estimate that

$$\begin{aligned} & \|\sqrt{\varrho^\delta + \bar{\rho}}u_t^d(t)\|_{L^2}^2 + 2\mu \int_0^t \|\nabla u_\tau^d\|_{L^2}^2 d\tau \\ & \leq \Lambda^2 \|\sqrt{\varrho^\delta + \bar{\rho}}u^d(t)\|_{L^2}^2 + \Lambda\mu \|\nabla u^d(t)\|_{L^2}^2 + C\delta^3 e^{3\Lambda t}. \end{aligned}$$

Based on the estimate above, by more-refined analysis, we can further infer the following Gronwall’s inequality

$$\begin{aligned} & \frac{d}{dt} \|\sqrt{\varrho^\delta + \bar{\rho}}u^d(t)\|_{L^2}^2 + \mu \|\nabla u^d(t)\|_{L^2}^2 \\ & \leq 2\Lambda \left(\|\sqrt{\varrho^\delta + \bar{\rho}}u^d(t)\|_{L^2}^2 + \mu \int_0^t \|\nabla u^d\|_{L^2}^2 d\tau \right) + C\delta^3 e^{3\Lambda t}. \end{aligned}$$

Since $\varrho^\delta + \bar{\rho}$ possesses a positive lower bound, we immediately get the desired error estimate (1.10) from the Gronwall’s inequality above and the mass equation. We mention that the author and Jiang [16] used another energy functional and the restrictive condition (1.8) to deduce the following Gronwall’s inequality

$$\frac{d}{dt} \int \left(\frac{|\varrho^d|^2}{\bar{\rho}'} + \frac{\bar{\rho}|u^d|^2}{g} \right) dx \leq 2\Lambda \int \left(\frac{|\varrho^d|^2}{\bar{\rho}'} + \frac{\bar{\rho}|u^d|^2}{g} \right) dx, + C\delta^3 e^{3\Lambda t}$$

and thus obviously got (1.10) under (1.8).

Finally, we end this section by explaining the notations used throughout the rest of this article. For simplicity, we drop the domain Ω in Sobolev spaces and the corresponding norms as well as in integrands over Ω , for example,

$$L^p := L^p(\Omega), H^k := W^{k,2}(\Omega), H_\sigma^1 := H_\sigma^1(\Omega), \int := \int_\Omega.$$

In addition, we denote $I_T := (0, T)$ and $\bar{I}_T := [0, T]$ for simplicity.

2. Preliminaries

This section is devoted to introduction of two auxiliary results, which were established in [16] and will be used to prove Theorem 1.1 in next section. The first result is about the instability result of the linearized problem (1.5)–(1.7).

PROPOSITION 2.1. *Assume that the steady density $\bar{\rho}$ satisfies (1.3). Then the steady-state (1.2) of the linearized system (1.5)–(1.7) is unstable. That is, there exists an unstable solution*

$$(\varrho, u, q) := e^{\Lambda t}(-\bar{\rho}'\tilde{v}_3/\Lambda, \tilde{v}, \bar{p})$$

to (1.5)–(1.7), where $(\tilde{v}, \bar{p}) \in H^2 \times H^1$ solves the following boundary problem

$$\begin{cases} \Lambda^2 \bar{\rho}\tilde{v} + \Lambda \nabla \bar{p} = \Lambda \mu \Delta \tilde{v} + g\bar{\rho}'\tilde{v}_3 e_3, \\ \operatorname{div} \tilde{v} = 0, \quad \tilde{v}|_{\partial\Omega} = 0 \end{cases}$$

with the positive constant growth rate Λ defined by

$$\Lambda^2 = \sup_{\tilde{w} \in H^1_\sigma} \frac{g \int \bar{\rho}' \tilde{w}_3^2 dx - \Lambda \mu \int_\Omega |\nabla \tilde{w}|^2 dx}{\int \bar{\rho} |\tilde{w}|^2 dx}. \tag{2.1}$$

Moreover, \tilde{v} satisfies $\tilde{v}_3 \neq 0$, $\tilde{v}_1^2 + \tilde{v}_2^2 \neq 0$ and

$$\bar{\rho}'\tilde{v}_3 \neq 0, \tag{2.2}$$

where \tilde{v}_i denotes the i th component of \tilde{v} for $i = 1, 2, 3$.

REMARK 2.2. The linear instability was showed in [16, Theorem 1.1] except (2.2). However, we can easily get (2.2) by contradiction. Suppose that $\bar{\rho}'\tilde{v}_3 \equiv 0$, then

$$0 < \Lambda^2 = \frac{g \int \bar{\rho}' \tilde{v}_3^2 dx - \Lambda \mu \int |\nabla \tilde{v}|^2 dx}{\int \bar{\rho} |\tilde{v}|^2 dx} = -\Lambda \mu \frac{\int |\nabla \tilde{v}|^2 dx}{\int \bar{\rho} |\tilde{v}|^2 dx} < 0,$$

which contradicts. Therefore, (2.2) holds.

The second result is about a local existence result of a unique strong solution to the perturbed problem (1.4)–(1.6), which enjoys an energy estimate of Gronwall-type, see [16, Proposition 3.3] for the detailed proof.

PROPOSITION 2.3. *Assume that the steady density $\bar{\rho} := \bar{\rho}(x)$ satisfies (1.3). For any given initial data $(\varrho_0, u_0) \in H^2 \times (H^2 \cap H^1_\sigma)$ satisfying $\inf_{x \in \Omega} \{\varrho_0(x) + \bar{\rho}\} > 0$, there exist a unique strong solution $(\varrho, u) \in C^0([0, T^{\max}), H^2 \times H^2)$ to the perturbed problem (1.4)–(1.6) with an associated pressure $q \in C^0([0, T^{\max}), H^1)$, where T^{\max} denotes the maximal time of existence. Moreover,*

(1) $u_t \in C^0([0, T^{\max}), L^2)$ and

$$0 < \inf_{x \in \Omega} \{\varrho_0(x) + \bar{\rho}\} \leq \inf_{x \in \Omega} \{\varrho(t, x) + \bar{\rho}\} \leq \sup_{x \in \Omega} \{\varrho(t, x) + \bar{\rho}\} \leq \sup_{x \in \Omega} \{\varrho_0(x) + \bar{\rho}\} < +\infty$$

for any $t \in [0, T^{\max})$.

(2) there is a constant $\bar{\delta}_0 \in (0, 1)$, such that if $\mathcal{E}(t) \leq \bar{\delta}_0$ on some interval $\bar{I}_T \subset [0, T^{\max})$, then the strong solution satisfies

$$\begin{aligned} & \mathcal{E}^2(t) + \|(u_t, \nabla q)(t)\|_{L^2}^2 + \int_0^t \|(\nabla u, u_\tau, \nabla u_\tau)\|_{L^2}^2 d\tau \\ & \leq C_1 \left(\mathcal{E}_0^2 + \int_0^t \|(\varrho, u)\|_{L^2}^2 d\tau \right) \end{aligned} \tag{2.3}$$

for any $t \in \bar{I}_T$, where we have defined that

$$\begin{aligned} \mathcal{E}(t) &:= \mathcal{E}((\varrho, u)(t)) = \sqrt{\|\varrho(t)\|_{L^2}^2 + \|u(t)\|_{H^2}^2}, \\ \mathcal{E}_0 &:= \mathcal{E}((\varrho, u)(0)) = \sqrt{\|\varrho_0\|_{L^2}^2 + \|u_0\|_{H^2}^2}, \end{aligned}$$

and the constant $C_1 > 0$ only depends on $\mu, g, \bar{\rho}$ and Ω .

3. Proof of Theorem 1.1

Now we are in a position to prove Theorem 1.1. To begin with, in view of Proposition 2.1, we can construct a linear solution

$$(\varrho^1, u^1) = e^{\Lambda t} (\bar{\varrho}_0, \bar{u}_0) \in H^2 \times (H^2 \cap H^1_\sigma) \text{ for each } t > 0 \tag{3.1}$$

to the equation (1.5) with an associated pressure $q^1 = e^{\Lambda t} \bar{q}_0$, where $\bar{q}_0 \in H^1$, and $(\bar{\varrho}_0, \bar{u}_0) \in H^2 \times (H^2 \cap H^1_\sigma)$ satisfy

$$\begin{aligned} \|\bar{\varrho}_0\|_{L^2} \|\bar{u}_{03}\|_{L^2} \|(\bar{u}_{01}, \bar{u}_{02})\|_{L^2} &> 0, \\ \mathcal{E}((\bar{\varrho}_0, \bar{u}_0)) &= \sqrt{\|\bar{\varrho}_0\|_{L^2}^2 + \|\bar{u}_0\|_{H^2}^2} = 1, \end{aligned} \tag{3.2}$$

where \bar{u}_{0i} stands for the i th component of \bar{u}_0 for $i = 1, 2, 3$.

Denote $(\varrho_0^\delta, u_0^\delta) := \delta(\bar{\varrho}_0, \bar{u}_0)$, and $C_2 := \|(\bar{\varrho}_0, \bar{u}_0)\|_{L^2}$. Keeping in mind that the condition $\inf_{x \in \Omega} \{\bar{\rho}(x)\} > 0$ and the embedding $H^2 \hookrightarrow L^\infty$, we can choose a sufficiently small $\tilde{\delta} \in (0, 1)$, such that

$$\frac{\inf_{x \in \Omega} \{\bar{\rho}(x)\}}{2} \leq \inf_{x \in \Omega} \{\varrho_0^\delta(x) + \bar{\rho}(x)\} \text{ for any } \delta \in (0, \tilde{\delta}).$$

Thus, by virtue of Proposition 2.3, for any $\delta < \tilde{\delta}$, there exists a unique local solution $(\varrho^\delta, u^\delta) \in C^0([0, T^{\max}), H^2 \times H^2)$ to the perturbed problem (1.4)–(1.6) with an associated pressure $q^\delta \in C^0([0, T^{\max}), H^1)$, emanating from the initial data $(\varrho_0^\delta, u_0^\delta)$ with $\mathcal{E}((\varrho_0^\delta, u_0^\delta)) = \delta$, where T^{\max} denotes the maximal time of existence. Moreover,

$$0 < \frac{\inf_{x \in \Omega} \{\bar{\rho}(x)\}}{2} \leq \inf_{x \in \Omega} \{\varrho^\delta(t, x) + \bar{\rho}\} \tag{3.3}$$

and

$$\sup_{x \in \Omega} \{\varrho^\delta(t, x) + \bar{\rho}\} \leq \sup_{x \in \Omega} \{\bar{\varrho}_0(x) + \bar{\rho}\} \leq C_3 \|\bar{\varrho}_0\|_{H^2} + \|\bar{\rho}\|_{L^\infty} \tag{3.4}$$

for any $t \in [0, T^{\max})$, where C_3 is the constant from the imbedding $H^2 \hookrightarrow L^\infty$.

Let $C_1 > 0$ and $\bar{\delta}_0 > 0$ be the same constants as in Proposition 2.3, and $\varepsilon_0 \in (0, 1)$ be a constant, which will be defined in (3.33). Denote $\delta_0 = \min\{\tilde{\delta}, \bar{\delta}_0\}$, for given $\delta \in (0, \delta_0)$, we define

$$T^\delta := \frac{1}{\Lambda} \ln \frac{2\varepsilon_0}{\delta} > 0, \quad \text{i.e., } \delta e^{\Lambda T^\delta} = 2\varepsilon_0, \tag{3.5}$$

$$T^* := \sup \{t \in I_{T^{\max}} \mid \mathcal{E}((\varrho^\delta, u^\delta)(t)) \leq \delta_0\} > 0$$

and

$$T^{**} := \sup \{t \in I_{T^{\max}} \mid \|(\varrho^\delta, u^\delta)(t)\|_{L^2} \leq 2\delta C_2 e^{\Lambda t}\} > 0.$$

Then T^* and T^{**} may be finite, and furthermore,

$$\mathcal{E}((\varrho^\delta, u^\delta)(T^*)) = \delta_0 \quad \text{if } T^* < \infty, \tag{3.6}$$

$$\|(\varrho^\delta, u^\delta)(T^{**})\|_{L^2} = 2\delta C_2 e^{\Lambda T^{**}} \quad \text{if } T^{**} < T^{\max}. \tag{3.7}$$

Now, we denote $T_{\min} := \min\{T^\delta, T^*, T^{**}\}$, then for all $t \in \bar{I}_{T_{\min}}$, we deduce from the estimate (2.3) and the definitions of T^* and T^{**} that

$$\begin{aligned} & \mathcal{E}^2((\varrho^\delta, u^\delta)(t)) + \|u_t^\delta(t)\|_{L^2}^2 + \int_0^t \|\nabla u_\tau^\delta\|_{L^2}^2 d\tau \\ & \leq C_1 \delta^2 \mathcal{E}^2((\bar{\varrho}_0, \bar{u}_0)) + C_1 \int_0^t \|(\varrho^\delta, u^\delta)\|_{L^2}^2 d\tau \\ & \leq C_1 \delta^2 + 4C_1 C_2^2 \delta^2 e^{2\Lambda t} / (2\Lambda) \leq C_4 \delta^2 e^{2\Lambda t} \end{aligned} \tag{3.8}$$

where $C_4 := C_1 + 4C_1 C_2^2 / (2\Lambda)$ is independent of δ .

Let $(\varrho^d, u^d) = (\varrho^\delta, u^\delta) - \delta(\varrho^1, u^1)$. Noting that $(\varrho^a, u^a) := \delta(\varrho^1, u^1) \in C^0([0, +\infty), H^2 \times H^2)$ is also a linear solution to (1.5)–(1.7) with the initial data $(\varrho_0^\delta, u_0^\delta) \in H^2 \times H^2$ and with an associated pressure $q^a = \delta q^1 \in C^0([0, +\infty), H^1)$, we find that (ϱ^d, u^d) satisfies the following error equations:

$$\begin{cases} \varrho_t^d + \bar{\rho}' u_3^d = -u^\delta \cdot \nabla \varrho^\delta, \\ (\varrho^\delta + \bar{\rho}) u_t^d - \mu \Delta u^d + \nabla q^d = f^\delta - g \varrho^d e_3, \\ \operatorname{div} u^d = 0 \end{cases} \tag{3.9}$$

with initial and boundary conditions

$$(\varrho^d(0), u^d(0)) = 0, \quad u^d|_{\partial\Omega} = 0$$

and compatibility conditions

$$u^d(0)|_{\partial\Omega} = 0, \quad \operatorname{div} u^d(0) = 0,$$

where we have defined that

$$q^d := q^\delta - q^a \in C^0(\bar{I}_{T_{\min}}, H^1) \quad \text{and} \quad f^\delta := -(\varrho^\delta + \bar{\rho}) u^\delta \cdot \nabla u^\delta - \varrho^\delta u_t^a.$$

Next, we shall establish an error estimate for (ϱ^d, u^d) in L^2 -norm.

LEMMA 3.1. *There is a constant C_4 , such that for all $t \in \bar{I}_{T_{\min}}$,*

$$\|(\varrho^d, u^d)(t)\|_{L^2}^2 \leq C_4 \delta^3 e^{3\Lambda t}. \tag{3.10}$$

Proof. Recalling that $(\varrho^d, u^d) = (\varrho^\delta, u^\delta) - (\varrho^a, u^a)$, in view of the regularity of $(\varrho^\delta, u^\delta)$ and (ϱ^a, u^a) , we can deduce from (3.9)₂ that for a.e. $t \in I_{T_{\min}}$,

$$\begin{aligned} & \frac{d}{dt} \int (\varrho^\delta + \bar{\rho}) |u_t^d|^2 dx = 2 \langle (\varrho^\delta + \bar{\rho}) u_t^d \rangle_t, u_t^d \rangle - \int \varrho_t^\delta |u_t^d|^2 dx \\ & = 2 \int (f_t^\delta - g \varrho_t^d e_3) u_t^d dx - 2\mu \int |\nabla u_t^d|^2 dx - \int \varrho_t^\delta |u_t^d|^2 dx, \end{aligned} \tag{3.11}$$

and $\|\sqrt{\varrho^\delta + \bar{\rho}} u_t^d\|_{L^2} \in C^0(\bar{I}_{T_{\min}})$, please refer to [2, Remark 6]. Noting that

$$\frac{d}{dt} \int \bar{\rho}' |u_3^d|^2 dx = 2 \int \bar{\rho}' u_3^d \partial_t u_3^d dx,$$

thus, using (3.9)₁, we can rewrite the equality (3.11) as

$$\begin{aligned} & \frac{d}{dt} \int [(\varrho^\delta + \bar{\rho})|u_t^d|^2 - g\bar{\rho}'|u_3^d|^2] dx + 2\mu \int |\nabla u_t^d|^2 dx \\ &= \int (2f_t + 2gu^\delta \cdot \nabla \varrho^\delta e_3 - \varrho_t^\delta u_t^d) \cdot u_t^d dx. \end{aligned} \tag{3.12}$$

Recalling that $u_3^d(0) = 0$, thus, integrating (3.12) in time from 0 to t , we get

$$\|\sqrt{\varrho^\delta + \bar{\rho}}u_t^d(t)\|_{L^2}^2 + 2\mu \int_0^t \|\nabla u_\tau^d\|_{L^2}^2 d\tau = \int g\bar{\rho}'|u_3^d(t)|^2 dx + R_1 + R_2(t), \tag{3.13}$$

where

$$R_1 = \left[\int (\varrho^\delta + \bar{\rho})|u_t^d|^2 dx \right] \Big|_{t=0} \tag{3.14}$$

and

$$R_2(t) = \int_0^t \int (2f_\tau + 2gu^\delta \cdot \nabla \varrho^\delta e_3 - \varrho_\tau^\delta u_\tau^d) \cdot u_\tau^d dx d\tau. \tag{3.15}$$

Next, we control the two higher-order terms R_1 and $R_2(t)$. In what follows, we denote by C a generic positive constant which may depend on $\mu, g, \bar{\rho}, \Lambda, \Omega$, and $(\bar{\varrho}_0, \bar{u}_0)$. The symbol $a \lesssim b$ means that $a \leq Cb$.

Multiplying (3.9)₂ by u_t^d in L^2 , we get

$$\int (\varrho^\delta + \bar{\rho})|u_t^d|^2 dx = \int (f^\delta - g\varrho^d e_3 + \mu \Delta u^d) \cdot u_t^d dx.$$

Exploiting (3.3) and Cauchy’s inequality, we get

$$\int (\varrho^\delta + \bar{\rho})|u_t^d|^2 dx \lesssim \|f^\delta - g\varrho^d e_3\|_{L^2}^2 + \|\Delta u^d\|_{L^2}^2. \tag{3.16}$$

By the definition of u_t^a , it holds that

$$\|\partial_t^j u^a\|_{H^k} = \Lambda^j \delta e^{\Lambda t} \|\bar{u}_0\|_{H^k} \text{ for } 0 \leq k, j \leq 2, \tag{3.17}$$

thus, using (3.4), (3.8), Hölder’s inequality, and the imbedding $H^2 \hookrightarrow L^\infty$, we have

$$\begin{aligned} \|f^\delta - g\varrho^d e_3\|_{L^2}^2 &\lesssim \|\varrho^d\|_{L^2}^2 + \|(\varrho^\delta + \bar{\rho})\|_{L^\infty}^2 \|u^\delta\|_{H^2}^4 + \|\varrho^\delta\|_{L^2}^2 \|u_t^a\|_{H^2}^2 \\ &\lesssim \|\varrho^d\|_{L^2}^2 + \delta^4 e^{4\Lambda t}. \end{aligned} \tag{3.18}$$

Noting that $\varrho^d(0) = 0, \Delta u^d(0) = 0$, and $\delta \in (0, 1)$, chaining the estimates (3.16) with (3.18) together, and taking limit for $t \rightarrow 0$, we immediately obtain the following estimate for the first higher-order term R_1 :

$$\begin{aligned} R_1 &= \lim_{t \rightarrow 0} \int (\varrho^\delta + \bar{\rho})|u_t^d(t)|^2 dx \\ &\lesssim \lim_{t \rightarrow 0} (\|\varrho^d(t)\|_{L^2}^2 + \|\Delta u^d(t)\|_{L^2}^2 + \delta^4 e^{4\Lambda t}) = \delta^4 \leq \delta^3. \end{aligned} \tag{3.19}$$

Now we turn to estimate the most complicated higher-order term $R_2(t)$. Recalling the definition of $R_2(t)$, we see that

$$\begin{aligned} R_2(t) &= -2 \int_0^t \int [\varrho^\delta u_{\tau\tau}^a + (\varrho^\delta + \bar{\rho})u_\tau^\delta \cdot \nabla u^\delta + (\varrho^\delta + \bar{\rho})u^\delta \cdot \nabla u_\tau^\delta] \cdot u_\tau^d dx d\tau \\ &\quad + \int_0^t \int [2gu^\delta \cdot \nabla \varrho^\delta e_3 - \varrho_\tau^\delta (2u_\tau^a + u_\tau^d + 2u^\delta \cdot \nabla u^\delta)] \cdot u_\tau^d dx d\tau \\ &:= R_{2,1}(t) + R_{2,2}(t). \end{aligned}$$

Using (3.4), (3.8), (3.17), Hölder’s inequality, and the imbeddings $H^2 \hookrightarrow L^\infty$ and $H^1 \hookrightarrow L^4$, the integral term $R_{2,1}(t)$ can be estimated as follows:

$$\begin{aligned} R_{2,1}(t) &\lesssim \int_0^t (\|\varrho^\delta\|_{L^2} \|u_{\tau\tau}^a\|_{H^2} + \|(\varrho^\delta + \bar{\rho})\|_{L^\infty} \|u^\delta\|_{H^2} \|u_\tau^\delta\|_{H^1}) \|u_\tau^d\|_{L^2} d\tau \\ &\lesssim \int_0^t \delta^2 e^{2\Lambda\tau} (\delta e^{\Lambda\tau} + \|\nabla u_\tau^\delta\|_{L^2}) d\tau \\ &\lesssim \delta^3 e^{3\Lambda t} + \left(\int_0^t \delta^4 e^{4\Lambda\tau} d\tau \right)^{\frac{1}{2}} \left(\int_0^t \|\nabla u_\tau^\delta\|_{L^2}^2 d\tau \right)^{\frac{1}{2}} \lesssim \delta^3 e^{3\Lambda t}. \end{aligned} \tag{3.20}$$

To estimate the second term $R_{2,2}(t)$, we use the mass equation (i.e. $\varrho_t^\delta = -(u^\delta \cdot \nabla \varrho^\delta + \bar{\rho}' u_3^\delta)$) and the formula of integration by parts to rewrite $R_{2,2}(t)$ as follows:

$$\begin{aligned} R_{2,2}(t) &= \int_0^t \int [(u^\delta \cdot \nabla \varrho^\delta + \bar{\rho}' u_3^\delta) (2u_\tau^a + u_\tau^d + 2u^\delta \cdot \nabla u^\delta) + 2gu^\delta \cdot \nabla \varrho^\delta e_3] \cdot u_\tau^d dx d\tau \\ &= \int_0^t \int [\bar{\rho}' u_3^\delta (2u_\tau^a + u_\tau^d + 2u^\delta \cdot \nabla u^\delta) u_\tau^d - 2g\varrho^\delta u^\delta \cdot \nabla \partial_\tau u_3^d] dx d\tau \\ &\quad - 2 \int_0^t \int [\varrho^\delta u^\delta \cdot \nabla (u_\tau^\delta + u^\delta \cdot \nabla u^\delta) \cdot u_\tau^d + \varrho^\delta u^\delta \cdot \nabla u_\tau^d \cdot (u_\tau^a + u^\delta \cdot \nabla u^\delta)] dx d\tau \\ &= R_{2,2,1}(t) + R_{2,2,2}(t). \end{aligned}$$

Similarly to (3.20), we can estimate that

$$\begin{aligned} R_{2,2,1}(t) &\lesssim \int_0^t [\|u_3^\delta\|_{H^2} (\|u_\tau^a\|_{L^2} + \|u_\tau^d\|_{L^2} + \|u^\delta\|_{H^2}^2) \|u_\tau^d\|_{L^2} + \|\varrho^\delta\|_{L^2} \|u^\delta\|_{H^2} \|\nabla \partial_\tau u_3^d\|_{L^2}] d\tau \\ &\lesssim \int_0^t [\delta^3 e^{3\Lambda\tau} (1 + \delta e^{\Lambda\tau}) + \delta^2 e^{2\Lambda\tau} \|\nabla \partial_\tau u_3^\delta\|_{L^2}] d\tau \lesssim \delta^3 e^{3\Lambda t} (1 + \delta e^{\Lambda t}), \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} R_{2,2,2}(t) &\lesssim \int_0^t \|\varrho^\delta\|_{L^\infty} \|u^\delta\|_{H^2} (\|\nabla u_\tau^\delta\|_{L^2} \|u_\tau^d\|_{L^2} + \|u^\delta\|_{H^2}^2 \|u_\tau^d\|_{L^2} \\ &\quad + \|u_\tau^a\|_{L^2} \|\nabla u_\tau^d\|_{L^2} + \|u^\delta\|_{H^2}^2 \|\nabla u_\tau^d\|_{L^2}) d\tau \\ &\lesssim \int_0^t [\delta^3 e^{3\Lambda\tau} (1 + \delta e^{\Lambda\tau}) + \delta^2 e^{2\Lambda\tau} \|\nabla u_\tau^\delta\|_{L^2}] d\tau \lesssim \delta^3 e^{3\Lambda t} (1 + \delta e^{\Lambda t}). \end{aligned} \tag{3.22}$$

By the definition of $\varepsilon_0 \in (0, 1)$ in (3.5),

$$\delta \leq \delta e^{\Lambda t} \leq \delta e^{\Lambda T^\delta} \leq 2 \text{ for any } t \in \bar{I}_{T_{\min}}. \tag{3.23}$$

Thus, summing up the estimates (3.19)–(3.22), we get

$$R_1 + R_2(t) = R_1 + R_{2,1}(t) + R_{2,2,1}(t) + R_{2,2,2}(t) \lesssim \delta^3 e^{3\Lambda t}, \tag{3.24}$$

which, together with (3.13), yields that

$$\|\sqrt{\varrho^\delta + \bar{\rho}} u_t^d(t)\|_{L^2}^2 + 2\mu \int_0^t \|\nabla u_\tau^d\|_{L^2}^2 d\tau \leq \int g\bar{\rho}'|u_3^d|^2 dx + C\delta^3 e^{3\Lambda t}.$$

Thanks to (2.1), we have

$$\begin{aligned} \int g\bar{\rho}'|u_3^d|^2 dx &\leq \Lambda^2 \int \bar{\rho}|u^d|^2 dx + \Lambda\mu \int |\nabla u^d|^2 dx \\ &= \Lambda^2 \int (\varrho^\delta + \bar{\rho})|u^d|^2 dx + \Lambda\mu \int |\nabla u^d|^2 dx - \Lambda^2 \int \varrho^\delta |u^d|^2 dx \\ &\leq \Lambda^2 \int (\varrho^\delta + \bar{\rho})|u^d|^2 dx + \Lambda\mu \int |\nabla u^d|^2 dx + C\delta^3 e^{3\Lambda t}. \end{aligned}$$

Chaining the previous two inequalities together, we obtain

$$\begin{aligned} &\|\sqrt{\varrho^\delta + \bar{\rho}} u_t^d(t)\|_{L^2}^2 + 2\mu \int_0^t \|\nabla u_\tau^d\|_{L^2}^2 d\tau \\ &\leq \Lambda^2 \|\sqrt{\varrho^\delta + \bar{\rho}} u^d(t)\|_{L^2}^2 + \Lambda\mu \|\nabla u^d(t)\|_{L^2}^2 + C\delta^3 e^{3\Lambda t}. \end{aligned} \tag{3.25}$$

Recalling that $u^d \in C^0(\bar{I}_{T_{\min}}, H^2)$ and $\nabla u^d(0) = 0$, thus, using Newton–Leibniz formula and Cauchy–Schwarz inequality, we find that

$$\begin{aligned} \Lambda\mu \|\nabla u^d(t)\|_{L^2}^2 &= 2\Lambda\mu \int_0^t \int_{\Omega} \sum_{1 \leq i, j \leq 3} \partial_{x_i} u_j^d \partial_{x_i} u_{j\tau}^d dx d\tau \\ &\leq \Lambda^2 \mu \int_0^t \|\nabla u^d\|_{L^2}^2 d\tau + \mu \int_0^t \|\nabla u_\tau^d\|_{L^2}^2 d\tau, \end{aligned} \tag{3.26}$$

where $u_{j\tau}^d$ denotes the j th component of u_τ^d . Putting (3.25) and (3.26) together, we have

$$\begin{aligned} &\frac{1}{\Lambda} \|\sqrt{\varrho^\delta + \bar{\rho}} u_t^d(t)\|_{L^2}^2 + \mu \|\nabla u^d(t)\|_{L^2}^2 \\ &\leq \Lambda \|\sqrt{\varrho^\delta + \bar{\rho}} u^d(t)\|_{L^2}^2 + 2\Lambda\mu \int_0^t \|\nabla u^d\|_{L^2}^2 d\tau + C\delta^3 e^{3\Lambda t}. \end{aligned} \tag{3.27}$$

On the other hand,

$$\begin{aligned} \frac{d}{dt} \|\sqrt{\varrho^\delta + \bar{\rho}} u^d\|_{L^2}^2 &= 2 \int (\varrho^\delta + \bar{\rho}) u^d \cdot u_t^d dx + \int \varrho_t^\delta |u^d|^2 dx \\ &\leq \frac{1}{\Lambda} \|\sqrt{(\varrho^\delta + \bar{\rho})} u_t^d\|_{L^2}^2 + \Lambda \|\sqrt{\varrho^\delta + \bar{\rho}} u^d\|_{L^2}^2 + \int \varrho_t^\delta |u^d|^2 dx \end{aligned}$$

and

$$\begin{aligned} \int \varrho_t^\delta |u^d|^2 dx &= - \int (u^\delta \cdot \nabla \varrho^\delta + \bar{\rho}' u_3^\delta) |u^d|^2 dx \\ &= \int (2\varrho^\delta u^\delta \cdot \nabla u^d - \bar{\rho}' u_3^\delta u^d) \cdot u^d dx \\ &\lesssim \delta^3 e^{3\Lambda t} \end{aligned}$$

Putting the previous three estimates together, we get the differential inequality

$$\begin{aligned} & \frac{d}{dt} \|\sqrt{\varrho^\delta + \bar{\rho}u^d}(t)\|_{L^2}^2 + \mu \|\nabla u^d(t)\|_{L^2}^2 \\ & \leq 2\Lambda \left(\|\sqrt{\varrho^\delta + \bar{\rho}u^d}(t)\|_{L^2}^2 + \mu \int_0^t \|\nabla u^d\|_{L^2}^2 d\tau \right) + C\delta^3 e^{3\Lambda t}. \end{aligned} \tag{3.28}$$

Recalling that $u^d = 0$, thus, applying Gronwall’s inequality to (3.28), one obtains

$$\|\sqrt{\varrho^\delta + \bar{\rho}u^d}(t)\|_{L^2}^2 + \mu \int_0^t \|\nabla u^d\|_{L^2}^2 d\tau \leq e^{2\Lambda t} \int_0^t (C\delta^3 e^{3\Lambda\tau}) e^{-2\Lambda\tau} d\tau \lesssim \delta^3 e^{3\Lambda t} \tag{3.29}$$

for all $t \leq \bar{T}_{\min}$, which, together with (3.4) and (3.27), yields that

$$\|u^d(t)\|_{H^1}^2 + \|u_t^d(t)\|_{L^2}^2 + \int_0^t \|\nabla u^d\|_{L^2}^2 d\tau \lesssim \delta^3 e^{3\Lambda t}. \tag{3.30}$$

Finally, using the estimates (3.8), (3.23), and (3.30), we can deduce from the equation (3.9)₁, that

$$\begin{aligned} \|\varrho^d(t)\|_{L^2} & \leq \int_0^t \|\varrho_\tau^d\|_{L^2} d\tau \\ & \lesssim \int_0^t (\|u^d\|_{H^1}^2 + \|u^\delta \cdot \nabla \varrho^\delta\|_{L^2}) d\tau \\ & \lesssim \int_0^t (\delta^{\frac{3}{2}} e^{\frac{3\Lambda}{2}\tau} + \delta^2 e^{2\Lambda\tau}) d\tau \lesssim \delta^{\frac{3}{2}} e^{\frac{3\Lambda}{2}t}, \end{aligned} \tag{3.31}$$

which, together with (3.30), yields (3.10). This completes the proof of Lemma 3.1. \square

Now, we claim that

$$T^\delta = T_{\min}, \tag{3.32}$$

provided that small ε_0 is taken to be

$$\varepsilon_0 = \min \left\{ \frac{\delta_0}{4}, \frac{C_2^2}{8C_4}, \frac{m_0^2}{C_4} \right\}, \tag{3.33}$$

where we have defined that $m_0 =: \min\{\|\bar{\varrho}_0\|_{L^2}, \|\bar{u}_{03}\|_{L^2}, \|(\bar{u}_{01}, \bar{u}_{02})\|_{L^2}\} > 0$ due to (3.2).

Indeed, if $T^* = T_{\min}$, then $T^* < \infty$. Moreover, from (3.5) and (3.8) we get

$$\mathcal{E}((\varrho^\delta, u^\delta)(T^*)) \leq \delta e^{\Lambda T^*} \leq \delta e^{\Lambda T^\delta} = 2\varepsilon_0 < \delta_0,$$

which contradicts with (3.6). On the other hand, if $T^{**} < T_{\min}$, then $T^{**} < T^* \leq T^{\max}$. Moreover, in view of (3.1), (3.5), and (3.10), we see that

$$\begin{aligned} \|(\varrho^\delta, u^\delta)(T^{**})\|_{L^2} & \leq \|(\varrho_\delta^a, u_\delta^a)(T^{**})\|_{L^2} + \|(\varrho^d, u^d)(T^{**})\|_{L^2} \\ & \leq \delta \|(\varrho^1, u^1)(T^{**})\|_{L^2} + \sqrt{C_4} \delta^{3/2} e^{3\Lambda T^{**}/2} \\ & \leq \delta C_2 e^{\Lambda T^{**}} + \sqrt{C_4} \delta^{3/2} e^{3\Lambda T^{**}/2} \leq \delta e^{\Lambda T^{**}} (C_2 + \sqrt{2C_4\varepsilon_0}) \\ & < 2\delta C_2 e^{\Lambda T^{**}}, \end{aligned}$$

which also contradicts with (3.7). Therefore, (3.32) holds.

Since $T^\delta = T_{\min}$, (3.10) holds for $t = T^\delta$. Thus, we can use (3.33) and (3.10) with $t = T^\delta$ to deduce that

$$\begin{aligned} \|\varrho^\delta(T^\delta)\|_{L^2} &\geq \|\varrho_\delta^a(T^\delta)\|_{L^2} - \|\varrho^d(T^\delta)\|_{L^2} = \delta \|\varrho^1(T^\delta)\|_{L^2} - \|\varrho^d(T^\delta)\|_{L^2} \\ &\geq \delta e^{\Lambda T^\delta} \|\bar{\varrho}_0\|_{L^2} - \sqrt{C_4} \delta^{3/2} e^{3\Lambda^* T^\delta/2} \\ &\geq 2\varepsilon_0 \|\bar{\varrho}_0\|_{L^2} - \sqrt{C_4} \varepsilon_0^{3/2} \geq 2m_0\varepsilon_0 - \sqrt{C_4} \varepsilon_0^{3/2} \geq m_0\varepsilon_0, \end{aligned}$$

Similar, we also have

$$\|u_3^\delta(T^\delta)\|_{L^2} \geq 2m_0\varepsilon_0 - \sqrt{C_4} \varepsilon_0^{3/2} \geq m_0\varepsilon_0,$$

and

$$\|(u_1^\delta, u_2^\delta)(T^\delta)\|_{L^2} \geq 2m_0\varepsilon_0 - \sqrt{C_4} \varepsilon_0^{3/2} \geq m_0\varepsilon_0,$$

where $u_i^\delta(T^\delta)$ denote the i th component of $u^\delta(T^\delta)$ for $i=1,2,3$. This completes the proof of Theorem 1.1 by defining $\varepsilon := m_0\varepsilon_0$.

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