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MULTIPLICITIES OF TENSOR EIGENVALUES*

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Abstract. We study in this article multiplicities of tensor eigenvalues. There are two natural multiplicities associated to an eigenvalue of a tensor: algebraic multiplicity and geometric multiplicity. The former is the multiplicity of the eigenvalue as a root of the characteristic polynomial, and the latter is the dimension of the eigenvariety (i.e., the set of eigenvectors) corresponding to the eigenvalue.

We show that the algebraic multiplicity could change along the orbit of tensors by the orthogonal linear group action, while the geometric multiplicity of the zero eigenvalue is invariant under this action, which is the main difficulty in studying their relationships. However, we show that for a generic tensor, every eigenvalue has a unique (up to scaling) eigenvector, and both the algebraic multiplicity and geometric multiplicity are one. In general, we suggest for an mth order n-dimensional tensor an inequality relating the algebraic multiplicity and geometric multiplicity. We show that it is true for several cases, especially when the eigenvariety contains a linear subspace of dimension the geometric multiplicity of the given eigenvalue in coordinate form. As both multiplicities are invariants under the orthogonal linear group action in the matrix counterpart, this generalizes the classical result for a matrix: the algebraic multiplicity is not smaller than the geometric multiplicity.

Key words. Tensor, eigenvalue, eigenvector, algebraic multiplicity, geometric multiplicity.

AMS subject classifications. 15A18, 15A42, 15A69.

1. Introduction

Tensors are ubiquitous and inevitable generalizations of matrices. The concept of an eigenvalue of a tensor, as natural generalized notion of the eigenvalue of a square matrix, was proposed independently by Lim [17] and Qi [23]. Among others, the number of eigenvalues [1,6,16,19,23], Perron–Frobenius theorem for nonnegative tensors [2], and applications to spectral hypergraph theory [4,12–14,18,24] are the well-studied topics.

The eigenvalues of a tensor are the roots of the characteristic polynomial, which is a monic polynomial with degree being determined by the order and the dimension of the tensor [11,23]. Therefore, for an eigenvalue, we can define its algebraic multiplicity as the multiplicity of the eigenvalue as a root of the characteristic polynomial, and the geometric multiplicity as the dimension of the set of eigenvectors corresponding to this eigenvalue. However, the set of eigenvectors of an eigenvalue is not a linear subspace in general, which is due to the nonlinearity of the eigenvalue equations of tensors. The two multiplicities are generalizations of those for eigenvalues of matrices. In the matrix counterpart, classical linear algebra says that the algebraic multiplicity is always greater than the geometric multiplicity. While, from the literature of eigenvalues of tensors, it is not clear what is the relationship between the algebraic multiplicity and the geometric multiplicity of an eigenvalue. It can also be seen from applications [4,6,14,21] that the understanding of the multiplicities of an eigenvalue is a necessity.

This article is devoted to the study on this topic, the rest of which is organized as follows. Basic notions, such as determinant, eigenvalues and their multiplicities of tensors, are presented in the next section. We discuss the multiplicities of the zero eigenvalue of a tensor in sections 3 and 4. More precisely, in Section 3 we show that the

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algebraic multiplicity is not an invariant of a tensor under the orthogonal linear group action, whereas the geometric multiplicity is. Therefore, it seems that the general relation between the two multiplicities is a subtle problem, but we make a reasonable conjecture on it in this section.

CONJECTURE 1.1. *Suppose that an mth order n-dimensional tensor $\mathcal{A}$ has an eigenvalue λ with the set of eigenvectors (i.e., eigenvariety) $V(\lambda)$ possessing κ irreducible components $V_1,\ldots,V_\kappa$. Then, the conjecture is that*

$$\mathrm{am}(\lambda)\geq\sum_{i=1}^{\kappa}\dim(V_i)(m-1)^{\dim(V_i)-1},$$

where $\mathrm{am}(\lambda)$ is the algebraic multiplicity of λ and $\dim(V_i)$ is the dimension of V_i as a subvariety in the affine space. Since the geometric multiplicity is $\dim(V_i)=\mathrm{gm}(\lambda)$ for some i, this general conjecture implies the following conjecture between the two multiplicities:

$$\mathrm{am}(\lambda)\geq\mathrm{gm}(\lambda)(m-1)^{\mathrm{gm}(\lambda)-1}. \tag{1.1}$$

The relation (1.1) is a generalization of that in the matrix counterpart (i.e., $m=2$): the algebraic multiplicity is greater than the geometric multiplicity, which can be proved by the basic case when the eigenspace is a linear subspace in coordinate form and the fact that in this situation both multiplicities are invariants under the orthogonal linear group action (cf. Section 3.1).

Conjecture 1.1 is shown to be true for several interesting cases in the sequel. Section 4 is for lower marginal rank symmetric tensors which include orthogonally decomposable symmetric tensors, and Section 5 is for tensors whose eigenvarieties contain linear subspaces of dimension $\mathrm{gm}(\lambda)$ in coordinate form, which generalizes the relation in the matrix counterpart. In Section 6, we discuss the generic case. We show that for a generic tensor, both the algebraic multiplicity and geometric multiplicity are one. Likewise, in this case, we have the relation (1.1). Moreover, we use the shape lemma to show that in this case, each eigenvalue has a unique (up to scaling) eigenvector. In Section 7, we discuss symmetric tensors, for which the determinantal hypersurface is the dual of the Veronese variety. We study the eigenvectors from the perspective of variety duality. Some final remarks are given in the last section.

2. Preliminaries

DEFINITION 2.1 (Classical Resultant [5, 7]). *For fixed positive degree d, let $\{u_{i,\alpha}:|\alpha|=d,\ i=1,\ldots,n\}$ be the set of indeterminants, and $f_i:=\sum_{|\alpha|=d}c_{i,\alpha}\mathbf{x}^\alpha$ be a homogeneous polynomial of degree d in $\mathbb{C}[\mathbf{x}]$ for $i\in\{1,\ldots,n\}$. Then there exists a unique polynomial $RES\in\mathbb{Z}[\{u_{i,\alpha}\}]$ called the* resultant *of degrees $(d,\ldots,d)$ satisfying the following properties:*

- *(i) The system of polynomial equations $f_1=\cdots=f_n=0$ has a nontrivial solution in $\mathbb{C}^n$ if and only if $RES(f_1,\ldots,f_n):=RES|_{u_{i,\alpha}=c_{i,\alpha}}=0$.*
- *(ii) $RES(x_1^{d_1},\ldots,x_n^{d_n})=1$.*
- *(iii) RES is an irreducible polynomial in $\mathbb{C}[\{u_{i,\alpha}\}]$.*

Let $\mathcal{T}=(t_{i_1\ldots i_m})$ be an mth order n-dimensional tensor, $\mathbf{x}=(x_i)\in\mathbb{C}^n$ (the n-dimensional complex space) and $\mathcal{T}\mathbf{x}^{m-1}$ be an n-dimensional vector with its ith element being

$$\sum_{i_2=1}^{n}\cdots\sum_{i_m=1}^{n}t_{ii_2\ldots i_m}x_{i_2}\cdots x_{i_m}.$$

It is actually a tensor contraction. The set of all mth order n-dimensional tensors is denoted as $\mathbb{T}(\mathbb{C}^n, m)$, which is $\otimes^m \mathbb{C}^n = \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n$ (m times). We adopt the definition of tensor determinant from [11].

DEFINITION 2.2 (Tensor Determinant). *Let RES be the resultant of degrees $(m-1, \dots, m-1)$ which is a polynomial in variables $\{u_{i,\alpha} \mid |\alpha| = m-1, i \in \{1, \dots, n\}\}$. The determinant DET of mth order n-dimensional tensors is defined as the polynomial with variables $\{v_{ii_2\dots i_m} \mid i, i_2, \dots, i_m \in \{1, \dots, n\}\}$ through replacing $u_{i,\alpha}$ in the polynomial RES by $\sum_{(i_2,\dots,i_m)\in\mathbb{X}(\alpha)} v_{ii_2\dots i_m}$. Here $\mathbb{X}(\alpha) := \{(i_2, \dots, i_m) \in \{1, \dots, n\}^{m-1} \mid x_{i_2} \cdots x_{i_m} = \mathbf{x}^\alpha\}$. The value of the determinant Det($\mathcal{T}$) of a specific tensor $\mathcal{T}$ is defined as the evaluation of DET at the point $\{v_{ii_2\dots i_m} = t_{ii_2\dots i_m}\}$.*

For the convenience of the subsequent analysis, we define $\mathrm{DET}(\mathcal{T})$ as the polynomial with variables $\{t_{ii_2\dots i_m} \mid i, i_2, \dots, i_m \in \{1, \dots, n\}\}$ through replacing $v_{ii_2\dots i_m}$ in DET by $t_{ii_2\dots i_m}$. There can be some specific relations on the variables $\{t_{ii_2\dots i_m}\}$, such as some being zero. In this case, $\mathcal{T}$ is considered as a tensor of indeterminate variables, while it is considered as a tensor of numbers in $\mathbb{C}$ when we talk about $\mathrm{Det}(\mathcal{T})$.

DEFINITION 2.3 (Eigenvalues and Eigenvectors [17, 23]). *Let tensor $\mathcal{T} = (t_{ii_2\dots i_m}) \in \mathbb{T}(\mathbb{C}^n, m)$. A number $\lambda \in \mathbb{C}$ is called an eigenvalue of $\mathcal{T}$, if there exists a vector $\mathbf{x} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ which is called an eigenvector such that*

$$\mathcal{T}\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]},$$

where $\mathbf{x}^{[m-1]}$ is an n-dimensional vector with its ith component being x_i^{m-1}.

The next proposition is a direct consequence of the definition, see also [11]. The set of eigenvalues of a given tensor $\mathcal{T}$ is always finite [11], which is denoted as $\sigma(\mathcal{T})$, and it is the set of roots of the univariate polynomial

$$\chi(\lambda) = \mathrm{Det}(\lambda\mathcal{I} - \mathcal{T})$$

which is called the *characteristic polynomial* of $\mathcal{T}$ [23].

PROPOSITION 2.4. *Let the tensor space being $\mathbb{T}(\mathbb{C}^n, m)$. The determinant* DET $\in \mathbb{C}[\{v_{i_1\dots i_m}\}]$ *is irreducible, and for any $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$*

$$\mathrm{Det}(\mathcal{T}) = 0 \iff 0 \in \sigma(\mathcal{T}).$$

For a given eigenvalue $\lambda \in \sigma(\mathcal{T})$, the set of eigenvectors corresponding to the eigenvalue λ is denoted as $V_{\mathcal{T}}(\lambda)$ (simplified as $V(\lambda)$ whenever it is clear from the content) [1]:

$$V_{\mathcal{T}}(\lambda) := \left\{\mathbf{x} \in \mathbb{C}^n : \mathcal{T}\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]}\right\}.$$

One can either treat $V(\lambda)$ as a variety in $\mathbb{PC}^{n-1}$ (the projective space of $\mathbb{C}^n$) or as a variety in $\mathbb{C}^n$. In this article, we will take $V(\lambda)$ as an affine variety. Instead of eigenspace as in linear algebra, we use the nomenclature *eigenvariety* for $V(\lambda)$, which can be sensed from the next example.

EXAMPLE 2.1. *Take the tensor $\mathcal{T} \in \mathbb{T}(\mathbb{C}^2, 3)$ as*

$$t_{111} = 2,\ t_{122} = 1,\ t_{222} = 1,\ \textit{and } t_{ijk} = 0 \textit{ for the others.}$$

[1] Here we include the vector $\mathbf{0}$ for convenience.

The eigenvalue equations are

$$2x_1^2+x_2^2=\lambda x_1^2,\ x_2^2=\lambda x_2^2.$$

Obviously, $1\in\sigma(\mathcal{T})$ *is an eigenvalue of* $\mathcal{T}$*, and*

$$V(1)=\left\{\alpha(\sqrt{-1},1),\beta(-\sqrt{-1},1),\ \alpha,\beta\in\mathbb{C}\right\}=\mathbb{C}(\sqrt{-1},1)\cup\mathbb{C}(-\sqrt{-1},1).$$

It is easy to see that $V(1)$ *is reducible and hence not a linear subspace, and the dimension is* 1.

Definition 2.5 (Multiplicity). *Let tensor* $\mathcal{T}=(t_{ii_2\dots i_m})\in\mathbb{T}(\mathbb{C}^n,m)$*. The algebraic multiplicity* am(λ) *of an eigenvalue* $\lambda\in\sigma(\mathcal{T})$ *is the multiplicity of* λ *as a root of the characteristic polynomial* $\chi(\lambda)$*. The geometric multiplicity* gm(λ) *of an eigenvalue* $\lambda\in\sigma(\mathcal{T})$ *is the dimension of the variety* $V(\lambda)\subseteq\mathbb{C}^n$*.*

Notice that we do not indicate the tensor $\mathcal{T}$ in the notation am(λ). The reason for this abbreviation is that when we talk about multiplicities, the tensor $\mathcal{T}$ is always fixed.

We refer to [5,7,9] for some basic algebraic geometry terminologies. The definitions of multiplicities boil down to the classical ones for matrices [10].

Proposition 2.6 (Disjoint). *Let tensor* $\mathcal{T}=(t_{ii_2\dots i_m})\in\mathbb{T}(\mathbb{C}^n,m)$*. For any* $\lambda,\mu\in\sigma(\mathcal{T})$*, we have*

$$V(\lambda)\cap V(\mu)=\{\mathbf{0}\}\iff\lambda\neq\mu,$$

and

$$V(\lambda)=V(\mu)\iff\lambda=\mu.$$

Proof. We only need to show that $V(\lambda)\cap V(\mu)=\{\mathbf{0}\}$ whenever $\lambda\neq\mu$. Suppose on the contrary that $(\lambda,\mathbf{x})$ and $(\mu,\mathbf{x})$ are two eigenpairs of $\mathcal{T}$ with $\lambda\neq\mu$ and $\mathbf{x}\neq\mathbf{0}$. Then, we have that

$$\lambda\mathbf{x}^{[m-1]}=\mathcal{T}\mathbf{x}^{m-1}=\mu\mathbf{x}^{[m-1]}$$

which is a contradiction, since $\mathbf{x}\neq\mathbf{0}$. □

In the matrix case, the geometric multiplicity over $\mathbb{R}$ of an eigenvalue is the same of that over $\mathbb{C}$. But in the tensor case ($m\geq 3$), the geometric multiplicity over $\mathbb{R}$ of an eigenvalue could be strictly smaller than that over $\mathbb{C}$. For instance, in Example 2.1, $V(1)$ has dimension one over $\mathbb{C}$ while it is even empty over $\mathbb{R}$. The reason for the difference is that the eigenvalue equation system in the matrix case is a linear system while in the tensor case the eigenvalue equation system becomes a system of at least two nonlinear polynomials. The following proposition should be obvious.

Proposition 2.7. *Let* $\mathcal{T}\in\mathbb{T}(\mathbb{R}^n,m)$ *be a tensor and let* $\lambda\in\mathbb{R}$ *be an eigenvalue of* $\mathcal{T}$*. Let* $\mathrm{gm}_{\mathbb{R}}(\lambda)$ *be the geometric multiplicity of* λ *over* $\mathbb{R}$ *and let* $\mathrm{gm}_{\mathbb{C}}(\lambda)$ *be the geometric multiplicity of* λ *over* $\mathbb{C}$*. Then*

$$\mathrm{gm}_{\mathbb{R}}(\lambda)\leq\mathrm{gm}_{\mathbb{C}}(\lambda).$$

We would like to thank the referee for pointing out such a difference between the matrix case and the tensor case.

3. The multiplicities of the zero eigenvalue

In this section, we discuss some properties of the multiplicities of the zero eigenvalue of a tensor. The general purpose is to have an intuition on the difficulty and the expected relationships between the two multiplicities. It begins with the matrix counterpart.

3.1. The matrix case. There are several ways to prove that the algebraic multiplicity of an eigenvalue of a matrix is no smaller than its geometric multiplicity, one standard approach is the following. It is essentially by choosing a basis with some eigenvectors to represent the matrix. Let $A\in\mathbb{C}^{n\times n}$ be a matrix with the property that the eigenspace associated to an eigenvalue $\lambda_*\in\sigma(A)$ is a coordinate subspace

$$V(\lambda_*)=\{\mathbf{x}\in\mathbb{C}^n : x_{k+1}=\cdots=x_n=0\}. \tag{3.1}$$

So, $\mathrm{gm}(\lambda_*)=k$. We can partition A according to $\{1,\dots,n\}=\{1,\dots,k\}\cup\{k+1,\dots,n\}$ as

$$A=\begin{bmatrix}A_1 & A_2\\ A_3 & A_4\end{bmatrix}$$

with $A_1\in\mathbb{C}^{k\times k}$ and $A_4\in\mathbb{C}^{(n-k)\times(n-k)}$. Since the eigenspace of A for λ_* has the form (3.1), we can get that

$$A_1=\lambda_* I \text{ and } A_3=\mathbf{0}.$$

Therefore,

$$\chi(\lambda)=\mathrm{Det}(\lambda I-A)=\mathrm{Det}(\lambda I-A_1)\mathrm{Det}(\lambda I-A_4)=(\lambda-\lambda_*)^k\,\mathrm{Det}(\lambda I-A_4).$$

Henceforth,

$$\mathrm{am}(\lambda_*)\geq k=\mathrm{gm}(\lambda_*). \tag{3.2}$$

For the general case when $V(\lambda_*)$ is not in the coordinate form (3.1), we can always adopt an orthogonal linear transformation $P\in\mathbb{O}(n,\mathbb{C})$ (the orthogonal linear group of order n over the field $\mathbb{C}$) such that

$$PV(\lambda_*):=\{P\mathbf{x}:\mathbf{x}\in V(\lambda_*)\}=\{\mathbf{x}\in\mathbb{C}^n : x_{k+1}=\cdots=x_n=0\}.$$

It is a matter of fact that the matrix

$$B=PAP^{\mathsf{T}}$$

has λ_* being an eigenvalue with the corresponding eigenspace being $PV(\lambda_*)$ and hence in the coordinate form (3.1). So, the preceding discussion implies that

$$\mathrm{Det}(\lambda I-B)=(\lambda-\lambda_*)^k p(\lambda)$$

for some monic polynomial $p\in\mathbb{C}[\lambda]$ of degree $n-k$. While,

$$\mathrm{Det}(\lambda I-B)=\mathrm{Det}(\lambda I-PAP^{\mathsf{T}})=\mathrm{Det}(P(\lambda I-A)P^{\mathsf{T}})=\mathrm{Det}(\lambda I-A).$$

Henceforth, (3.2) still holds for the general case.

In summary, the above technique relies heavily on the fact that the eigenvalues (or equivalently the zero eigenvalue) together with their multiplicities of a matrix are invariants under the orthogonal group action. Likewise, we can associate a group action to tensors like that for matrices. However, the eigenvalues of a tensor are not invariants under this action anymore [23]. On the other hand, the zero eigenvalue is an invariant of a tensor under this action [11].

3.2. The zero eigenvalue. Given a tensor $\mathcal{A}\in\mathbb{T}(\mathbb{C}^n,m)$, and matrices $P^{(i)}\in\mathbb{C}^{r\times n}$ for $i=1,\dots,m$, we can define the matrix-tensor multiplication $(P^{(1)},P^{(2)},\dots,P^{(m)})\cdot\mathcal{A}\in\mathbb{T}(\mathbb{C}^r,m)$ as (cf. [15, 18])

$$\begin{aligned}&\left[(P^{(1)},P^{(2)},\dots,P^{(m)})\cdot\mathcal{A}\right]_{i_1i_2\dots i_m}\\ :=&\sum_{j_1,j_2,\dots,j_m=1}^{n} a_{j_1j_2\dots j_m}p^{(1)}_{i_1j_1}p^{(2)}_{i_2j_2}\dots p^{(m)}_{i_mj_m} \text{ for all } i_1,i_2,\dots,i_m=1,\dots,r.\end{aligned}$$

This generalizes the matrix multiplication: it is easy to see that for a matrix A, we have $(P,P)\cdot A=PAP^{\mathsf{T}}$. In general, when

$$P^{(1)}=\dots=P^{(m)}=P,$$

we simplify $(P,P,\dots,P)\cdot\mathcal{A}$ as $P\cdot\mathcal{A}$. It is a direct calculation to see that for all $P\in\mathbb{C}^{r\times n}$,

$$\left(P\cdot\mathcal{A}\right)\mathbf{x}^{m-1}=P\left\{\left[(I,P,\dots,P)\cdot\mathcal{A}\right]\mathbf{x}^{m-1}\right\}=P\left[\mathcal{A}\left(P^{\mathsf{T}}\mathbf{x}\right)^{m-1}\right)\right] \text{ for all } \mathbf{x}\in\mathbb{C}^r, \tag{3.3}$$

where I is the identity matrix in $\mathbb{C}^{n\times n}$.

Particularly, when $r=n$, and $P\in\mathbb{GL}(n,\mathbb{C})$, it is easy to see that the system of polynomial equations

$$\mathcal{A}\mathbf{x}^{m-1}=\mathbf{0}$$

becomes

$$\left[(I,P^{-\mathsf{T}},\dots,P^{-\mathsf{T}})\cdot\mathcal{A}\right]\mathbf{y}^{m-1}=\mathbf{0}$$

under the coordinate change $\mathbb{C}^n\to\mathbb{C}^n$ through $\mathbf{x}\mapsto\mathbf{y}=P\mathbf{x}$.

When $P\in\mathbb{O}(n,\mathbb{C})$, the matrix-tensor multiplication $P\cdot\mathcal{A}$ becomes a natural orthogonal linear group action on the space $\mathbb{T}(\mathbb{C}^n,m)$. We have also the group action by $\mathbb{GL}(n,\mathbb{C})$, the general linear group. It is a matter of fact that the determinantal hypersurface $\mathbb{V}(\mathrm{DET})$ is invariant under this group action (see [7] or directly from Definition 2.2). The group action structure of the matrix-tensor multiplication for $\mathbb{O}(n,\mathbb{C})$ will only be encountered in this section, whereas in the others solely being an algebraic operation between matrices and tensors is understood.

The next result shows the subtlety of algebraic multiplicity from the geometric perspective.

PROPOSITION 3.1 (Non-Invariance of Algebraic Multiplicity). *The algebraic multiplicity of the zero eigenvalue of a tensor is not invariant under the group action by $\mathbb{O}(n,\mathbb{C})$ as above.*

Proof. We show by an example that the coefficients with codegree higher than one of the characteristic polynomial

$$\mathrm{Det}(\lambda\mathcal{I}-\mathcal{A})$$

of a tensor $\mathcal{A}$ are zero, while the codegree one coefficient is nonzero for that of $P\cdot\mathcal{A}$

$$\mathrm{Det}(\lambda\mathcal{I}-P\cdot\mathcal{A})$$

for some $P \in \mathbb{O}(n,\mathbb{C})$.

Let $\mathcal{A} \in \mathbb{T}(\mathbb{C}^2,3)$ be given as

$$a_{112} = 1 \text{ and } a_{ijk} = 0 \text{ for the other } i,j,k \in \{1,2\}.$$

$\mathcal{A}$ is a tensor in the upper triangular form and by [11, Proposition 5.1] we have that

$$\mathrm{Det}(\lambda\mathcal{I} - \mathcal{A}) = (\lambda - a_{111})^2(\lambda - a_{222})^2 = \lambda^4.$$

Therefore, the algebraic multiplicity of the zero eigenvalue of $\mathcal{A}$ is 4, and zero is the unique eigenvalue of $\mathcal{A}$.

Let $P \in \mathbb{C}^{2\times 2}$ be given by

$$P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}.$$

Then, $P \in \mathbb{O}(2,\mathbb{C})$. It is a direct calculation to show that

$$\mathcal{B} := P \cdot \mathcal{A} \in \mathbb{T}(\mathbb{C}^2,3)$$

with

$$b_{111} = \frac{-1}{2\sqrt{2}} \text{ and } b_{222} = \frac{-1}{2\sqrt{2}}.$$

Henceforth,

$$\mathrm{tr}(\mathcal{B}) := 2(b_{111} + b_{222}) = -\sqrt{2}.$$

It follows from [11, Section 6] that

$$\mathrm{Det}(\lambda\mathcal{I} - \mathcal{B}) = \lambda^4 - \mathrm{tr}(\mathcal{B})\lambda^3 + \text{lower order terms} = \lambda^4 + \sqrt{2}\lambda^3 + \text{lower order terms}.$$

So, $\mathcal{B} = P \cdot \mathcal{A}$ has a nonzero eigenvalue, and hence it has the algebraic multiplicity of the zero eigenvalue being strictly less than 4. ∎

A direct consequence of Proposition 3.1 is that the multiplicity of any eigenvalue of a tensor could change when we apply the above group action. So, the algebraic multiplicity is indeed “algebraic”, not a good “geometric” object; and it becomes quite mixed on the orbits in $\mathbb{T}(\mathbb{C}^n,m)$ by the natural action of $\mathbb{O}(n,\mathbb{C})$. However, we can still ask for the relationship between the algebraic multiplicity and the geometric multiplicity. In general, the philosophy, together with classical linear algebra, suggests that

$$\text{algebraic multiplicity} \geq \mathfrak{f}\big(\text{geometric multiplicity}\big) \geq \text{geometric multiplicity}, \qquad (3.4)$$

for some function $\mathfrak{f}$ which depends also on the order m of the tensors in the ambient space.

On the other hand, for the zero eigenvalue, the geometric multiplicity is “geometric”.

Proposition 3.2 (Zero Geometric Multiplicity). *The number of irreducible components of the eigenvariety of the zero eigenvalue and their dimensions are invariants under the group action by $\mathbb{O}(n,\mathbb{C})$ as above. Especially, the geometric multiplicity of the zero eigenvalue of a tensor is an invariant.*

Proof. The eigenvariety $V_{P\cdot\mathcal{A}}(0)$ of the tensor $P\cdot\mathcal{A}$ for the eigenvalue zero is the transformation $PV_{\mathcal{A}}(0)$ of the eigenvariety $V_{\mathcal{A}}(0)$ of $\mathcal{A}$ for the eigenvalue zero. Since both the dimension and the number of irreducible components of a variety are invariants under coordinate changes, the result follows. ∎

In linear algebra, since the eigenvariety is an eigenspace, i.e., a linear subspace, under the orthogonal group action, the essential geometric characterization is unique, which is the dimension of that eigenspace. However, the eigenvariety for the tensor case is much more complicated, as nonlinear algebra to linear algebra. Nevertheless, the first two essential characterizations of the eigenvariety would be the number of irreducible components and their dimensions.

3.3. A conjecture on the relationship. As suggested by Proposition 3.2, we continue with the example in Proposition 3.1, and the other entries of $\mathcal{B}$ are

$$b_{112}=-\frac{1}{2\sqrt{2}},\ b_{121}=\frac{1}{2\sqrt{2}},\ b_{122}=\frac{1}{2\sqrt{2}},\ b_{211}=\frac{1}{2\sqrt{2}},\ b_{212}=\frac{1}{2\sqrt{2}}\text{ and }b_{221}=-\frac{1}{2\sqrt{2}}.$$

So, the equations for the eigenvalue problem of the tensor $\mathcal{B}$ are

$$\frac{1}{2\sqrt{2}}\big(-x^2+y^2\big)=\lambda x^2,\ \text{and}\ \frac{1}{2\sqrt{2}}\big(x^2-y^2\big)=\lambda y^2.$$

With Macaulay2 [8], we can compute the characteristic polynomial of $\mathcal{B}$ which is

$$\chi(\lambda)=\lambda^4+\sqrt{2}\lambda^3+\frac{1}{2}\lambda^2.$$

Therefore, the eigenvalues of $\mathcal{B}$ are

$$0(\text{with am}(0)=2)\ \text{and}\ -\frac{1}{\sqrt{2}}(\text{with am}(-\frac{1}{\sqrt{2}})=2).$$

Since $\mathcal{A}\mathbf{x}^2\not\equiv\mathbf{0}$, it can be shown that the geometric multiplicity of $\mathcal{A}$ for the zero eigenvalue cannot be two, and hence it is one. By direct calculation, we have that the eigenvariety of $\mathcal{A}$ for eigenvalue zero is

$$V(0)=\mathbb{C}\{(1,0)\}\cup\mathbb{C}\{(0,1)\},$$

which has two irreducible components. We also see that the algebraic multiplicity of the zero eigenvalue of $\mathcal{B}$ is the number of irreducible components of the eigenvariety of the zero eigenvalue with each irreducible component being a point in the projective space and hence dimension one in the affine space.

In general, based on Proposition 3.2, some examples and the case where the tensor is a matrix, one should expect the following: (3.4) should hold for every irreducible component V_i of the eigenvariety of an eigenvalue λ of a tensor

$$\text{am}(\lambda)\ge\mathfrak{f}\big(\dim(V_i)\big)\ge\dim(V_i);$$

and integrally

$$\text{am}(\lambda)\ge\sum_{V_i\subset V(\lambda)\text{ is an irreducible component}}\mathfrak{f}\big(\dim(V_i)\big). \tag{3.5}$$

Likewise, $\mathfrak{f}\big(\dim(V_i)\big)$ depends on the order m of the ambient tensor space in which the eigenvalue problem is studied. Whenever we have (3.5), immediately we have a realization of (3.4) since

$$\mathrm{am}(\lambda)\geq \sum_{V_i\subset V(\lambda)\text{ is an irreducible component}} \mathfrak{f}\big(\dim(V_i)\big)\geq \mathfrak{f}\big(\dim(V_*)\big)\geq \mathrm{gm}(\lambda)$$

with $\dim(V_*)=\mathrm{gm}(\lambda)$. In linear algebra (i.e., $m=2$), $V(\lambda)$ is connected, and

$$\mathfrak{f}\big(\dim(V(\lambda))\big)=\dim(V(\lambda)).$$

It is conjectured that

$$\mathfrak{f}\big(\dim(V_i)\big)=\dim(V_i)(m-1)^{\dim(V_i)-1},$$

which is satisfied by the former examples, and will be supported by all the cases studied subsequently. Putting these together, we formally arrive at Conjecture 1.1. While, we will defer the discussion on the number of irreducible components of the eigenvariety to future work, and in this article mainly focus on the relation between $\mathrm{am}(\lambda)$ and $\mathrm{gm}(\lambda)$ (cf. (1.1)).

The trivial case serves as the first taste of the conjecture.

Proposition 3.3 (Identity Tensor). *Let tensor $\mathcal{I}\in\mathbb{T}(\mathbb{C}^n,m)$ be the identity tensor. Then, the characteristic polynomial of $\mu\mathcal{I}$ is*

$$\chi(\lambda)=(\lambda-\mu)^{n(m-1)^{n-1}}$$

for any $\mu\in\mathbb{C}$.

For the tensor $\mu\mathcal{I}$, from Definition 2.3, we see that $V(\mu)=\mathbb{C}^n$, $\sigma(\mu\mathcal{I})=\{\mu\}$ and hence $\mathrm{gm}(\mu)=n$; and from Proposition 3.3 that $\mathrm{am}(\mu)=n(m-1)^{n-1}=\mathrm{gm}(\mu)(m-1)^{\mathrm{gm}(\mu)-1}$.

4. Lower marginal rank symmetric tensors

We continue to study the zero eigenvalue for Conjecture 1.1 in this section. The underlying tensors are lower marginal rank symmetric tensors.

A tensor $\mathcal{T}\in\mathbb{T}(\mathbb{C}^n,m)$ is *symmetric*, if $t_{i_1\dots i_m}=t_{i_{\tau(1)}\dots i_{\tau(m)}}$ for all $\tau\in\mathfrak{S}(m)$, the permutation group on m elements. Let $\mathcal{S}^m(\mathbb{C}^n)\subset\mathbb{T}(\mathbb{C}^n,m)$ be the space of mth order n-dimensional symmetric tensors and $\mathcal{A}\in\mathcal{S}^m(\mathbb{C}^n)$.

Every tensor $\mathcal{A}\in\mathcal{S}^m(\mathbb{C}^n)$ can be regarded as a linear map $L_{\mathcal{A}}:\mathcal{S}^{m-1}(\mathbb{C}^n)\to\mathbb{C}^n$, which maps $\mathbf{x}^{\otimes(m-1)}$ to $\mathcal{A}\mathbf{x}^{m-1}$. The range of the linear map $L_{\mathcal{A}}$ is the *marginal space* $M_{\mathcal{A}}$ of the tensor $\mathcal{A}$. The rank of the matrix representation of the linear map $L_{\mathcal{A}}$, or the dimension of the marginal space $M_{\mathcal{A}}$, is called the *marginal rank* of the tensor $\mathcal{A}$, which is denoted by $\mathrm{mrank}(\mathcal{A})$. Let $P\in\mathbb{C}^{n\times n}$ be the orthogonal projection from $\mathbb{C}^n$ onto $M_{\mathcal{A}}$, then it holds that $P\cdot\mathcal{A}=\mathcal{A}$ (cf. [18]).

Every tensor $\mathcal{A}\in\mathcal{S}^m(\mathbb{C}^n)$ has a minimal symmetric rank one decomposition [15]. Moreover, the decomposition can be chosen as

$$\mathcal{A}=\sum_{i=1}^R\mathbf{a}_i^{\otimes m},\ \mathbf{a}_i\in M_{\mathcal{A}},\ R\in\mathbb{N}. \tag{4.1}$$

Actually, for any given decomposition $\mathcal{A}=\sum_{i=1}^R\mathbf{a}_i^{\otimes m}$, one can get (4.1) by the fact that

$$\mathcal{A}=P\cdot\mathcal{A}=\sum_{i=1}^R(P\mathbf{a}_i)^{\otimes m}.$$

Let

$$A = [\mathbf{a}_1, \dots, \mathbf{a}_R], \text{ and } \mathbf{d} = \big((\mathbf{a}_1^\mathsf{T}\mathbf{x})^{m-1}, \dots, (\mathbf{a}_R^\mathsf{T}\mathbf{x})^{m-1}\big)^\mathsf{T}.$$

Then $\mathcal{A}\mathbf{x}^{m-1} = A\mathbf{d}$, and hence $\mathrm{range}(A) \subseteq M_{\mathcal{A}} \subseteq \mathrm{range}(A)$. It follows that the rank of the matrix $A \in \mathbb{C}^{n\times R}$ is the marginal rank of the tensor $\mathcal{A}$, and the column space of the matrix A is the marginal space of the tensor $\mathcal{A}$, which can be adopted as the definitions for the marginal rank and the marginal space respectively. Obviously, $\mathrm{mrank}(\mathcal{A}) \le n$; while it always happens that $\mathrm{mrank}(\mathcal{A}) < R$. We refer to [15, Chapter 3.4] and [3], and references therein for more on symmetric tensors. The phrase "lower marginal rank symmetric tensors" means symmetric tensors with marginal rank being smaller than n.

With the rank one decomposition (4.1), the eigenvalue equations of the tensor $\mathcal{A}$ become

$$\sum_{i=1}^{R} (\mathbf{a}_i^\mathsf{T}\mathbf{x})^{m-1}\mathbf{a}_i = \lambda \mathbf{x}^{[m-1]}. \tag{4.2}$$

A tensor $\mathcal{A}$ is called *essentially orthogonally decomposable*, if its marginal rank is R, i.e., the matrix A is of full column rank. Implicitly, we have $\mathrm{mrank}(\mathcal{A}) = R \le n$. It also follows from [3, Lemma 5.1] that the rank of this tensor is $\mathrm{mrank}(\mathcal{A})$. In this situation, since A consists of linear independent column vectors, the tensor $\mathcal{A}$ can be decomposed as $P \cdot \mathcal{B}$ with $P \in \mathbb{GL}(n, \mathbb{C})$ and $\mathcal{B}$ an *orthogonally decomposable tensor*, i.e., a tensor possessing a rank one decomposition (4.1) with A consisting of orthogonal columns. Orthogonally decomposable tensors have important applications in machine learning, see [25] and references herein.

Let $\mathrm{mrank}(\mathcal{A}) = s < n$. An immediate consequence of (4.2) is that the eigenvariety $V(0)$ of $\mathcal{A}$ contains the kernel of A^T. Hence, $\mathrm{gm}(0) \ge n - s$.

PROPOSITION 4.1 (Generic Geometric Multiplicity of the Zero Eigenvalue). *Let $s \le n$, and $\mathcal{A} \in \mathcal{S}^m(\mathbb{C}^n)$ with $\mathrm{mrank}(\mathcal{A}) \le s$ be generic. Then*

$$V(0) = \ker(A^\mathsf{T}),$$

and hence $\mathrm{gm}(0) = n - s$.

Proof. The case when $s = n$ is trivial. We show the other cases. Let $P \in \mathbb{O}(n, \mathbb{C})$ be a nonsingular matrix such that $P \cdot \mathcal{A}$ is in an upper block diagonal form $\mathcal{B}$ with the nonzero block being a tensor $\mathcal{C}$ in $\mathcal{S}^m(\mathbb{C}^s)$, i.e.,

$$b_{i_1\dots i_m} = \begin{cases} c_{i_1\dots i_m} & \text{if } i_1, \dots, i_m \in \{1, \dots, s\}, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$PA = \begin{bmatrix} B \\ \mathbf{0} \end{bmatrix} \tag{4.3}$$

with some $B \in \mathbb{C}^{s\times R}$. Let

$$P = [\mathbf{p}_1, \dots, \mathbf{p}_n]^\mathsf{T}.$$

We adopt the coordinate change $\mathbf{x} \mapsto P^\mathsf{T}\mathbf{y}$. Under this transformation, by (3.3), the equations

$$P\big[\mathcal{A}\mathbf{x}^{m-1}\big] = \mathbf{0}$$

become

$$P\left[\mathcal{A}(P^{\mathsf{T}}\mathbf{y})^{m-1}\right]=(P\cdot\mathcal{A})(\mathbf{y})^{m-1}=\mathcal{B}\mathbf{y}^{m-1}=\mathbf{0},$$

or explicitly as

$$\mathcal{C}\mathbf{z}^{m-1}=\mathbf{0} \tag{4.4}$$

with $\mathbf{y}=(\mathbf{z},\mathbf{w})$ and $\mathbf{z}\in\mathbb{C}^s$. We see that

$$\mathbf{x}\in V_{\mathcal{A}}(0) \text{ if and only if } P\mathbf{x}\in V_{\mathcal{B}}(0) \text{ if and only if } P\mathbf{x}=(\mathbf{z},\mathbf{w}) \text{ with } \mathbf{z}\in V_{\mathcal{C}}(0).$$

Note that the set of symmetric tensors with marginal rank t such that $t\leq s$ is a variety in the total space $\mathcal{S}^m(\mathbb{C}^n)$, which contains the set of symmetric tensors with marginal rank $t=s$ as a nonempty Zariski open subset. Moreover, it is invariant under the group action by $\mathbb{O}(n,\mathbb{C})$. For a generic tensor $\mathcal{A}$ in this variety, it has the maximal marginal rank s and

$$\operatorname{Det}(\mathcal{C})\neq 0,$$

since the tensor $\binom{\mathcal{I}_s}{\mathbf{0}}$ obviously has marginal rank s and $\operatorname{Det}(\mathcal{I}_s)=1$[2]. Here $\mathcal{I}_s$ is the identity tensor in $\mathcal{S}^m(\mathbb{C}^s)$.

Therefore, for a generic $\mathcal{A}\in\mathcal{S}^m(\mathbb{C}^n)$ of marginal rank s, the unique solution to (4.4) is $\mathbf{z}=\mathbf{0}$. So,

$$\mathbf{x}\in V_{\mathcal{A}}(0) \text{ if and only if } P\mathbf{x}=(\mathbf{0},\mathbf{w}) \text{ with } \mathbf{w}\in\mathbb{C}^{n-s}.$$

By the shape of PA (cf. (4.3)), it follows that

$$V_{\mathcal{A}}(0)=\ker(A^{\mathsf{T}}),$$

and the results follow. ∎

Notice that we use the term "generic tensors" in both the statement and the proof of Proposition 4.1. We would like to clarify such a term formally. The statement "a generic symmetric tensor with marginal rank at most s has property that $V(0)=\operatorname{Ker}(A^T)$" means that the subset of all symmetric tensors with marginal rank at most s such that $V(0)=\operatorname{Ker}(A^T)$ contains a dense (in Zariski topolgy) subset of the set of all symmetric tensors with marginal rank at most s.

The next theorem says that Conjecture 1.1 is true for the zero eigenvalue of generic symmetric tensors of a fixed marginal rank.

[2]Let X be the set of symmetric tensors with marginal rank t such that $t\leq s$ and $Y:=\{\mathcal{B}\in X: b_{i_1\ldots i_m}=c_{i_1\ldots i_m}$ if $i_1,\ldots,i_m\in\{1,\ldots,s\}$ and $b_{i_1\ldots i_m}=0$ otherwise$\}$, i.e., Y is the subset of X of tensors in block diagonal form with the nonzero block tensor being in $\mathcal{S}^m(\mathbb{C}^s)$. Let $f:\mathbb{O}(n,\mathbb{C})\times Y\to X$ be given as $f(P,\mathcal{B})=P\cdot\mathcal{B}$ for $P\in\mathbb{O}(n,\mathbb{C})$ and $\mathcal{B}\in Y$. It follows that $\operatorname{image}(f)=X$, and then $\overline{f(\mathbb{O}(n,\mathbb{C}),U)}=X$ for any Zariski open subset $U\subseteq Y$. Therefore, $f(\mathbb{O}(n,\mathbb{C}),U)$ contains a Zariski open subset $V\subseteq X$. It is easy to see that both the marginal rank and the geometric multiplicity of the zero eigenvalue (cf. Proposition 3.2) are invariants under the group action by $\mathbb{O}(n,\mathbb{C})$. Then the conclusion follows if we find a Zariski open subset $U\subseteq Y$ such that all points in U have the maximal marginal rank s and $\operatorname{Det}(\mathcal{C})\neq 0$. Since both the subsets of tensors of marginal rank s and $\operatorname{Det}(\mathcal{C})\neq 0$ are Zariski open in Y (notice that the determinant is a polynomial so it is a continuous function in the Zariski topology), we only need to show that the two subsets have a nonempty intersection, which boils down to finding a point, which both has marginal rank s and nonzero determinant.

Theorem 4.2 (Lower Marginal Rank Symmetric Tensors). *Let* $\mathcal{A}\in\mathcal{S}^m(\mathbb{C}^n)$ *and* $\mathrm{mrank}(\mathcal{A})=s\le n$. *Then, the number* $\mathrm{nnz}(\mathcal{A})$ *of nonzero eigenvalues of* $\mathcal{A}$ *satisfies*

$$\mathrm{nnz}(\mathcal{A})\le s(m-1)^{n-1} \tag{4.5}$$

with equality holding for generic tensors of marginal rank $\mathrm{mrank}(\mathcal{A})=s$. *Therefore, we have that generically*

$$\mathrm{am}(0)\ge(n-s)(m-1)^{n-1}\ge\mathrm{gm}(0)(m-1)^{\mathrm{gm}(0)-1},$$

and a generic tensor of marginal rank s *has* $\mathrm{am}(0)=(n-s)(m-1)^{n-1}$.

Proof. Similarly as in the proof for Proposition 4.1, let $P\in\mathbb{O}(n,\mathbb{C})$ be a nonsingular matrix such that $P\cdot\mathcal{A}$ is in an upper block diagonal form $\mathcal{B}$ with the nonzero block being a tensor $\mathcal{C}$ in $\mathcal{S}^m(\mathbb{C}^s)$, and

$$P=[\mathbf{p}_1,\dots,\mathbf{p}_n]^{\mathsf{T}}.$$

Under the coordinate change $\mathbf{x}\mapsto P^{\mathsf{T}}\mathbf{y}$, the equations

$$P\left[\mathcal{A}\mathbf{x}^{m-1}-\lambda\mathbf{x}^{[m-1]}\right]=\mathbf{0}$$

become (cf. (3.3))

$$P\left[\mathcal{A}(P^{\mathsf{T}}\mathbf{y})^{m-1}-\lambda(P^{\mathsf{T}}\mathbf{y})^{[m-1]}\right]=\mathcal{B}\mathbf{y}^{m-1}-\lambda P\left[(P^{\mathsf{T}}\mathbf{y})^{[m-1]}\right]=\mathbf{0}.$$

Let $\mathcal{D}\in\mathcal{S}^m(\mathbb{C}^n)$ be the symmetric tensor associated to $P\left[(P^{\mathsf{T}}\mathbf{y})^{[m-1]}\right]$, and be partitioned as

$$\mathcal{D}=\begin{bmatrix}\mathcal{D}_1\\ \mathcal{D}_2\end{bmatrix}$$

with $\mathcal{D}_1$ corresponding to the first s slices of $\mathcal{D}$ and $\mathcal{D}_2$ the others. It follows from (3.3) that $\mathcal{D}=P\cdot\mathcal{I}$. Then,

$$\begin{aligned}\mathrm{Det}(P)^{m(m-1)^{n-1}}\mathrm{Det}(\lambda\mathcal{I}-\mathcal{A})&=\mathrm{Det}\left(P\cdot(\lambda\mathcal{I}-\mathcal{A})\right)\\&=\mathrm{Det}(\lambda\mathcal{D}-\mathcal{B})\\&=\mathrm{Det}\left(\begin{bmatrix}\lambda\mathcal{D}_1-\mathcal{C}\\ \lambda\mathcal{D}_2\end{bmatrix}\right)\\&=\lambda^{(n-s)(m-1)^{n-1}}\mathrm{Det}\left(\begin{bmatrix}\lambda\mathcal{D}_1-\mathcal{C}\\ \mathcal{D}_2\end{bmatrix}\right)\end{aligned}$$

where the equalities follow from [5, Theorems 3.3.5 and 3.3.3.1]. Therefore,

$$\mathrm{Det}(\lambda\mathcal{I}-\mathcal{A})$$

has a factor of λ^t with $t\ge(n-s)(m-1)^{n-1}$, since $\mathrm{Det}(P)\neq0$. Henceforth, we have the bound for the number of nonzero eigenvalues (4.5).

Since $\mathrm{Det}(\lambda\mathcal{I}-\mathcal{A})$ is a monic polynomial in λ of degree $n(m-1)^{n-1}$ with constant term being $\mathrm{Det}(-\mathcal{A})$, we have that

$$\mathrm{Det}\left(\begin{bmatrix}\lambda\mathcal{D}_1-\mathcal{C}\\ \mathcal{D}_2\end{bmatrix}\right)=\mathrm{Det}(P)^{-m(m-1)^{n-1}}\lambda^{s(m-1)^{n-1}}+\text{lower order terms}.$$

Moreover, $\mathrm{nnz}(\mathcal{A})$ attains the upper bound if and only if

$$\mathrm{Det}\left(\begin{bmatrix}\mathcal{C}\\ \mathcal{D}_2\end{bmatrix}\right)\neq 0. \tag{4.6}$$

Note that for a generic $\mathcal{A}$, $\mathrm{Det}(\mathcal{C})\neq 0$ (since $\mathcal{I}_s\in\mathcal{S}^m(\mathbb{C}^s)$ has determinant 1), therefore generically

$$\mathrm{Det}\left(\begin{bmatrix}\mathcal{C}\\ \mathcal{D}_2\end{bmatrix}\right)=0$$

if and only if

$$\mathcal{D}_2\mathbf{y}^{m-1}=\mathbf{0}$$

has a nonzero solution in $\{\mathbf{y}\in\mathbb{C}^n : y_1=\cdots=y_s=0\}$. Let P^T be partitioned accordingly as

$$P^\mathsf{T}=\begin{bmatrix}P_l^\mathsf{T} & P_r^\mathsf{T}\end{bmatrix}.$$

Then

$$\mathcal{D}_2\mathbf{y}^{m-1}=\mathbf{0}$$

has a nonzero solution in $\{\mathbf{y}\in\mathbb{C}^n : y_1=\cdots=y_s=0\}$ if and only if

$$P_r(P_r^\mathsf{T}\mathbf{w})^{[m-1]}=\mathbf{0} \tag{4.7}$$

has a nonzero solution in $\mathbb{C}^{n-s}$. Since P_r^T has full rank, we can assume without loss of generality that the block consisting of the first $n-s$ rows of P_r^T has full rank. This matrix is denoted by P_0. Then,

$$P_r^\mathsf{T}=\begin{bmatrix}P_0\\ P_1\end{bmatrix}.$$

Let $\mathbf{z}=P_0\mathbf{w}\in\mathbb{C}^{n-s}$. It follows that (4.7) is equivalent to

$$[P_0^\mathsf{T},P_1^\mathsf{T}]\big((\mathbf{z}^{[m-1]})^\mathsf{T},((P_1P_0^{-1}\mathbf{z})^{[m-1]})^\mathsf{T}\big)^\mathsf{T}=\mathbf{0}$$

and furthermore

$$[I,(P_0^{-1})^\mathsf{T}P_1^\mathsf{T}]\big((\mathbf{z}^{[m-1]})^\mathsf{T},((P_1P_0^{-1}\mathbf{z})^{[m-1]})^\mathsf{T}\big)^\mathsf{T}=\mathbf{0} \tag{4.8}$$

has a nonzero solution $\mathbf{z}$ in $\mathbb{C}^{n-s}$. Let $M=P_1P_0^{-1}$. Then, we have that (4.8) can be further reformulated as

$$\mathbf{z}^{[m-1]}+M^\mathsf{T}(M\mathbf{z})^{[m-1]}=\mathbf{0}.$$

By (3.3), it is

$$\big(\mathcal{I}+M^\mathsf{T}\cdot\mathcal{I}\big)\mathbf{z}^{m-1}=\mathbf{0}.$$

For a generic P, we show that (4.7) does not have a nontrivial solution in $\mathbb{C}^{n-s}$. To this end, it sufficient to find a P such that (4.7) does not have a nontrivial solution[3], since

[3]Let $X\subset\mathbb{C}^n$ be a variety and $Y\subset\mathbb{C}^n$ be an open subset. Then, whenever $Y\cap Z\neq\emptyset$ for a nonempty open subset $Z\subseteq X$, $Y\cap Z$ is a nonempty open subset in the induced Zariski topology on the variety X, which is a dense subset of X.

$n-s$ homogeneous polynomials of degree $m-1$ in $n-s$ variables which do not have a nontrivial solution form a Zariski open subset. Let $P=[\mathbf{e}_n,\dots,\mathbf{e}_1]$ with $\mathbf{e}_i\in\mathbb{C}^n$ the ith standard basis vector. Then, $P_0=I\in\mathbb{C}^{(n-s)\times(n-s)}$ and $P_1=\mathbf{0}\in\mathbb{C}^{s\times(n-s)}$. For this P, the above reformulation results in

$$\mathcal{I}\mathbf{z}^{m-1}=\mathbf{0},$$

since $M=P_1P_0^{-1}=\mathbf{0}$. Therefore, (4.7) does not have a nontrivial solution generically, and hence (4.6) holds generically. ∎

The next proposition is in deterministic form.

PROPOSITION 4.3 (Essentially Orthogonally Decomposable Tensors). *Let $\mathcal{A}\in\mathcal{S}^m(\mathbb{C}^n)$ be such that the matrix A from* (4.1) *has full column rank, i.e., $R=\operatorname{mrank}(\mathcal{A})=s\le n$. Then,*

$$\operatorname{gm}(0)=n-s,$$

and therefore

$$\operatorname{am}(0)\ge(n-s)(m-1)^{n-1}\ge\operatorname{gm}(0)(m-1)^{\operatorname{gm}(0)-1}.$$

Proof. First we recall the definition of $\mathbf{d}$:

$$\mathbf{d}=\big((\mathbf{a}_1^{\mathsf{T}}\mathbf{x})^{m-1},\dots,(\mathbf{a}_R^{\mathsf{T}}\mathbf{x})^{m-1}\big)^{\mathsf{T}}.$$

Sine the matrix A is of full column rank, it follows from (4.1) and (4.2) that $\mathbf{x}\in V(0)$ if and only if

$$\mathbf{d}=\mathbf{0}$$

if and only if

$$A^{\mathsf{T}}\mathbf{x}=\mathbf{0}.$$

Therefore, $V(0)$ is the kernel of the matrix A^{T}, which is a linear subspace of dimension $n-s$. ∎

5. Coordinate linear subspace as eigenvectors

In this section, we generalize the relation between the two multiplicities for matrices to tensors. As we saw from Section 3.1, the general relation for matrices follows from the case when the eigenspace is a linear subspace in coordinate form and the fact that both multiplicities are invariants under the orthogonal linear group action. Since the algebraic multiplicity for a tensor is not an invariant any more (cf. Proposition 3.1), we show the coordinate case. To this end, properties on quasi-triangular tensors and symmetrizations of tensors are necessary.

5.1. Quasi-triangular tensors.

DEFINITION 5.1 (Sub-Tensor). *Let $\mathcal{T}\in\mathbb{T}(\mathbb{C}^n,m)$ and $1\le k\le n$. Tensor $\mathcal{U}\in\mathbb{T}(\mathbb{C}^k,m)$ is called a sub-tensor of $\mathcal{T}$ associated to the index set $\{j_1,\dots,j_k\}\subseteq\{1,\dots,n\}$ if and only if $u_{i_1\dots i_m}=t_{j_{i_1}\dots j_{i_m}}$ for all $i_1,\dots,i_m\in\{1,\dots,k\}$.*

PROPOSITION 5.2 (Quasi-Triangular Tensor). *Let tensor $\mathcal{T}=(t_{ii_2\dots i_m})\in\mathbb{T}(\mathbb{C}^n,m)$. Suppose that there exists some $k\in\{1,\dots,n\}$ such that $\sum_{(i_2,\dots,i_m)\in\mathbb{X}(\alpha)}t_{ii_2\dots i_m}=0$ for all*

$i>k$ and $\alpha\in\{0,\dots,m-1\}^k\times\{0\}^{n-k}$. Let $\mathcal{U}\in\mathbb{T}(\mathbb{C}^k,m)$ be the sub-tensor of $\mathcal{T}$ associated to $\{1,\dots,k\}$. Then,

$$\mathrm{DET}(\mathcal{T})=\mathrm{DET}(\mathcal{U})p(\mathcal{T})$$

for some polynomial $p\in\mathbb{C}[\mathcal{T}]$.

Proof. Let $\mathcal{T}=(t_{ii_2\dots i_m})\in\mathbb{T}(\mathbb{C}^n,m)$ be a tensor such that the hypothesis are satisfied and $\mathrm{Det}(\mathcal{U})=0$ for the sub-tensor $\mathcal{U}$. Then, by Proposition 2.4, we see that there is a vector $\mathbf{y}\in\mathbb{C}^k\setminus\{\mathbf{0}\}$ such that

$$\mathcal{U}\mathbf{y}^{m-1}=\mathbf{0}.$$

Define a vector $\mathbf{x}\in\mathbb{C}^n$ as $\mathbf{x}=(\mathbf{y}^\mathsf{T},\mathbf{0}^\mathsf{T})^\mathsf{T}$. Then $\mathbf{x}\neq\mathbf{0}$, and by the assumption on $\mathcal{T}$

$$\mathcal{T}\mathbf{x}^{m-1}=\left((\mathcal{U}\mathbf{y}^{m-1})^\mathsf{T},\mathbf{0}^\mathsf{T}\right)^\mathsf{T}=\mathbf{0}.$$

Therefore, by Proposition 2.4 again, $\mathrm{Det}(\mathcal{T})=0$.

The set of tensors satisfying the hypothesis is a linear subspace $\mathbb{L}$ of the total space $\mathbb{T}(\mathbb{C}^n,m)$. Let $\mathbb{M}$ be the set of sub-tensors of elements of $\mathbb{L}$ associated to $\{1,\dots,k\}$. There is a projection

$$\delta:\mathbb{T}(\mathbb{C}^n,m)\to\mathbb{T}(\mathbb{C}^k,m)$$

sending $\mathcal{T}\in\mathbb{T}(\mathbb{C}^n,m)$ to its sub-tensor $\mathcal{U}\in\mathbb{T}(\mathbb{C}^k,m)$ associated to $\{1,\dots,k\}$. In particular, δ maps $\mathbb{L}$ onto $\mathbb{M}$. Let $\delta^*:\mathbb{C}[\mathcal{U}]\to\mathbb{C}[\mathcal{T}]$ be the induced homomorphism. It is easy to see that $\mathbb{M}$ is in fact simply $\mathbb{T}(\mathbb{C}^k,m)$.

Hence we have

$$\mathbb{L}\cap\mathbb{V}(\delta^*\,\mathrm{DET}(\mathcal{U}))\subseteq\mathbb{V}(\mathrm{DET}(\mathcal{T})).$$

This implies that on the linear space $\mathbb{L}$ we have

$$\mathbb{V}(\delta^*\,\mathrm{DET}(\mathcal{U}))\subseteq\mathbb{V}(\mathrm{DET}(\mathcal{T})).$$

It is not hard to see that $\delta^*\,\mathrm{DET}(\mathcal{U})$ is an irreducible polynomial even on $\mathbb{L}$, since $\mathbb{L}$ is the linear subspace consisting of quasi-triangular tensors. Therefore by Nullstellensatz [5] we conclude that

$$\mathrm{DET}(\mathcal{T})=\mathrm{DET}(\mathcal{U})p(\mathcal{T})$$

for some polynomial $p\in\mathbb{C}[\mathcal{T}]$. ∎

Tensors in $\mathbb{T}(\mathbb{C}^n,m)$ satisfying the hypothesis in Proposition 5.2 are *quasi-triangular tensors*. It is a class of structured tensors broader than the class of upper triangular tensors introduced in [11]. We note that both notions reduce to upper triangularity in the matrix counterpart.

5.2. Symmetrization. The topic of this article is eigenvalues of tensors, which are closely and solely related to $\mathcal{T}\mathbf{x}^{m-1}$. It is easy to see that the ith slice $\mathcal{T}_i:=(t_{ii_2\dots i_m})_{1\le i_2,\dots,i_m\le n}$ of $\mathcal{T}$ is an $(m-1)$th order n-dimensional tensor, and for all $i=1,\dots,n$

$$\left(\mathcal{T}\mathbf{x}^{m-1}\right)_i=\langle\mathrm{Sym}(\mathcal{T}_i),\mathbf{x}^{\otimes(m-1)}\rangle:=\sum_{i_2,\dots,i_m=1}^{n}\left(\mathrm{Sym}(\mathcal{T}_i)\right)_{i_2\dots i_m}x_{i_2}\dots x_{i_m}\text{ for all }\mathbf{x}\in\mathbb{C}^n,$$

where $\mathrm{Sym}(\mathcal{T}_i)$ is the *symmetrization* of the tensor $\mathcal{T}_i$ as a symmetric tensor in the sense of the above equalities. Therefore, for every tensor $\mathcal{T}\in\mathbb{T}(\mathbb{C}^n,m)$, we associate it an element $\mathrm{eSym}(\mathcal{T})$ in $\mathbb{TS}(\mathbb{C}^n,m):=\mathbb{C}^n\otimes\mathcal{S}^{m-1}(\mathbb{C}^n)$ by symmetrizing its slices. Hence,

$$\mathcal{T}\mathbf{x}^{m-1}=\mathrm{eSym}(\mathcal{T})\mathbf{x}^{m-1} \text{ for all } \mathbf{x}\in\mathbb{C}^n.$$

We see that all tensors in the fibre of the above map have the same defining equations for the eigenvalue problem.

Proposition 5.3 (Symmetrization). *Let $\mathcal{T}\in\mathbb{T}(\mathbb{C}^n,m)$. Then,*

$$\mathrm{Det}(\mathcal{T}-\lambda\mathcal{I})=\mathrm{Det}(\mathrm{eSym}(\mathcal{T})-\lambda\mathcal{I}).$$

Proof. Suppose that $\mathcal{T}\in\mathbb{T}(\mathbb{C}^n,m)$ with symbolic entries, then we have that

$$\mathrm{Det}(\mathcal{T})=\mathrm{DET}.$$

By Definition 2.2, we have

$$\mathrm{Det}(\mathcal{T})=\mathrm{Det}(\mathrm{eSym}(\mathcal{T})).$$

Henceforth, the result follows from the fact that $\mathrm{eSym}(\mathcal{I})=\mathcal{I}$. ∎

Proposition 5.4 (Correspondence). *There is a one to one correspondence between the tensor space $\mathbb{TS}(\mathbb{C}^n,m)$ and the set of systems of n homogeneous polynomials in n variables of degree $m-1$.*

Proof. The correspondence is indicated by $\mathcal{T}\mathbf{x}^{m-1}$ for $\mathcal{T}\in\mathbb{TS}(\mathbb{C}^n,m)$. ∎

5.3. The coordinate case. The eigenvariety can determine the tensor in certain cases.

Proposition 5.5 (Uniqueness). *Let tensor $\mathcal{T}=(t_{ii_2\dots i_m})\in\mathbb{TS}(\mathbb{C}^n,m)$. If $V(\lambda)=\mathbb{C}^n$ for some λ, then*

$$\mathcal{T}=\lambda\mathcal{I}.$$

Proof. Note that $V(\lambda)=\mathbb{C}^n$ is equivalent to say that the polynomial system

$$(\mathcal{T}-\lambda\mathcal{I})\mathbf{x}^{m-1}\equiv\mathbf{0}.$$

Therefore, every polynomial $\big((\mathcal{T}-\lambda\mathcal{I})\mathbf{x}^{m-1}\big)_i\equiv 0$. By Proposition 5.4 and Hilbert's Nullstellensatz [5], we have that all the coefficients of this polynomial are zero. Henceforth, we must have $\mathcal{T}=\lambda\mathcal{I}$. ∎

The next theorem is the main theorem in this section.

Theorem 5.6 (Coordinate Case). *Let tensor $\mathcal{T}\in\mathbb{T}(\mathbb{C}^n,m)$. If for some $\lambda\in\sigma(\mathcal{T})$, $V(\lambda)\supseteq P\{\mathbf{x}\in\mathbb{C}^n: x_{\mathrm{gm}(\lambda)+1}=\cdots=x_n=0\}$ for some permutation matrix $P\in\mathbb{O}(\mathbb{C},n)$, then we have that*

$$\mathrm{am}(\lambda)\geq\mathrm{gm}(\lambda)(m-1)^{\mathrm{gm}(\lambda)-1}.$$

Proof. Let $P\in\mathbb{O}(n,\mathbb{C})$ be a permutation matrix. It is easy to see that

$$P\cdot\mathcal{I}=\mathcal{I}.$$

Therefore,

$$\operatorname{Det}(\lambda\mathcal{I}-P\cdot\mathcal{A})=\operatorname{Det}(P\cdot(\lambda\mathcal{I}-\mathcal{A}))=\operatorname{Det}(\lambda\mathcal{I}-\mathcal{A}).$$

Henceforth, we can assume that $P=I$, the identity matrix, without loss of any generality.

Let $\mathcal{B}=\operatorname{eSym}(\mathcal{T})$ be the tensor in $\mathbb{TS}(\mathbb{C}^n,m)$ corresponding to $\mathcal{T}$. Since $\mathcal{B}$ and $\mathcal{T}$ have the same defining equations for the eigenvalue problem, they have the same eigenvalues and eigenvectors. Let $\mathcal{U}\in\mathbb{T}(\mathbb{C}^{\operatorname{gm}(\lambda)},m)$ and $\mathcal{C}\in\mathbb{TS}(\mathbb{C}^{\operatorname{gm}(\lambda)},m)$ be the sub-tensors of $\mathcal{T}$ and $\mathcal{B}$ associated to the index set $\{1,\dots,\operatorname{gm}(\lambda)\}$ respectively (cf. Definition 5.1), e.g.,

$$u_{i_1\dots i_m}=t_{i_1\dots i_m}\ \text{for all}\ i_1,\dots,i_m\in\{1,\dots,\operatorname{gm}(\lambda)\}.$$

It is easy to see that $\mathcal{C}=\operatorname{eSym}(\mathcal{U})$.

By the hypothesis,

$$\mathcal{B}\mathbf{x}^{m-1}=\lambda\mathbf{x}^{[m-1]}\ \text{for all}\ \mathbf{x}\in\{\mathbf{x}\in\mathbb{C}^n:x_{\operatorname{gm}(\lambda)+1}=\cdots=x_n=0\}.$$

By a direct calculation, we see that

$$\mathcal{C}\mathbf{y}^{m-1}=\lambda\mathbf{y}^{[m-1]}\ \text{for all}\ \mathbf{y}\in\mathbb{C}^{\operatorname{gm}(\lambda)}.$$

Hence, $\mathcal{C}$ has an eigenvalue λ with the eigenvectors being the whole $\mathbb{C}^{\operatorname{gm}(\lambda)}\setminus\{\mathbf{0}\}$. Therefore, it follows from Proposition 5.5 that

$$\mathcal{C}=\lambda\mathcal{I}.$$

It follows from Propositions 3.3 and 5.3 that

$$\operatorname{Det}(\mathcal{U}-\mu\mathcal{I})=\operatorname{Det}(\mathcal{C}-\mu\mathcal{I})=(\lambda-\mu)^{\operatorname{gm}(\lambda)(m-1)^{\operatorname{gm}(\lambda)-1}}.$$

Likewise, by the hypothesis, we have that for all $i>\operatorname{gm}(\lambda)$,

$$(\mathcal{B}\mathbf{x}^{m-1})_i=\sum_{i_2,\dots,i_m}b_{ii_2\dots i_m}x_{i_2}\dots x_{i_m}=\lambda x_i^{m-1}=0$$
$$\text{for all}\ \mathbf{x}\in\{\mathbf{x}\in\mathbb{C}^n:x_{\operatorname{gm}(\lambda)+1}=\cdots=x_n=0\}.$$

Then, it follows from Proposition 5.4 that

$$b_{ii_2\dots i_m}=0\ \text{for all}\ i>\operatorname{gm}(\lambda),\ i_2,\dots,i_m\in\{1,\dots,\operatorname{gm}(\lambda)\}.$$

This is the same to say that $\sum_{(i_2,\dots,i_m)\in\mathbb{X}(\alpha)}t_{ii_2\dots i_m}=0$ for all $i>\operatorname{gm}(\lambda)$ and $\alpha\in\{0,\dots,m-1\}^{\operatorname{gm}(\lambda)}\times\{0\}^{n-\operatorname{gm}(\lambda)}$. In other words, both $\mathcal{B}$ and $\mathcal{T}$ are quasi-triangular and the associated index set is $\{1,\dots,\operatorname{gm}(\lambda)\}$.

Therefore, it is easy to see that $\mathcal{T}-\mu\mathcal{I}$ satisfies the hypothesis in Proposition 5.2 with $k=\operatorname{gm}(\lambda)$. Henceforth,

$$\operatorname{Det}(\mathcal{T}-\mu\mathcal{I})=\operatorname{Det}(\mathcal{U}-\mu\mathcal{I})p(\mathcal{T}-\mu\mathcal{I})=(\lambda-\mu)^{\operatorname{gm}(\lambda)(m-1)^{\operatorname{gm}(\lambda)-1}}p(\mathcal{T}-\mu\mathcal{I})$$

for some polynomial p. So, λ is an eigenvalue of $\mathcal{T}$ with algebraic multiplicity at least $\operatorname{gm}(\lambda)(m-1)^{\operatorname{gm}(\lambda)-1}$. The conclusion then follows. ∎

6. Generic tensors

In this section, we consider generic tensors.

LEMMA 6.1 (Generic Tensor Eigenvalue). *Let tensor $\mathcal{T}\in\mathbb{T}(\mathbb{C}^n,m)$ be generic. Then,*

$$\mathrm{am}(\lambda)=1$$

for all $\lambda\in\sigma(\mathcal{T})$.

Proof. The polynomial $\mathrm{Det}(\lambda\mathcal{I}-\mathcal{T})$ is monic and irreducible when we regard $\mathcal{T}$ symbolically [11]. Therefore, for a generic tensor $\mathcal{T}$, each root of the characteristic polynomial is algebraically simple. It then follows from Proposition 2.4 that for generic tensor $\mathcal{T}$, every eigenvalue has algebraic multiplicity one. □

We remark here again for the use of the term "generic tensor". The statement "a generic tensor has algebraic multiplicity one for its every eigenvalue" means that the set consisting of all tensors of algebraic multiplicity one for its every eigenvalue contains a dense (in Zariski topology) subset of $\mathbb{T}(\mathbb{C}^n,m)$. This remark should automatically apply to other use of the term "generic tensor" in this section.

LEMMA 6.2 (Generic tensor eigenvector). *Let tensor $\mathcal{T}\in\mathbb{T}(\mathbb{C}^n,m)$ be generic. Then,*

$$\mathrm{gm}(\lambda)=1$$

for all $\lambda\in\sigma(\mathcal{T})$.

Proof. Given a polynomial system of n equations $p_1,\dots,p_n$ of degree m in $(n+1)$ variables $x_1,\dots,x_n,\lambda$, the dimension of the variety

$$V:=\{(\mathbf{x},\lambda)\in\mathbb{C}^{n+1}:p_1(\mathbf{x},\lambda)=\cdots=p_n(\mathbf{x},\lambda)=0\}$$

is equal to (cf. [27])

$$n+1-\mathrm{rank}_V(D\mathbf{P}),$$

where $\mathbf{P}=(p_1,\dots,p_n)^{\mathsf{T}}:\mathbb{C}^{n+1}\to\mathbb{C}^n$, $D\mathbf{P}$ being the Jacobian map of $\mathbf{P}$, and

$$\mathrm{rank}_V(D\mathbf{P}):=\max_{(\mathbf{x},\lambda)\in V}\mathrm{rank}(D\mathbf{P})(\mathbf{x},\lambda).$$

It is known that for a generic $\mathbf{P}$ (over the space of all polynomial systems of n equations with degree m in $(n+1)$ variables),

$$\dim(V)=1.$$

Since $D\mathbf{P}:\mathbb{C}^{n+1}\to\mathbb{C}^{n\times(n+1)}$, it follows that $\dim(V)=1$ if and only if there is a point $(\mathbf{x},\lambda)\in V$ such that a maximal minor of $D\mathbf{P}(\mathbf{x},\lambda)$ is nonzero. Therefore, by the continuity of V depending on $\mathbf{P}$ and the fact that the full rank matrices in $\mathbb{C}^{n\times(n+1)}$ form a Zariski open subset, we conclude that the set of all polynomial systems possessing $\dim(V)=1$ is a Zariski open subset.

Given a tensor $\mathcal{T}\in\mathbb{T}(\mathbb{C}^n,m)$, we consider the eigenvalue equation system

$$\mathcal{T}x^{m-1}=\lambda x^{[m-1]}.$$

All such systems generated by tensors in $\mathbb{T}(\mathbb{C}^n,m)$ form a subvariety of the space of polynomial systems of n equations of degree m in $(n+1)$ variables $x_1,\dots,x_n,\lambda$. Let Z be the set of all solutions to the system. Namely,

$$Z:=\{(x_1,\dots,x_n,\lambda)\,|\,\mathcal{T}x^{m-1}=\lambda x^{[m-1]}\}.$$

If we can find a tensor $\mathcal{T}$ such that the corresponding Z has dimension one, then we can conclude that a generic tensor will have $\dim(Z)=1$. For this, we consider the tensor $\mathcal{T}$ such that the eigenvalue equation $\mathcal{T}x^{m-1}=\lambda x^{[m-1]}$ is simply

$$ix_i^{m-1}=\lambda x_i^{m-1}, i=1,2,\dots,n.$$

It is easy to solve this system, and the set of solutions are

$$\{(t,0,0,\dots,0,1):t\in\mathbb{C}\}\cup\{(0,t,0,\dots,0,2):t\in\mathbb{C}\}\cup\cdots\cup\{(0,0,0,\dots,t,n):t\in\mathbb{C}\}.$$

Hence Z is a union of n curves and $\dim(Z)=1$. Therefore, given a generic tensor $\mathcal{T}$, Z is one dimensional, i.e., a curve.

By Proposition 2.6, we see that Z is a disjoint union of eigenvarieties, i.e.,

$$Z=\sqcup_{\lambda\in\sigma(\mathcal{T})}V(\lambda).$$

Since $\sigma(\mathcal{T})$ is a finite set, we conclude that each $V(\lambda)$ has dimension one for a generic tensor. ∎

To prove the next result about eigenvectors of a generic tensor, we need the following proposition.

PROPOSITION 6.3 (Shape Lemma, [26, 29]). *Let I be a zero-dimensional radical ideal in $\mathbb{C}[x_1,\dots,x_n]$ such that all d complex roots of I have distinct x_n-coordinates. Then the reduced Gröbner basis of I in the lexicographic term order has the shape*

$$\mathcal{G}=\{x_1-q_1(x_n),x_2-q_2(x_n),\dots,x_{n-1}-q_{n-1}(x_n),r(x_n)\}$$

where r is a polynomial of degree d and the q_i are polynomials of degree $\le d-1$.

LEMMA 6.4 (Unique Eigenvector). *Let tensor $\mathcal{T}\in\mathbb{T}(\mathbb{C}^n,m)$ be generic. Then $V(\lambda)$ has dimension one and is irreducible for all $\lambda\in\sigma(\mathcal{T})$, i.e., $\mathcal{T}$ has a unique (up to scaling) eigenvector for every $\lambda\in\sigma(\mathcal{T})$.*

Proof. Let

$$I=\left\langle f_i(\mathbf{x},\lambda):=(\mathcal{T}\mathbf{x}^{m-1})_i-\lambda x_i^{m-1},\ i=1,\dots,n\right\rangle$$

be the ideal generated by the eigenvalue-eigenvector polynomials. In the following, we will switch between the homogeneous and non-homogeneous setting. Generically, the homogeneous ideal $I\subset\mathbb{C}[x_1,\dots,x_n,\lambda]$ is 1-dimensional. We first dehomogenize it so it has dimension zero to use the shape lemma for the uniqueness. When getting back to the homogeneous setting, we get the uniqueness up to scaling as desired.

Define $t_i:=\frac{x_i}{x_n}$ for $i=1,\dots,n-1$. Let

$$g_i(\mathbf{t},\lambda)=f_i((\mathbf{t},1),\lambda),\text{ for all } i=1,\dots,n,$$

and

$$\mathbb{V}_0=\mathbb{V}(g_1,\dots,g_n).$$

Then, $\mathbb{V}_0$ is the intersection of $\mathbb{V}(I)$ and $\{(\mathbf{x},\lambda):x_n=1\}$.

Since $\mathcal{T}$ is generic, $\mathbb{V}(I)$ has dimension one and $\sigma(\mathcal{T})$ is of cardinality $n(m-1)^{n-1}$, i.e., all the eigenvalues are distinct (cf. Lemma 6.1). Therefore, $\mathbb{V}_0$ is zero-dimensional and has distinct λ-coordinate (Notice that a point in $\mathbb{V}_0$ has coordinate $(x_1,\dots,x_n,\lambda)$).

Since we only care about $\mathbb{V}_0$ and $\mathbb{V}(I)=\mathbb{V}(\sqrt{I})$ for any ideal I, we may assume that the ideal $(g_1,\dots,g_n)$ is a radical ideal (If $(g_1,\dots,g_n)$ is not radical then we just replace $(g_1,\dots,g_n)$ by its radical). Henceforth, by Proposition 6.3 we know that the reduced Gröbner basis of $\langle g_1,\dots,g_n\rangle$ in the lexicographic term order has the following shape

$$\mathcal{G}=\{t_1-q_1(\lambda),\dots,t_{n-1}-q_{n-1}(\lambda),q_n(\lambda)\}.$$

Therefore, for each fixed λ, there is a unique point $(\mathbf{t},\lambda)$ in V_0 as $\mathbf{t}$ is uniquely determined. By Proposition 2.6, different λ should have different $\mathbf{t}$'s. We then have $\#(\mathbb{V}_0)=\deg(q_n)$.

When $\mathcal{T}$ is generic, $\mathbb{V}(I)$ is a union of finite irreducible components of dimension one and many singletons which will be cut off by $\mathbb{V}_0$, and then we have that $\#(\mathbb{V}_0)=\deg(q_n)=n(m-1)^{n-1}$ and $q_n(\lambda)=\chi(\lambda)$, the characteristic polynomial of $\mathcal{T}$. Therefore, $V(\lambda)$ has dimension one and is irreducible for all $\lambda\in\sigma(\mathcal{T})$, i.e., it has a unique (up to scaling) eigenvector. ∎

In conclusion, we have the next theorem.

THEOREM 6.5 (Generic Tensor). *Let tensor $\mathcal{T}\in\mathbb{T}(\mathbb{C}^n,m)$ be generic. Then,*

$$\mathrm{am}(\lambda)=\mathrm{gm}(\lambda)=1$$

for all $\lambda\in\sigma(\mathcal{T})$.

Note that, in the generic case, we then always have the relation

$$\mathrm{am}(\lambda)=\mathrm{gm}(\lambda)(m-1)^{\mathrm{gm}(\lambda)-1}.$$

7. Duality

The set of symmetric tensors of order m and dimension n forms a strictly smaller subspace of the space of tensors of order m and dimension n. Therefore, results in Section 6 do not apply directly to generic symmetric tensors which will be considered in this section.

We consider the natural duality theory associated to eigenvalue theory in this section. We work in the space $\mathcal{S}^m(\mathbb{C}^n)$ of mth order and n-dimensional symmetric tensors.

For the eigenvalues of symmetric tensors, we have a geometric interpretation in terms of the duality between the determinantal hypersurface and the Veronese variety. We refer to [7, 9, 15] for basic definitions.

The mth Veronese map ν_m from $\mathbb{C}^n$ to $\mathcal{S}^m(\mathbb{C}^n)$ is

$$\nu_m(\mathbf{x})=\mathbf{x}^{\otimes m}\text{ for all }\mathbf{x}\in\mathbb{C}^n.$$

The image of ν_m over $\mathbb{C}^n$ is the *Veronese variety* $\nu_m(\mathbb{C}^n)$. This map is defined on $\mathbb{PC}^{n-1}$ naturally, and the image lies in the projective space $\mathbb{P}\mathcal{S}^m(\mathbb{C}^n)$. The corresponding variety $\nu_m(\mathbb{PC}^{n-1})$ is a smooth, non-degenerate, homogeneous and irreducible variety of dimension $n-1$ [15]. We know that the tangent space of $\nu_m(\mathbb{PC}^{n-1})$ at a point $[\mathbf{x}^{\otimes m}]$[4] is

$$T_{[\mathbf{x}^{\otimes m}]}\nu_m(\mathbb{PC}^{n-1})=\mathbb{P}\{\mathrm{Sym}(\mathbf{x}^{\otimes m-1}\otimes\mathbf{y}):\mathbf{y}\in\mathbb{C}^n\}.$$

If we identify the dual of $\mathbb{P}\mathcal{S}^m(\mathbb{C}^n)$ as itself, and a point $H\in\mathbb{P}\mathcal{S}^m(\mathbb{C}^n)$ as the hyperplane it defines in the tutorial way, the dual variety of $\nu_m(\mathbb{PC}^{n-1})$ is

$$\nu_m(\mathbb{PC}^{n-1})^\vee:=\{H\in\mathbb{P}\mathcal{S}^m(\mathbb{C}^n):T_{[\mathbf{x}^{\otimes m}]}\nu_m(\mathbb{PC}^{n-1})\subset H\text{ for some }[\mathbf{x}^{\otimes m}]\}.$$

[4]For a variety $X\in\mathbb{P}V$ over the projective space of a linear space V, we write $[\mathbf{y}]$ for the equivalent class of the point $\mathbf{y}$ in the affine cone of X.

Let SDET be the restriction of the determinant DET on the symmetric tensor space. It is called *symmetric hyperdeterminant* [23], as it is one of the irreducible factors of the hyperdeterminant over the symmetric space [20]. By the definitions of determinant and dual variety, it is easy to see that

$$\nu_m(\mathbb{P}\mathbb{C}^{n-1})^\vee = \mathbb{V}(\mathrm{DET}) \cap \mathbb{P}\mathcal{S}^m(\mathbb{C}^n) = \mathbb{V}(\mathrm{SDET}) \subset \mathbb{P}\mathcal{S}^m(\mathbb{C}^n).$$

Therefore, geometrically, a nonzero tensor $\mathcal{A} \in \mathcal{S}^m(\mathbb{C}^n)$ has an eigenvector $\mathbf{x}$ corresponding to eigenvalue zero if and only if the determinantal hypersurface (discriminant hypersurface in [15]) $\mathbb{V}(\mathrm{SDET})$ is tangent to the variety $\nu_m(\mathbb{P}\mathbb{C}^{n-1})$ at the pair $([\mathcal{A}],[\mathbf{x}^{\otimes m}])$.

Note that $(\lambda,\mathbf{x})$ is an eigenpair of $\mathcal{A}$ if and only if $\mathbf{x}$ is an eigenvector of $\mathcal{A}-\lambda\mathcal{I}$ corresponding to eigenvalue zero.

For every tensor $\mathcal{A} \in \mathcal{S}^m(\mathbb{C}^n)$,

$$l(\lambda) = [\mathcal{A} - \lambda\mathcal{I}]$$

defines a line in $\mathbb{P}\mathcal{S}^m(\mathbb{C}^n)$. The next proposition is immediate.

PROPOSITION 7.1 (Intersection Interpretation). *Let $\mathcal{A} \in \mathcal{S}^m(\mathbb{C}^n)$. Then, $\lambda \in \sigma(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$ if and only if the line $l(\gamma) = [\mathcal{A} - \gamma\mathcal{I}]$ has an intersection with the determinantal hypersurface* $\mathbb{V}(\mathrm{SDET})$ *at $\gamma = \lambda$.*

The hypersurface $\mathbb{V}(\mathrm{SDET})$ has degree $n(m-1)^{n-1}$ [7, 23]. With multiplicity, the intersections of the line $l(\lambda)$ with the hypersurface $\mathbb{V}(\mathrm{SDET})$ are exactly the eigenvalues of the tensor $\mathcal{A}$.

PROPOSITION 7.2 (Generic Symmetric Tensor). *Let $\mathcal{A} \in \mathcal{S}^m(\mathbb{C}^n)$ be generic. Then, every eigenvalue has both algebraic multiplicity and geometric multiplicity one.*

Proof. The symmetric hyperdeterminant SDET is still irreducible [1, 23]. The characteristic polynomial $\mathrm{Det}(\mathcal{T} - \lambda\mathcal{I})$ for a symmetric tensor is still monic. Therefore, we can show that for a generic tensor all the eigenvalues have algebraic multiplicity one.

The dual of $\nu_m(\mathbb{P}\mathbb{C}^{n-1})$ is an irreducible hypersurface [7, Proposition 1.1.1.3], then at a generic point (thus a smooth point) on this hypersurface, the corresponding tensor has a unique zero eigenvector [7, Theorem 1.1.1.5]. Since the eigenvectors of a tensor $\mathcal{T}$ for the eigenvalue λ are exactly the eigenvectors of the tensor $\mathcal{T} - \lambda\mathcal{I}$ for the eigenvalue zero, the result now follows from Proposition 7.1. ∎

8. Final remarks

In this article, we studied the relationship between the algebraic multiplicity and the geometric multiplicity of an eigenvalue of a given tensor. Unlike the case for a matrix, the set of eigenvectors of an eigenvalue for a tensor does not have to be a linear subspace. In several cases, we show that the algebraic multiplicity is bounded below by a quantity determined by the geometric multiplicity and the order of the tensor (cf. (3.4)), which reduces to the geometric multiplicity in the matrix counterpart. In general, the algebraic multiplicity should be bounded below by a quantity given by the number of irreducible components of the eigenvariety and their dimensions (includes the geometric multiplicity), i.e., (3.5).

The study of (3.4) and (3.5) would be interesting: the right-hand sides of which should be invariants under the orthogonal linear group action, while the left-hand side is not. One can also consider (3.5) at the scheme level, for example, if there is an irreducible component V of an eigenvariety being a double line as a scheme, then one should count this line twice in the summand. It would not be surprising that the inequality in (3.5) becomes equality in the scheme level. Moreover, the conjecture does not involve degrees of eigenvarieties because it concerns only the inequality. If one consider degrees of eigenvarieties one might get a better inequality. But we definitely agree, as a referee of this paper has pointed to us, that it is reasonable to consider degrees of eigenvarieties. The referee also pointed out that as Proposition 3.1 indicates that the algebraic multiplicity is not invariant under the orthogonal group action, it may be invariant under the action of another group. In fact, it is obvious that the algebraic multiplicity is invariant under the action of a torus, but this group is too small to be useful, since we are not able to use this small group to normalize our tensor (One should recall that in the matrix case, we are allowed to use the orthogonal group to diagonalize our matrices). Our point of view is that if one can find a larger group such that algebraic multiplicities are invariant under this group, then one should be able to normalize tensors to some good forms. This should be our next research topic.

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REFERENCES

[1] D. Cartwright and B. Sturmfels, *The number of eigenvalues of a tensor*, Linear Algebra Appl., 438, 942–952, 2013.

[2] K.C. Chang, K. Pearson, and T. Zhang, *Perron–Frobenius theorem for nonnegative tensors*, Commun. Math. Sci., 6, 507–520, 2008.

[3] P. Comon, G. Golub, L.-H. Lim, and B. Mourrain, *Symmetric tensors and symmetric tensor rank*, SIAM J. Matrix Anal. Appl., 30, 1254–1279, 2008.

[4] J. Cooper and A. Dutle, *Spectra of uniform hypergraphs*, Linear Algebra Appl., 436, 3268–3292, 2012.

[5] D. Cox, J. Little, and D. O´Shea, *Using Algebraic Geometry*, New York: Springer-Verlag, 1998.

[6] S. Friedland and G. Ottaviani, *The number of singular vector tuples and uniqueness of best rank one approximation of tensors*, Found. Comput. Math., 2014. doi: 10.1007/s10208-014-9194-z

[7] I.M. Gelfand, M.M. Kapranov, and A.V. Zelevinsky, *Discriminants, Resultants and Multidimensional Determinants*, Birkhäuser, Boston, 1994.

[8] D.R. Grayson and M.E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, http://www.math.uiuc.edu/Macaulay2/.

[9] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics 52, Springer, New York, 1977.

[10] R.A. Horn and C.R. Johnson, *Matrix Analysis*, Cambridge University Press, New York, 1985.

[11] S. Hu, Z.H. Huang, C. Ling, and L. Qi, *On determinants and eigenvalue theorem of tensors*, J. Symbolic Comput., 50, 508–531, 2013.

[12] S. Hu and L. Qi, *Algebraic connectivity of an even uniform hypergraph*, J. Comb. Optim., 24, 564–579, 2012.

[13] S. Hu and L. Qi, *The Laplacian of a uniform hypergraph*, J. Comb. Optim., 29, 331–366, 2015.

[14] S. Hu and L. Qi, *The eigenvectors of the zero Laplacian and signless Laplacian eigenvalues of a uniform hypergraph*, Discrete Appl. Math., 169, 140–151, 2014.

[15] J.M. Landsberg, *Tensors: Geometry and Applications*, AMS, Providence, RI, 2012.

[16] A.-M. Li, L. Qi, and B. Zhang, *E-characteristic polynomials of tensors*, Commun. Math. Sci., 11, 33–53, 2013.

[17] L.-H. Lim, *Singular values and eigenvalues of tensors: a variational approach*, Computational

Advances in Multi-Sensor Adaptive Processing, 1st IEEE International Workshop on, 129–132, 2005.

[18] L.-H. Lim, *Foundations of Numerical Multilinear Algebra: Decomposition and Approximation of Tensors*, PhD thesis, Standford University, USA, 2007.

[19] G. Ni, L. Qi, F. Wang, and Y. Wang, *The degree of the E-characteristic polynomial of an even order tensor*, J. Math. Anal. Appl., 329, 1218–1229, 2007.

[20] L. Oeding, *Hyperdeterminants of polynomials*, Advances in Mathematics, 231, 1308–1326, 2012.

[21] L. Oeding and G. Ottaviani, *Eigenvectors of tensors and algorithms for Waring decomposition*, J. Symbolic Comput., 64, 9–35, 2013.

[22] G. Ottaviani and B. Sturmfels, *Matrices with eigenvectors in a given subspace*, Proc. of the American Math. Soc., 141, 1219–1232, 2013.

[23] L. Qi, *Eigenvalues of a real supersymmetric tensor*, J. Symbolic Comput., 40, 1302–1324, 2005.

[24] L. Qi, *H^+-eigenvalues of Laplacian and signless Lapaclian tensors*, Commun. Math. Sci., 12, 1045–1064, 2014.

[25] E. Robeva, *Orthogonal decomposition of symmetric tensors*, arXiv:1409.6685.

[26] F. Rouillier, *Solving zero-dimensional systems through the rational univariate representation*, J. Appl. Algebra in Engineering, Commun. Comput., 9, 433–461, 1999.

[27] I. R. Shafarevich, *Basic Algebraic Geometry*, Springer-Verlag, Berlin, 1977.

[28] J.-Y. Shao, H.-Y. Shan, and L. Zhang, *On some properties of the determinants of tensors*, Linear Algebra Appl., 439, 3057–3069, 2013.

[29] B. Sturmfels, *Solving Systems of Polynomial Equations*, American Math. Soc., Providence, 2002.