Commun. Math. Sci.
Vol. 14, No. 5, pp. 1189–1212

DECAY ESTIMATES OF SOLUTIONS TO THE COMPRESSIBLE NAVIER–STOKES–MAXWELL SYSTEM IN $\mathbb{R}^3$*

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Abstract. The compressible Navier–Stokes–Maxwell system with linear damping is investigated in $\mathbb{R}^3$, and the global existence and large-time behavior of solutions are established. We first construct the global unique solution under the assumptions that the H^3 norm of the initial data is small but that the higher-order derivatives can be arbitrarily large. Further, if the initial data belongs to $\dot{H}^{-s}$ $(0\le s<3/2)$ or $\dot{B}^{-s}_{2,\infty}$ $(0<s\le 3/2)$, by a regularity interpolation trick, we obtain the various decay rates of the solution and its higher-order derivatives. As an immediate byproduct, the L^p–L^2 $(1\le p\le 2)$ type of the decay rates follow without requiring that the L^p norm of initial data is small.

Key words. Compressible Navier–Stokes–Maxwell system, global solution, time decay rate, energy method, interpolation.

AMS subject classifications. 35Q35, 35Q30, 35Q61, 82D37, 76N10, 35B40.

1. Introduction

In this paper, we consider the three-dimensional isentropic compressible Navier–Stokes–Maxwell equations with the linear damping [17]

$$\begin{cases} \partial_t\rho+\operatorname{div}(\rho u)=0,\\ \partial_t(\rho u)+\operatorname{div}(\rho u\otimes u)+\nabla p(\rho)=-\rho(E+u\times M)+\nu\Delta u-a\rho u,\\ \partial_t E-\nabla\times M=\rho u,\\ \partial_t M+\nabla\times E=0,\\ \operatorname{div}E=\rho_\infty-\rho,\quad \operatorname{div}M=0. \end{cases} \tag{1.1}$$

The unknown functions ρ,u,E,M represent the density, velocity, electric field, and magnetic field of the fluid, respectively. The pressure $p=p(\rho)$ is smooth, and $p'(\rho)>0$ for $\rho>0$. The constant $\nu>0$ is the viscosity coefficient, the constant $a>0$ models friction, and $\rho_\infty>0$ is a constant denoting the uniform background density (e.g., of ions).

This system (1.1) is supplemented by the following initial and compatible conditions:

$$\begin{cases} (\rho,u,E,M)(x,t)|_{t=0}=(\rho_0,u_0,E_0,M_0)(x), \quad (x,t)\in\mathbb{R}^3\times[0,+\infty),\\ \operatorname{div}E_0=\rho_\infty-\rho_0,\quad \operatorname{div}M_0=0. \end{cases} \tag{1.2}$$

Despite its physical importance, due to the mathematical complexity, a small number of mathematical studies on the Navier–Stokes–Maxwell system have been obtained. For isentropic compressible Navier–Stokes–Maxwell equations, Duan [1] proved the global existence and asymptotic behavior of smooth solutions around a constant steady state by using Green's function argument. Hong et al. [8] considered the initial boundary value problem to the Navier–Stokes–Maxwell system with large initial data and initial vacuum in a bounded annulus Ω of $\mathbb{R}^3$ and obtained the global spherically symmetric classical solutions. For the non-isentropic case, by using careful energy estimates and the techniques of symmetrizer, Feng–Peng–Wang [3] established the large-time behavior of global smooth solutions in $\mathbb{R}^3$. [14] dealt with the low Much number limit of the

*Received: April 5, 2015; accepted: June 6, 2016. Communicated by Yaguang Wang.
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full compressible Navier–Stokes–Maxwell system in the framework of [12]. For incompressible Navier–Stokes–Maxwell equations, with the help of Fujita–Kato's method in l^1-based (for the Fourier coefficients) functional spaces, Ibrahim–Yoneda [10] established the existence of a local unique solution and loss of smoothness of the velocity and magnetic field for the periodic problem. Ibrahim–Keraani [9] also obtained the existence of global small mild solutions in three dimensions and the same results in spaces as close as possible to the energy space in two dimensions. Masmoudi [15] obtained the existence and uniqueness of global strong solutions in $2D$. Fan–Li [2] established the uniform local existence and uniqueness of classical solutions to the density-dependent Navier–Stokes–Maxwell system in $\mathbb{R}^3$. [25] studied the combined quasi-neutral and nonrelativistic limit in a three-dimensional torus. Germain–Ibrahim–Masmoudi [6] proved the local existence of mild solutions for arbitrarily large data in a space similar to the scale-invariant spaces classically used for Navier–Stokes by using an a priori $L^2_t(L^\infty_x)$ estimate for solutions of the forced Navier–Stokes equations and refined the results in [9].

The main purpose of this paper is to derive various time decay rates of the solutions as well as their spatial derivatives of any order by using this refined energy method together with the interpolation trick in [7,22]. We also establish a refined global existence of smooth solutions near the constant equilibrium state $(\rho_\infty,0,0,B_\infty)$ to the compressible Navier–Stokes–Maxwell system. We reformulate the Cauchy problem (1.1)–(1.2) of the compressible Navier–Stokes–Maxwell system. Without loss of generality, we set the constants ρ_∞, ν, a, $p'(1)=p''(1)$ to be 1. Denote $n=\rho-1$, $B=M-B_\infty$. Then the Cauchy problem (1.1)–(1.2) is transformed into the following:

$$\begin{cases} \partial_t n+\mathrm{div}u=-u\cdot\nabla n-n\mathrm{div}u,\\ \partial_t u-\Delta u+u+u\times B_\infty+\nabla n+E=-u\cdot\nabla u-f(n)\nabla n-u\times B-g(n)\Delta u,\\ \partial_t E-\nabla\times B-u=nu,\\ \partial_t B+\nabla\times E=0,\\ \mathrm{div}E=-n,\quad \mathrm{div}B=0, \end{cases} \tag{1.3}$$

with

$$\begin{cases} (n,u,E,B)(x,t)|_{t=0}=(n_0,u_0,E_0,B_0)(x), \quad (x,t)\in\mathbb{R}^3\times[0,+\infty),\\ \mathrm{div}E_0=-n_0,\quad \mathrm{div}B_0=0. \end{cases} \tag{1.4}$$

Here the nonlinear functions $f(n),g(n)$ are defined by

$$f(n):=\frac{p'(n+1)}{n+1}-1,\ g(n):=\frac{n}{n+1}. \tag{1.5}$$

Notation. In this paper, we use $H^s(\mathbb{R}^3),s\in\mathbb{R}$ to denote the usual Sobolev spaces with norm $\|\cdot\|_{H^s}$ and $L^p(\mathbb{R}^3),1\leqslant p\leqslant\infty$ to denote the usual L^p spaces with norm $\|\cdot\|_{L^p}$. In particular, we denote the L^2 spaces with norm $\|\cdot\|$, and ∇^l with an integer $l\geqslant 0$ stands for, as usual, any spatial derivatives of order l. When $l<0$ or l is not a positive integer, ∇^l stands for Λ^l defined by $\Lambda^l f:=\mathbf{F}^{-1}(|\xi|^l\mathbf{F}f)$, where $\mathbf{F}$ is the usual Fourier transform operator and $\mathbf{F}^{-1}$ is its inverse. We use $\dot{H}^s(\mathbb{R}^3),s\in\mathbb{R}$ to denote the homogeneous Sobolev spaces on $\mathbb{R}^3$ with norm $\|\cdot\|_{\dot{H}^s}$ defined by $\|f\|_{\dot{H}^s}:=\|\Lambda^s f\|$. We then recall the homogeneous Besov spaces. Let $\phi\in C_0^\infty(R^3_\xi)$ be such that $\phi(\xi)=1$ when $|\xi|\le 1$ and $\phi(\xi)=0$ when $|\xi|\ge 2$. Let $\varphi(\xi)=\phi(\xi)-\phi(2\xi)$ and $\varphi_j(\xi)=\varphi(2^{-j}\xi)$ for $j\in\mathbb{Z}$. Then, by the construction, $\sum_{j\in\mathbb{Z}}\varphi_j(\xi)=1$ if $\xi\neq 0$. We define $\dot{\Delta}_j f:=\mathbf{F}^{-1}(\varphi_j)*f$. Then, for $s\in\mathbb{R}$, we define the homogeneous Besov spaces $\dot{B}^s_{2,\infty}(\mathbb{R}^3)$ with norm $\|\cdot\|_{\dot{B}^s_{2,\infty}}$ defined by

$$\|f\|_{\dot{B}^s_{2,\infty}} := \sup_{j\in\mathbb{Z}} 2^{sj} \left\|\dot{\Delta}_j f\right\|_{L^2}. \tag{1.6}$$

Throughout this paper, we let C denote some positive (generally large) universal constants which *do not* depend on either k or N; otherwise, we will denote them by C_k, C_N, etc. We will use $a \lesssim b$ if $a \le Cb$, and $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$. We use C_0 to denote the constants depending on the initial data and k, N, s. For simplicity, we write $\|(A,B)\|_X := \|A\|_X + \|B\|_X$ and $\int f := \int_{\mathbb{R}^3} f\,dx$.

For $N \ge 3$, we define the energy functional by

$$\mathcal{E}_N(t) := \sum_{l=0}^{N} \left\|\nabla^l (n,u,E,B)\right\|^2 \tag{1.7}$$

and the corresponding dissipation rate by

$$\mathcal{D}_N(t) := \sum_{l=0}^{N} \left\|\nabla^l n\right\|^2 + \sum_{l=0}^{N+1} \left\|\nabla^l u\right\|^2 + \sum_{l=0}^{N-1} \left\|\nabla^l E\right\|^2 + \sum_{l=1}^{N-1} \left\|\nabla^l B\right\|^2. \tag{1.8}$$

Our first main result about the global unique solution to the system (1.3) is stated as follows.

Theorem 1.1. *There exists a sufficiently small constant $\delta_0 > 0$ such that, if $\mathcal{E}_3(0) \leqslant \delta_0$, then the system* (1.3) *has a unique global solution satisfying*

$$\sup_{0\leqslant t\leqslant\infty} \mathcal{E}_3(t) + \int_0^\infty \mathcal{D}_3(\tau)d\tau \leqslant C\mathcal{E}_3(0). \tag{1.9}$$

Moreover, if $\mathcal{E}_N(0) < +\infty$ for any $N \geqslant 3$, there exists an increasing continuous function $P_N(\cdot)$ with $P_N(0) = 0$ such that the unique solution satisfies

$$\sup_{0\leqslant t\leqslant\infty} \mathcal{E}_N(t) + \int_0^\infty \mathcal{D}_N(\tau)d\tau \leqslant CP_N(\mathcal{E}_N(0)). \tag{1.10}$$

The proof of Theorem 1.1 is inspired by the works of [5, 23, 24]. The major difficulty here is the regularity-loss of the electromagnetic field. We will do the refined energy estimates stated in Lemma 2.8–2.9, which allow us to deduce

$$\frac{d}{dt}\mathcal{E}_3 + \mathcal{D}_3 \lesssim \sqrt{\mathcal{E}_3}\mathcal{D}_3$$

and, for $N \geqslant 4$,

$$\frac{d}{dt}\mathcal{E}_N + \mathcal{D}_N \le C_N \mathcal{D}_{N-1}\mathcal{E}_N.$$

Then Theorem 1.1 follows in the fashion of [5, 23, 24].

Our second main result is on some various decay rates of the solution to the system (1.3) by making the much stronger assumption on the initial data.

Theorem 1.2. *Assume that $(n,u,E,B)(t)$ is the solution to the Cauchy problem* (1.3)–(1.4) *constructed in Theorem 1.1 with $N \geqslant 5$. There exists a sufficiently small $\delta_0 = \delta_0(N)$*

such that, if $\mathcal{E}_N(0)\le\delta_0$, and assuming that $(u_0,E_0,B_0)\in\dot H^{-s}$ for some $s\in[0,3/2)$ or $(u_0,E_0,B_0)\in\dot B^{-s}_{2,\infty}$ for some $s\in(0,3/2]$, then we have

$$\|(u,E,B)(t)\|_{\dot H^{-s}}\le C_0 \tag{1.11}$$

or

$$\|(u,E,B)(t)\|_{\dot B^{-s}_{2,\infty}}\le C_0. \tag{1.12}$$

Moreover, for any fixed integer $k\ge 0$, if $N\ge 2k+2+s$, then

$$\left\|\nabla^k(n,u,E,B)(t)\right\|\le C_0(1+t)^{-\frac{k+s}{2}}. \tag{1.13}$$

Furthermore, for any fixed integer $k\ge 0$, if $N\ge 2k+4+s$, then

$$\left\|\nabla^k(n,u,E)(t)\right\|\le C_0(1+t)^{-\frac{k+1+s}{2}}; \tag{1.14}$$

if $N\ge 2k+6+s$, then

$$\left\|\nabla^k n(t)\right\|\le C_0(1+t)^{-\frac{k+2+s}{2}}; \tag{1.15}$$

and, if $N\ge 2k+12+s$ and $B_\infty=0$, then

$$\left\|\nabla^k(n,\mathrm{div}u)(t)\right\|\le C_0(1+t)^{-(\frac{k}{2}+\frac{7}{4}+s)}. \tag{1.16}$$

The proof of Theorem 1.2 is based on the regularity interpolation method developed in Strain and Guo [19], Guo and Wang [7], and Sohinger and Strain [20]. To prove the optimal decay rate of the dissipative equations in the whole space, Guo and Wang [7] developed a general energy method of using a family of scaled energy estimates with minimum derivative counts and interpolations among them. The method of [7,20] can be applied to many dissipative equations in the whole space. However, it cannot be applied directly to the compressible Navier–Stokes–Maxwell system which is of regularity-loss. In addition to the regularity-loss of the electromagnetic field, there is another difficulty, that is, the dissipation $\mathcal{D}_k^{k+2}$ contains $\left\|\nabla^{k+3}u\right\|^2$ which cannot be contained in $\mathcal{E}_k^{k+2}$. Based on the refined energy estimates stated in Lemma 2.8–2.9, we deduce

$$\frac{d}{dt}\mathcal{E}_k^{k+2}+\mathcal{D}_k^{k+2}\le C_k\|(n,u)\|_{L^\infty}\left\|\nabla^{k+2}(n,u)\right\|\left\|\nabla^{k+2}(E,B)\right\|, \tag{1.17}$$

where $\mathcal{E}_k^{k+2}$ and $\mathcal{D}_k^{k+2}$ with minimum derivative counts are defined by (3.5) and (3.6), respectively. Then, combining the methods of [7,20] and a trick of Strain and Guo [19] to treat the electromagnetic field and the term $\left\|\nabla^{k+3}u\right\|^2$, we are able to conclude the decay rate (1.13). If in view of the whole solution, the decay rate (1.13) can be regarded as optimal. The faster decay rates (1.14)–(1.16) follow by revisiting the equations carefully. In particular, we will use a bootstrap argument to derive (1.16).

As quoted above, by Theorem 1.2, we have the following corollary of the usual L^p–L^2 type of the decay results.

Corollary 1.1. *Under the assumptions of Theorem 1.2 except that we replace the $\dot H^{-s}$ or $\dot B^{-s}_{2,\infty}$ assumption by that $(u_0,E_0,B_0)\in L^p$ for some $p\in[1,2]$, then, for any fixed integer $k\ge 0$, if $N\ge 2k+2+s_p$, then*

$$\left\|\nabla^k(n,u,E,B)(t)\right\|\le C_0(1+t)^{-\frac{k+s_p}{2}}. \tag{1.18}$$

Here, the number $s_p := 3\left(\frac{1}{p} - \frac{1}{2}\right)$.

Furthermore, for any fixed integer $k \geq 0$, *if* $N \geq 2k+4+s_p$, *then*

$$\left\|\nabla^k(n,u,E)(t)\right\| \leq C_0(1+t)^{-\frac{k+1+s_p}{2}}; \tag{1.19}$$

if $N \geq 2k+6+s_p$, *then*

$$\left\|\nabla^k n(t)\right\| \leq C_0(1+t)^{-\frac{k+2+s_p}{2}}; \tag{1.20}$$

and, if $N \geq 2k+12+s_p$ *and* $B_\infty = 0$, *then*

$$\left\|\nabla^k(n,\mathrm{div}u)(t)\right\| \leq C_0(1+t)^{-(\frac{k}{2}+\frac{7}{4}+s_p)}. \tag{1.21}$$

The following are remarks for Theorem 1.1, Theorem 1.2, and Corollary 1.1.

REMARK 1.1. In Theorem 1.1, we only assume the H^3 norm of the initial data is small, but the higher-order derivatives can be arbitrarily large. Notice that, in Theorem 1.2, the $\dot{H}^{-s}$ and $\dot{B}^{-s}_{2,\infty}$ norms of the solution are preserved along the time evolution but that in Corollary 1.1 it is difficult to show that the L^p norm of the solution can be preserved. Note that the L^2 decay rate of the higher-order spatial derivatives of the solution are obtained. Then the general optimal L^q $(2 \leq q \leq \infty)$ decay rates of the solution follow by the Sobolev interpolation.

REMARK 1.2. We remark that, in Theorem 1.2, the homogeneous Sobolev space $\dot{H}^{-s}$ was introduced there to enhance the decay rates. By the usual embedding theorem, we know that, for $p \in (1,2]$, $L^p \subset \dot{H}^{-s}$ with $s = 3\left(\frac{1}{p} - \frac{1}{2}\right) \in [0,3/2)$. Hence the L^p–L^2 type of the optimal decay results follows as a corollary. However, this does not cover the case $p=1$. To amend this, Sohinger and Strain [20] instead introduced the homogeneous Besov space $\dot{B}^{-s}_{2,\infty}$ due to the fact that the endpoint embedding $L^1 \subset \dot{B}^{-3/2}_{2,\infty}$ holds.

The rest of our paper is organized as follows. In Section 2, we establish the refined energy estimates for the solution and derive the negative Sobolev and Besov estimates. Theorem 1.1 and Theorem 1.2 are proved in Section 3.

2. Nonlinear energy estimates

In this section, we will do the a priori estimate by assuming that $\|(n,u,E,B)(t)\|_{H^3} \leq \delta \ll 1$. Then, by Sobolev's inequality, we have

$$\frac{1}{2} \leqslant 1+n \leqslant \frac{3}{2}. \tag{2.1}$$

Since the Navier–Stokes–Maxwell system is a symmetrizable hyperbolic system, the Cauchy problem in $\mathbb{R}^3$ has a unique local smooth solution when the initial data is smooth; see Kato [13] and Jerome [11] for instance. According to [18], the global existence of smooth solutions follows from the local existence and the a priori estimates via a standard continuity argument.

2.1. Preliminary.

In this subsection, we collect the analytic tools which will be used in the paper.

LEMMA 2.1. *Let* $2 \leq p \leq +\infty$ *and* $\alpha, m, \ell \geq 0$. *Then we have*

$$\|\nabla^\alpha f\|_{L^p} \leq C_p \|\nabla^m f\|^{1-\theta} \left\|\nabla^\ell f\right\|^\theta.$$

Here $0\le\theta\le 1$ *(if* $p=+\infty$*, then we require that* $0<\theta<1$*) and* α *satisfy*

$$\alpha+3\left(\frac{1}{2}-\frac{1}{p}\right)=m(1-\theta)+\ell\theta.$$

Proof. For the case $2\le p<+\infty$, we refer to Lemma A.1 in [7]; for the case $p=+\infty$, we refer to Exercise 6.1.2 in [4]. ∎

LEMMA 2.2. *Assume* $\|n\|_{H^3}\le 1$ *and the function* $h(n)$ *satisfies*

$$h(n)\sim n \text{ and } \left|h^{(k)}(n)\right|\le C_k \text{ for any } k\ge 1. \tag{2.2}$$

Then, for any integer $k\ge 0$ *and* $p\ge 2$*, we have*

$$\left\|\nabla^k h(n)\right\|_{L^\infty}\le C_k\left\|\nabla^k n\right\|^{1/4}\left\|\nabla^{k+2} n\right\|^{3/4} \tag{2.3}$$

and

$$\left\|\nabla^k h(n)\right\|_{L^p}\le C_k\left\|\nabla^k n\right\|_{L^p}. \tag{2.4}$$

Proof. The proof is based on Lemma 2.1. For (2.3), we refer to Lemma 3.1 in [7]. For (2.4), in light of (2.2), it suffices to prove that, when $k\ge 1$, (2.4) holds for all $h(n)$ with bounded derivatives. We will use an induction on $k\ge 1$. If $k=1$, we have

$$\|\nabla h(n)\|_{L^p}=\|h'(n)\nabla n\|_{L^p}\lesssim\|\nabla n\|_{L^p}.$$

Assume (2.4) holds for from 1 to $k-1$. We use the Leibniz formula to have

$$\begin{aligned}\left\|\nabla^k h(n)\right\|_{L^p}&=\left\|\nabla^{k-1}(h'(n)\nabla n)\right\|_{L^p}\\&\le C_k\left(\left\|h'(n)\nabla^k n\right\|_{L^p}+\left\|\nabla h'(n)\nabla^{k-1}n\right\|_{L^p}+\sum_{\ell=2}^{k-1}\left\|\nabla^\ell h'(n)\nabla^{k-\ell}n\right\|_{L^p}\right).\end{aligned} \tag{2.5}$$

Here, if $k=2$, then the summing term in (2.5) is nothing, etc. By Hölder's and Sobolev's inequalities, we have

$$\left\|h'(n)\nabla^k n\right\|_{L^p}\lesssim\left\|\nabla^k n\right\|_{L^p}$$

and

$$\begin{aligned}\left\|\nabla h'(n)\nabla^{k-1}n\right\|_{L^p}&\lesssim\|\nabla n\|_{L^{2p}}\left\|\nabla^{k-1}n\right\|_{L^{2p}}\\&\lesssim\|\nabla^\alpha n\|^{\frac{2pk+p-3}{2pk+3p-6}}\left\|\nabla^k n\right\|_{L^p}^{\frac{2p-3}{2pk+3p-6}}\|n\|^{\frac{2p-3}{2pk+3p-6}}\left\|\nabla^k n\right\|_{L^6}^{\frac{2pk+p-3}{2pk+3p-6}}\\&\lesssim\left\|\nabla^k n\right\|_{L^p},\end{aligned}$$

where α is defined by

$$\frac{1}{3}-\frac{1}{2p}=\left(\frac{\alpha}{3}-\frac{1}{2}\right)\times\frac{2pk+p-3}{2pk+3p-6}+\left(\frac{k}{3}-\frac{1}{p}\right)\times\frac{2p-3}{2pk+3p-6}\Longrightarrow\alpha=\frac{6pk+9p-18}{4pk+2p-6}<3.$$

For the summing term, we use the induction hypothesis to obtain that, for $2 \le \ell \le k-1$,

$$\left\|\nabla^{\ell} h'(n) \nabla^{k-\ell} n\right\|_{L^p} \le \left\|\nabla^{\ell} h'(n)\right\|_{L^p} \left\|\nabla^{k-\ell} n\right\|_{L^\infty} \lesssim \left\|\nabla^{\ell} n\right\|_{L^p} \left\|\nabla^{k-\ell} n\right\|_{L^\infty}.$$

By Lemma 2.1, if $\ell \le \left[\frac{k-1}{2}\right]$, then we have

$$\begin{aligned}\left\|\nabla^{\ell} n\right\|_{L^p} \left\|\nabla^{k-\ell} n\right\|_{L^\infty} &\lesssim \left\|\nabla^{\alpha} n\right\|^{\frac{2pk-2p\ell+3p}{2pk+3p-6}} \left\|\nabla^{k} n\right\|_{L^p}^{\frac{2p\ell-6}{2pk+3p-6}} \|n\|^{\frac{2p\ell-6}{2pk+3p-6}} \left\|\nabla^{k} n\right\|_{L^p}^{\frac{2pk-2p\ell+3p}{2pk+3p-6}} \\ &\lesssim \left\|\nabla^{k} n\right\|_{L^p},\end{aligned}$$

where α is defined by

$$\begin{aligned}\frac{\ell}{3} - \frac{1}{p} &= \left(\frac{\alpha}{3} - \frac{1}{2}\right) \times \frac{2pk-2p\ell+3p}{2pk+3p-6} + \left(\frac{k}{3} - \frac{1}{p}\right) \times \frac{2p\ell-6}{2pk+3p-6} \\ &\Longrightarrow \alpha = \frac{6pk+9p-18}{4pk-4p\ell+6p} < 3,\end{aligned}$$

and, if $\ell \ge \left[\frac{k-1}{2}\right]+1$, then we have

$$\begin{aligned}\left\|\nabla^{\ell} n\right\|_{L^p} \left\|\nabla^{k-\ell} n\right\|_{L^\infty} &\lesssim \|n\|^{\frac{2pk-2p\ell}{2pk+3p-6}} \left\|\nabla^{k} n\right\|_{L^p}^{\frac{2p\ell+3p-6}{2pk+3p-6}} \left\|\nabla^{\alpha} n\right\|^{\frac{2p\ell+3p-6}{2pk+3p-6}} \left\|\nabla^{k} n\right\|_{L^p}^{\frac{2pk-2p\ell}{2pk+3p-6}} \\ &\lesssim \left\|\nabla^{k} n\right\|_{L^p},\end{aligned}$$

where α is defined by

$$\frac{k-\ell}{3} = \left(\frac{\alpha}{3} - \frac{1}{2}\right) \times \frac{2p\ell+3p-6}{2pk+3p-6} + \left(\frac{k}{3} - \frac{1}{p}\right) \times \frac{2pk-2p\ell}{2pk+3p-6} \Longrightarrow \alpha = \frac{6pk+9p-18}{4p\ell+6p-12} < 3.$$

We thus conclude the lemma. ∎

We recall the following commutator and product estimates.

Lemma 2.3. *Let $k \ge 1$ be an integer and define the commutator*

$$\left[\nabla^k, g\right] h = \nabla^k (gh) - g \nabla^k h. \tag{2.6}$$

Then we have

$$\left\|\left[\nabla^k, g\right] h\right\|_{L^{p_0}} \le C_k \left(\|\nabla g\|_{L^{p_1}} \left\|\nabla^{k-1} h\right\|_{L^{p_2}} + \left\|\nabla^k g\right\|_{L^{p_3}} \|h\|_{L^{p_4}}\right).$$

In addition, we have that, for $k \ge 0$,

$$\left\|\nabla^k (gh)\right\|_{L^{p_0}} \le C_k \left(\|g\|_{L^{p_1}} \left\|\nabla^k h\right\|_{L^{p_2}} + \left\|\nabla^k g\right\|_{L^{p_3}} \|h\|_{L^{p_4}}\right). \tag{2.7}$$

In the above, $p_0, p_2, p_3 \in (1, +\infty)$ such that

$$\frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Proof. It can be proved by using Lemma 2.1; see Lemma 3.4 in [16] for instance. ∎

We have the following L^p embeddings.

Lemma 2.4. *Let $0 \le s < 3/2$ and $1 < p \le 2$ with $1/2 + s/3 = 1/p$. Then*

$$\|f\|_{\dot{H}^{-s}} \lesssim \|f\|_{L^p}. \tag{2.8}$$

Proof. It follows from the Hardy–Littlewood–Sobolev theorem; see [4]. ☐

Lemma 2.5. *Let $0<s\le 3/2$ and $1\le p<2$ with $1/2+s/3=1/p$. Then*

$$\|f\|_{\dot{B}^{-s}_{2,\infty}} \lesssim \|f\|_{L^p}. \tag{2.9}$$

Proof. See Lemma 4.6 in [20]. ☐

It is important to use the following special interpolation estimates:

Lemma 2.6. *Let $s\ge 0$ and $\ell\ge 0$. Then we have*

$$\left\|\nabla^\ell f\right\| \le \left\|\nabla^{\ell+1} f\right\|^{1-\theta} \|f\|^{\theta}_{\dot{H}^{-s}}, \ \textit{where } \theta=\frac{1}{\ell+1+s}. \tag{2.10}$$

Proof. It follows directly by the Parseval theorem and Hölder's inequality. ☐

Lemma 2.7. *Let $s>0$ and $\ell\ge 0$. Then we have*

$$\left\|\nabla^\ell f\right\| \le \left\|\nabla^{\ell+1} f\right\|^{1-\theta} \|f\|^{\theta}_{\dot{B}^{-s}_{2,\infty}}, \ \textit{where } \theta=\frac{1}{\ell+1+s}. \tag{2.11}$$

Proof. See Lemma 4.5 in [20]. ☐

2.2. Energy estimates. In this subsection, we derive the basic energy estimates for the solution to the Navier–Stokes–Maxwell system (1.3). We begin with the standard energy estimates.

Lemma 2.8. *For any integer $k\ge 0$, we have*

$$\begin{aligned}
&\frac{d}{dt}\sum_{l=k}^{k+2}\left\|\nabla^l(n,u,E,B)\right\|^2+\sum_{l=k}^{k+3}\left\|\nabla^l u\right\|^2\\
&\lesssim (\|n\|_{H^2}+\|u\|_{H^3}+\|\nabla B\|_{H^1})\\
&\qquad\times\left(\sum_{l=k}^{k+2}\left\|\nabla^l n\right\|^2+\sum_{l=k}^{k+3}\left\|\nabla^l u\right\|^2+\sum_{l=k}^{k+1}\left\|\nabla^l E\right\|^2+\left\|\nabla^{k+1}B\right\|^2\right)\\
&\qquad+\|(n,u)\|_{L^\infty}\left\|\nabla^{k+2}(n,u)\right\|\left\|\nabla^{k+2}(E,B)\right\|.
\end{aligned} \tag{2.12}$$

Proof. The standard ∇^l $(l=k,k+1,k+2)$ energy estimates on the system (1.3) yield

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\left|\nabla^l(n,u,E,B)\right|^2+\left\|\nabla^l u\right\|^2+\left\|\nabla^{l+1}u\right\|^2\\
&=-\int\nabla^l(u\cdot\nabla n+n\mathrm{div}u)\nabla^l n-\int\nabla^l(u\cdot\nabla u+f(n)\nabla n)\cdot\nabla^l u\\
&\qquad-\int\nabla^l(g(n)\Delta u)\cdot\nabla^l u-\int\nabla^l(u\times B)\cdot\nabla^l u+\int\nabla^l(nu)\cdot\nabla^l E\\
&:=I_1+I_2+I_3+I_4+I_5.
\end{aligned} \tag{2.13}$$

We shall estimate the terms I_1–I_5 on the right-hand side of (2.13). First, we estimate I_1–I_3. We consider the case $l=k=0$. By Hölder's, Sobolev's, and Cauchy's inequalities, we obtain

$$\begin{aligned}
&-\int (u\cdot\nabla n+n\mathrm{div}u)n-\int (u\cdot\nabla u+f(n)\nabla n)\cdot u-\int g(n)\Delta u\cdot u\\
&\lesssim (\|u\|_{L^6}\|\nabla n\|_{L^3}+\|n\|_{L^\infty}\|\mathrm{div}u\|)\|n\|+\|u\|_{L^\infty}\|\nabla u\|\|u\|\\
&\qquad +\|f(n)\|\|\nabla n\|_{L^3}\|u\|_{L^6}+\|\nabla g(n)\|_{L^3}\|\nabla u\|\|u\|_{L^6}+\|g(n)\|_{L^\infty}\|\nabla u\|^2\\
&\lesssim \|(n,\nabla u)\|_{H^2}\left(\|n\|^2+\|u\|_{H^1}^2\right).
\end{aligned}\tag{2.14}$$

For the case $l\geq 1$ ($l=k,k+1,k+2$), by the integration by parts, lemmas 2.2–2.3, and Hölder's, Sobolev's, and Cauchy's inequalities, we obtain

$$\begin{aligned}
I_1&=-\int\left([\nabla^l,u]\cdot\nabla n+u\cdot\nabla\nabla^l n\right)\nabla^l n-\int\nabla^l(n\mathrm{div}u)\nabla^l n\\
&\lesssim \left\|[\nabla^l,u]\cdot\nabla n\right\|\left\|\nabla^l n\right\|+\|\mathrm{div}u\|_{L^\infty}\left\|\nabla^l n\right\|^2+\left\|\nabla^l(n\mathrm{div}u)\right\|\left\|\nabla^l n\right\|\\
&\lesssim \left(\|\nabla u\|_{L^\infty}\left\|\nabla^l n\right\|+\left\|\nabla^l u\right\|_{L^6}\|\nabla n\|_{L^3}\right)\left\|\nabla^l n\right\|+\|\nabla u\|_{H^2}\left\|\nabla^l n\right\|^2\\
&\quad+\left(\|n\|_{L^\infty}\left\|\nabla^{l+1}u\right\|+\left\|\nabla^l n\right\|\|\nabla u\|_{L^\infty}\right)\left\|\nabla^l n\right\|\\
&\lesssim \|(n,\nabla u)\|_{H^2}\left(\left\|\nabla^l n\right\|^2+\left\|\nabla^{l+1}u\right\|^2\right),
\end{aligned}\tag{2.15}$$

$$\begin{aligned}
I_2&=\int\nabla^{l-1}(u\cdot\nabla u+f(n)\nabla n)\cdot\nabla^{l+1}u\\
&\lesssim\left(\left\|\nabla^{l-1}(u\cdot\nabla u)\right\|+\left\|\nabla^{l-1}(f(n)\nabla n)\right\|\right)\left\|\nabla^{l+1}u\right\|\\
&\lesssim\left(\|u\|_{L^\infty}\left\|\nabla^l u\right\|+\left\|\nabla^{l-1}u\right\|_{L^6}\|\nabla u\|_{L^3}\right)\left\|\nabla^{l+1}u\right\|\\
&\quad+\left(\|f(n)\|_{L^\infty}\left\|\nabla^l n\right\|+\left\|\nabla^{l-1}f(n)\right\|_{L^6}\|\nabla n\|_{L^3}\right)\left\|\nabla^{l+1}u\right\|\\
&\lesssim\|(n,\nabla u)\|_{H^2}\left(\left\|\nabla^l(n,u)\right\|^2+\left\|\nabla^{l+1}u\right\|^2\right),
\end{aligned}\tag{2.16}$$

and

$$\begin{aligned}
I_3&=-\int\nabla^l(g(n)\Delta u)\cdot\nabla^l u=\int\nabla^{l-1}(g(n)\Delta u)\cdot\nabla^{l+1}u\\
&\lesssim\left\|\nabla^{l-1}(g(n)\Delta u)\right\|\left\|\nabla^{l+1}u\right\|\\
&\lesssim\left(\|g(n)\|_{L^\infty}\left\|\nabla^{l-1}\Delta u\right\|+\left\|\nabla^{l-1}g(n)\right\|_{L^6}\|\Delta u\|_{L^3}\right)\left\|\nabla^{l+1}u\right\|\\
&\lesssim\left(\|n\|_{H^2}\left\|\nabla^{l+1}u\right\|+\left\|\nabla^l n\right\|\|\nabla u\|_{H^2}\right)\left\|\nabla^{l+1}u\right\|\\
&\lesssim\left(\|n\|_{H^2}+\|\nabla u\|_{H^2}\right)\left(\left\|\nabla^l n\right\|^2+\left\|\nabla^{l+1}u\right\|^2\right).
\end{aligned}\tag{2.17}$$

In light of the estimates (2.14)–(2.17), we have

$$I_1+I_2+I_3\lesssim\|(n,\nabla u)\|_{H^2}\left(\left\|\nabla^l(n,u)\right\|^2+\left\|\nabla^{l+1}u\right\|^2\right).$$

Next, we estimate the term I_4, and we must be much more careful with this term since the magnetic field B has the weakest dissipative estimates. First of all, by Hölder's inequality, we have

$$I_4=-\int\nabla^l(u\times B)\cdot\nabla^l u\leq\left\|\nabla^l(u\times B)\right\|\left\|\nabla^l u\right\|.\tag{2.18}$$

We have to distinguish the arguments by the value of l. We make good use of the product estimates (2.7) of Lemma 2.3 to bound

$$\begin{aligned}
\left\|\nabla^l(u\times B)\right\| &\le C_l\left(\|B\|_{L^\infty}\left\|\nabla^l u\right\|+\|u\|_{L^3}\left\|\nabla^l B\right\|_{L^6}\right)\\
&\le C_l\left(\|\nabla B\|_{H^1}\left\|\nabla^l u\right\|+\|u\|_{H^1}\left\|\nabla^{l+1}B\right\|\right)\quad\text{for}\quad l=k; \qquad (2.19)\\
\left\|\nabla^l(u\times B)\right\| &\le C_l\left(\|B\|_{L^\infty}\left\|\nabla^l u\right\|+\|u\|_{L^\infty}\left\|\nabla^l B\right\|\right)\\
&\le C_l\left(\|\nabla B\|_{H^1}\left\|\nabla^l u\right\|+\|u\|_{H^2}\left\|\nabla^l B\right\|\right)\quad\text{for}\quad l=k+1;\\
\left\|\nabla^l(u\times B)\right\| &\le C_l\left(\|B\|_{L^\infty}\left\|\nabla^l u\right\|+\|u\|_{L^\infty}\left\|\nabla^l B\right\|\right)\\
&\le C_l\left(\|\nabla B\|_{H^1}\left\|\nabla^l u\right\|+\|u\|_{L^\infty}\left\|\nabla^l B\right\|\right)\quad\text{for}\quad l=k+2.
\end{aligned}$$

Hence, by Young's inequality, we deduce from (2.18) that, for $l=k$,

$$I_4\le C_k\left(\|u\|_{H^1}+\|\nabla B\|_{H^1}\right)\left(\left\|\nabla^k u\right\|^2+\left\|\nabla^{k+1}B\right\|^2\right), \qquad (2.20)$$

for $l=k+1$,

$$I_4\le C_k\left(\|u\|_{H^2}+\|\nabla B\|_{H^1}\right)\left(\left\|\nabla^{k+1} u\right\|^2+\left\|\nabla^{k+1}B\right\|^2\right),$$

and, for $l=k+2$,

$$I_4\le C_k\|\nabla B\|_{H^1}\left\|\nabla^{k+2}u\right\|^2+C_k\|u\|_{L^\infty}\left\|\nabla^{k+2}B\right\|\left\|\nabla^{k+2}u\right\|.$$

Finally, we estimate the last term I_5. First, by Hölder's inequality, we obtain

$$I_5=\int\nabla^l(nu)\cdot\nabla^l E\lesssim\left\|\nabla^l(nu)\right\|\left\|\nabla^l E\right\|. \qquad (2.21)$$

We next again have to distinguish the arguments by the value of l. By the product estimates (2.7) of Lemma 2.3 and Sobolev's inequality, we have

$$\begin{aligned}
\left\|\nabla^l(nu)\right\| &\le C_l\left(\|n\|_{L^3}\left\|\nabla^l u\right\|_{L^6}+\|u\|_{L^\infty}\left\|\nabla^l n\right\|\right)\\
&\le C_l\left(\|n\|_{H^1}\left\|\nabla^{l+1}u\right\|+\|\nabla u\|_{H^1}\left\|\nabla^l n\right\|\right)\quad\text{for}\quad l=k,k+1;\\
\left\|\nabla^l(nu)\right\| &\le C_l\left(\|n\|_{L^\infty}\left\|\nabla^l u\right\|+\|u\|_{L^\infty}\left\|\nabla^l n\right\|\right)\quad\text{for}\quad l=k+2.
\end{aligned}$$

Hence, by Young's inequality, we deduce from (2.21) that, for $l=k,\,k+1$,

$$I_5\le C_k\left(\|n\|_{H^1}+\|\nabla u\|_{H^1}\right)\left(\left\|\nabla^l(n,E)\right\|^2+\left\|\nabla^{l+1}u\right\|^2\right)$$

and, for $l=k+2$,

$$I_5\le C_k\|(n,u)\|_{L^\infty}\left\|\nabla^{k+2}(n,u)\right\|\left\|\nabla^{k+2}E\right\|.$$

Consequently, plugging these estimates for I_1–I_5 into (2.13) with $l=k,\,k+1,\,k+2$ and then summing up, we deduce (2.12). ∎

Note that, in Lemma 2.8, we only derive the dissipative estimate of u. We now recover the dissipative estimates of n, E, and B by constructing some interactive energy functionals in the following lemma.

Lemma 2.9. *For any integer $k\geq 0$, we have*

$$\frac{d}{dt}\left(\sum_{l=k}^{k+1}\int\nabla^l u\cdot\nabla\nabla^l n+\sum_{l=k}^{k+1}\int\nabla^l u\cdot\nabla^l E-\int\nabla^k E\cdot\nabla^k\nabla\times B\right)$$
$$+\sum_{l=k}^{k+2}\left\|\nabla^l n\right\|^2+\sum_{l=k}^{k+1}\left\|\nabla^l E\right\|^2+\left\|\nabla^{k+1}B\right\|^2$$
$$\lesssim\left(\|n\|_{H^2}^2+\|u\|_{H^3}^2+\|\nabla B\|_{H^1}^2\right)\left(\sum_{l=k}^{k+2}\left\|\nabla^l n\right\|^2+\sum_{l=k}^{k+3}\left\|\nabla^l u\right\|^2+\left\|\nabla^{k+1}B\right\|^2\right)+\sum_{l=k}^{k+3}\left\|\nabla^l u\right\|^2. \tag{2.22}$$

Proof. We divide the proof into several steps.

Step 1: Dissipative estimate of n.

Applying ∇^l $(l=k,k+1)$ to $(1.3)_2$ and then taking the L^2 inner product with $\nabla\nabla^l n$, we obtain

$$\begin{aligned}&\int\partial_t\nabla^l u\cdot\nabla\nabla^l n+\left\|\nabla\nabla^l n\right\|^2\\ &\leq-\int\nabla^l E\cdot\nabla\nabla^l n+C\left\|\nabla^{l+2}u\right\|\left\|\nabla^{l+1}n\right\|+C\left\|\nabla^l u\right\|\left\|\nabla^{l+1}n\right\|\\ &\quad+\left\|\nabla^l\left(u\cdot\nabla u+f(n)\nabla n+u\times B+g(n)\Delta u\right)\right\|\left\|\nabla^{l+1}n\right\|.\end{aligned} \tag{2.23}$$

For the first term on the left-hand side of (2.23), we similarly obtain

$$\begin{aligned}\int\nabla^l\partial_t u\cdot\nabla\nabla^l n&=\frac{d}{dt}\int\nabla^l u\cdot\nabla\nabla^l n-\int\nabla^l u\cdot\nabla\nabla^l\partial_t n\\ &=\frac{d}{dt}\int\nabla^l u\cdot\nabla\nabla^l n+\int\nabla^l\mathrm{div}u\nabla^l\partial_t n\\ &=\frac{d}{dt}\int\nabla^l u\cdot\nabla\nabla^l n-\left\|\nabla^l\mathrm{div}u\right\|^2-\int\nabla^l\mathrm{div}u\nabla^l\mathrm{div}(nu).\end{aligned}$$

By the product estimates (2.7) of Lemma 2.3, we easily obtain

$$\left\|\nabla^l\mathrm{div}(nu)\right\|\leq C_l\|(n,u)\|_{H^2}\left\|\nabla^{l+1}(n,u)\right\|.$$

So, we obtain

$$\int\nabla^l\partial_t u\cdot\nabla\nabla^l n\geq\frac{d}{dt}\int\nabla^l u\cdot\nabla^l\nabla n-C\left\|\nabla^{l+1}u\right\|^2-C_l\|(n,u)\|_{H^2}^2\left\|\nabla^{l+1}(n,u)\right\|^2. \tag{2.24}$$

For the first term on the right-hand side of (2.23), by integrating by parts and using the equation $(1.3)_5$, we have

$$-\int\nabla^l E\cdot\nabla\nabla^l n=\int\nabla^l\mathrm{div}E\nabla^l n=-\left\|\nabla^l n\right\|^2. \tag{2.25}$$

By (2.7), (2.4), and Sobolev's inequality, we have

$$\begin{aligned}&\left\|\nabla^l\left(u\cdot\nabla u+f(n)\nabla n+g(n)\Delta u\right)\right\|\\ &\leq C_l\left(\|u\|_{L^\infty}\left\|\nabla^{l+1}u\right\|+\left\|\nabla^l u\right\|_{L^6}\|\nabla u\|_{L^3}\right)\end{aligned}$$

$$\begin{aligned}&+C_l\left(\|f(n)\|_{L^\infty}\left\|\nabla^{l+1}n\right\|+\left\|\nabla^l f(n)\right\|_{L^6}\|\nabla n\|_{L^3}\right)\\&+C_l\left(\|g(n)\|_{L^\infty}\left\|\nabla^{l+2}u\right\|+\left\|\nabla^l g(n)\right\|_{L^6}\left\|\nabla^2 u\right\|_{L^3}\right)\\ \le&C_l\left(\|n\|_{H^2}+\|u\|_{H^3}\right)\left(\left\|\nabla^{l+1}(n,u)\right\|+\left\|\nabla^{l+2}u\right\|\right).\end{aligned}\tag{2.26}$$

From the estimate of I_4 in Lemma 2.8, we have that, for $l=k$ or $k+1$,

$$\left\|\nabla^l(u\times B)\right\|\le C_k\left(\|u\|_{H^3}+\|\nabla B\|_{H^1}\right)\left(\left\|\nabla^l u\right\|+\left\|\nabla^{k+1}B\right\|\right).\tag{2.27}$$

Plugging the estimates (2.24)–(2.27) into (2.23), by Cauchy's inequality, we obtain that

$$\begin{aligned}&\frac{d}{dt}\sum_{l=k}^{k+1}\int\nabla^l u\cdot\nabla\nabla^l n+\sum_{l=k}^{k+2}\left\|\nabla^l n\right\|^2\\&\lesssim\left(\|n\|_{H^2}^2+\|u\|_{H^3}^2+\|\nabla B\|_{H^1}^2\right)\left(\sum_{l=k}^{k+2}\left\|\nabla^l n\right\|^2+\sum_{l=k}^{k+3}\left\|\nabla^l u\right\|^2+\left\|\nabla^{k+1}B\right\|^2\right)+\sum_{l=k}^{k+3}\left\|\nabla^l u\right\|^2.\end{aligned}\tag{2.28}$$

This completes the dissipative estimate for n.

Step 2: Dissipative estimate of E.

Applying ∇^l $(l=k,k+1)$ to $(1.3)_2$ and then taking the L^2 inner product with $\nabla^l E$, we obtain

$$\begin{aligned}\int\nabla^l\partial_t u\cdot\nabla^l E+\left\|\nabla^l E\right\|^2\le&-\int\nabla\nabla^l n\cdot\nabla^l E+\int\nabla^l\Delta u\cdot\nabla^l E+C\left\|\nabla^l u\right\|\left\|\nabla^l E\right\|\\&+\left\|\nabla^l\left(u\cdot\nabla u+f(n)\nabla n+u\times B+g(n)\Delta u\right)\right\|\left\|\nabla^l E\right\|.\end{aligned}\tag{2.29}$$

First, for the first term on the left-hand side of (2.29), by integrating by parts in the t-variable and using the Equation $(1.3)_3$ in the Maxwell system, we obtain

$$\begin{aligned}\int\nabla^l\partial_t u\cdot\nabla^l E&=\frac{d}{dt}\int\nabla^l u\cdot\nabla^l E-\int\nabla^l u\cdot\nabla^l\partial_t E\\&=\frac{d}{dt}\int\nabla^l u\cdot\nabla^l E-\left\|\nabla^l u\right\|^2-\int\nabla^l u\cdot\nabla^l\left(nu+\nabla\times B\right).\end{aligned}\tag{2.30}$$

By the product estimates (2.7) of Lemma 2.3, we have that

$$\left\|\nabla^l(nu)\right\|\le C_l\left(\|n\|_{L^\infty}\left\|\nabla^l u\right\|+\left\|\nabla^l n\right\|\|u\|_{L^\infty}\right)\le C_l\|(n,u)\|_{H^2}\left\|\nabla^l(n,u)\right\|.\tag{2.31}$$

We must be much more careful with the remaining term in (2.30) since there is no small factor in front of it. The key is to use Cauchy's inequality and to distinguish the cases of $l=k$ and $l=k+1$ due to the weakest dissipative estimate of B. For $l=k$, we have

$$\int\nabla^k u\cdot\nabla\times\nabla^k B\le\varepsilon\left\|\nabla^{k+1}B\right\|^2+C_\varepsilon\left\|\nabla^k u\right\|^2;\tag{2.32}$$

for $l=k+1$, integrating by parts, we obtain

$$\int\nabla^{k+1}u\cdot\nabla\times\nabla^{k+1}B=\int\nabla\times\nabla^{k+1}u\cdot\nabla^{k+1}B\le\varepsilon\left\|\nabla^{k+1}B\right\|^2+C_\varepsilon\left\|\nabla^{k+2}u\right\|^2.\tag{2.33}$$

By Hölder's and Young's inequalities, we have

$$\int\nabla^l\Delta u\cdot\nabla^l E\lesssim\left\|\nabla^{l+2}u\right\|\left\|\nabla^l E\right\|\lesssim C_\varepsilon\left\|\nabla^{l+2}u\right\|^2+\varepsilon\left\|\nabla^l E\right\|^2.\tag{2.34}$$

Plugging the estimates (2.30)–(2.34) and (2.25)–(2.27) from Step 1 into (2.29), by Cauchy's inequality, we obtain

$$\begin{aligned}
&\frac{d}{dt}\sum_{l=k}^{k+1}\int\nabla^l u\cdot\nabla^l E+\sum_{l=k}^{k+1}\left\|\nabla^l E\right\|^2\\
&\lesssim\varepsilon\left\|\nabla^{k+1}B\right\|^2+\sum_{l=k}^{k+3}\left\|\nabla^l u\right\|^2\\
&\quad+\left(\|n\|_{H^2}^2+\|u\|_{H^3}^2+\|\nabla B\|_{H^1}^2\right)\left(\sum_{l=k}^{k+2}\left\|\nabla^l n\right\|^2+\sum_{l=k}^{k+3}\left\|\nabla^l u\right\|^2+\left\|\nabla^{k+1}B\right\|^2\right). \qquad (2.35)
\end{aligned}$$

This completes the dissipative estimate for E.

Step 3: Dissipative estimate of B.

Applying ∇^k to $(1.3)_3$ and then taking the L^2 inner product with $-\nabla\times\nabla^k B$, we obtain

$$\begin{aligned}
&-\int\nabla^k\partial_t E\cdot\nabla\times\nabla^k B+\left\|\nabla\times\nabla^k B\right\|^2\\
&\le\left\|\nabla^k u\right\|\left\|\nabla\times\nabla^k B\right\|+\left\|\nabla^k(nu)\right\|\left\|\nabla\times\nabla^k B\right\|. \qquad (2.36)
\end{aligned}$$

For the first term on the left-hand side of (2.36), integrating by parts for both the t- and x-variables and using the Equation $(1.3)_4$, we have

$$\begin{aligned}
-\int\nabla^k\partial_t E\cdot\nabla\times\nabla^k B&=-\frac{d}{dt}\int\nabla^k E\cdot\nabla\times\nabla^k B+\int\nabla\times\nabla^k E\cdot\nabla^k\partial_t B\\
&=-\frac{d}{dt}\int\nabla^k E\cdot\nabla\times\nabla^k B-\left\|\nabla\times\nabla^k E\right\|^2. \qquad (2.37)
\end{aligned}$$

By (2.34), we obtain

$$\left\|\nabla^k(nu)\right\|\le C_k\|(n,u)\|_{H^2}\left\|\nabla^k(n,u)\right\|. \qquad (2.38)$$

Plugging the estimates (2.37)–(2.38) into (2.36), by Cauchy's inequality, since $\operatorname{div}B=0$, we obtain

$$\begin{aligned}
&-\frac{d}{dt}\int\nabla^k E\cdot\nabla^k\nabla\times B+\left\|\nabla^{k+1}B\right\|^2\\
&\lesssim\left\|\nabla^k u\right\|^2+\left\|\nabla^{k+1}E\right\|^2+\|(n,u)\|_{H^2}^2\left\|\nabla^k(n,u)\right\|^2. \qquad (2.39)
\end{aligned}$$

This completes the dissipative estimate for B.

Step 4: Conclusion.

Multiplying (2.39) by a small enough but fixed constant, adding it with (2.35) so that the second term on the right-hand side of (2.39) can be absorbed, and choosing ε small enough so that the first term in (2.35) can be absorbed, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\sum_{l=k}^{k+1}\int\nabla^l u\cdot\nabla^l E-\int\nabla^k E\cdot\nabla^k\nabla\times B\right)+\sum_{l=k}^{k+1}\left\|\nabla^l E\right\|^2+\left\|\nabla^{k+1}B\right\|^2\\
&\lesssim\left(\|n\|_{H^2}^2+\|u\|_{H^3}^2+\|\nabla B\|_{H^1}^2\right)\left(\sum_{l=k}^{k+2}\left\|\nabla^l n\right\|^2+\sum_{l=k}^{k+3}\left\|\nabla^l u\right\|^2+\left\|\nabla^{k+1}B\right\|^2\right)+\sum_{l=k}^{k+3}\left\|\nabla^l u\right\|^2. \qquad (2.40)
\end{aligned}$$

Adding the inequality above with (2.28), we get (2.22). □

2.3. Negative Sobolev estimates. In this subsection, we derive the evolution of the negative Sobolev norms of the solution (u,E,B). In order to estimate the nonlinear terms, we need to restrict ourselves to that $s\in(0,3/2)$. We will establish the following lemma.

LEMMA 2.10. *For $s\in(0,1/2]$, we have*

$$\frac{d}{dt}\|(u,E,B)\|_{\dot H^{-s}}^2+\|u\|_{\dot H^{-s}}^2+\|\nabla u\|_{\dot H^{-s}}^2$$
$$\lesssim\Big(\|(n,u)\|_{H^2}^2+\|\nabla B\|_{H^1}^2\Big)\|(u,E,B)\|_{\dot H^{-s}}+\|E\|_{H^2}^2; \tag{2.41}$$

for $s\in(1/2,3/2)$, we have

$$\frac{d}{dt}\|(u,E,B)\|_{\dot H^{-s}}^2+\|u\|_{\dot H^{-s}}^2+\|\nabla u\|_{\dot H^{-s}}^2$$
$$\lesssim\Big(\|n\|_{H^1}^2+\|u\|_{H^2}^2+\|B\|^{s-1/2}\|\nabla B\|^{3/2-s}\|u\|\Big)\|(u,E,B)\|_{\dot H^{-s}}+\|E\|_{H^2}^2. \tag{2.42}$$

Proof. The Λ^{-s} $(s>0)$ energy estimate of $(1.3)_2$–$(1.3)_4$ yield

$$\frac12\frac{d}{dt}\|(u,E,B)\|_{\dot H^{-s}}^2+\|u\|_{\dot H^{-s}}^2+\|\nabla u\|_{\dot H^{-s}}^2$$
$$=-\int\Lambda^{-s}(u\cdot\nabla u+f(n)\nabla n)\cdot\Lambda^{-s}u$$
$$-\int\Lambda^{-s}\nabla n\cdot\Lambda^{-s}u-\int\Lambda^{-s}(u\times B+g(n)\Delta u)\cdot\Lambda^{-s}u+\int\Lambda^{-s}(nu)\cdot\Lambda^{-s}E$$
$$\lesssim\|u\cdot\nabla u+f(n)\nabla n\|_{\dot H^{-s}}\|u\|_{\dot H^{-s}}+\|\nabla n\|_{\dot H^{-s}}\|u\|_{\dot H^{-s}}$$
$$+\|u\times B+g(n)\Delta u\|_{\dot H^{-s}}\|u\|_{\dot H^{-s}}+\|nu\|_{\dot H^{-s}}\|E\|_{\dot H^{-s}}. \tag{2.43}$$

We now restrict the value of s in order to estimate the nonlinear terms on the right-hand side of (2.43). If $s\in(0,1/2]$, then $1/2+s/3<1$ and $3/s\geqslant 6$. Then, applying Lemma 2.4 together with Hölder's, Sobolev's, and Young's inequalities, we obtain

$$\|u\cdot\nabla u\|_{\dot H^{-s}}\lesssim\|u\cdot\nabla u\|_{L^{\frac{1}{1/2+s/3}}}\lesssim\|u\|_{L^{3/s}}\|\nabla u\|$$
$$\lesssim\|\nabla u\|^{1/2+s}\left\|\nabla^2u\right\|^{1/2-s}\|\nabla u\|$$
$$\lesssim\|\nabla u\|_{H^1}^2.$$

Similarly, we can bound

$$\|f(n)\nabla n\|_{\dot H^{-s}}\lesssim\|\nabla n\|_{H^1}^2+\|\nabla n\|^2;$$
$$\|g(n)\Delta u\|_{\dot H^{-s}}\lesssim\|\nabla n\|_{H^1}^2+\left\|\nabla^2u\right\|^2;$$
$$\|nu\|_{\dot H^{-s}}\lesssim\|\nabla u\|_{H^1}^2+\|n\|^2;$$
$$\|u\times B\|_{\dot H^{-s}}\lesssim\|\nabla B\|_{H^1}^2+\|u\|^2.$$

Now, if $s\in(1/2,3/2)$, we shall estimate the right-hand side of (2.43) in a different way. Since $s\in(1/2,3/2)$, we have that $1/2+s/3<1$ and $2<3/s<6$. Then, applying Lemma 2.4 and using (a different) Sobolev's inequality, we have

$$\|u\cdot\nabla u\|_{\dot H^{-s}}\lesssim\|u\|_{L^{3/s}}\|\nabla u\|\lesssim\|u\|^{s-1/2}\|\nabla u\|^{3/2-s}\|\nabla u\|\lesssim\|u\|_{H^1}^2;$$

$$\|f(n)\nabla n\|_{\dot{H}^{-s}} \lesssim \|n\|_{H^1}^2 + \|\nabla n\|^2;$$
$$\|u\times B\|_{\dot{H}^{-s}} \lesssim \|B\|^{s-1/2}\|\nabla B\|^{3/2-s}\|u\|;$$
$$\|g(n)\Delta u\|_{\dot{H}^{-s}} \lesssim \|n\|_{H^1}^2 + \left\|\nabla^2 u\right\|^2;$$
$$\|nu\|_{\dot{H}^{-s}} \lesssim \|u\|_{H^1}^2 + \|n\|^2.$$

Note that we fail to estimate the remaining last term on the right-hand side of (2.43) as above. To overcome this obstacle, the key point is to make full use of $(1.3)_5$. This idea was also used in [21]. Indeed, using $(1.3)_5$, we have

$$\|\nabla n\|_{\dot{H}^{-s}} \lesssim \left\|\Lambda^{-s}\nabla \operatorname{div} E\right\| \lesssim \|E\|_{H^2}.$$

Now, collecting all the estimates we have derived, by Cauchy's inequality, we deduce (2.41) for $s\in(0,1/2]$ and (2.42) for $s\in(1/2,3/2)$. ∎

2.4. Negative Besov estimates. In this section, we derive the evolution of the negative Besov norms of (u,E,B). The argument is similar to the previous subsection.

LEMMA 2.11. *For $s\in(0,1/2]$, we have*

$$\begin{aligned}&\frac{d}{dt}\|(u,E,B)\|_{\dot{B}_{2,\infty}^{-s}}^2 + \|u\|_{\dot{B}_{2,\infty}^{-s}}^2 + \|\nabla u\|_{\dot{B}_{2,\infty}^{-s}}^2 \\ &\lesssim \left(\|(n,u)\|_{H^2}^2 + \|\nabla B\|_{H^1}^2\right)\|(u,E,B)\|_{\dot{B}_{2,\infty}^{-s}} + \|E\|_{H^2}^2;\end{aligned}$$

for $s\in(1/2,3/2]$, we have

$$\begin{aligned}&\frac{d}{dt}\|(u,E,B)\|_{\dot{B}_{2,\infty}^{-s}}^2 + \|u\|_{\dot{B}_{2,\infty}^{-s}}^2 + \|\nabla u\|_{\dot{B}_{2,\infty}^{-s}}^2 \\ &\lesssim \left(\|n\|_{H^1}^2 + \|u\|_{H^2}^2 + \|B\|^{s-1/2}\|\nabla B\|^{3/2-s}\|u\|\right)\|(u,E,B)\|_{\dot{B}_{2,\infty}^{-s}} + \|E\|_{H^2}^2.\end{aligned}$$

Proof. The $\dot{\Delta}_j$ energy estimates of $(1.3)_2$–$(1.3)_4$ yield, with multiplication of 2^{-2sj} and then taking the supremum over $j\in\mathbb{Z}$,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(u,E,B)\|_{\dot{B}_{2,\infty}^{-s}}^2 + \|u\|_{\dot{B}_{2,\infty}^{-s}}^2 + \|\nabla u\|_{\dot{B}_{2,\infty}^{-s}}^2 \\ &\lesssim \sup_{j\in\mathbb{Z}} 2^{-2sj}\left(-\int \dot{\Delta}_j\left(u\cdot\nabla u + f(n)\nabla n + g(n)\Delta u + u\times B\right)\cdot\dot{\Delta}_j u\right) \\ &\quad + \sup_{j\in\mathbb{Z}} 2^{-2sj}\left(\int \dot{\Delta}_j(nu)\cdot\dot{\Delta}_j E - \int \dot{\Delta}_j\nabla n\cdot\dot{\Delta}_j u\right) \\ &\lesssim \|u\cdot\nabla u + f(n)\nabla n + g(n)\Delta u + u\times B\|_{\dot{B}_{2,\infty}^{-s}}\|u\|_{\dot{B}_{2,\infty}^{-s}} \\ &\quad + \|nu\|_{\dot{B}_{2,\infty}^{-s}}\|E\|_{\dot{B}_{2,\infty}^{-s}} + \|\nabla n\|_{\dot{B}_{2,\infty}^{-s}}\|u\|_{\dot{B}_{2,\infty}^{-s}}.\end{aligned}$$

Then, the proof is exactly the same as the proof of Lemma 2.10 except that we should apply Lemma 2.5 instead to estimate the $\dot{B}_{2,\infty}^{-s}$ norm. Note that we allow $s=3/2$. ∎

3. Proof of theorems

3.1. Proof of Theorem 1.1. In this subsection, we will prove the unique global solution to the system (1.3), and the key point is that we only assume the H^3 norm of the initial data is small.

Step 1. Global small $\mathcal{E}_3$ solution.

We close the energy estimates at the H^3 level by assuming a priori that $\sqrt{\mathcal{E}_3(t)} \leq \delta$ is sufficiently small. Taking $k=0,1$ in (2.12) of Lemma 2.8 and then summing up, we obtain

$$\frac{d}{dt}\sum_{l=0}^{3}\left\|\nabla^l(n,u,E,B)\right\|^2+\sum_{l=0}^{4}\left\|\nabla^l u\right\|^2 \lesssim \sqrt{\mathcal{E}_3}\mathcal{D}_3+\sqrt{\mathcal{D}_3}\sqrt{\mathcal{D}_3}\sqrt{\mathcal{E}_3} \lesssim \delta\mathcal{D}_3. \tag{3.1}$$

Taking $k=0,1$ in (2.22) of Lemma 2.9 and then summing up, we obtain

$$\begin{aligned}&\frac{d}{dt}\left(\sum_{l=0}^{2}\int\nabla^l u\cdot\nabla\nabla^l n+\sum_{l=0}^{2}\int\nabla^l u\cdot\nabla^l E-\sum_{l=0}^{1}\int\nabla^l E\cdot\nabla^l\nabla\times B\right)\\ &\qquad+\sum_{l=0}^{3}\left\|\nabla^l n\right\|_{L^2}^2+\sum_{l=0}^{2}\left\|\nabla^l E\right\|_{L^2}^2+\sum_{l=1}^{2}\left\|\nabla^l B\right\|_{L^2}^2\\ &\lesssim\sum_{l=0}^{4}\left\|\nabla^l u\right\|_{L^2}^2+\delta^2\mathcal{D}_3.\end{aligned} \tag{3.2}$$

Multiplying (3.2) by ε and then adding it to (3.1), since δ is small, we deduce that there exists an instant energy functional $\tilde{\mathcal{E}}_3$ equivalent to $\mathcal{E}_3$ such that

$$\frac{d}{dt}\tilde{\mathcal{E}}_3+\mathcal{D}_3 \leqslant 0.$$

Integrating the inequality above directly in time, we obtain (1.9). By a standard continuity argument, we then close the a priori estimates if we assume at initial time that $\mathcal{E}_3(0) \leqslant \delta_0$ is sufficiently small. This concludes the unique global small $\mathcal{E}_3$ solution.

Step 2: Global $\mathcal{E}_N$ solution.

We will prove this by an induction on $N \geqslant 3$. We assume (1.10) holds for $N-1$ (now $N \geqslant 4$) since we have proved in step 1 that (1.10) is valid for $N=3$. Summing up (2.12) of Lemma 2.8 from $k=0,1,\cdots N-2$, we obtain

$$\begin{aligned}&\frac{d}{dt}\sum_{l=0}^{N}\left\|\nabla^l(n,u,E,B)\right\|^2+\sum_{l=0}^{N+1}\left\|\nabla^l u\right\|^2\\ &\lesssim\sqrt{\mathcal{D}_{N-1}}\sqrt{\mathcal{E}_N}\sqrt{\mathcal{D}_N}+\sqrt{\mathcal{D}_{N-1}}\sqrt{\mathcal{D}_N}\sqrt{\mathcal{E}_N}+\sqrt{\mathcal{E}_3}\sum_{l=0}^{N+1}\left\|\nabla^l u\right\|^2\\ &\lesssim\sqrt{\mathcal{D}_{N-1}}\sqrt{\mathcal{E}_N}\sqrt{\mathcal{D}_N}+\sqrt{\mathcal{E}_3}\sum_{l=0}^{N+1}\left\|\nabla^l u\right\|^2.\end{aligned} \tag{3.3}$$

Here, we have used the fact that $3 \leqslant N-2+1=N-1$ since $N \geqslant 4$. Note that it is important that we have put the first two factors in (2.12) into the dissipation.

Taking $k=0,\cdots,N-2$ in (2.22) of Lemma 2.9 and then summing up, we obtain

$$\begin{aligned}&\frac{d}{dt}\Big(\sum_{l=0}^{N-1}\int\nabla^l u\cdot\nabla\nabla^l n+\sum_{l=0}^{N-1}\int\nabla^l u\cdot\nabla^l E-\sum_{l=0}^{N-2}\int\nabla^l E\cdot\nabla\times\nabla^l B\Big)\\&\quad+\sum_{l=0}^{N}\left\|\nabla^l n\right\|_{L^2}^2+\sum_{l=0}^{N-1}\left\|\nabla^l E\right\|_{L^2}^2+\sum_{l=1}^{N-1}\left\|\nabla^l B\right\|_{L^2}^2\\&\lesssim\sum_{l=0}^{N+1}\left\|\nabla^l u\right\|_{L^2}^2+\sqrt{\mathcal{D}_{N-1}}\sqrt{\mathcal{D}_N}\sqrt{\mathcal{E}_N}+\sqrt{\mathcal{E}_3}\sum_{l=0}^{N+1}\left\|\nabla^l u\right\|^2.\end{aligned}\tag{3.4}$$

Since $\sqrt{\mathcal{E}_3}\lesssim\delta$ is sufficiently small, we deduce from (3.4)$\times\epsilon+$ (3.3) (for sufficiently small ϵ) that there exists an instant energy functional $\tilde{\mathcal{E}}_N$ equivalently to $\mathcal{E}_N$ such that, by Cauchy's inequality, we have

$$\frac{d}{dt}\tilde{\mathcal{E}}_N+\mathcal{D}_N\lesssim\sqrt{\mathcal{D}_{N-1}}\sqrt{\mathcal{E}_N}\sqrt{\mathcal{D}_N}\lesssim\epsilon\mathcal{D}_N+\mathcal{D}_{N-1}\mathcal{E}_N.$$

Thus, we have

$$\frac{d}{dt}\tilde{\mathcal{E}}_N+\frac{1}{2}\mathcal{D}_N\lesssim\mathcal{D}_{N-1}\mathcal{E}_N.$$

We then conclude

$$\mathcal{E}_N(t)+\int_0^t\mathcal{D}_N(\tau)d\tau\lesssim\mathcal{E}_N(0)e^{\int_0^t\mathcal{D}_{N-1}(\tau)d\tau}\lesssim\mathcal{E}_N(0)e^{P_{N-1}(\mathcal{E}_N(0))}=P_N(\mathcal{E}_N(0))$$

upon on application of the Gronwall inequality and the induction hypothesis. This concludes the global $\mathcal{E}_N$ solution. The proof of Theorem 1.1 is completed.

3.2. Proof of Theorem 1.2. In this subsection, we will prove the various time decay rates of the unique global solution to the system (1.3) obtained in Theorem 1.1. Fix $N\geq 5$. We need to assume that $\mathcal{E}_N(0)\leq\delta_0=\delta_0(N)$ is small. Then Theorem 1.1 implies that there exists a unique global $\mathcal{E}_N$ solution, and $\mathcal{E}_N(t)\leq P_N\mathcal{E}_N(0)\leq\delta_0$ is small for all time t. Since now our δ_0 is relative small with respect to N, we just ignore the N dependence of the constants in the energy estimates in the previous section.

Step 1. Basic decay.

For the convenience of presentations, we define a family of energy functionals and the corresponding dissipation rates with *minimum derivative counts* as

$$\mathcal{E}_k^{k+2}=\sum_{l=k}^{k+2}\left\|\nabla^l(n,u,E,B)\right\|^2\tag{3.5}$$

and

$$\mathcal{D}_k^{k+2}=\sum_{l=k}^{k+2}\left\|\nabla^l n\right\|^2+\sum_{l=k}^{k+3}\left\|\nabla^l u\right\|^2+\sum_{l=k}^{k+1}\left\|\nabla^l E\right\|^2+\left\|\nabla^{k+1}B\right\|^2.\tag{3.6}$$

By Lemma 2.8, we have that, for $k=0,\ldots,N-2$,

$$\frac{d}{dt}\sum_{l=k}^{k+2}\left\|\nabla^l(n,u,E,B)\right\|^2+\sum_{l=k}^{k+3}\left\|\nabla^l u\right\|^2$$

$$\lesssim \sqrt{\delta_0}\mathcal{D}_k^{k+2} + \|(n,u)\|_{L^\infty}\left\|\nabla^{k+2}(n,u)\right\|\left\|\nabla^{k+2}(E,B)\right\|. \tag{3.7}$$

By Lemma 2.9, we have that, for $k=0,\dots,N-2$,

$$\begin{aligned}&\frac{d}{dt}\left(\sum_{l=k}^{k+1}\int \nabla^l u\cdot\nabla\nabla^l n + \sum_{l=k}^{k+1}\int \nabla^l u\cdot\nabla^l E - \int \nabla^k E\cdot\nabla^k\nabla\times B\right)\\ &\quad + \sum_{l=k}^{k+2}\left\|\nabla^l n\right\|^2 + \sum_{l=k}^{k+1}\left\|\nabla^l E\right\|^2 + \left\|\nabla^{k+1}B\right\|^2\\ &\lesssim \sum_{l=k}^{k+3}\left\|\nabla^l u\right\|^2 + \delta_0\mathcal{D}_k^{k+2}.\end{aligned} \tag{3.8}$$

Multiplying (3.8) by a sufficiently small but fixed factor ε and then adding it with (3.7), since δ_0 is small, we deduce that there exists an instant energy functional $\widetilde{\mathcal{E}}_k^{k+2}$ equivalent to $\mathcal{E}_k^{k+2}$ such that

$$\frac{d}{dt}\widetilde{\mathcal{E}}_k^{k+2} + \mathcal{D}_k^{k+2} \lesssim \|(n,u)\|_{L^\infty}\left\|\nabla^{k+2}(n,u)\right\|\left\|\nabla^{k+2}(E,B)\right\|. \tag{3.9}$$

Note that we cannot absorb the right-hand side of (3.9) by the dissipation $\mathcal{D}_k^{k+2}$ since it does not contain $\left\|\nabla^{k+2}(E,B)\right\|^2$. We will distinguish the arguments by the value of k. If $k=0$, we bound $\left\|\nabla^{k+2}(E,B)\right\|$ by the energy. Then we have that, for $k=0,1$,

$$\frac{d}{dt}\widetilde{\mathcal{E}}_k^{k+2} + \mathcal{D}_k^{k+2} \lesssim \sqrt{\mathcal{D}_k^{k+2}}\sqrt{\mathcal{D}_k^{k+2}}\sqrt{\mathcal{E}_3} \lesssim \sqrt{\delta_0}\mathcal{D}_k^{k+2}, \tag{3.10}$$

which implies

$$\frac{d}{dt}\widetilde{\mathcal{E}}_k^{k+2} + \mathcal{D}_k^{k+2} \le 0. \tag{3.11}$$

If $k\ge 2$, we have to bound $\left\|\nabla^{k+2}(E,B)\right\|$ in term of $\left\|\nabla^{k+1}(E,B)\right\|$ since $\sqrt{\mathcal{D}_k^{k+2}}$ cannot control $\|(n,u)\|_{L^\infty}$. The key point is to use the regularity interpolation method developed in [7, 19]. By Lemma 2.1, we have

$$\begin{aligned}&\|(n,u)\|_{L^\infty}\left\|\nabla^{k+2}(n,u)\right\|\left\|\nabla^{k+2}(E,B)\right\|\\ &\lesssim \|(n,u)\|^{1-\frac{3}{2k}}\left\|\nabla^k(n,u)\right\|^{\frac{3}{2k}}\left\|\nabla^{k+2}(n,u)\right\|\left\|\nabla^{k+1}(E,B)\right\|^{1-\frac{3}{2k}}\|\nabla^\alpha(E,B)\|^{\frac{3}{2k}},\end{aligned} \tag{3.12}$$

where α is defined by

$$k+2 = (k+1)\times\left(1-\frac{3}{2k}\right) + \alpha\times\frac{3}{2k} \Longrightarrow \alpha = \frac{5}{3}k+1. \tag{3.13}$$

Hence, for $k\ge 2$, if $N\ge \frac{5}{3}k+1 \Longleftrightarrow 2\le k\le \frac{3}{5}(N-1)$, then by (3.12), we deduce from (3.9) that

$$\frac{d}{dt}\widetilde{\mathcal{E}}_k^{k+2} + \mathcal{D}_k^{k+2} \lesssim \sqrt{\mathcal{E}_N}\mathcal{D}_k^{k+2} \lesssim \sqrt{\delta_0}\mathcal{D}_k^{k+2}, \tag{3.14}$$

which allow us to find that, for any integer k with $0\le k\le \frac{3}{5}(N-1)$ (note that $N-2\ge \frac{3}{5}(N-1)\ge 2$ since $N\ge 5$), we have

$$\frac{d}{dt}\widetilde{\mathcal{E}}_k^{k+2} + \mathcal{D}_k^{k+2} \le 0. \tag{3.15}$$

The fact that $\mathcal{D}_k^{k+2}$ is weaker than $\mathcal{E}_k^{k+2}$ prevents the exponential decay of the solution. In order to effectively derive the decay rate from (3.15), we still manage to bound the term $\left\|\nabla^{k+3}u\right\|^2$ in $\mathcal{D}_k^{k+2}$ and the missing terms in the energy, that is, $\left\|\nabla^k B\right\|^2$ and $\left\|\nabla^{k+2}(E,B)\right\|^2$ in terms of $\mathcal{E}_k^{k+2}$ in (3.15). We again use the regularity interpolation method, but now we need to use the negative Sobolev or Besov norms. Assume for the moment that we have proved (1.11) or (1.12). Using Lemma 2.6, we have that, for $s \geq 0$ and $k+s>0$,

$$\left\|\nabla^k B\right\| \leq \|B\|_{\dot{H}^{-s}}^{\frac{1}{k+1+s}} \left\|\nabla^{k+1}B\right\|^{\frac{k+s}{k+1+s}} \leq C_0 \left\|\nabla^{k+1}B\right\|^{\frac{k+s}{k+1+s}} \tag{3.16}$$

and

$$\left\|\nabla^{k+2}u\right\| \leq \|u\|_{\dot{H}^{-s}}^{\frac{1}{k+3+s}} \left\|\nabla^{k+3}u\right\|^{\frac{k+2+s}{k+3+s}} \leq C_0 \left\|\nabla^{k+3}u\right\|^{\frac{k+2+s}{k+3+s}}.$$

Similarly, using Lemma 2.7, we have that, for $s>0$ and $k+s>0$,

$$\left\|\nabla^k B\right\| \leq \|B\|_{\dot{B}_{2,\infty}^{-s}}^{\frac{1}{k+1+s}} \left\|\nabla^{k+1}B\right\|^{\frac{k+s}{k+1+s}} \leq C_0 \left\|\nabla^{k+1}B\right\|^{\frac{k+s}{k+1+s}} \tag{3.17}$$

and

$$\left\|\nabla^{k+2}u\right\| \leq \|u\|_{\dot{B}_{2,\infty}^{-s}}^{\frac{1}{k+3+s}} \left\|\nabla^{k+3}u\right\|^{\frac{k+2+s}{k+3+s}} \leq C_0 \left\|\nabla^{k+3}u\right\|^{\frac{k+2+s}{k+3+s}}.$$

On the other hand, for $k+2<N$, we have

$$\left\|\nabla^{k+2}(E,B)\right\| \leq \left\|\nabla^{k+1}(E,B)\right\|^{\frac{N-k-2}{N-k-1}} \left\|\nabla^{N}(E,B)\right\|^{\frac{1}{N-k-1}} \leq C_0 \left\|\nabla^{k+1}(E,B)\right\|^{\frac{N-k-2}{N-k-1}}. \tag{3.18}$$

Then, we deduce from (3.15) that

$$\frac{d}{dt}\widetilde{\mathcal{E}}_k^{k+2} + \left\{\mathcal{E}_k^{k+2}\right\}^{1+\vartheta} \leq 0, \tag{3.19}$$

where $\vartheta = \max\left\{\frac{1}{k+s}, \frac{1}{N-k-2}\right\}$. Solving this inequality directly, we obtain in particular that

$$\mathcal{E}_k^{k+2}(t) \leq \left\{\left[\mathcal{E}_k^{k+2}(0)\right]^{-\vartheta} + \vartheta t\right\}^{-1/\vartheta} \leq C_0(1+t)^{-1/\vartheta} = C_0(1+t)^{-\min\{k+s,N-k-2\}}. \tag{3.20}$$

Notice that (3.20) holds also for $k+s=0$ or $k+2=N$. So, if we want to obtain the optimal decay rate of the whole solution for the spatial derivatives of order k, we only need to assume N large enough (for fixed k and s) that $k+s \leq N-k-2$. Thus, we should require that

$$N \geq \max\left\{k+2, \frac{5}{3}k+1, 2k+2+s\right\} = 2k+2+s. \tag{3.21}$$

This proves the optimal decay (1.13).

Finally, we turn back to prove (1.11) and (1.12). First, we prove (1.11) by using Lemma 2.10. However, we are not able to prove them for all $s \in [0,3/2)$ at this moment. We must distinguish the arguments by the value of s. First, for $s \in (0,1/2]$, integrating (2.41) in time, by (1.9) we obtain that, for $s \in (0,1/2]$,

$$\|(u,E,B)(t)\|_{\dot{H}^{-s}}^2 \lesssim \|(u_0,E_0,B_0)\|_{\dot{H}^{-s}}^2 + \int_0^t \mathcal{D}_3(\tau)\left(1+\|(u,E,B)(\tau)\|_{\dot{H}^{-s}}\right) d\tau$$

$$\leq C_0\left(1+\sup_{0\leqslant\tau\leqslant t}\|(u,E,B)(\tau)\|_{\dot H^{-s}}\right). \tag{3.22}$$

By Cauchy's inequality, this together with (1.9) gives (1.11) for $s\in[0,1/2]$ and thus verifies (1.12) for $s\in[0,1/2]$. Next, we let $s\in(1/2,1)$. Observing that we have $(u_0,E_0,B_0)\in\dot H^{-1/2}$ since $\dot H^{-s}\cap L^2\subset\dot H^{-s'}$ for any $s'\in[0,s]$, we then deduce from what we have proved for (1.13) with $s=1/2$ that the following decay result holds:

$$\left\|\nabla^k(n,u,E,B)(t)\right\|_{L^2}\leq C_0(1+t)^{-\frac{k+1/2}{2}}\quad\text{for } k=0,1. \tag{3.23}$$

Here, since we have required $N\geq5$ and now $s=1/2$, we could have taken $k=1$ in (1.13). Thus, by (3.23), (1.9), and Hölder's inequality, we deduce from (2.42) that, for $s\in(1/2,1)$,

$$\begin{aligned}\|(u,E,B)(t)\|_{\dot H^{-s}}^2\lesssim&\|(u_0,E_0,B_0)\|_{\dot H^{-s}}^2+\int_0^t\mathcal D_3(\tau)(1+\|(u,E,B)(\tau)\|_{\dot H^{-s}})\,d\tau\\&+\int_0^t\|B(\tau)\|^{s-1/2}\|\nabla B(\tau)\|^{3/2-s}\sqrt{\mathcal D_3(\tau)}\|(u,E,B)(\tau)\|_{\dot H^{-s}}\,d\tau\\\leq&C_0\left(1+\left(1+\int_0^t(1+\tau)^{-2(1-s/2)}\,d\tau\right)\sup_{0\leq\tau\leq t}\|(u,E,B)(\tau)\|_{\dot H^{-s}}\right)\\\leq&C_0\left(1+\sup_{0\leq\tau\leq t}\|(u,E,B)(\tau)\|_{\dot H^{-s}}\right).\end{aligned} \tag{3.24}$$

Here, we have used the fact $s\in(1/2,1)$ so that the time integral in (3.24) is finite. This gives (1.11) for $s\in(1/2,1)$ and thus verifies (1.13) for $s\in(1/2,1)$. Now, let $s\in[1,3/2)$. We choose s_0 so that $s-1/2<s_0<1$. Hence, $(u_0,E_0,B_0)\in\dot H^{-s_0}$. We then deduce from what we have proved for (1.13) with $s=s_0$ that the following decay result holds:

$$\left\|\nabla^k(n,u,E,B)(t)\right\|_{L^2}\leq C_0(1+t)^{-\frac{k+s_0}{2}}\quad\text{for } k=0,1. \tag{3.25}$$

Here, since we have required $N\geq5$ and now $s=s_0<1$, we could have taken $k=1$ in (1.13). Thus, by (3.25) and Hölder's inequality, we deduce from (2.42) that, for $s\in[1,3/2)$, similarl to in (3.24),

$$\begin{aligned}\|(u,E,B)(t)\|_{\dot H^{-s}}^2\leq&C_0\left(1+\left(1+\int_0^t(1+\tau)^{-(s_0+3/2-s)}\,d\tau\right)\sup_{0\leq\tau\leq t}\|(u,E,B)(\tau)\|_{\dot H^{-s}}\right)\\\leq&C_0\left(1+\sup_{0\leq\tau\leq t}\|(u,E,B)(\tau)\|_{\dot H^{-s}}\right).\end{aligned} \tag{3.26}$$

Here, we have used the fact $s-s_0<1/2$ so that the time integral in (3.26) is finite. This gives (1.11) for $s\in[1,3/2)$ and thus verifies (1.13) for $s\in[1,3/2)$. Note that (1.12) can be proved similarly except that we use instead Lemma 2.11.

Step 2. Further decay.

We first prove (1.14) and (1.15). First, noticing that $\operatorname{div}E=-n$, by (1.13) and Lemma 2.2, if $N\geq2k+4+s$, then

$$\left\|\nabla^k n(t)\right\|\lesssim\left\|\nabla^k\operatorname{div}E(t)\right\|\lesssim\left\|\nabla^{k+1}E(t)\right\|\lesssim C_0(1+t)^{-\frac{k+1+s}{2}}. \tag{3.27}$$

Next, applying ∇^k to $(1.3)_2$ and $(1.3)_3$, multiplying the resulting identities by $\nabla^k u$ and $\nabla^k E$ respectively, summing up, and integrating over $\mathbb R^3$, we obtain

$$\frac12\frac{d}{dt}\int\left|\nabla^k(u,E)\right|^2+\left\|\nabla^k u\right\|^2+\left\|\nabla^{k+1}u\right\|^2$$

$$
\begin{aligned}
&= -\int \nabla^k (\nabla n + u\cdot\nabla u + f(n)\nabla n + u\times B)\cdot\nabla^k u \\
&\qquad + \int \nabla^{k-1}(g(n)\Delta u)\cdot\nabla^{k+1}u + \int \nabla^k(\nabla\times B + nu)\cdot\nabla^k E \\
&\lesssim \left\|\nabla^{k+1}n\right\|\left\|\nabla^k u\right\| + \left\|\nabla^k(u\cdot\nabla u + f(n)\nabla n + u\times B)\right\|\left\|\nabla^k u\right\| \\
&\qquad + \left\|\nabla^{k-1}(g(n)\Delta u)\right\|\left\|\nabla^{k+1}u\right\| + \left\|\nabla^k(\nabla\times B + nu)\right\|\left\|\nabla^k E\right\|. \qquad (3.28)
\end{aligned}
$$

On the other hand, taking $l=k$ in (2.29), by the integration by parts and Hölder's inequality, we may have

$$
\begin{aligned}
&\int \nabla^k\partial_t u\cdot\nabla^k E + \left\|\nabla^k E\right\|^2 \\
&\lesssim \left\|\nabla^{k+1}n\right\|^2 + \left\|\nabla^k u\right\|^2 + \left\|\nabla^k(u\cdot\nabla u + f(n)\nabla n + u\times B)\right\|^2 \\
&\qquad + \left\|\nabla^{k-1}(g(n)\Delta u)\right\|\left\|\nabla^{k+1}E\right\| + \left\|\nabla^{k+1}u\right\|\left\|\nabla^{k+1}E\right\|. \qquad (3.29)
\end{aligned}
$$

Substituting (2.30) with $l=k$ into (3.29), we may then have

$$
\begin{aligned}
&\frac{d}{dt}\int \nabla^k u\cdot\nabla^k E + \left\|\nabla^k E\right\|^2 \\
&\lesssim \left\|\nabla^k u\right\|^2 + \left\|\nabla^{k+1}(n,u,E)\right\|^2 + \left\|\nabla^k(u\cdot\nabla u + f(n)\nabla n + u\times B)\right\|^2 \\
&\qquad + \left\|\nabla^k(\nabla\times B + nu)\right\|^2 + \left\|\nabla^{k-1}(g(n)\Delta u)\right\|^2. \qquad (3.30)
\end{aligned}
$$

Multiplying (3.30) by a sufficiently small but fixed factor ε and then adding it with (3.28), since ε is small, we deduce that there exists $\mathcal{F}_k(t)$ equivalent to $\left\|\nabla^k(u,E)(t)\right\|^2$ such that, by Cauchy's inequality, (2.26), (2.19), (1.13), and (3.27),

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{F}_k(t) + \mathcal{F}_k(t) \\
&\lesssim \left\|\nabla^{k+1}(n,E)\right\|^2 + \left\|\nabla^{k+1}B\right\|^2 + \left\|\nabla^k(u\cdot\nabla u + f(n)\nabla n)\right\|^2 \\
&\qquad + \left\|\nabla^k(u\times B)\right\|^2 + \left\|\nabla^k(nu)\right\|^2 + \left\|\nabla^{k-1}(g(n)\Delta u)\right\|^2 \\
&\lesssim \left\|\nabla^{k+1}(n,E,B)\right\|^2 + (\|n\|_{H^2} + \|u\|_{H^3})^2\left\|\nabla^{k+1}(n,u)\right\|^2 \\
&\qquad + \|(n,u,B)\|_{L^\infty}^2\left\|\nabla^k(n,u)\right\|^2 + \left\|\nabla^2 u\right\|_{L^\infty}^2\left\|\nabla^{k-1}n\right\|^2 \\
&\le C_0(1+t)^{-(k+1+s)}, \qquad (3.31)
\end{aligned}
$$

where we required $N\ge 2k+4+s$. Applying the standard Gronwall lemma to (3.31), we obtain

$$
\mathcal{F}_k(t) \le \mathcal{F}_k(0)e^{-t} + C_0\int_0^t e^{-(t-\tau)}(1+\tau)^{-(k+1+s)}\,d\tau \lesssim C_0(1+t)^{-(k+1+s)}. \qquad (3.32)
$$

This implies

$$
\left\|\nabla^k(u,E)(t)\right\| \lesssim \sqrt{\mathcal{F}_k(t)} \lesssim C_0(1+t)^{-\frac{k+1+s}{2}}. \qquad (3.33)
$$

We thus complete the proof of (1.14). Notice that (1.15) now follows by (3.27) with the improved decay rate of E in (1.14), just requiring $N\ge 2k+6+s$.

Now we prove (1.16). Assuming $B_\infty = 0$, we can extract the following system from $(1.3)_1$–$(1.3)_2$, denoting $\psi = \operatorname{div} u$:

$$\begin{cases} \partial_t n + \psi = -u\cdot\nabla n - n\operatorname{div} u, \\ \partial_t \psi + \psi - \Delta\psi - n = -\Delta n - \operatorname{div}(u\cdot\nabla u + f(n)\nabla n + g(n)\Delta u + u\times B). \end{cases} \tag{3.34}$$

Applying ∇^k to (3.34), multiplying the resulting identities by $\nabla^k n$ and $\nabla^k\psi$, respectively, summing up, and integrating over $\mathbb{R}^3$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int\left(\left|\nabla^k n\right|^2 + \left|\nabla^k\psi\right|^2\right) + \left\|\nabla^k\psi\right\|^2 + \left\|\nabla^{k+1}\psi\right\|^2 \\ &= -\int \nabla^k(u\cdot\nabla n + n\operatorname{div} u)\nabla^k n - \int \nabla^k \Delta n \nabla^k \psi \\ &\quad - \int \nabla^k\left[\operatorname{div}(u\cdot\nabla u + f(n)\nabla n + g(n)\Delta u + u\times B)\right]\nabla^k\psi \\ &= -\int \nabla^k(u\cdot\nabla n + n\operatorname{div} u)\nabla^k n - \int \nabla^k\Delta n\nabla^k\psi + \int \nabla^k\left(g(n)\Delta u\right)\cdot\nabla^{k+1}\psi \\ &\quad - \int \nabla^k\left[\operatorname{div}(u\cdot\nabla u + f(n)\nabla n + u\times B)\right]\nabla^k\psi. \end{aligned} \tag{3.35}$$

Applying ∇^k to $(3.34)_2$, multiplying by $-\nabla^k n$, integrating by parts over t and x variables as before, and using the equation $(3.34)_1$, we may obtain

$$\begin{aligned} &-\frac{d}{dt}\int \nabla^k\psi\nabla^k n + \left\|\nabla^k n\right\|^2 \\ &= \left\|\nabla^k\psi\right\|^2 + \left\|\nabla^{k+2}\psi\right\|^2 + \int \nabla^k(u\cdot\nabla n + n\operatorname{div} u)\nabla^k\psi \\ &\quad + \int \nabla^k\left[\Delta n + \operatorname{div}(u\cdot\nabla u + f(n)\nabla n + g(n)\Delta u + u\times B)\right]\nabla^k n \\ &= \left\|\nabla^k\psi\right\|^2 + \left\|\nabla^{k+2}\psi\right\|^2 + \int \nabla^k(u\cdot\nabla n + n\operatorname{div} u)\nabla^k\psi + \int \nabla^{k-1}\left(g(n)\Delta u\right)\nabla^{k+2} n \\ &\quad + \int \nabla^k\left[\Delta n + \operatorname{div}(u\cdot\nabla u + f(n)\nabla n + u\times B)\right]\nabla^k n. \end{aligned} \tag{3.36}$$

Multiplying (3.36) by a sufficiently small but fixed factor ε and then adding it with (3.35), since ε is small, we deduce that there exists $\mathcal{G}_k(t)$ equivalent to $\left\|\nabla^k(n,\psi)\right\|^2$ such that, by Cauchy's inequality,

$$\begin{aligned} &\frac{d}{dt}\mathcal{G}_k(t) + \mathcal{G}_k(t) \\ &\lesssim \left\|\nabla^{k+2}(n,\psi)\right\|^2 + \left\|\nabla^{k+1}(u\cdot\nabla u)\right\|^2 + \left\|\nabla^{k+1}(f(n)\nabla n)\right\|^2 \\ &\quad + \left\|\nabla^k(g(n)\Delta u)\right\|^2 + \left\|\nabla^{k-1}(g(n)\Delta u)\right\|^2 + \left\|\nabla^{k+1}(u\times B)\right\|^2 \\ &\quad + \left\|\nabla^k(u\cdot\nabla n)\right\|^2 + \left\|\nabla^k(n\operatorname{div} u)\right\|^2. \end{aligned} \tag{3.37}$$

By Lemma 2.3 and Cauchy's inequality, we obtain

$$\begin{aligned} &\left\|\nabla^{k+1}(u\times B)\right\|^2 \\ &= \left\|u\times\nabla^{k+1}B + \left[\nabla^{k+1}, u\right]\times B\right\|^2 \end{aligned}$$

$$\begin{aligned}&\lesssim \left\|u\times\nabla^{k+1}B\right\|^2+\left\|\left[\nabla^{k+1},u\right]\times B\right\|^2\\&\lesssim \|u\|_{L^\infty}^2\left\|\nabla^{k+1}B\right\|^2+\|\nabla u\|_{L^\infty}^2\left\|\nabla^kB\right\|^2+\left\|\nabla^{k+1}u\right\|^2\|B\|_{L^\infty}^2. \qquad (3.38)\end{aligned}$$

The other nonlinear terms on the right-hand side of (3.37) can be estimated similarly. Hence, we deduce from (3.37) that, by (1.13)–(1.15),

$$\begin{aligned}&\frac{d}{dt}\mathcal{G}_k(t)+\mathcal{G}_k(t)\\&\lesssim \left\|\nabla^{k+2}(n,\psi)\right\|^2+\|u\|_{L^\infty}^2\left\|\nabla^{k+1}(n,B)\right\|^2+\|\nabla u\|_{L^\infty}^2\left\|\nabla^k(n,B)\right\|^2\\&\quad+\|B\|_{L^\infty}^2\left\|\nabla^{k+1}u\right\|^2+\|(n,u)\|_{L^\infty}^2\left\|\nabla^{k+2}(n,u)\right\|^2+\|\nabla(n,u)\|_{L^\infty}^2\left\|\nabla^{k+1}(n,u)\right\|^2\\&\quad+\|\nabla n\|_{L^\infty}^2\left\|\nabla^ku\right\|^2+\|n\|_{L^\infty}^2\left\|\nabla^{k+1}u\right\|^2+\left\|\nabla^2u\right\|_{L^\infty}^2\left\|\nabla^kn\right\|^2\\&\quad+\left\|\nabla^2u\right\|_{L^\infty}^2\left\|\nabla^{k-1}n\right\|^2\\&\le C_0\left((1+t)^{-(k+3+s)}+(1+t)^{-(k+7/2+2s)}+(1+t)^{-(k+11/2+2s)}\right)\\&\le C_0(1+t)^{-(k+3+s)},\end{aligned}$$

where we required $N\ge 2k+8+s$. Applying the Gronwall lemma to (3.39) again, we obtain

$$\mathcal{G}_k(t)\le\mathcal{G}_k(0)e^{-t}+C_0\int_0^t e^{-(t-\tau)}(1+\tau)^{-(k+3+s)}\,d\tau\le C_0(1+t)^{-(k+3+s)}. \qquad (3.39)$$

This implies

$$\left\|\nabla^k(n,\psi)(t)\right\|\lesssim\sqrt{\mathcal{G}_k(t)}\le C_0(1+t)^{-\frac{k+3+s}{2}}. \qquad (3.40)$$

If required that $N\ge 2k+12+s$, then, by (3.40), we have

$$\left\|\nabla^{k+2}(n,\psi)(t)\right\|\lesssim C_0(1+t)^{-\frac{k+5+s}{2}}. \qquad (3.41)$$

Having obtained such faster decay, we can then improve (3.39) to be

$$\frac{d}{dt}\mathcal{G}_k(t)+\mathcal{G}_k(t)\le C_0\left((1+t)^{-(k+5+s)}+(1+t)^{-(k+7/2+2s)}\right)\le C_0(1+t)^{-(k+7/2+2s)}. \qquad (3.42)$$

Applying the Gronwall lemma again, we obtain

$$\left\|\nabla^k(n,\psi)(t)\right\|\lesssim\sqrt{\mathcal{G}_k(t)}\le C_0(1+t)^{-(k/2+7/4+s)}. \qquad (3.43)$$

We thus complete the proof of (1.16). The proof of Theorem 1.2 is completed.

Acknowledgement. This research is supported by the National Natural Science Foundation of China (No. 11271305, 11531010) and the Fundamental Research Funds for Xiamen University (No. 201412G004).

REFERENCES

[1] R.J. Duan, *Green's function and large time behavior of the Navier–Stokes–Maxwell system*, Anal. Appl. (Singap.), 10, 133–197, 2012.

[2] J.S. Fan and F.C. Li, *Uniform local well-posedness to the density-dependent Navier–Stokes–Maxwell dystem*, Acta Appl. Math., 133, 19–32, 2014.

[3] Y.H. Feng, Y.J. Peng, and S. Wang, *Asymptotic behavior of global smooth solutions for full compressible Navier–Stokes–Maxwell equations*, Nonlinear Anal. Real World Appl., 19, 105–116, 2014.
[4] L. Grafakos, *Classical and Modern Fourier Analysis*, Pearson Education, Inc., Prentice Hall, 2004.
[5] Y. Guo, *The Vlasov–Poisson–Landau system in a periodic box*, J. Amer. Math. Soc., 25, 759–812, 2012.
[6] P. Germain, S. Ibrahim, and N. Masmoudi, *Well-posedness of the Navier–Stokes–Maxwell equations*, Proc. Roy. Soc. Edinburgh Sect., A144, 71–86, 2014.
[7] Y. Guo and Y.J. Wang, *Decay of dissipative equations and negative Sobolev spaces*, Commun. Part. Diff. Eqs., 37, 2165–2208, 2012.
[8] G.Y. Hong, X.F. Hou, H.J. Peng, and C.J. Zhu, *Global spherically symmetric classical solution to the Navier–Stokes–Maxwell system with large initial data and vacuum*, Sci. China Math., 57(12), 2463–2484, 2014.
[9] S. Ibrahim and S. Keraani, *Global small solutions of the Navier–Stokes–Maxwell equations*, SIAM J. Math. Anal., 43, 2275–2295, 2011.
[10] S. Ibrahim and T. Yoneda, *Local solvability and loss of smoothness of the Navier–Stokes–Maxwell equations with large initial data*, J. Math. Anal. Appl., 396, 555–561, 2012.
[11] J.W. Jerome, *The Cauchy problem for compressible hydrodynamic-Maxwell systems: A local theory for smooth solutions*, Differential Integral Equations, 16, 1345–1368, 2003.
[12] S. Jiang, Q.C. Ju, and F.C. Li, *Low Mach number limit for the multi-dimensional full magnetohydrodynamic equations*, Nonlinearity, 25, 1351–1365, 2012.
[13] T. Kato, *The Cauchy problem for quasi-linear symmetric hyperbolic systems*, Arch. Rat. Mech. Anal., 58, 181–205, 1975.
[14] F.C. Li and Y.M. Mu, *Low Mach number limit of the full compressible Navier–Stokes–Maxwell system*, J. Math. Anal. Appl., 412, 334–344, 2014.
[15] N. Masmoudi, *Global well posedness for the Maxwell-Navier-Stokes system in* $2D$, J. Math. Pures Appl., 93(6), 559–571, 2010.
[16] A.J. Majda and A.L. Bertozzi, *Vorticity and Incompressible Flow, Cambridge University Press*, Cambridge, 2002.
[17] P.A. Markowich, C. Ringhofer, and C. Schmeiser, *Semiconductor Equations*, Springer-Verlag: Vienna, 1990.
[18] T. Nishida, *Nonlinear hyperbolic equations and related topics in fluids dynamics*, Publications Mathématiques d'Orsay, Université Paris-Sud, Orsay, 78–02, 1978.
[19] R.M. Strain and Y. Guo, *Almost exponential decay near Maxwellian*, Commun. Part. Diff. Eqs., 31, 417–429, 2006.
[20] V. Sohinger and R.M. Strain, *The Boltzmann equation, Besov spaces, and optimal time decay rates in* $\mathbb{R}^n_x$, Advances in Mathematics, 261, 274–332, 2014.
[21] Z. Tan and Y.J. Wang, *Global existence and large-time behavior of weak solutions to the compressible magnetohydrodynamic equations with Coulomb force*, Nonlinear Anal., 71, 5866–5884, 2009.
[22] Z. Tan and Y. Wang, *Global solution and large-time behavior of the* $3D$ *compressible Euler equations with damping*, J. Diff. Eqs., 254, 1686–1704, 2013.
[23] Z. Tan, Y.J. Wang, and Y. Wang, *Global solution and time decay of the compressible Euler–Maxwell system in* $\mathbb{R}^3$, J. Diff. Eqs., 257, 2846–2873, 2014.
[24] Y.J. Wang, *Global solution and time decay of the Vlasov–Poisson–Landau system in* $\mathbb{R}^3$, SIAM J. Math. Anal., 44 (5), 3281–3323, 2012.
[25] J.W. Yang and S. Wang, *Convergence of compressible Navier–Stokes–Maxwell equations to incompressible Navier-Stokes equations*, Sci. China Math., 57 (10), 2153–2162, 2014.