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EMERGENCE OF PHASE-LOCKED STATES FOR THE KURAMOTO MODEL IN A LARGE COUPLING REGIME*

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Abstract. We study the emergence of phase-locked states to the finite-dimensional Kuramoto model from generic initial configurations which are not phase-locked states in a large coupling regime. In the literature of physics and engineering, it has often been argued that complete synchronization may occur for a generic initial configuration in a large coupling regime. Such arguments are generally based on the results of numerical simulations. Unfortunately, this plausible scenario has not been completely verified by rigorous mathematical arguments, although there are several partial results available for a restricted class of initial configurations. In this paper, we provide a sufficient framework for complete synchronization from a *generic initial configuration* in a large coupling regime. Our analysis depends on the gradient flow structure of the Kuramoto model and the uniform boundedness of the phase configuration.

Key words. Complete synchronization, Kuramoto model, order parameter, phase-locked states.

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1. Introduction

The synchronization of weakly coupled limit-cycle oscillators was first reported by Huygens in the mid-17th century via two pendulum clocks hanging over a common bar. Since then, the phenomenon was observed by several researchers [24] before Kuramoto's systematic studies [19, 20] based on a coupled first-order ODE system (see [1, 11, 24, 25] for a brief history and mathematical results). In the last forty years, the Kuramoto model has been extensively studied by physicists [1, 24, 25] and engineers in control theory [7, 12, 18] in relation to a kind of phase-transition from a disordered to ordered state at some critical coupling strength. For simplicity of presentation, we can visualize Kuramoto oscillators as rotors moving on the unit circle $\mathbb{S}^1 \subset \mathbb{C}^1$ in the complete plane. Let $z_j = e^{\mathrm{i}\theta_j}$ be the position of the jth rotor on the unit circle. In the following, θ_j and $\dot{\theta}_j$ denote the phase and frequency, respectively, of the jth oscillator. Then, the phase dynamics of Kuramoto oscillators are governed by the following system of ODEs:

$$\dot{\theta}_j = \Omega_j + \frac{K}{N}\sum_{k=1}^{N} \sin(\theta_k - \theta_j), \quad t > 0, \qquad \theta_j(0) = \theta_{j0}, \tag{1.1}$$

where K is the uniform positive coupling strength, and Ω_j represents the intrinsic natural frequency of the jth oscillator drawn from some distribution function $g = g(\Omega)$. Without loss of generality, we assume that the natural frequencies and initial phases both have an average of zero:

$$\frac{1}{N}\sum_{j=1}^{N} \Omega_j = 0, \qquad \frac{1}{N}\sum_{j=1}^{N} \theta_{j0} = 0, \qquad \theta_{j0} \in [-\pi, \pi), \quad 1 \le j \le N. \tag{1.2}$$

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Note that the right-hand side of (1.1) encodes the competing mechanisms of *randomness* in the distributed natural frequencies and *nonlinear coupling* via the mean-field-type sinusoidal interactions between oscillators. It is well known [1,11] that, as the coupling strength K increases from zero to infinity, spontaneous synchronous dynamics emerges among the ensemble of Kuramoto oscillators. However, this plausible scenario has not been completely understood from a rigorous mathematical viewpoint, although several mathematical results [7,11,12,14,15,17,18] are available for restricted classes of initial configurations.

In this paper, we are mainly concerned with "*(asymptotic) complete synchronization*," whereby all oscillators asymptotically move on the unit circle with the same phase velocity (frequency). The resulting asymptotic configuration (phase-locked state) looks like a train moving on the circle. Thus, complete synchronization is sometimes called an entrainment [1]. More precisely, for system (1.1) with finite N and a "*generic initial configuration*" Θ_0, the complete synchronization problem (CSP) is to show that the solution $\Theta(t)$ with initial configuration Θ_0 converges to a phase-locked state for a large coupling strength K. A detailed discussion on the CSP is given in Subsection 2.3. Throughout this paper, we will consider the Kuramoto model whose phase space is $\mathbb{R}^N$, rather than $\mathbb{T}^N$, although the right-hand side of (1.1) is 2π-periodic in the θ-variable, and thus the configuration space is normally regarded as $\mathbb{T}^N$. This extended view is more convenient for the analysis in later sections.

It is well known [17,26] that the Kuramoto model (1.1) can be rewritten as a gradient system with an analytical potential. Thus, as long as the configuration is bounded in $\mathbb{R}^N$, it converges to a unique phase-locked state (see Theorem 2.1). For system (1.1) with the same natural frequencies $\Omega_j=\Omega$, $j=1,\dots,N$, Jadbabaie et al. [18] and Dong–Xue [10] verified that all generic initial configurations lead to complete synchronization. In particular, Dong–Xue showed that the phase configuration is uniformly bounded so that the initial configuration tends asymptotically to the phase-locked state.

The main results of this paper are twofold. First, we provide a framework that guarantees the uniform boundedness of solutions to (1.1)–(1.2). In Proposition 4.1, we show the existence of a time-varying trapping set for dynamic solutions under a sufficiently large coupling strength. This leads to the uniform boundedness of solutions to (1.1)–(1.2). Second, we show that an initial configuration Θ_0 with a positive Kuramoto order parameter $r(\Theta_0)$, where

$$r_0=r(\Theta_0):=\Big|\frac{1}{N}\sum_{j=1}^{N}e^{\mathrm{i}\theta_{j0}}\Big|,$$

will evolve toward an admissible set in our framework described in Proposition 4.1 in finite time. For this, we use the monotonicity of the order parameter for the ensemble of identical oscillators and the closeness of dynamic trajectories of identical and non-identical oscillators with small natural frequencies in a finite time interval (see Proposition 4.1 and Lemma 4.1). Thus, our main result on the CSP can be stated as follows (a detailed proof will be given in Section 4.2):

THEOREM 1.1. *Suppose that the initial configuration Θ_0 and natural frequencies Ω_i satisfy the condition* (1.2) *and*

$$r_0>0,\quad \theta_{j0}\neq\theta_{k0},\quad 1\le j\neq k\le N,\qquad \max_{1\le j\le N}|\Omega_j|<\infty. \tag{1.3}$$

Then there exists a large coupling strength $K_\infty>0$ such that if $K\ge K_\infty$ then there exists *a phase-locked state Θ^∞ such that the solution with initial data Θ_0 satisfies*

$$\lim_{t\to\infty} ||\Theta(t)-\Theta^\infty||_\infty = 0,$$

where the norm $||\cdot||_\infty$ is the standard ℓ^∞-norm in $\mathbb{R}^N$.

Remark 1.1.

1. In [5,7,12,15], complete synchronization estimates have been obtained for some restricted classes of initial configurations that can be confined to the half-circle in finite time. However, the result presented in Theorem 1.1 covers all initial configurations, except that with $r_0=0$. Note that the set of all configurations with $r=0$ has measure zero in the configurations space $\mathbb{R}^N$.
2. (Non-uniqueness of phase-locked states): If we consider the phase-locked states confined in a half circle region, it is unique up to phase-shift by orbital ℓ^1-stability in a large coupling regime as shown in [5]. However in general, phase-locked states are not unique even up to phase-shift as discussed in [3,22]. Thus, our result in Theorem 1.1 is simply an existence theorem on the phase-locked states.
3. Since we are mainly interested in the emergent phase-locked states from initial configurations that are not phase-locked states, we imposed the first two conditions in (1.3) to avoid trivial phase-locked states such as bipolar configurations (see Definition 3.1), splay states which are uniformly distributed on the unit circle as initial configurations in the case of identical oscillators.

The remainder of this paper consists of four sections. In Section 2, we briefly review the gradient flow formulation of the Kuramoto model, Kuramoto order parameters, and the state-of-the-art regarding the CSP. In Section 3, we study the structure of bipolar configurations that emerge from generic initial configurations in an ensemble of identical oscillators. In Section 4, we provide a proof of Theorem 1.1. Finally, Section 5 is devoted to a summary of our main results and a discussion of possible future work.

2. Preliminaries

In this section, we review the gradient flow formulation of the Kuramoto model, the dynamics of Kuramoto order parameters, and some state-of-the-art results on the CSP. The first two subjects are crucial in later sections.

We first recall the definitions of a phase-locked state and complete synchronization for the Kuramoto system (1.1).

Definition 2.1.

1. *We say that $\Theta^\infty = (\theta_1^\infty,\dots,\theta_N^\infty)$ is a phase-locked state of (1.1)–(1.2) if and only if it is an equilibrium solution:*

$$\Omega_j + \frac{K}{N}\sum_{k=1}^N \sin(\theta_k^\infty - \theta_j^\infty) = 0, \quad j=1,\dots,N.$$

2. *Let $\Theta=\Theta(t)$ be a dynamic solution to system (1.1). Then, Θ exhibits complete synchronization (in an asymptotic sense) if and only if the relative frequencies tend to zero asymptotically, i.e.,*

$$\lim_{t\to\infty} |\dot\theta_j(t) - \dot\theta_k(t)| = 0, \quad 1\le j,k\le N.$$

2.1. Kuramoto model as a gradient flow. For a given natural frequency set $\{\Omega_j\}_{j=1}^N$ and phase configuration Θ, we introduce the analytical potential $V = V(\Theta)$

$$V[\Theta] := -\sum_{k=1}^N \Omega_k \theta_k + \frac{K}{2N} \sum_{k,l=1}^N \big(1 - \cos(\theta_k - \theta_l)\big). \tag{2.1}$$

Then, it is easy to see [26] that system (1.1) can be rewritten as a gradient system with potential V

$$\dot{\Theta} = -\nabla_\Theta V(\Theta), \quad t > 0. \tag{2.2}$$

As a gradient system, the Kuramoto model (2.1)–(2.2) has the following property regarding its asymptotic dynamics.

THEOREM 2.2. [17] *Let $\Theta = \Theta(t)$ be a uniformly bounded global solution to* (1.1)–(1.2) *in $\mathbb{R}^N$:*

$$\sup_{0 \le t < \infty} ||\Theta(t)||_\infty < \infty.$$

Then the phase configuration $\Theta(t)$ and the frequency vector $\dot{\Theta}(t)$ converge to a phase-locked state and the zero vector, respectively as $t \to \infty$, i.e., there exists a phase-locked state Θ^∞ such that

$$\lim_{t\to\infty} ||\Theta(t) - \Theta^\infty||_\infty = 0 \quad \textit{and} \quad \lim_{t\to\infty} ||\dot{\Theta}(t)||_\infty = 0.$$

REMARK 2.1. In dynamical systems theory, uniform boundedness does not generally imply convergence. This is essentially due to the gradient flow structure of the Kuramoto flow with an analytical potential.

2.2. Order parameters. In this subsection, we review the dynamics of the order parameters following the presentation in [15]. For a configuration $\Theta = \Theta(t)$ governed by (1.1), the Kuramoto order parameters r and ϕ are defined by the following relation:

$$re^{\mathrm{i}\phi} := \frac{1}{N} \sum_{k=1}^N e^{\mathrm{i}\theta_k}. \tag{2.3}$$

Note that the modulus r is always bounded by 1, and is invariant under uniform rotation. The state $r(\Theta) = 1$ corresponds to the state in which all phases are the same, i.e., phase synchronization:

$$r(\Theta) = 1 \quad \Longleftrightarrow \quad \Theta = (\alpha, \ldots, \alpha), \quad \text{for some } \alpha \in \mathbb{R}.$$

We next derive the dynamics of the order parameters r and ϕ. For this, we divide (2.3) by $e^{\mathrm{i}\theta_j}$ to obtain:

$$re^{\mathrm{i}(\phi - \theta_j)} = \frac{1}{N} \sum_{k=1}^N e^{\mathrm{i}(\theta_k - \theta_j)},$$

and compare the real and imaginary parts of the above relation to find

$$\begin{aligned} r\cos(\phi-\theta_j) &= \frac{1}{N}\sum_{k=1}^{N}\cos(\theta_k-\theta_j), \\ r\sin(\phi-\theta_j) &= \frac{1}{N}\sum_{k=1}^{N}\sin(\theta_k-\theta_j). \end{aligned} \tag{2.4}$$

By comparing the second relation in (2.4) and the coupling terms in (1.1), it is easy to see that the Kuramoto system (1.1) can be rewritten in mean-field form:

$$\dot{\theta}_j = \Omega_j - Kr\sin(\theta_j-\phi), \quad t>0.$$

LEMMA 2.3. [15] *For identical oscillators with $\Omega_i = 0$, the Kuramoto order parameter r defined by relation* (2.3) *is monotonically increasing, i.e.,*

$$\dot{r} \geq 0, \quad t>0.$$

REMARK 2.2. For identical oscillators, the order parameter r is non-decreasing, but may not be strictly increasing. For example, let Θ_0 be a bipolar configuration such that $m(\neq \frac{N}{2})$ identical oscillators are located at 0 and $N-m$ are located at π. Then, it is easy to see that this configuration is an equilibrium for (1.1), and

$$z_c = \frac{me^{\mathrm{i}0}+(N-m)e^{\mathrm{i}\pi}}{N} = \frac{2m-N}{N} \neq 0, \quad r = \left|\frac{2m-N}{N}\right| > 0.$$

Thus, we have

$$r(t) = r_0, \qquad \forall\ t \geq 0.$$

2.3. A complete synchronization problem. For the Kuramoto model (1.1)–(1.2) with a randomly chosen initial configuration, as we increase the coupling strength K from zero to a sufficiently large number compared to the size of Ω_i, ensembles of Kuramoto oscillators tend to phase-locked states as $t \to \infty$ in numerical simulations. Thus, natural questions regarding this phenomenon are as follows.

- Problem A (emergence of phase-locked states): Can we verify the emergence of phase-locked states for generic initial configurations, when the coupling strength K is sufficiently large ?
- Problem B (existence of a critical coupling strength): Does a critical coupling strength exist in Problem A ? i.e., Is there a coupling strength K_c such that if $K > K_c$, then the Kuramoto model always exhibit the emergence of phase-locked states for generic initial phase configurations, whereas if $K < K_c$, then there exists a set $\mathcal{S}$ of initial configurations with a positive Lebesgue measure in $\mathbb{R}^N$ such that phase-locked states do not emerge from initial configurations in the set $\mathcal{S}$?

Of course, these two problems are closely related to each other. For the first problem, there has been many previous works (see [11] for most updated results). The synchronization problem [1, 8] has been studied using different approaches. Ermentrout [13] found a critical coupling at which all oscillators become phase-locked, independent of their number, and the linear and nonlinear stabilities of this phase-locked state have been studied in several papers (see [2,4,5,9,18,21,23,26]) using tools such as Lyapunov functionals, spectral graph theory, and control theory. The studies most closely related to this paper are those of Chopra and Spong [7], Choi et al. [5], Dórfler and Bullo [12], Ha et al. [15]. These papers use the phase diameter $D(\Theta) := \max_{1\le j,k\le N} |\theta_j - \theta_k|$ as a Lyapunov functional, and study its temporal evolution via Gronwall's inequality. In fact, these papers only deal with initial configurations whose phase diameter is at most $\pi+\varepsilon$ with $\varepsilon \ll 1$. More precisely, Ha–Kim–Park [15] extended the previous work of Choi et al. [5] to allow initial configurations whose diameter is slightly larger than π for sufficiently large coupling strength. In fact, they showed that sufficiently large coupling can push initial configurations into configurations confined in the half circle so that they can use the result in [5]. For this, they used the dynamics of order parameters r and ϕ (see [15] for details). On the other hand, for Problem B, there are few works available in [6,12,18]. In particular, Dórfler and Bullo [12] showed that $K = D(\Omega)$ is a critical coupling strength for the set of initial configurations which can be confined in the half circle. So far, the available necessary condition for a critical coupling strength is $K_c > \frac{D(\Omega)}{2}$. In this paper, we focus on the first problem on the emergence of phase-locked states. We now recall a most recent result on complete synchronization from [5,15]. By a slight modification of the arguments in [5,15], we obtain the following estimate for the Problem A.

Theorem 2.4. [5,15] *Suppose that the coupling strength K satisfies*

$$K > D(\Omega) := \max_{1\le j,k\le N} |\Omega_k - \Omega_j|.$$

and let $\Theta = \Theta(t)$ be a solution to (1.1)–(1.2) such that there exists a positive time $T \in (0,\infty)$ such that

$$0 < D(\Theta(T)) < \pi - \arcsin\left(\frac{D(\Omega)}{K}\right).$$

Then, there exist positive constants $C_0(T)$ and Λ such that

$$D(\dot{\Theta}(t)) \le C_0 \exp(-\Lambda(t-T)), \quad \text{as } t \to \infty.$$

Remark 2.3.

1. For identical oscillators $D(\Omega) = 0$, complete synchronization has been shown in [10] for an arbitrary initial configuration with $D(\Theta_0) < 2\pi$. Of course, the synchronization estimate given in [10] does not yield the detailed relaxation process toward a phase-locked state as it is.
2. The result in [5] corresponds to the case $T = 0$, i.e., $D(\Theta_0) < \pi$, and the result in [15] deals with the case $D(\Theta_0) < \pi + \varepsilon$, $\varepsilon \ll 1$ and $K \gg D(\Omega)$.

Before we close this section, we provide an elementary estimate of the dynamics of Adler's equation in $\mathbb{R}$:

$$\dot{\theta} = \Omega - K\sin\theta, \quad t > 0, \qquad \theta(0) = \theta_0, \tag{2.5}$$

where Ω and K are positive constants.

An explicit representation of the solution to (2.5) can be found in Appendix D of [5]. Note that for $K > \Omega$, equation (2.5) has the roots θ_u^1 and θ_s^1:

$$0 < \theta_s^1 < \frac{\pi}{2} < \theta_u^1 < \pi, \quad \theta_s^n = 2(n-1)\pi + \theta_s^1, \quad \theta_u^n = 2(n-1)\pi + \theta_u^1, \quad n \neq 1,$$

and a simple stability analysis shows that θ_s^n and θ_u^n are stable and unstable equilibrium points, respectively.

LEMMA 2.5. *Suppose the coefficients Ω and K in (2.5) and initial data θ_0 satisfy*

$$0 < \Omega < K, \qquad \theta_0 \in [-\pi, \pi), \quad \theta_0 \neq \theta_u^1.$$

Then, the solution θ is uniformly bounded. More precisely, there are two cases:

1. *If $-\pi < \theta_0 < \theta_u^1$, then*

$$\lim_{t \to \infty} \theta(t) = \theta_s^1.$$

2. *If $\theta_u^1 < \theta_0 \leq \pi$, then*

$$\lim_{t \to \infty} \theta(t) = \theta_s^1 + 2\pi.$$

3. Structure of the emergent bipolar configuration

In this section, we study the structure of the emergent bipolar configuration from the ensemble of identical Kuramoto oscillators.

We first define a bipolar configuration as follows.

DEFINITION 3.1. *We say that $\Theta = (\theta_1, \dots, \theta_N)$ is a bipolar configuration if and only if the following two conditions hold:*

$$(i)\ |\theta_k - \theta_l| \equiv 0,\ \pi \quad mod\ 2\pi, \quad 1 \leq k, l \leq N \qquad and$$

$$(ii)\ \exists\ k, j \in \{1, \dots, N\}\ such\ that \quad |\theta_k - \theta_j| \equiv \pi\ mod\ 2\pi.$$

It is clear that all bipolar configurations are equilibrium solutions to the following system:

$$\dot{\theta}_j = \frac{K}{N} \sum_{k=1}^{N} \sin(\theta_k - \theta_j), \quad \text{or equivalently} \quad \dot{\theta}_j = K r \sin(\phi - \theta_j). \tag{3.1}$$

However, not all such bipolar configurations are emergent, i.e., they cannot be formed from non-bipolar configurations via the Kuramoto flow. We first recall a result on the possible asymptotic behavior of identical Kuramoto oscillators.

PROPOSITION 3.2. [3, 15] *Let $\Theta = (\theta_1, \dots, \theta_N)$ be a solution to* (3.1) *with initial data Θ_0 satisfying* (1.2) *and $r_0 := r(\Theta_0) > 0$. Then, we have*

$$\lim_{t \to \infty} |\theta_j(t) - \phi(t)| = 0 \quad or \quad \pi, \quad for\ all\ j = 1, \dots, N.$$

In the following lemma, we will refine the estimate of Proposition 3.1. For a dynamic solution $\Theta(t)$ to (3.1), we divide the oscillator set $\mathcal{N} := \{1, \dots, N\}$ into synchronous and anti-synchronous oscillators with respect to the overall phase:

$$\mathcal{I}_s := \{j : \lim_{t \to \infty} |\theta_j(t) - \phi(t)| = 0\}, \qquad \mathcal{I}_b := \{j : \lim_{t \to \infty} |\theta_j(t) - \phi(t)| = \pi\}.$$

Then, $\{\mathcal{I}_s, \mathcal{I}_b\}$ is a partition of $\mathcal{N}$. First, we consider a two-oscillator system:

$$\dot{\theta}_1 = \frac{K}{2}\sin(\theta_2 - \theta_1), \qquad \dot{\theta}_2 = \frac{K}{2}\sin(\theta_1 - \theta_2).$$

The phase difference $\theta := \theta_1 - \theta_2$ satisfies

$$\dot{\theta} = -K\sin\theta.$$

Unless $\theta_0 = 0, \pi$, the phase difference θ will decrease or increase according to whether $\theta_0 \in (0,\pi)$ or $\theta_0 \in (\pi, 2\pi)$ to asymptotically reach the point of phase synchronization. Below, we study the phase synchronization of oscillators in the set $\mathcal{I}_s$.

LEMMA 3.3. *Let $\Theta = (\theta_1, \dots, \theta_N)$ be a solution to* (3.1) *with initial configuration Θ_0 satisfying* (1.2) *and following conditions:*

$$r_0 > 0, \qquad \theta_{k0} \neq \theta_{j0}, \qquad 1 \leq k, j \leq N. \tag{3.2}$$

Then, we have

$$|\mathcal{I}_b| \leq 1,$$

where $|A|$ is the cardinality of the set A.

Proof. We first note that the condition (3.2) and the uniqueness of the Kuramoto flow imply

$$\theta_j(t) \neq \theta_k(t) \quad \forall\, t > 0.$$

For the desired estimate, we will show that the assumption $|\mathcal{I}_b| \geq 2$ leads to a contradiction.

Suppose that the bipolar set $\mathcal{I}_b$ contains at least two oscillators, say the first and second particles. Without loss of generality, we further assume that

$$\phi - \frac{\pi}{2} < \theta_1 < \theta_2 < \phi + \frac{3\pi}{2}.$$

Since $\lim_{t\to\infty} |\theta_j - \phi| = \pi$, $j = 1, 2$, for any given $0 < \varepsilon \ll 1$, there exists $T_1 = T_1(\varepsilon) > 0$ such that

$$\pi - \varepsilon < \theta_1 - \phi < \theta_2 - \phi < \pi + \varepsilon \qquad \forall\, t > T_1. \tag{3.3}$$

On the other hand, it follows from (3.1) and the mean-value theorem that

$$\frac{d}{dt}(\theta_2 - \theta_1) = Kr[\sin(\phi - \theta_2) - \sin(\phi - \theta_1)] = Kr(\cos\xi_{12})(\theta_1 - \theta_2), \tag{3.4}$$

where ξ_{12} is between $\phi - \theta_1$ and $\phi - \theta_2$ and satisfies

$$|\xi_{12} - \pi| \leq \varepsilon \quad \text{and} \quad \theta_2 - \theta_1 > 0. \tag{3.5}$$

It follows from (3.4) and (3.5) that we have

$$\frac{d}{dt}(\theta_2 - \theta_1) \geq Kr(-\cos(\pi - \varepsilon))(\theta_2 - \theta_1) = Kr\cos(\varepsilon)(\theta_2 - \theta_1), \quad t > T_1. \tag{3.6}$$

On the other hand, since

$$r(t) \geq r_0 > 0,$$

it follows from (3.6) that

$$|\theta_2(t) - \theta_1(t)| \geq |\theta_2(T_1) - \theta_1(T_1)| e^{r_0(\cos\varepsilon)(t-T_1)}, \quad t \geq T_1.$$

This contradicts (3.3). Thus, we have

$$|\mathcal{I}_b| \leq 1.$$

□

Below, we show that the diameter of the configurations in the set $\mathcal{I}_s$ will collapse to zero exponentially fast. For this, we set

$$\Theta_s = (\theta_{i_1}, \ldots, \theta_{i_{|\mathcal{I}_s|}}), \quad i_j \in \mathcal{I}_s, \qquad D(\Theta_s) := \max_{j,k \in \mathcal{I}_s} |\theta_j - \theta_k|.$$

From the definition of the synchronizing set $\mathcal{I}_s$, it is clear that, for $\theta_j, \theta_k \in \mathcal{I}_s$,

$$|\theta_j - \theta_k| \leq |\theta_j - \phi| + |\theta_k - \phi| \to 0, \quad \text{as } t \to \infty.$$

The following lemma states that this convergence to zero is at least exponential.

LEMMA 3.4. *Let $\Theta = (\theta_1, \ldots, \theta_N)$ be a solution to* (3.1) *with initial data Θ_0 satisfying* (1.2) *and following conditions:*

$$r_0 > 0, \qquad \theta_{j0} \neq \theta_{k0}, \qquad 1 \leq j,k \leq N.$$

Then, there exists $T_2 \geq 0$ such that

$$D(\Theta_s(t)) \leq e^{-K\lambda(t-T_2)} D(\Theta_s(T_2)), \quad t \geq T_2,$$

where λ is a positive constant to be defined later.

Proof. It suffices to show that the desired estimate holds for $N \geq 3$. It follows from the definition of $\mathcal{I}_s$ that, for a given $\varepsilon > 0$, there is $T_2 = T_2(\varepsilon) \geq 0$ such that

$$\max_{j \in \mathcal{I}_s} |\theta_j(t) - \phi(t)| \leq \varepsilon \quad t \geq T_2.$$

We set extremal phases:

$$\theta_M := \max_{j \in \mathcal{I}_s} \theta_j \qquad \theta_m := \min_{j \in \mathcal{I}_s} \theta_j.$$

Then, we have

$$D(\Theta_s) = \theta_M - \theta_m.$$

We use (3.1) to obtain a Gronwall-type inequality:

$$\begin{aligned} &\frac{d}{dt}(\theta_M - \theta_m) \\ &= \frac{K}{N} \sum_{k=1}^{N} (\sin(\theta_k - \theta_M) - \sin(\theta_k - \theta_m)) \end{aligned}$$

$$
\begin{aligned}
&= \frac{K}{N}\sum_{k\in\mathcal{I}_s}\Big(\sin(\theta_k-\theta_M)-\sin(\theta_k-\theta_m)\Big)+\frac{K}{N}\sum_{k\in\mathcal{I}_b}\Big(\sin(\theta_k-\theta_M)-\sin(\theta_k-\theta_m)\Big)\\
&\le \frac{K(N-1)}{N}\frac{\sin 2\varepsilon}{2\varepsilon}(\theta_m-\theta_M)+\frac{K}{N}(\sin(\theta_j-\theta_M)-\sin(\theta_j-\theta_m))\Big|_{k\in\mathcal{I}_b}\\
&\le -\frac{K(N-1)}{N}\frac{\sin 2\varepsilon}{2\varepsilon}(\theta_M-\theta_m)+\frac{K}{N}(\theta_M-\theta_m)\\
&= -\frac{K}{N}\left[(N-1)\frac{\sin 2\varepsilon}{2\varepsilon}-1\right](\theta_M-\theta_m), \quad t\ge t_0.
\end{aligned}
\tag{3.7}
$$

Note that since $\lim\limits_{\varepsilon\to 0}\dfrac{\sin\varepsilon}{\varepsilon}=1$, for a positive constant $\delta\ll 1$, there exists $\varepsilon_0>0$ such that

$$
\frac{\sin 2\varepsilon}{2\varepsilon}>1-\delta, \quad \forall\ \varepsilon<\varepsilon_0.
$$

Thus, we have

$$
(N-1)\frac{\sin 2\varepsilon}{2\varepsilon}-1\ge (N-1)(1-\delta)-1. \tag{3.8}
$$

Note that

$$
(N-1)(1-\delta)-1>0 \quad\Longleftrightarrow\quad N>1+\frac{1}{1-\delta}, \quad \text{i.e.,}\quad N\ge 3. \tag{3.9}
$$

Finally, we can combine (3.7), (3.8), and (3.9) and then apply Gronwall's inequality to conclude

$$
\begin{aligned}
D(\Theta_s(t)) &\le D(\Theta_s(T_2))\exp\Big[-K\Big(\frac{(N-1)(1-\delta)}{N}-\frac{1}{N}\Big)(t-T_2)\Big]\\
&=: D(\Theta_s(T_2))\exp\Big[-K\lambda(t-T_2)\Big] \qquad t\ge T_2.
\end{aligned}
$$

□

4. Complete synchronization estimate

In this section, we provide a complete synchronization estimate for an initial configuration Θ_0 with the corresponding order parameter $r_0>0$.

Consider the synchronization of the two-oscillator system:

$$
\begin{aligned}
\dot\theta_1 &= \Omega_1+\frac{K}{2}\sin(\theta_2-\theta_1), \quad t>0,\\
\dot\theta_2 &= \Omega_2+\frac{K}{2}\sin(\theta_1-\theta_2).
\end{aligned}
$$

We now consider the differences θ and Ω:

$$
\theta:=\theta_2-\theta_1, \quad \Omega:=\Omega_2-\Omega_1.
$$

The difference θ satisfies the Adler equation:

$$
\dot\theta=\Omega-K\sin\theta, \quad t>0.
$$

Then, it follows from the explicit formula in [5] that, for $K>|\Omega|$, complete synchronization occurs for any initial configuration:

$$
\lim_{t\to\infty}||\dot\theta_1(t)-\dot\theta_2(t)||_\infty=0. \tag{4.1}
$$

Again, we use $\sum_{j=1}^{2}\dot\theta_j=0$ to find

$$\lim_{t\to\infty}||\dot\Theta(t)||_\infty=0.$$

4.1. Preparatory estimates. In this part, we present several preparatory estimates for the proof of Theorem 1.1, and for the simplicity of presentation, we restrict the range of initial phase configuration in the box $[-\pi,\pi)^N$ in $\mathbb{R}^N$.

Proposition 4.1. *Suppose that the initial configuration Θ_0 satisfies*

$$\theta_{j0}\in[-\pi,\pi),\quad 1\le j\le N,$$

and let n_0, ℓ, and K satisfy

$$\begin{aligned} &n_0\in\mathbb{Z}_+\cap\Big(\frac{N}{2},N\Big],\quad \ell\in\Big(0,2\cos^{-1}\frac{N-n_0}{n_0}\Big),\\ &\max_{1\le j,k\le n_0}|\theta_{j0}-\theta_{k0}|<\ell,\quad K>\frac{D(\Omega)}{\frac{n_0}{N}\sin\ell-\frac{2(N-n_0)}{N}\sin\frac{\ell}{2}}. \end{aligned}\tag{4.2}$$

Let Θ be a global solution to system (1.1)–(1.2). *Then, we have*

$$\sup_{0\le t<\infty}D(\Theta(t))\le4\pi+\ell,\qquad \lim_{t\to\infty}||\dot\Theta(t)||_\infty=0.$$

Proof. (i) (Uniform boundedness of $D(\Theta(t))$): Let $\Theta=(\theta_1,\dots,\theta_N)$ be a solution to system (1.1)–(1.2) satisfying the condition on Θ_0 in (4.2):

$$\max_{1\le j,k\le n_0}|\theta_{j0}-\theta_{k0}|<\ell.$$

• Case A (Dynamics of $\{\theta_1,\dots,\theta_{n_0}\}$): For this, we set

$$\begin{aligned} &\Sigma_0(t):=\{\theta_1(t),\dots,\theta_{n_0}(t)\}\subset\mathbb{R},\\ &I_0:=\text{an interval containing the set }\ \Sigma_0\text{ with length }\ell. \end{aligned}$$

We first study the dynamics of the set $\Sigma_0(t)$. We claim:

$$\mathrm{diam}\Sigma_0(t)\le\ell,\quad t\ge0.\tag{4.3}$$

Proof of claim (4.3): For this, we define

$$T:=\sup\{\ t\in[0,\infty)|\ \max_{1\le j,k\le n_0}|\theta_j(s)-\theta_k(s)|\le\ell,\qquad \forall\ 0\le s\le t\}.$$

If $T=\infty$, we are done. Otherwise, we derive a contradiction. Let us introduce the notation

$$\theta_{M_s}:=\max_{1\le j\le n_0}\theta_j,\qquad \theta_{m_s}:=\min_{1\le j\le n_0}\theta_j.$$

If $T<\infty$, then it is easy to see that

$$\theta_{M_s}(T)-\theta_{m_s}(T)=\ell,\qquad \frac{d}{dt}(\theta_{M_s}-\theta_{m_s})\Big|_{t=T}\ge0.\tag{4.4}$$

However, we can use (1.1) to obtain

$$
\begin{aligned}
\frac{d}{dt}(\theta_{M_s}-\theta_{m_s})\Big|_{t=T} =&\Omega_{M_s}-\Omega_{m_s}+\frac{K}{N}\sum_{k=1}^{N}\Big[\sin(\theta_k-\theta_{M_s})-\sin(\theta_k-\theta_{m_s})\Big]\Big|_{t=T}\\
\le& D(\Omega)-\frac{2K}{N}\sin\frac{\theta_{M_s}-\theta_{m_s}}{2}\left[\sum_{k=1}^{N}\cos(\theta_k-\frac{\theta_{M_s}+\theta_{m_s}}{2})\right]\Big|_{t=T}\\
=&D(\Omega)-\frac{2K}{N}\sin\frac{\theta_{M_s}-\theta_{m_s}}{2}\\
&\times\left[\sum_{k=1}^{n_0}\cos(\theta_k-\frac{\theta_{M_s}+\theta_{m_s}}{2})+\sum_{k=n_0+1}^{N}\cos(\theta_k-\frac{\theta_{M_s}+\theta_{m_s}}{2})\right]\Big|_{t=T}.
\end{aligned}
\tag{4.5}
$$

Using

$$
\begin{aligned}
\left|\theta_k(T)-\frac{\theta_{M_s}(T)+\theta_{m_s}(T)}{2}\right| &\le\frac{1}{2}|\theta_k(T)-\theta_{M_s}(T)+\theta_k(T)-\theta_{m_s}(T)|\\
&\le\frac{1}{2}\max\Big\{|\theta_k(T)-\theta_{M_s}(T)|,|\theta_k(T)-\theta_{m_s}(T)|\Big\}\\
&\le\frac{\ell}{2},\quad k=1,\dots,n_0,
\end{aligned}
$$

and

$$
\cos(\theta_k-\frac{\theta_{M_s}+\theta_{m_s}}{2})\ge-1,\quad k=n_0+1,\dots,N,
$$

we have that

$$
\sum_{k=1}^{n_0}\cos(\theta_k-\frac{\theta_{M_s}+\theta_{m_s}}{2})+\sum_{k=n_0+1}^{N}\cos(\theta_k-\frac{\theta_{M_s}+\theta_{m_s}}{2})\Big|_{t=T}\ge n_0\cos\frac{\ell}{2}-(N-n_0). \tag{4.6}
$$

Thus, it follows from (4.5), (4.6), and the condition on K in (4.2) that

$$
\begin{aligned}
\frac{d}{dt}(\theta_M-\theta_m)|_{t=T} &\le D(\Omega)-\frac{2K}{N}\sin\frac{\ell}{2}\left[n_0\cos\frac{\ell}{2}-(N-n_0)\right]\\
&=D(\Omega)-K\left[\frac{n_0}{N}\sin\ell-\frac{2(N-n_0)}{N}\sin\frac{\ell}{2}\right]\\
&<0.
\end{aligned}
\tag{4.7}
$$

This gives a contradiction to (4.4), and we have

$$
\mathrm{diam}\Sigma_0(t)\le\ell,\quad t\ge0.
$$

• Case B (Dynamics of $\{\theta_{n_0+1},\dots,\theta_N\}$): We assume that

$$
\theta_{n_0+1}(0),\dots,\theta_N(0)\in[-\pi,\pi)\cap(\Sigma_0(0))^c,
$$

and set

$$
\Sigma_{-1}:=\Sigma_0-2\pi,\qquad \Sigma_1:=\Sigma_0+2\pi,\qquad I_{-1}:=I_0-2\pi,\qquad I_1:=I_0+2\pi.
$$

Below, we will show that the oscillator phases $\mathcal{N}\cap(\Sigma_0)^c := \theta_{n_0+1},\dots,\theta_N$ are confined to a bounded neighborhood of the set $\Sigma_0(t)$. Suppose that the trajectory of one oscillators in the set $\mathcal{N}\cap(\Sigma_0)^c$ is unbounded in $\mathbb{R}$ (say θ_{n_0+1} plays such a role). Then, before its trajectory goes to ∞ or $-\infty$, this oscillator should enter at least one bounded neighborhood of $\Sigma_k,\ k=-1,0,1$ in finite time. Without loss of generality, we assume that there exists a time $t_e \geq 0$ such that

$$\max_{1\leq k\leq n_0} |\theta_{n_0+1}(t_e) - (\theta_k(t_e)+2\pi)| < \ell.$$

We claim:

$$\max_{1\leq k\leq n_0} |\theta_{n_0+1}(t) - (\theta_k(t)+2\pi)| \leq \ell, \qquad \forall\, t > t_e. \tag{4.8}$$

Proof of (4.8): Suppose that (4.8) does not hold, i.e.,

$$T := \sup\{\ t \mid \max_{1\leq k\leq n_0} |\theta_{n_0+1} - (\theta_k+2\pi)|(s) \leq \ell, \qquad \forall\, t_e \leq s \leq t\} < \infty.$$

Then, either

$$\theta_{n_0+1}(T) - (\theta_m(T)+2\pi) = \ell, \qquad \frac{d}{dt}(\theta_{n_0+1} - (\theta_m+2\pi))|_{t=T} \geq 0, \tag{4.9}$$

or

$$(\theta_M(T)+2\pi) - \theta_{n_0+1}(T) = \ell, \qquad \frac{d}{dt}((\theta_M+2\pi) - \theta_{n_0+1})|_{t=T} \geq 0. \tag{4.10}$$

We use the same argument as in (4.7) to derive a contradiction, i.e., by the same argument, we have

$$\text{Either} \quad \frac{d}{dt}(\theta_{n_0+1} - (\theta_m+2\pi))|_{t=T} < 0 \qquad \text{or} \qquad \frac{d}{dt}((\theta_M+2\pi) - \theta_{n_0+1})|_{t=T} < 0,$$

which are contradictory to (4.9) and (4.10), respectively. Thus, we have (4.8).

Finally, we combine the results of Cases A and B to obtain

$$\sup_{0\leq t<\infty} D(\Theta(t)) \leq 4\pi + \ell. \tag{4.11}$$

(ii) Note that the total phase is zero for any solutions to (1.1) and (1.2):

$$\sum_{j=1}^N \dot\theta_j(t) = 0 \quad \Rightarrow \quad \sum_{j=1}^N \theta_j(t) = \sum_{j=1}^N \theta_{j0} = 0, \quad t \geq 0.$$

Then, we combine the above estimate and our first result (4.11) to obtain

$$|\theta_j(t)| \leq |\theta_j(t) - \theta_c(t)| + |\theta_c(t)| \leq \sup_{t\geq 0} D(\Theta(t)) + |\theta_c(0)| < \infty, \qquad \theta_c(t) := \frac{1}{N}\sum_{k=1}^N \theta_k(t).$$

This clearly yields

$$\sup_{0\leq t<\infty} ||\Theta(t)||_\infty \leq \sup_{0\leq t<\infty} D(\Theta(t)) + ||\Theta_0||_\infty \leq 4\pi + \ell + ||\Theta_0||_\infty < \infty.$$

We now apply Theorem 2.1 to derive the condition of complete synchronization. $\square$

REMARK 4.1. As a special case of Lemma 4.1, we choose $n_0 = N$. Then, we have $\ell \in (0,\pi)$. Thus, we may choose $\ell = D(\Theta_0) < \pi$ so that condition (4.2) becomes

$$D(\Theta_0) < \pi, \quad K > \frac{D(\Omega)}{\sin D(\Theta_0)},$$

which is exactly the same as in the framework in [5]. Thus, Lemma 4.1 improves the estimate of Theorem 3.1 in [5].

Let Θ^I and Θ^{NI} be the phases of identical and non-identical oscillators, respectively, whose dynamics are governed by the following systems:

$$\dot\theta_j^I = \frac{K}{N}\sum_{k=1}^{N}\sin(\theta_k^I - \theta_j^I), \quad t > 0, \tag{4.12}$$

and

$$\dot\theta_j^{NI} = \Omega_j + \frac{K}{N}\sum_{k=1}^{N}\sin(\theta_k^{NI} - \theta_j^{NI}), \quad \sum_{j=1}^{N}\Omega_j = 0, \quad t > 0, \tag{4.13}$$

subject to the same initial data

$$\theta_j^I(0) = \theta_j^{NI}(0) = \theta_{j0}. \tag{4.14}$$

LEMMA 4.2. *Let Θ^I and Θ^{NI} be solutions to systems* (4.12) *and* (4.13), *respectively, for the same initial data* (4.14). *Then, we have*

$$||\Theta^{NI}(t) - \Theta^I(t)||_\infty \le \frac{||\Omega||_\infty}{2K}(e^{2Kt} - 1) \qquad t > 0. \tag{4.15}$$

Here, $||\Omega||_\infty$ is defined as the ℓ^∞-norm of Ω_i:

$$||\Omega||_\infty := \max_{1\le j\le N}|\Omega_j|.$$

Proof. It follows from (4.12), (4.13), and the mean-value theorem that

$$\begin{aligned}\frac{d}{dt}(\theta_j^{NI} - \theta_j^I) =& \Omega_j + \frac{K}{N}\sum_{k=1}^{N}\Big(\sin(\theta_k^{NI} - \theta_j^{NI}) - \sin(\theta_k^I - \theta_j^I)\Big)\\ =& \Omega_j + \frac{K}{N}\sum_{k=1}^{N}\cos\xi_{kj}^*\Big[(\theta_k^{NI} - \theta_k^I) - (\theta_j^{NI} - \theta_j^I)\Big],\end{aligned} \tag{4.16}$$

where ξ_{kj}^* is a value on the line segment between $\theta_k^{NI} - \theta_j^{NI}$ and $\theta_k^I - \theta_j^I$.

We multiply $\mathrm{sgn}(\theta_j^{NI} - \theta_j^I)$ by (4.16) to find

$$\frac{d}{dt}|\theta_j^{NI}(t) - \theta_j^I(t)| \le ||\Omega||_\infty + 2K||\Theta^{NI}(t) - \Theta^I(t)||_\infty. \tag{4.17}$$

Note that the difference $|\Theta^{NI}(t) - \Theta^I(t)|$ is a Lipschitz continuous function, so it is differentiable almost everywhere. For a given $t > 0$, there exist j_t such that

$$||\Theta^{NI}(t) - \Theta^I(t)||_\infty = |\theta_{j_t}^{NI} - \theta_{j_t}^I|.$$

Then, for such i_t, we apply the estimate in (4.17) to obtain

$$\frac{d}{dt}||\Theta^{NI}(t)-\Theta^I(t)||_\infty \le ||\Omega||_\infty + 2K||\Theta^{NI}(t)-\Theta^I(t)||_\infty, \quad \text{a.e. } t>0,$$

with the initial condition

$$||\Theta^{NI}(0)-\Theta^I(0)||_\infty = 0.$$

The standard form of Gronwall's lemma yields

$$||\Theta^{NI}(t)-\Theta^I(t)||_\infty \le \frac{||\Omega||_\infty}{2K}(e^{2Kt}-1), \quad t\ge 0.$$

□

REMARK 4.2. In a finite-time interval $[0,T)$, the estimate in (4.15) yields convergence from the dynamics of (4.13) to (4.12) in a zero natural frequency limit:

$$\lim_{||\Omega||_\infty \to 0} \sup_{0\le t<T} ||\Theta^{NI}(t)-\Theta^I(t)||_\infty = 0.$$

4.2. Emergence of phase locked states. In this part, we first consider a special case with $K=1$, and then reduce the general case to this special case using a scaling argument.

LEMMA 4.3. *Suppose that the initial data, natural frequencies and coupling strength satisfy* (1.2) *and following conditions:*

$$r(\Theta_0)>0, \quad \theta_{j0}\ne\theta_{k0}, \quad 1\le j,k\le N, \qquad ||\Omega||_\infty \le L<\infty, \quad K=1.$$

Then, there exists $L_\infty>0$ such that, if $L\le L_\infty$, any solution $\Theta=\Theta(t)$ of system (1.1)–(1.2) *has*

$$\lim_{t\to\infty} ||\dot\Theta(t)||_\infty = 0. \tag{4.18}$$

Proof.

• Case A ($N=2$): In this case, we choose $L_\infty < \frac{1}{2}$. Then,

$$|\Omega_j-\Omega_k| \le 2||\Omega||_\infty \le 2L_\infty < 1 = K.$$

Thus, it follows from (4.1) that

$$\lim_{t\to\infty} |\dot\theta_1(t)-\dot\theta_2(t)| = 0.$$

We now combine the above relation and $\sum_{j=1}^{2}\dot\theta_j = 0$ to find

$$\lim_{t\to\infty} |\dot\theta_j(t)| = 0, \quad j=1,2,$$

which gives the desired estimate (4.18).

• Case B ($N\ge 3$): In this case, we use the approximation of (4.13) by (4.12) for sufficiently small $||\Omega||_\infty$.

$\diamond$ Step B.1 : Let Θ^I be the solution to

$$\dot{\theta}_j^I = \frac{1}{N}\sum_{k=1}^{N}\sin(\theta_k^I - \theta_j^I), \quad t>0, \qquad \Theta^I(0)=\Theta_0.$$

It follows from Lemma 3.4 that there exists $T_1>0$ such that

$$\max_{i,j\in\mathcal{I}_s}|\theta_i^I(t)-\theta_j^I(t)| \le e^{-\lambda(t-T_1)}\max_{i,j\in\mathcal{I}_s}|\theta_i^I(T_1)-\theta_j^I(T_1)|, \qquad \forall\ t\ge T_1,$$

where $N-1\le|\mathcal{I}_s|\le N$. For any $\eta\ll 1$, we choose a sufficiently large T_2 such that

$$2\pi e^{-\lambda(T_2-T_1)}\le\eta.$$

Then, this gives

$$\max_{i,j\in\mathcal{I}_s}|\theta_i^I(t)-\theta_j^I(t)|\le 2\pi e^{-\lambda(t-T_1)}\le 2\pi e^{-\lambda(T_2-T_1)}\le\eta \qquad \forall\ t\ge T_2, \tag{4.19}$$

where we used

$$\max_{i,j}|\theta_i^I(t)-\theta_j^I(t)|\le 2\pi \qquad \forall\ t\ge 0,$$

for identical Kuramoto oscillators.

$\diamond$ Step B.2: For notational simplicity, we set

$$\Delta_j\Theta := \theta_j^{NI}-\theta_j^I, \qquad \Delta\phi := \phi^{NI}-\phi^I.$$

It follows from Lemma 4.2 with $K=1$ that, for any $\zeta\ll 1$, there exists L_1 such that, if $L\le L_1$,

$$\max_{1\le j\le N}|\Delta_j\Theta(T_2)| = ||\Theta^{NI}(T_2)-\Theta^I(T_2)||_\infty \le \frac{L}{2}(e^{2T_2}-1)\le\frac{L_1}{2}(e^{2T_2}-1)\le\zeta. \tag{4.20}$$

We can combine (4.19) and (4.20) to get

$$\begin{aligned}\max_{i,j\in\mathcal{I}_s}|\theta_i^{NI}(T_2)-\theta_j^{NI}(T_2)| &\le \max_{i,j\in\mathcal{I}_s}|\theta_i^I(T_2)-\theta_j^I(T_2)|+2\max_{j\in\mathcal{I}_s}|\Delta_j\Theta(T_2)|\\ &=\eta+2\zeta.\end{aligned} \tag{4.21}$$

Note that $N\ge 3$ implies $n_0 := N-1>\frac{N}{2}$. Hence, we can use Proposition 4.1 with $K=1$ and let ℓ be a positive number such that

$$0<\ell<2\cos^{-1}\frac{1}{N-1}.$$

We choose $L_2>0$ such that

$$L_2<\frac{1}{2}\Big(\frac{N-1}{N}\sin\ell-\frac{1}{N}\sin\frac{\ell}{2}\Big).$$

Then, if $||\Omega||_\infty\le L_2$, we have

$$\frac{D(\Omega)}{\frac{N-1}{N}\sin\ell-\frac{1}{N}\sin\frac{\ell}{2}}\le\frac{2||\Omega||_\infty}{\frac{N-1}{N}\sin\ell-\frac{1}{N}\sin\frac{\ell}{2}}\le\frac{2L_2}{\frac{N-1}{N}\sin\ell-\frac{1}{N}\sin\frac{\ell}{2}}<1=K.$$

Now, we choose

$$\eta := \frac{\ell}{2} \qquad \zeta := \frac{\ell}{4} \qquad L_\infty := \min\{L_1, L_2\}.$$

If $||\Omega||_\infty \le L_\infty$, then (4.21) gives $N-1$ oscillators lying on an arc of length ℓ at time T_2. Now, we apply Proposition 4.1 with $n_0 = N-1$ and T_2 as the initial time. This gives the desired estimate

$$\lim_{t\to\infty} ||\dot{\Theta}(t)||_\infty = 0.$$

□

We are now ready to prove Theorem 1.1 in full generality. Recall the initial value problem for the Kuramoto model:

$$\frac{d\theta_j}{dt} = \Omega_j + \frac{K}{N}\sum_{k=1}^{N}\sin(\theta_k - \theta_j), \quad t>0, \qquad \theta_j(0) = \theta_{j0}. \tag{4.22}$$

We introduce a slow time scale $\tau := Kt$, and rescale the natural frequency $\tilde{\Omega}_j = \frac{\Omega_j}{K}$ and corresponding phase variable $\tilde{\theta}_j$:

$$\tau := Kt, \quad \tilde{\theta}_j(\tau) := \theta_j\left(\frac{\tau}{K}\right) \quad \text{and} \quad \tilde{\Omega}_j := \frac{\Omega_j}{K}.$$

Then, system (4.22) can be rewritten as

$$\frac{d\tilde{\theta}_j}{d\tau} = \tilde{\Omega}_j + \frac{1}{N}\sum_{k=1}^{N}\sin(\tilde{\theta}_k - \tilde{\theta}_j), \quad t>0, \qquad \tilde{\theta}_j(0) = \theta_{j0}.$$

Note that the scaled natural frequencies $\tilde{\Omega}_j$ satisfy

$$|\tilde{\Omega}_j| \le \frac{L}{K}.$$

We choose K_∞ sufficiently large so that

$$\frac{L}{K_\infty} < L_\infty.$$

This completes the proof of Theorem 1.1.

5. Conclusion

In this paper, we have presented a sufficient framework for the complete synchronization to the Kuramoto model. Kuramoto oscillators can be visualized as point rotors moving on the unit circle. In numerical simulations, it has been observed that the relative phase velocities (frequencies) of an ensemble of Kuramoto oscillators tend to zero, regardless of initial configuration, so that the ensemble of rotors behave like a train on the unit circle (the so-called emergence of entrainment). However, such numerical results have not been confirmed in full generality by rigorous mathematical arguments, although there are several partial results available in the literature. These previous works restrict the diameter of the initial phase set to less than $\pi+\varepsilon$ with $\varepsilon \ll 1$. Thus, previous studies do not cover initial configurations that are scattered around the whole circle. In this work, we used three key arguments to the complete synchronization for generic initial configurations. First, we used that the ensemble of identical oscillators is a good approximation for non-identical oscillators with small natural frequencies in finite time. Second, we used the fact that the asymptotics of the ensemble of identical oscillators with a nonzero initial order parameter is a phase sync or a bipolar configuration that synchronized with the overall phase, except for at most one oscillator. Third, we generalized the global analysis on the diameter of the whole configuration to the local condition dealing with an oscillator set of more than half of the configuration. This new localized version of the boundedness estimate yielded a uniform bound for the diameter of the configuration. Combining the gradient flow structure of the Kuramoto model with these three key arguments, we were able to derive a state of complete synchronization for a generic initial configuration. Of course, there are many further questions in relation to the complete synchronization problem. In this paper, we have only scratched the surface, basically confirming that the complete synchronization results given by numerical simulations are rigorous in a large coupling regime. In our main result in Theorem 1.1, we did not optimize the coupling strength leading to complete synchronization. Thus, in the ongoing work [16], we will pursue this issue.

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