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TRANSONIC SHOCK SOLUTIONS TO THE EULER–POISSON SYSTEM IN QUASI-ONE-DIMENSIONAL NOZZLES*

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Abstract. In this paper, we study the transonic shock solutions to the Euler–Poisson system in quasi-one-dimensional nozzles. For a given supersonic flow at the entrance of the nozzle, under some proper assumptions on the data and nozzle length we first obtain a class of steady transonic shock solutions for the exit pressure lying in a suitable range. The shock position is monotonically determined by the exit pressure. More importantly, by the estimates on the coupled effects of the electric field and the geometry of the nozzle, we prove the dynamic stability of the transonic shock solutions under suitable physical conditions. As a consequence, there indeed exist dynamically stable transonic shock solutions for the Euler–Poisson system in convergent nozzles, which is not true for the Euler system [T.-P. Liu, Commun. Math. Phys., 83, 243–260, 1982].

Key words. Euler–Poisson system, transonic shock, dynamic stability.

AMS subject classifications. 35L65, 35B35, 35L67, 76H05, 82D37.

1. Introduction

We consider the compressible isentropic Euler–Poisson system in quasi-one-dimensional nozzle

$$\begin{cases} (A(x)\rho)_t + (A(x)\rho u)_x = 0, \\ (A(x)\rho u)_t + (A(x)\rho u^2)_x + A(x)p_x = A(x)\rho E, \qquad t>0, \quad l<x<L, \\ (A(x)E)_x = A(x)(\rho - b(x)). \end{cases} \tag{1.1}$$

System (1.1) characterizes the propagation of electrons in submicron semiconductor devices and plasmas (see [24]). Here u, ρ, and p represent the macroscopic particle velocity, electron density, and pressure, respectively. E is the electric field, $b(x)>0$ stands for the density of fixed positively charged background ions, and $A(x)>0$ is the cross-section area of the given nozzle (semiconductor device). The typical examples are 1D, 2D rotationally symmetric, and 3D spherically symmetric Euler–Poisson systems, where $A(x)$ are 1, x, and x^2, respectively.

We assume $A(x)$ is $\mathbb{C}^1$ smooth, $b(x)$ is continuous, and the pressure p satisfies:

$$p(0)=p'(0)=0, \quad p'(\rho)>0, \quad p''(\rho)\geq 0, \quad \text{for } \rho>0, \quad p(+\infty)=+\infty.$$

For polytropic gas, $p(\rho)=a\rho^\gamma$ with constant $a>0$ and adiabatic exponent $\gamma\geq 1$. The sound speed is $c(\rho):=\sqrt{p'(\rho)}$. The flow is called *supersonic* if $|u|>c(\rho)$; *subsonic* if $|u|<c(\rho)$; *sonic* if $|u|=c(\rho)$. The Mach number M is defined to be $M=\frac{|u|}{c(\rho)}$.

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First, we study the steady transonic shock solutions to

$$\begin{cases} (A(x)\rho u)_x = 0, \\ (A(x)\rho u^2)_x + A(x)p_x = A(x)\rho E, & l < x < L, \\ (A(x)E)_x = A(x)(\rho - b(x)), \end{cases} \tag{1.2}$$

with boundary conditions

$$(\rho, u, E)(l) = (\rho_l, u_l, E_l), \tag{1.3}$$

and

$$\rho(L) = \rho_r. \tag{1.4}$$

Suppose at the entrance of the nozzle the flow is supersonic with positive density and velocity, i.e.

$$u_l > \sqrt{p'(\rho_l)}, \quad \rho_l > 0, \quad u_l > 0. \tag{1.5}$$

The transonic shock solution means a piecewise smooth solution of (1.2) with two smooth solutions separated by a shock connecting a supersonic state on the left to a subsonic state on the right. We give the strict definition as follows.

DEFINITION 1.1. *Set*

$$(\rho, u, E) = \begin{cases} \rho_-(x), u_-(x), E_-(x), & l \leqslant x < r_s, \\ \rho_+(x), u_+(x), E_+(x), & r_s < x \leqslant L. \end{cases}$$

We call (ρ, u, E) *a steady transonic shock solution to (1.2)–(1.4) if*

(i) $(\rho_\pm, u_\pm, E_\pm)$ *satisfy (1.2)–(1.4) piecewisely, with* $M_- > 1$, $M_+ < 1$, *here* $M_\pm$ *are the Mach numbers.*

(ii) The following Rankine–Hugoniot conditions hold at $x = r_s$,

$$\begin{cases} [\rho u] = 0, \\ [\rho u^2 + p] = 0, \\ [E] = 0. \end{cases} \tag{1.6}$$

In the previous works, many purely subsonic and supersonic solutions are obtained for both one-dimensional and multi-dimensional Euler–Poisson system (cf. [2, 3, 5, 6, 26, 34] and references therein). However, for transonic shock solutions, there are only a few results even for one-dimensional Euler–Poisson system. In the one-dimensional case, $A(x) = 1$, a transonic shock problem with a linear pressure $p(\rho) = k\rho$ and special boundary conditions was discussed in [1]. To study a general case, phase plane analysis was given in [32], however, there is no result for transonic shock solutions. In [7], Gamba constructed a transonic shock solution, which may contain boundary layers due to a technical limit. A thorough study of the transonic shock solutions for one-dimensional Euler–Poisson equations with a constant background charge $b(x) = b_0$ was given by Luo–Xin in [20]. The existence, non-existence, uniqueness, and non-uniqueness of solutions with transonic shock were obtained and the discussion is based on the different cases in category with respect to the boundary data and the physical interval length. Later on, Luo–Rauch–Xie–Xin [19] investigated the structural stability of one type of transonic shock solutions obtained in [20] under small perturbations of the background charge $b(x)$, and they also proved that transonic shock is dynamically stable if the electric field E is not too negative at the shock position. For a viscous approximation of transonic solutions in the two-dimensional case (see Gamba–Morawetz [8]).

In this paper, we extend the results on one-dimensional transonic shocks in [19, 20] to the quasi-one-dimensional case and we expect this may be helpful for studying the multi-dimensional transonic shock problem for the Euler–Poisson system. We first prove that under specific assumptions on the supersonic data (1.3) and the nozzle length L, there exists a suitable range $[\rho_{min}, \rho_{max}]$ such that the steady transonic shock solution to (1.2)–(1.4) exists for any exit density ρ_r lying in $[\rho_{min}, \rho_{max}]$. Moreover, the shock position is monotonically dependent on the exit pressure (density). (See Theorem 2.8 for the details.) One major difficulty compared with the one-dimensional problem in [20] is that the powerful phase plane analysis technique is not applicable due to the effect of the geometry of the nozzle $A(x)$ and the non-constant background charge $b(x)$. We rewrite the system (1.2) into an ODE system, which is degenerate in the sonic state, and derive the existence of purely supersonic/subsonic solutions via the ODE theory. The key comparison Lemma 2.5 implies the monotone relation between the shock position and the exit pressure as well as the uniqueness of the transonic shock solution.

Secondly, we study the unsteady transonic shock solutions to the initial boundary value problem (1.1), (1.3), and (1.4) with given initial condition which is a small perturbation of any steady transonic shock solution $(\bar{\rho}, \bar{u}, \bar{E})(x)$ with shock position x_0. Suppose at the shock position x_0, the coupled effect of the geometry of the nozzle $A'(x_0)$ and the electric field $\bar{E}(x_0)$ is not too negative, then the unsteady transonic shock solution exists globally and approaches $(\bar{\rho}, \bar{u}, \bar{E})(x)$ at an exponential rate as time goes to infinity, which implies the transonic shock solution is dynamically stable in this nozzle. (See Theorem 3.2 for the details.) The main idea of the proof is inspired by Section 3 [19] and also see Remark 7 [19]. First, we introduce a non-trivial transformation to reformulate the problem into a second-order quasilinear hyperbolic equation. The linearized problem resembles a Klein–Gordon equation. Under our assumptions on the coupled effect of the geometry and the electric field, we are able to prove the exponential decay of a non-trivial energy functional for the solution to the linearized problem. Finally the uniform a priori energy estimates for the original nonlinear problem, together with the local existence result yields the global existence of the unsteady transonic shock solutions.

It is interesting to compare our result with the transonic shock solution for quasi-one-dimensional Euler system

$$\begin{cases} (A(x)\rho)_t + (A(x)\rho u)_x = 0, \\ (A(x)\rho u)_t + (A(x)\rho u^2)_x + A(x)p_x = 0. \end{cases} \tag{1.7}$$

where ρ, u, and p denote, respectively, the density, velocity and pressure, and $A(x)$ is the cross-sectional area of the nozzle. In [16], a wave front tracking variant of Glimm's scheme was used by Liu to prove that when $|A'(x)/A(x)|$ is small, a weak transonic shock is dynamically stable if $A'(x_0) > 0$ and dynamically unstable if $A'(x_0) < 0$, where x_0 is the shock position. The smallness assumption was removed later by Rauch–Xie–Xin [28] and they proved the exponential decay estimates for the transonic shock solution in divergent nozzle. Since the effect of the electric force in the Euler–Poisson system has a similar stabilizing effect as geometry of divergent domain in the Euler system, we prove the dynamic stability without restriction $A'(x) > 0$ in [16] for Euler system. The key issue in the analysis is the comparison between the stabilizing effect and destabilizing effect from the electric force and the geometry of the nozzle, and the balance of the two effects make the transonic shock solution stable.

The rest of this paper is organized as follows. In Section 2, we construct a class of transonic shock solutions to (1.2)–(1.4), and the monotonic dependence between the shock location and the exit density is shown in Theorem 2.8. Section 3 will be devoted to the dynamic stability of the transonic shock solutions and Theorem 3.2 will be proved.

2. Steady transonic shock solutions

In this section, we investigate the existence of steady transonic shock solution to (1.2)–(1.4) under suitable assumptions on the boundary data and the nozzle length. To obtain the main existence result we need the following lemmas.

Lemma 2.1 (Local existence of supersonic/subsonic solutions). *For any fixed $y \geq l$ and for any given supersonic/subsonic state (ρ_*, u_*, E_*) at y with $\rho_* > 0, u_* > 0$, there exists L_y determined by ρ_*, u_*, E_*, l, $b(x)$, $A(x)$, and y such that the initial value problem*

$$\begin{cases} \text{equation (1.2)}, \text{ for } y < x < L, \\ (\rho, u, E)(y) = (\rho_*, u_*, E_*), \end{cases} \tag{2.1}$$

has a unique supersonic/subsonic solution $(\rho, u, E)(x)$ on $[y, L_y)$.

Proof. A Direct computation from (2.1) shows that, a smooth non-sonic solution (ρ, u, E) satisfies

$$\begin{cases} \rho_x = \dfrac{\rho E}{c^2 - u^2} + \dfrac{A'(x)}{A(x)} \dfrac{\rho u^2}{c^2 - u^2}, \\ u_x = \dfrac{-uE}{c^2 - u^2} - \dfrac{A'(x)}{A(x)} \dfrac{uc^2}{c^2 - u^2}, \\ E_x = -\dfrac{A'(x)}{A(x)} E + \rho - b, \\ (\rho, u, E)(y) = (\rho_*, u_*, E_*). \end{cases} \tag{2.2}$$

By the local existence theory of ODE system, one can defined L_y to be the lifespan of the supersonic/subsonic solution (ρ, u, E) to (2.2). ∎

Remark 2.2. Consider problem (2.1) with given supersonic initial data (ρ_l, u_l, E_l) at l satisfying (1.5) and let L_1 be the lifespan of the corresponding supersonic smooth solution $(\rho, u, E)(x)$. Then $(1.2)_1$ implies that

$$A(x)\rho(x)u(x) \equiv A(l)\rho_l u_l =: J > 0. \tag{2.3}$$

Therefore, any smooth solution must satisfy $\rho(x) > 0$, $u(x) > 0$. Furthermore, we claim that for finite L_1, the smooth solution $(\rho, u, E)(x)$ is sonic at $x = L_1$, which can be proved by contradiction.

More precisely, suppose the smooth solution is supersonic on $[l, L_1]$, then the solution blows up at $x = L_1$ by the property of lifespan. Solve the equation $\frac{J}{A(x)\rho} = c(\rho)$ for ρ and denote the smooth solution as $\rho_s(x)$, which is the density for the sonic state. Then the flow being supersonic is equivalent to $\rho < \rho_s(x)$. Note that $\rho_s(x)$ is bounded on $[l, L_1]$, thus ρ and E are uniformly bounded at L_1. Next, from $(2.2)_2$, the velocity u satisfies the estimate $|u_x| \leq \frac{uC_1}{u^2 - C_2} \leq uC_3$ where C_1, C_2, C_3 are constants depending on the bound for ρ, E, and $A(x)$. This estimate implies u grows at most exponentially, and therefore is bounded at L_1. The boundedness of the smooth solution $(\rho,u,E)(x)$ at L_1 gives a contradiction to the blow up phenomena, by which, we finish the proof of the claim.

LEMMA 2.3. *Let $(\rho_-,u_-,E_-)(\cdot)$ be the smooth supersonic solution to problem (2.1) with initial data (ρ_l,u_l,E_l) at $x=l$. For any point $x\in[l,L_1)$, the left supersonic state $(\rho_-,u_-,E_-)(x)$ is fixed. Then there exists a unique right subsonic state $(\rho_+,u_+,E_+)(x)$ such that the Rankine–Hugoniot conditions (1.6) hold at x.*

Proof. According to Rankine–Hugoniot condition at x, we have

$$\begin{cases} \rho_+(x)u_+(x)=\rho_-(x)u_-(x)=\dfrac{J}{A(x)}, \\ \rho_+(x)u_+(x)^2+p(\rho_+)=\rho_-(x)u_-(x)^2+p(\rho_-), \\ E_+(x)=E_-(x), \end{cases}$$

that is,

$$\frac{J^2}{A^2(x)\rho_+}+p(\rho_+)=\frac{J^2}{A^2(x)\rho_-}+p(\rho_-) \quad \text{and} \quad E_+(x)=E_-(x),$$

where J is as in (2.3). For fixed x, define $F(\rho)=\frac{J^2}{A(x)\rho}+p(\rho)$, which is decreasing in $(0,\rho_s(x))$ and increasing in $(\rho_s(x),+\infty)$, where $\rho_s(x)$ is the same as the one in Remark 2.2. Moreover, F increases to infinity as ρ increases to infinity. Thus, for given $\rho_-<\rho_s$, there exists an unique ρ_+, such that $F(\rho_+)=F(\rho_-)$ and $\rho_-<\rho_s(x)<\rho_+$. Denote the solution ρ_+ by $\mathfrak{s}(\rho_-(x),x)$ and set $\mathfrak{t}(u_-(x),x)=\frac{J}{A(x)\mathfrak{s}(\rho_-(x),x)}$, then the unique right state is $(\mathfrak{s}(\rho_-(x),x),\mathfrak{t}(u_-(x),x),E_-(x))$ which is subsonic, since $\mathfrak{s}(\rho_-(x),x)>\rho_s$. $\square$

REMARK 2.4. For fixed supersonic initial data (ρ_l,u_l,E_l) satisfying (1.3), by Lemma 2.1 we can always solve the IVP (1.2)–(1.3) to get a supersonic solution $(\rho_-(x),u_-(x),E_-(x))$ on $[l,s]$ for any $s\in[l,L_1]$. Then we obtain the right state $(\rho_+(s),u_+(s),E_+(s))=(\mathfrak{s}(\rho_-(s),s),\mathfrak{t}(u_-(s),s),E_-(s))$ according to Lemma 2.3. Now fix a $s\in[l,L_1]$, by Lemma 2.1, we can solve the problem (1.2) with initial data $(\mathfrak{s}(\rho_-(s),s),\mathfrak{t}(u_-(s),s),E_-(s))$ at s to get a subsonic solution $(\rho_+(x),u_+(x),E_+(x))$.

In the rest of this section, we will suppose $L<L_1$, where L_1 is defined as in Remark 2.2. Next, we will discuss the relation between the exist density $\rho_+(L)$ and the shock position s.

LEMMA 2.5 (Monotone Dependence). *Let*

$$(\rho^{(i)},u^{(i)},E^{(i)})(x)=\begin{cases} (\rho_-^{(i)},u_-^{(i)},E_-^{(i)})(x) & l<x<x_i, \\ (\rho_+^{(i)},u_+^{(i)},E_+^{(i)})(x) & x_i<x<L, \end{cases}$$

$i=1,2$ be two transonic shock solutions to (1.2) with given supersonic initial data (ρ_l,u_l,E_l) at $x=l$, and $\rho^{(1)}(l)=\rho^{(2)}(l)=\rho_l$, $u^{(1)}(l)=u^{(2)}(l)=u_l$, $E^{(1)}(l)=E^{(2)}(l)=E_l$, $x_1<x_2$. Assume

$$E_-^{(2)}(x)+\frac{A'(x)}{A(x)}u_-^{(2)}(x)\mathfrak{t}(u_-^{(2)}(x),x)>0, \quad \forall x\in[x_1,x_2], \tag{2.4}$$

then we have

$$\rho^{(1)}(L)>\rho^{(2)}(L).$$

Proof. Define $E_\alpha(x)$ on $[x_1,x_2]$ to be the solution to

$$\begin{cases} (A(x)E_\alpha)_x = A(x)(\mathfrak{s}(\rho_-^{(2)}(x),x)-b(x)), \quad x\in[x_1,x_2],\\ E_\alpha(x_1)=E_-^{(1)}(x_1)=E_+^{(1)}(x_1)=E_-^{(2)}(x_1). \end{cases}$$

Then we have

$$\begin{cases} (A(x)(E_-^{(2)}-E_\alpha))_x = A(x)(\rho_-^{(2)}-\mathfrak{s}(\rho_-^{(2)}(x),x))<0, \quad x\in[x_1,x_2],\\ (E_-^{(2)}-E_\alpha)(x_1)=0, \end{cases}$$

since $\rho_-^{(2)}<\rho_s<\mathfrak{s}(\rho_-^{(2)},x)$ and therefore $E_\alpha(x)>E_-^{(2)}(x)$, $x\in[x_1,x_2]$.

Differentiating the equation of $\mathfrak{s}(\rho_-^{(2)}(x),x)$,

$$p(\mathfrak{s}(\rho_-^{(2)}(x),x))+\frac{J^2}{A^2(x)\mathfrak{s}(\rho_-^{(2)}(x),x)} = p(\rho_-^{(2)}(x))+\frac{J^2}{A^2(x)\rho_-^{(2)}(x)},$$

with respect to x, it holds for $x\in[x_1,x_2]$ that,

$$\begin{aligned} &\frac{d\mathfrak{s}(\rho_-^{(2)}(x),x)}{dx}\left(p'(\mathfrak{s}(\rho_-^{(2)}(x),x))-\frac{J^2}{A^2\mathfrak{s}(\rho_-^{(2)}(x),x)^2}\right)\\ =&\rho_-^{(2)}E_-^{(2)}+\frac{2A'J^2}{A^3\mathfrak{s}(\rho_-^{(2)}(x),x)}-\frac{A'J^2}{A^3\rho_-^{(2)}(x)}\\ =&\rho_-^{(2)}\left(E_-^{(2)}+\frac{A'(x)}{A(x)}u_-^{(2)}(x)\mathfrak{t}(u_-^{(2)}(x),x)\right)+\frac{A'J^2}{A^3\mathfrak{s}(\rho_-^{(2)}(x),x)}-\frac{A'J^2}{A^3\rho_-^{(2)}(x)}\\ <&\mathfrak{s}(\rho_-^{(2)}(x),x)\left(E_-^{(2)}+\frac{A'(x)}{A(x)}u_-^{(2)}(x)\mathfrak{t}(u_-^{(2)}(x),x)\right)+\frac{A'J^2}{A^3\mathfrak{s}(\rho_-^{(2)}(x),x)}-\frac{A'J^2}{A^3\rho_-^{(2)}(x)}\\ <&\mathfrak{s}(\rho_-^{(2)}(x),x)\left(E_\alpha+\frac{A'(x)}{A(x)}u_-^{(2)}(x)\mathfrak{t}(u_-^{(2)}(x),x)\right)+\frac{A'J^2}{A^3\mathfrak{s}(\rho_-^{(2)}(x),x)}-\frac{A'J^2}{A^3\rho_-^{(2)}(x)}\\ =&\mathfrak{s}(\rho_-^{(2)}(x),x)E_\alpha+\frac{A'J^2}{A^3\mathfrak{s}(\rho_-^{(2)}(x),x)}, \end{aligned}$$

where the assumption (2.4), the fact $\rho_-^{(2)}(x)<\mathfrak{s}(\rho_-^{(2)}(x),x)$ and $E_-^{(2)}(x)<E_\alpha(x)$ are used. Therefore,

$$\begin{cases} \dfrac{d\mathfrak{s}(\rho_-^{(2)}(x),x)}{dx}<\dfrac{1}{p'(\mathfrak{s}(\rho_-^{(2)}(x),x))-\frac{J^2}{A^2\mathfrak{s}(\rho_-^{(2)}(x),x)^2}}(\mathfrak{s}(\rho_-^{(2)}(x),x)E_\alpha+\dfrac{A'J^2}{A^3\mathfrak{s}(\rho_-^{(2)}(x),x)}),\\ \dfrac{dE_\alpha}{dx}=-\dfrac{A'(x)}{A(x)}E_\alpha+\mathfrak{s}(\rho_-^{(2)}(x),x)-b(x),\\ \mathfrak{s}(\rho_-^{(2)}(x_1),x_1)=\rho_+^{(1)}(x_1),\ E_\alpha(x_1)=E_+^{(1)}(x_1). \end{cases} \tag{2.5}$$

Note that $(\rho_+^{(1)}(x),E_+^{(1)}(x))$ satisfies

$$\begin{cases} \dfrac{d\rho_+^{(1)}}{dx} = \dfrac{1}{p'(\rho_+^{(1)}) - \frac{J^2}{A^2\rho_+^{(1)}}}(\rho_+^{(1)}E_+^{(1)} + \dfrac{A'J^2}{A^3\rho_+^{(1)}}), \\ \dfrac{dE_+^{(1)}}{dx} = -\dfrac{A'(x)}{A(x)}E_+^{(1)} + \rho_+^{(1)} - b(x), \\ \rho_+^{(1)}(x_1) = \rho_+^{(1)}(x_1),\ E_+^{(1)}(x_1) = E_+^{(1)}(x_1), \end{cases} \tag{2.6}$$

then by the idea in [19], one can apply the comparison principle for ODE system to (2.5)–(2.6) and obtain

$$\mathfrak{s}(\rho_-^{(2)}(x),x)) < \rho_+^{(1)}(x), \quad E_\alpha(x) < E_+^{(1)}(x), \quad x \in (x_1,x_2]. \tag{2.7}$$

Therefore, $\rho_+^{(2)}(x_2) = \mathfrak{s}(\rho_-^{(2)}(x_2),x_2)) < \rho_+^{(1)}(x_2)$ and $E_+^{(2)}(x_2) = E_-^{(2)}(x_2) < E_\alpha(x_2) < E_+^{(1)}(x_2)$. Note that $(\rho_-^{(1)},E_-^{(1)})$ and $(\rho_-^{(2)},E_-^{(2)})$ satisfy the same ODE system on $[x_2,L]$, thus

$$\rho_+^{(1)}(L) > \rho_+^{(2)}(L) \quad \text{and} \quad E_+^{(1)}(L) > E_+^{(2)}(L) \tag{2.8}$$

by the comparison principle again. □

Remark 2.6. We give another proof to (2.7). First, systems (2.5)–(2.6) give

$$\frac{d\mathfrak{s}(\rho_-^{(2)}(x),x)}{dx}(x_1) < \frac{d\rho_+^{(1)}}{dx}(x_1),$$

which implies that there exist a $x_3 \in (x_1,x_2]$ such that $\mathfrak{s}(\rho_-^{(2)}(x),x) < \rho_+^{(1)}(x)$ for $x \in (x_1,x_3]$. Thus,

$$(A(E_\alpha - E_+^{(1)}))'(x) = A(\mathfrak{s}(\rho_-^{(2)}(x),x) - \rho_+^{(1)}) < 0, \quad x \in (x_1,x_3],$$

and therefore $E_\alpha(x) < E_+^{(1)}(x)$ for $x \in (x_1,x_3]$. Define

$$x^* = \sup\{x_3 \in (x_1,x_2] : \mathfrak{s}(\rho_-^{(2)}(x),x) < \rho_+^{(1)}(x), \text{for } x \in (x_1,x_3]\}.$$

If $x^* < x_2$, then (2.7) holds for $x \in (x_1,x^*)$ with $\mathfrak{s}(\rho_-^{(2)}(x^*),x^*)) = \rho_+^{(1)}(x^*)$, $E_\alpha(x^*) \le E_+^{(1)}(x^*)$, and $\frac{d\mathfrak{s}(\rho_-^{(2)}(x),x)}{dx}(x^*) - \frac{d\rho_+^{(1)}}{dx}(x^*) \ge 0$. On the other hand, (2.5) and (2.6) imply

$$\frac{d\mathfrak{s}(\rho_-^{(2)}(x),x)}{dx}(x^*) - \frac{d\rho_+^{(1)}}{dx}(x^*) < 0,$$

which leads to a contradiction. Thus $x^* = x_2$ and (2.7) holds for $x \in (x_1,x_2)$. By using the same contradiction argument, (2.7) is true at $x = x_2$.

Finally, (2.8) can be proved by the same contradiction argument again.

Remark 2.7. The assumption (2.4) is essential to guarantee the monotone relation between the position of transonic shock and the exit density. Thus, in order to obtain the following main existence theorem, we assume the upstream boundary data satisfies the following condition,

$$E_l + \frac{A'(l)}{A(l)}u_l\mathfrak{t}(u_l,l) > 0. \tag{2.9}$$

Now we state our main result for this section:

THEOREM 2.8. *Assume the upstream boundary data (ρ_l,u_l,E_l) satisfies (1.5) and (2.9). Then there exists a constant $L^*=L^*(\rho_l,u_l,E_l,A(x),b(x),l)$ such that if the nozzle length $L<L^*$, then there exist two constants ρ_{min},ρ_{max} such that the boundary value problem (1.2)–(1.4) with $\rho_r\in[\rho_{min},\rho_{max}]$ has a unique transonic shock solution on $[l,L]$. Moreover, the position of the transonic shock depends on the exit density ρ_r monotonically.*

Proof. For given supersonic state (ρ_l,u_l,E_l) at $x=l$, let L_1 be defined as in Remark 2.2 and the corresponding supersonic solution be $(\rho_-(x),u_-(x),\,E_-(x))$. Assume $L<L_1$ and for any x in $[l,L]$, the function $E_-(x)+\frac{A'(x)}{A(x)}u_-(x)\mathfrak{t}(u_-(x),x)$ is continuous with respect to x. Since the upstream boundary data satisfies (2.9), we can define

$$L_2:=\sup_{[l,L]}\left\{y:\ E_-(x)+\frac{A'(x)}{A(x)}u_-(x)\mathfrak{t}(u_-(x),x)>0 \text{ for all } x\in[l,y]\right\}.$$

It is clear that $L_2>l$.

Next, suppose $l<L<\min\{L_1,L_2\}$, and the shock occurs at the entrance l. By Lemma 2.1, we obtain the local subsonic solution $(\rho_+^l(x),u_+^l(x),E_+^l(x))$ to the problem (1.2) with boundary data $(\mathfrak{s}(\rho_l,l),\mathfrak{t}(u_l,l),E_l)$, whose lifespan is denoted by $L_3>l$.

Set $L^*=\min\{L_1,L_2,L_3\}$, and let $l<L<L^*$. As in Remark 2.4, for any shock position $s\in[l,L]$, one can solve the IVP (1.2)–(1.3) to get the supersonic solution $(\rho_-(x),u_-(x),E_-(x))$ on $[l,s]$, and then obtain the right subsonic state $(\mathfrak{s}(\rho_-(s),s),\mathfrak{t}(u_-(s),s),E_-(s))$ by Lemma 2.3. Next, we solve the problem (1.2) with boundary data $(\mathfrak{s}(\rho_-(s),s),\mathfrak{t}(u_-(s),s),E_-(s))$ at s to get a local subsonic solution $(\rho_+^s(x),u_+^s(x),E_+^s(x))$. We claim that such local subsonic solution exists on $[s,L]$. To this end, we apply Lemma 2.5 to compare $(\rho_+^s(x),u_+^s(x),E_+^s(x))$ with $(\rho_+^l(x),u_+^l(x),E_+^l(x))$ and deduce that $\rho_+^s(x)\le\rho_+^l(x)$. Similarly, we have $\rho_+^s(x)\ge\mathfrak{s}(\rho_-(x),x))>\rho_s(x)$ for $x\in[s,L]$. Therefore, $\rho_+^s(x)$ never blows up or touches sonic state for $x\in[s,L]$, and as a consequence, the subsonic solution exist upto L.

Denote by ρ_{min} the exit density corresponding to $s=L$ and by ρ_{max} to be the exit density corresponding to $s=l$. Then, if $\rho_r\in[\rho_{min},\rho_{max}]$, we can find a unique shock front s such that (1.2) -(1.4) has a transonic shock solution on $[l,L]$ and the shock position depends monotonically on the exit density ρ_r by Lemma 2.5. ∎

REMARK 2.9. In the proof of Theorem 2.8, L_3 is the lifespan of the subsonic solution. It is not clear whether the solution blows up at L_3 or the sonic state occurs first. In contrast, for the supersonic solution, by Remark 2.2, the sonic state occurs before the blow up of the solution.

REMARK 2.10. The background charge $b(x)$ does not need to be a constant, and there is no restriction on $b(x)<\rho_s(x)$ or $b(x)>\rho_s(x)$ as in [20].

3. Dynamical stability of transonic shock solutions

Denote $(\bar\rho,\bar u,\bar E)(x)$ to be the steady transonic shock solution to (1.2)–(1.4) with shock position x_0. Suppose the solution is away from vacuum

$$\inf_{x\in[l,L]}\bar\rho(x)>0. \tag{3.1}$$

Consider the initial boundary value problem of system (1.1) with boundary conditions

$$\begin{cases} (\rho,u,E)(t,l)=(\rho_l,u_l,E_l), \\ \rho(t,L)=\rho_r, \end{cases} \tag{3.2}$$

and the initial condition

$$(\rho,u,E)(0,x)=(\rho_0,u_0,E_0)(x). \tag{3.3}$$

Assume the initial data $(\rho_0,u_0,E_0)(x)$ with the form

$$(\rho_0,u_0)(x)=\begin{cases} (\rho_{0-},u_{0-})(x), \text{ if } l<x<\tilde{x}_0, \\ (\rho_{0+},u_{0+})(x), \text{ if } \tilde{x}_0<x<L, \end{cases} \tag{3.4}$$

and

$$E_0(x)=E_l+\int_l^x (\rho_0(s)-b(s))ds, \tag{3.5}$$

is a small perturbation of $(\bar{\rho},\bar{u},\bar{E})(x)$ in the sense that

$$\begin{aligned} |x_0-\tilde{x}_0|&+\|(\rho_{0+},u_{0+})-(\bar{\rho}_+,\bar{u}_+)\|_{H^{k+2}([\check{x}_0,L])} \\ &+\|(\rho_{0-},u_{0-})-(\bar{\rho}_-,\bar{u}_-)\|_{H^{k+2}([l,\hat{x}_0])}<\varepsilon, \end{aligned} \tag{3.6}$$

for some small constant $\varepsilon>0$, and some integer $k\geq 15$, where $\check{x}_0=\min\{x_0,\tilde{x}_0\}$ and $\hat{x}_0=\max\{x_0,\tilde{x}_0\}$. Moreover, (ρ_0,u_0,E_0) is assumed to satisfy the Rankine–Hugoniot conditions as $x=\tilde{x}_0$,

$$\begin{aligned} &\big((p(\rho_{0+})+\rho_{0+}u_{0+}^2-(p(\rho_{0-})+\rho_{0-}u_{0-}^2)\big)\cdot(\rho_{0+}-\rho_{0-})(\tilde{x}_0) \\ =&(\rho_{0+}u_{0+}-\rho_{0-}u_{0-})^2(\tilde{x}_0). \end{aligned} \tag{3.7}$$

We will study the global existence and asymptotic behavior of the transonic shock solutions to the initial boundary value problem (1.1), (3.2), and (3.3). The transonic shock solutions are time-dependent piecewise smooth entropy solutions, which are defined as follows.

Definition 3.1. *Assume*

$$(\rho,u,E)(t,x)=\begin{cases} \rho_-(t,x),u_-(t,x),E_-(t,x), & l\leqslant x<s(t), \\ \rho_+(t,x),u_+(t,x),E_+(t,x), & s(t)<x\leqslant L. \end{cases}$$

Then we call $(\rho,u,E)(t,x)$ a piecewise smooth entropy solution to (1.1), (3.2)–(3.3) if:

(i) *$(\rho_\pm,u_\pm,E_\pm)$ satisfy (1.1), (3.2)–(3.3) piecewisely, with $M_->1$, $M_+<1$, where M is the Mach number.*

(ii) *the following Rankine–Hugoniot conditions hold at $x=s(t)$,*

$$\begin{cases} (p(\rho)+\rho u^2)(t,s(t)+)-(p(\rho)+\rho u^2)(t,s(t)-) \\ \quad =(\rho u(t,s(t)+)-\rho u(t,s(t)-))\dot{s}(t), \\ \rho u(t,s(t)+)-\rho u(t,s(t)-)=(\rho(t,s(t)+)-\rho(t,s(t)-))\dot{s}(t), \\ E(s(t)+,t)=E(s(t)-,t). \end{cases} \tag{3.8}$$

(iii) the Lax geometric entropy condition holds,

$$(u-\sqrt{p'(\rho)})(t,s(t)-)>\dot{s}(t)>(u-\sqrt{p'(\rho)})(t,s(t)+),$$
$$(u+\sqrt{p'(\rho)})(t,s(t)+)>\dot{s}(t).$$

The dynamical stability result in this paper is the following theorem.

THEOREM 3.2. *Let $(\bar\rho,\bar u,\bar E)$ be any steady transonic shock solution to (1.2)–(1.4) satisfying (3.1). Then there exist positive constants δ, ϵ_0 depending on $(\bar\rho,\bar u,\bar E,A,b,l,L)$ such that if $\epsilon\le\epsilon_0$ and*

$$\bar E_-(x_0)+\frac{A'(x_0)}{A(x_0)}\bar u_-(x_0)\bar u_+(x_0)>-\delta, \tag{3.9}$$

and if the initial data (ρ_0,u_0,E_0) satisfies (3.4)–(3.7) and the $(k+2)$th order compatibility conditions hold at $x=l$, $x=x_0$ and $x=L$, then the initial boundary value problem (1.1), (3.2)–(3.3) admits a unique piecewise smooth entropy solution $(\rho,u,E)(x,t)$ for $(t,x)\in[0,\infty)\times[l,L]$ containing a single transonic shock $x=s(t)$ $(l<s(t)<L)$ with $s(0)=\tilde x_0$. Furthermore, there exist $T_0>0$ and $\lambda>0$ such that

$$(\rho_-,u_-,E_-)(t,x)=(\bar\rho_-,\bar u_-,\bar E_-)(x),\ \textit{for}\ l\le x<s(t),\ t>T_0$$

and

$$\|(\rho_+,u_+,)(\cdot,t)-(\bar\rho_+,\bar u_+)(\cdot)\|_{W^{k-7,\infty}(s(t),L)}+\|E_+(\cdot,t)-\bar E_+(\cdot)\|_{W^{k-6,\infty}(s(t),L)}\le C\varepsilon e^{-\lambda t},$$
$$\sum_{m=0}^{k-6}|\partial_t^m(s(t)-x_0)|\le C\varepsilon e^{-\lambda t},$$

for $t\ge0$, where $(\bar\rho_\pm,\bar u_\pm,\bar E_\pm)$ are the solutions of the Euler–Poisson equations in the associated regions.

REMARK 3.3. The compatibility conditions for the initial boundary value problems for hyperbolic equations were discussed in detail in [21, 25, 29].

3.1. Formulation of the problem. It follows from the argument in [14] that there exists a local piecewise smooth solution containing a single shock $x=s(t)$ (with $s(0)=\tilde x_0$) satisfying the Rankine-Hugoniot conditions and Lax geometric shock condition (3.8) of the initial boundary value problem (1.1),(3.2)–(3.3) on $[0,\bar T]$ for some $\bar T>0$, which can be written as

$$(\rho,u,E)(x,t)=\begin{cases}(\rho_-,u_-,E_-),\ \text{if}\ l<x<s(t),\\(\rho_+,u_+,E_+),\ \text{if}\ s(t)<x<L.\end{cases}$$

Note that, when $t>T_0$ for some $T_0>0$, (ρ_-,u_-,E_-) depends only on the boundary conditions at $x=l$. Moreover, when ε is small, by the standard lifespan argument, we have $T_0<\bar T$ (see [14]). Therefore,

$$(\rho_-,u_-,E_-)=(\bar\rho_-,\bar u_-,\bar E_-)\ \text{for}\ t>T_0. \tag{3.10}$$

Without loss of generality we assume $T_0=0$. Then the key point to extend the local solution to global solution is to obtain uniform estimates in the region $x>s(t),t>0$. First, from the Rankine–Hugoniot conditions (3.8) one has

$$[\rho u^2+p][\rho]=[\rho u]^2,$$

which is equivalent to

$$
\left(p(\rho_+)(t,s(t))+\frac{J_+^2(t,s(t))}{A^2(s(t))\rho_+(t,s(t))}-p(\rho_-)(t,s(t))-\frac{J_-^2(t,s(t))}{A^2(s(t))\rho_-(t,s(t))}\right)\cdot(\rho_+-\rho_-)
$$
$$
=\left(\frac{J_+(t,s(t))-J_-(t,s(t))}{A(s(t))}\right)^2,
$$

where $J(t,x)=A(x)\rho(t,x)u(t,x)$. It follows from (3.10) that,

$$
J(t,s(t)-)=\bar{J}=A(l)\rho_l u_l.
$$

Then the Taylor expansions and the Rankine–Hugoniot conditions imply that

$$
\begin{aligned}
&\Big\{p'(\bar{\rho}_+)(s(t))(\rho_+(t,s(t))-\bar{\rho}_+(s(t)))-\frac{\bar{J}^2}{A^2\bar{\rho}_+^2}(s(t))\cdot(\rho_+(t,s(t))-\bar{\rho}_+(s(t)))\\
&\quad+\frac{2\bar{J}}{A^2\bar{\rho}_+}(s(t))\cdot(J_+(t,s(t))-\bar{J}(s(t)))+\partial_x(p(\bar{\rho}_+)+\frac{\bar{J}^2}{A^2\bar{\rho}_+})(x_0)\cdot(s(t)-x_0)\\
&\quad-\partial_x(p(\bar{\rho}_-)+\frac{\bar{J}^2}{A^2\bar{\rho}_-})(x_0)\cdot(s(t)-x_0)+R_1\Big\}(\bar{\rho}_+(x_0)-\bar{\rho}_-(x_0)+R_2)\\
&=\left(\frac{J_+(t,s(t))-\bar{J}(s(t))}{A(s(t))}\right)^2,
\end{aligned}
$$

with

$$
\begin{aligned}
&R_1=O((\rho_+-\bar{\rho}_+)^2+(J_+-\bar{J})^2+(s(t)-x_0)^2),\\
&R_2=O(|\rho_+-\bar{\rho}_+|+|(s(t)-x_0)|).
\end{aligned}
$$

Thus, by the implicit function theorem,

$$
(J_+-\bar{J})(t,s(t))=\mathcal{A}_1((\rho_+-\bar{\rho}_+)(t,s(t)),s(t)-x_0), \tag{3.11}
$$

where $\mathcal{A}_1$ is considered to be a function of two variables satisfying $\mathcal{A}_1(0,0)=0$ and

$$
\begin{aligned}
\frac{\partial\mathcal{A}_1}{\partial(\rho_+-\bar{\rho}_+)}|_{(0,0)}&=-\frac{A(p'(\bar{\rho}_+)-\bar{u}_+^2)}{2\bar{u}_+}(x_0),\\
\frac{\partial\mathcal{A}_1}{\partial(s-x_0)}|_{(0,0)}&=-(\bar{\rho}_+-\bar{\rho}_-)(\frac{A'\bar{u}_-}{2}+\frac{A\bar{E}}{2\bar{u}_+})(x_0).
\end{aligned}
$$

Substituting (3.11) into the Rankine–Hugoniot conditions yields

$$
s'(t)=\mathcal{A}_2(\rho_+-\bar{\rho}_+,s(t)-x_0), \tag{3.12}
$$

where $\mathcal{A}_2$ satisfies $\mathcal{A}_2(0,0)=0$ and

$$
\frac{\partial\mathcal{A}_2}{\partial(\rho_+-\bar{\rho}_+)}|_{(0,0)}=-\frac{p'(\bar{\rho}_+)-\bar{u}_+^2}{2\bar{u}_+(\bar{\rho}_+-\bar{\rho}_-)}(x_0),\ \frac{\partial\mathcal{A}_2}{\partial(s(t)-x_0)}|_{(0,0)}=-(\frac{A'\bar{u}_-}{2A}+\frac{\bar{E}_+}{2\bar{u}_+})(x_0).
$$

It follows from $(1.1)_3$ that

$$
A(x)E_+(x,t)=A(l)E_l+\int_l^{s(t)}A(y)(\rho_--b)(y)dy+\int_{s(t)}^x A(y)(\rho_+-b)(y)dy
$$

for $s(t) < x \le L$. The equation $(1.1)_1$ and the Rankine–Hugoniot conditions (3.8) give

$$\partial_t(A(x)E_+) = -J_+(t,x) + \bar{J}.$$

Let $Y = A(x)(E_+(x,t) - \bar{E}_+(x))$. Then

$$Y_t = \bar{J} - J_+,\ Y_x = A(\rho_+ - \bar{\rho}_+).$$

Therefore, it follows from $(1.1)_2$ that

$$\partial_{tt}Y + \partial_x\left(\frac{\bar{J}^2}{A\bar{\rho}_+} - \frac{(\bar{J}-Y_t)^2}{A\bar{\rho}_+ + Y_x}\right) + A\partial_x\left(p(\bar{\rho}_+) - p(\bar{\rho}_+ + \frac{Y_x}{A})\right) + Y_x\bar{E}_+ + \bar{\rho}_+Y + \frac{YY_x}{A} = 0. \tag{3.13}$$

Set $\xi = (\xi_0,\xi_1) = (t,x)$, then we have

$$\sum_{0\le i,j\le 1} \bar{a}_{ij}(x,Y_t,Y_x)\partial_{ij}Y + \sum_{0\le i\le 1} \bar{b}_i(x,Y_t,Y_x)\partial_iY + \bar{c}(x,Y_t,Y_x)Y = 0, \tag{3.14}$$

where $\bar{a}_{ij}$, $\bar{b}_i$ and $\bar{c}$ are smooth functions of their arguments, and satisfy

$$\begin{aligned}\mathcal{L}_0Y &= \sum_{0\le i,j\le 1} \bar{a}_{ij}(x,0,0)\partial_{ij}Y + \sum_{0\le i\le 1} \bar{b}_i(x,0,0)\partial_iY + \bar{c}(x,0,0)Y \\ &= \partial_{tt}Y + A\partial_x(p'(\bar{\rho}_+)\frac{Y_x}{A}) - \partial_x(\bar{u}_+^2Y_x) + \partial_x(\frac{2\bar{J}}{\bar{\rho}_+}Y_t) + \bar{E}_+Y_x + \bar{\rho}_+Y.\end{aligned}$$

Furthermore, the Rankine–Hugoniot conditions (3.11) and (3.12) yield

$$Y_t = -\mathcal{A}_1(\frac{Y_x}{A}, s(t) - x_0), \tag{3.15}$$

and

$$s' = \mathcal{A}_2(\frac{Y_x}{A}, s - x_0). \tag{3.16}$$

Direct computation yields

$$\begin{aligned}Y(s(t),t) &= A(s(t))(E_+(s(t),t) - \bar{E}_+(s(t))) \\ &= (\partial_x(A\bar{E}_-) - \partial_x(A\bar{E}_+))(x_0)\cdot(s(t) - x_0) + O((s(t)-x_0)^2).\end{aligned}$$

Then from $(1.1)_3$ one has

$$s(t) - x_0 = \mathcal{A}_3(Y(t,s(t))) \tag{3.17}$$

with $\mathcal{A}_3(0) = 0$ and $\frac{\partial\mathcal{A}_3}{\partial Y}(0) = \frac{1}{A(\bar{\rho}_- - \bar{\rho}_+)}(x_0)$. It follows from (3.15) and (3.17) that

$$\partial_tY = \mathcal{A}_4(Y_x,Y),\ \text{at } x = s(t), \tag{3.18}$$

where

$$\mathcal{A}_4(0,0) = 0,\ \frac{\partial\mathcal{A}_4}{\partial Y_x}(0,0) = \frac{c^2(\bar{\rho}_+) - \bar{u}_+^2}{2\bar{u}_+}(x_0),\ \frac{\partial\mathcal{A}_4}{\partial Y}(0,0) = -\frac{\bar{E}_+}{2\bar{u}_+}(x_0) - \frac{A'\bar{u}_-}{2A}(x_0).$$

Note that on the right boundary, $x=L$, Y satisfies

$$\partial_x Y = 0, \text{ at } x = L. \tag{3.19}$$

Our goal is to derive uniform estimates for Y and s which satisfy (3.14), (3.17)–(3.19).

To transform the problem to the fixed domain $[x_0, L]$, we introduce the transformation

$$\tilde{t} = t,\ \tilde{x} = (L - x_0)\frac{x - s(t)}{L - s(t)} + x_0,\ \sigma(\tilde{t}) = s(t) - x_0.$$

Set

$$q_1(\tilde{x}, \sigma) = \frac{L - \tilde{x}}{L - x_0 - \sigma(\tilde{t})},\ q_2(\sigma) = \frac{L - x_0}{L - x_0 - \sigma(\tilde{t})}.$$

Then (3.13) can be rewritten as

$$\begin{aligned}
&\sigma''(\tilde{t}) q_1 \partial_{\tilde{x}} Y \\
=&\partial_{\tilde{t}\tilde{t}} Y + (q_1\sigma'(\tilde{t}))^2 \partial_{\tilde{x}\tilde{x}} Y - 2\sigma'(t) q_1 \partial_{\tilde{x}\tilde{t}} Y - q_1 \frac{2(\sigma'(\tilde{t}))^2}{L - x_0 - \sigma(\tilde{t})} \partial_{\tilde{x}} Y \\
&+ A q_2 \partial_{\tilde{x}} \left(p(\bar{\rho}_+) - p(\bar{\rho}_+ + \frac{q_2 Y_{\tilde{x}}}{A}) \right) + q_2 \partial_{\tilde{x}} \left(\frac{\bar{J}^2}{A\bar{\rho}_+} - \frac{(\bar{J} - Y_{\tilde{t}} + \sigma'(\tilde{t}) q_1 Y_{\tilde{x}})^2}{\bar{\rho}_+ + q_2 Y_{\tilde{x}}} \right) \\
&+ \bar{\rho}_+ Y + \bar{E}_+ q_2 \partial_{\tilde{x}} Y + \frac{q_2 Y Y_{\tilde{x}}}{A}.
\end{aligned}$$

By straightforward computation, the equation (3.17) becomes

$$\sigma = \mathcal{A}_3(Y(t, \tilde{x} = x_0)). \tag{3.20}$$

The equation for the shock front, (3.16), becomes

$$\frac{d\sigma}{d\tilde{t}} = \mathcal{A}_2\left(\frac{q_2(\sigma) Y_{\tilde{x}}}{A},\ \sigma(\tilde{t})\right).$$

Applying (3.20) in order to represent the quadratic terms for σ in terms of Y, at $\tilde{x} = x_0$, we have

$$\frac{d\sigma}{d\tilde{t}} + \sigma\left(\frac{\bar{E}_+}{2\bar{u}_+} + \frac{A'\bar{u}_-}{2A}\right)(x_0) = \mathcal{C}_2(Y_{\tilde{x}}, Y), \tag{3.21}$$

where $\mathcal{C}_2$ satisfies

$$\left| \mathcal{C}_2(Y_{\tilde{x}}, Y) + \frac{c^2(\bar{\rho}_+) - \bar{u}_+^2}{2(\bar{\rho}_+ - \bar{\rho}_-)\bar{u}_+ A}(x_0) Y_{\tilde{x}} \right| \le C(Y_{\tilde{x}}^2 + Y^2).$$

It follows from (3.20) and (3.21) that one can represent (σ, σ') in terms of Y and the derivative $Y_{\tilde{x}}$ at $\tilde{x} = x_0$. Thus, by (3.18), (3.20), and (3.21)

$$Y_{\tilde{t}} = \mathcal{C}_1(Y_{\tilde{x}}, Y), \quad \text{at } \tilde{x} = x_0.$$

Or, equivalently,

$$Y_{\tilde{x}} = \mathcal{C}_3(Y_{\tilde{t}}, Y), \quad \text{at } \tilde{x} = x_0,$$

where $\mathcal{C}_3$ satisfies

$$\left|\mathcal{C}_3(Y_{\tilde{t}},Y)-\frac{2\bar{u}_+}{c^2(\bar{\rho}_+)-\bar{u}_+^2}(x_0)Y_{\tilde{t}}-\frac{\bar{E}_++\frac{A'\bar{u}_+\bar{u}_-}{A}}{c^2(\bar{\rho}_+)-\bar{u}_+^2}(x_0)Y\right|\le C(Y_{\tilde{t}}^2+Y^2).$$

We drop $\tilde{}$ both in $\tilde{x}$ and $\tilde{t}$ for simplicity. Then the original problem can be formulated into the following compact form

$$\begin{cases}\mathcal{L}(x,Y,\sigma)Y=\sigma''(t)q_1\partial_x Y, \quad (t,x)\in[0,\infty)\times[x_0,L],\\ \partial_x Y=d_1(Y_t,Y)Y_t+e_1(Y_t,Y)Y, \text{ at } x=x_0,\\ \partial_x Y=0, \text{ at } x=L,\\ \sigma(t)=\mathcal{A}_3(Y(t,x_0)),\end{cases}\tag{3.22}$$

where, by using ξ_0 and ξ_1 to denote t and x, respectively,

$$\mathcal{L}(x,Y,\sigma)Z=\sum_{i,j=0}^{1}a_{ij}(x,Y,\nabla Y,\sigma,\sigma')\partial_{ij}Z+\sum_{i=0}^{1}b_i(x,Y,\nabla Y,\sigma,\sigma')\partial_i Z$$
$$+g(x,Y,\nabla Y,\sigma,\sigma')Z$$

with

$$d_1(Y_t,Y)=\int_0^1\frac{\partial\mathcal{C}_3}{\partial Y_t}(\theta Y_t,\theta Y)d\theta,\ e_1(Y_t,Y)=\int_0^1\frac{\partial\mathcal{C}_3}{\partial Y}(\theta Y_t,\theta Y)d\theta.$$

Furthermore, one has $\mathcal{L}(x,0,0)Z=\mathcal{L}_0 Z$, and

$$a_{00}(x,Y,\nabla Y,\sigma,\sigma')=1,\ a_{01}(x,0,0,0,0)=a_{10}(x,0,0,0,0)=\frac{\bar{J}}{A\bar{\rho}_+}=\bar{u}_+,$$
$$a_{11}(x,0,0,0,0)=-\left(c^2(\bar{\rho}_+)-\bar{u}_+^2\right),$$
$$b_0(x,0,0,0,0)=\partial_x\left(2\bar{u}_+\right),\ b_1(x,0,0,0,0)=-\partial_x\left(c^2(\bar{\rho}_+)-\bar{u}_+^2\right)+\bar{E}_++\frac{A'c^2(\bar{\rho}_+)}{A},$$
$$g(x,0,0,0,0)=\bar{\rho}_+,\ d_1(0,0)=\frac{2\bar{u}_+}{c^2(\bar{\rho}_+)-\bar{u}_+^2}(x_0),\ e_1(0,0)=\frac{\bar{E}_++\frac{A'\bar{u}_+\bar{u}_-}{A}}{c^2(\bar{\rho}_+)-\bar{u}_+^2}(x_0).$$

3.2. Linearized problem. This subsection is devoted to the study of the linearized problem.

LEMMA 3.4. *Let Y be a smooth solution of the linearized problem*

$$\begin{cases}\mathcal{L}(x,0,0)Y=0, & x_0<x<L,\ t>0,\\ \partial_x Y=d_1(0,0)\partial_t Y+e_1(0,0)Y, & \text{at } x=x_0,\\ \partial_x Y=0, & \text{at } x=L,\\ Y(0,x)=h_1(x),\ Y_t(0,x)=h_2(x), & x_0<x<L.\end{cases}\tag{3.23}$$

Then the following dissipation identity holds:

$$\varphi(Y,t)+D(Y,t)=\varphi(Y,0),\tag{3.24}$$

where

$$\varphi(Y,t)=\left(\bar{E}_+\bar{u}_++\frac{A'}{A}\bar{u}_+^2\bar{u}_-\right)(x_0)Y^2(t,x_0)$$
$$+\int_{x_0}^{L}\bar{u}_+\left\{\bar{\rho}_+Y^2+\left(p'(\bar{\rho}_+)-\bar{u}_+^2\right)(\partial_x Y)^2+(\partial_t Y)^2\right\}(t,x)dx$$

and

$$D(Y,t)=2\left(\int_0^t \bar{u}_+^2(\partial_t Y)^2(s,L)ds+\int_0^t \bar{u}_+^2(\partial_t Y)^2(s,x_0)ds\right).$$

Proof. Multiplying the first equation in (3.23) by $\bar{u}_+\partial_t Y$ on both sides and integrating the result equation on $[0,t]\times[x_0,L]$, we have

$$\begin{aligned}
0=&\frac{1}{2}\int_{x_0}^L \bar{u}_+\left\{(\partial_t Y)^2+\left(p'(\bar{\rho}_+)-\bar{u}_+^2\right)(\partial_x Y)^2+\bar{\rho}_+Y^2\right\}(t,x)dx\\
&-\frac{1}{2}\int_{x_0}^L \bar{u}_+\left\{(\partial_t Y)^2+\left(p'(\bar{\rho}_+)-\bar{u}_+^2\right)(\partial_x Y)^2+\bar{\rho}_+Y^2\right\}(0,x)dx\\
&+\int_0^t \left(\bar{u}_+^2Y_t^2+(\bar{u}_+^2-c^2(\bar{\rho}_+))\bar{u}_+Y_xY_t\right)|_{x=x_0}^{x=L}ds\\
&+\int_0^t\int_{x_0}^L\left(\bar{u}_+\bar{E}_+-\partial_x\bar{u}_+(\bar{u}_+^2-c^2(\bar{\rho}_+))+\frac{A'}{A}\bar{u}_+c^2(\bar{\rho}_+)\right)dxds\\
=&\sum_{i=1}^4 I_i.
\end{aligned}\tag{3.25}$$

Since $\partial_x\bar{u}_+=\frac{-\bar{u}_+\bar{E}_+}{c^2(\bar{\rho}_+)-\bar{u}_+^2}-\frac{A'(x)}{A(x)}\frac{\bar{u}_+c^2(\bar{\rho}_+)}{c^2(\bar{\rho}_+)-\bar{u}_+^2}$, we have

$$I_4=0,\tag{3.26}$$

which together with the boundary condition (3.23) implies that

$$I_3=\frac{1}{2}D(Y,t)+\frac{Y^2}{2}(\bar{u}_+\bar{E}_++\frac{A'}{A}\bar{u}_+^2\bar{u}_-)|_{(0,x_0)}^{(t,x_0)}.\tag{3.27}$$

Then the lemma follows from (3.25)–(3.27). ∎

The estimate in Lemma 3.4 implies the decay of the solution to (3.23) is exponential.

LEMMA 3.5. *There exists $\delta>0$ such that if (3.9) holds, then the solution Y of (3.23) satisfies*

$$\varphi(Y,t)\le Ce^{-\lambda_0 t}\varphi(Y,0),$$

and

$$\int_0^\infty e^{\frac{\lambda_0 t}{4}}(|\partial_t Y|^2(t,x_0)+|\partial_t Y|^2(t,L))dt\le C\varphi(Y,0),$$

for some constants $\lambda_0>0$ and $C>0$.

Proof.

Step 1: (The Rauch–Taylor-type estimates). Using the boundary conditions at $x=x_0$ and the fact that $\frac{c^2(\bar{\rho}_+)-\bar{u}_+^2}{2u_+}(x_0)\ge C$ for some constant $C>0$, it is easy to see that

$$D(Y,t)\ge C_1\int_0^t(Y_t^2+Y_x^2)(s,x_0)ds-C_2\int_0^t Y^2(s,x_0)ds,$$

for some positive constants C_1 and C_2 independent of t. Therefore,

$$\varphi(Y,t)+C_1\int_0^t (Y_t^2+Y_x^2)(s,x_0)ds\le\varphi(Y,0)+C_2\int_0^t Y^2(s,x_0)ds.$$

By the ideas in [19], there exist $T>0$ and $\delta\in(0,T/4)$ such that

$$\int_0^T (Y_t^2+Y_x^2)(t,x_0)dt\ge\int_{\frac{T}{2}-\delta}^{\frac{T}{2}+\delta}\varphi(Y,s)ds-C_3\int_0^T Y^2(t,x_0)dt.$$

Note that $\varphi(Y,t)$ is decreasing with respect to t by (3.24). Thus

$$\begin{aligned}\int_0^T (Y_t^2+Y_x^2)(t,x_0)dt\ge&\delta\varphi(Y,\frac{T}{2}+\delta)-C_3\int_0^T Y^2(t,x_0)dt\\ \ge&\delta\varphi(Y,T)-C_3\int_0^T Y^2(t,x_0)dt,\end{aligned}$$

which together with (3.24) and (3.2) gives

$$(1+C_4)\varphi(Y,T)\le\varphi(Y,0)+C_5\int_0^T Y^2(t,x_0)dt$$

for some positive constants C_4 and C_5 independent of t.

Step 2: (The spectrum of the solution operator). Define a new norm $\|\cdot\|_{\mathbf{X}}$ for the function $h=(h_1,h_2)\in H^1\times L^2([x_0,L])$,

$$\begin{aligned}\|h\|_{\mathbf{X}}^2=&(\bar{E}_+\bar{u}_++\frac{A'}{A}\bar{u}_+^2\bar{u}_-)(x_0)|h_1|^2(x_0)\\ &+\int_{x_0}^L \bar{u}_+\left\{|h_2|^2+(p'(\bar{\rho}_+)-\bar{u}_+^2)|h_1'|^2+\bar{\rho}_+|h_1|^2\right\}(x)dx.\end{aligned}$$

By Sobolev embedding theorems, there exists a constant $\delta>0$, such that if (3.9) holds, then the new norm $\|(h_1,h_2)\|_{\mathbf{X}}$ is equivalent to $\|h_1\|_{H^1}+\|h_2\|_{L^2}$.

The associated complex Hilbert space will be denoted by $(\mathbf{X},\|\cdot\|_{\mathbf{X}})$. Define the solution operator $S_t:\mathbf{X}\mapsto\mathbf{X}$ as

$$S_t(h)=(Y(t,\cdot),Y_t(t,\cdot)),$$

where Y is the solution of the problem (3.23) with the initial data $h=(h_1,h_2)$.

According to the standard Fredholm-type lemma in [27], there are only finitely many generalized eigenvalues for the operator S_T in the annulus $\{\frac{1}{1+C_6}<|z|\le1\}$ on the complex plane. Each of these eigenvalues has finite multiplicity. More precisely, by the computations in [19] we can show that the spectrum $\sigma(S_T)$ does not touch the unit circle. Therefore, there exists $0<\beta_0<1$ such that $\sigma(S_T)\subset\{|z|\le\sqrt{\beta_0}\}$, which implies that

$$\varphi(Y,T)\le\beta_0\varphi(Y,0).$$

Step 3: (Exponential decay). Noting that $\varphi(Y,t)$ is decreasing in t, for any $t\in[nT,(n+1)T)$, $n\in\mathbf{N}$, one has

$$\varphi(Y,t)\le\varphi(Y,nT)\le\beta_0^n\varphi(Y,0)\le\beta_0^{\frac{t}{T}-1}\varphi(Y,0)=e^{-\lambda_0 t}\beta_0^{-1}\varphi(Y,0),$$

where we have chosen $\lambda_0=-\frac{\ln\beta_0}{T}$.

According to (3.24),

$$\begin{aligned}&\int_{2^iT}^{2^{i+1}T} e^{\frac{\lambda_0 t}{4}}(|\partial_t Y|^2(t,x_0)+|\partial_t Y|^2(t,L))dt\\ \le& Ce^{\lambda_0 2^{i-1}T}(\varphi(Y,2^iT)-\varphi(Y,2^{i+1}T))\\ \le& Ce^{\lambda_0 2^{i-1}T}\beta_0^{-1}e^{-\lambda_0 2^iT}\varphi(Y,0)=C\beta_0^{-1}e^{-\lambda_0 2^{i-1}T}\varphi(Y,0).\end{aligned}$$

Thus

$$\begin{aligned}&\int_0^\infty e^{\frac{\lambda_0 t}{4}}(|\partial_t Y|^2(t,x_0)+|\partial_t Y|^2(t,L))dt\\ =&\sum_{i=0}^\infty\int_{2^iT}^{2^{i+1}T} e^{\frac{\lambda_0 t}{4}}\sum_{l=1}^{k+1}(|\partial_t Y|^2(t,x_0)+|\partial_t Y|^2(t,L))dt\\ \le& C\sum_{i=0}^\infty \beta_0^{-1}e^{-\lambda_0 T2^{i-1}}\varphi(Y,0)\le C\varphi(Y,0).\end{aligned}$$

Thus completing the proof of the lemma. □

Since the coefficients of the equation and the boundary conditions in (3.23) are independent of t, we can take differentiation with respect to t and apply Lemma 3.5 to $\partial_t^l Y$ to get the following corollary.

COROLLARY 3.6. *Suppose (3.9) holds and $\delta>0$ is chosen as in Lemma 3.5. For any $0\le k\in\mathbf{N}$, define $\varphi_k(Y,t)=\sum_{l=0}^k\varphi(\partial_t^lY,t)$, then*

$$\varphi_k(Y,t)\le Ce^{-\lambda_0 t}\varphi_k(Y,0),$$

and

$$\int_0^\infty e^{\frac{\lambda_0 t}{4}}\sum_{l=1}^{k+1}(|\partial_t^l Y|^2(t,x_0)+|\partial_t^l Y|^2(t,L))dt\le C\varphi_k(Y,0),$$

where $\lambda_0>0$ is the constant in Lemma 3.5.

3.3. Uniform a priori estimates. The existence of local-in-time solutions for the problem (1.1), (3.2)–(3.3) satisfying the conditions (3.1), (3.4)–(3.7) is standard, see [14]. In order to get the global existence, it suffices to derive the global a priori estimate for the problem (3.22) with the initial condition

$$Y(0,x)=h_1(x),\, Y_t(0,x)=h_2(x),\quad x_0<x<L,\quad \text{and}\quad \sigma(0)=\sigma_0. \tag{3.28}$$

The following computations are similar to the computations in [19]. First we choose T large enough such that

$$\alpha_0:=Ce^{-\lambda_0 T}<1,$$

where C,λ_0 are the constants in Lemma 3.5. Then choose an integer $k\ge 15$, and define

$$|||(Y,\sigma)|||=\tilde{|}(Y,\sigma)\tilde{|}+\tilde{\|}(Y,\sigma)\tilde{\|},$$

where

$$\tilde{|}(Y,\sigma)\tilde{|} = \sup_{\tau\in[0,t)} \sum_{0\le m\le k-6} \left(\sum_{0\le l\le m} e^{\frac{\lambda\tau}{16}} \|\partial_t^l \partial_x^{m-l} Y(\tau,\cdot)\|_{L^\infty([x_0,L])} + e^{\frac{\lambda\tau}{16}} \left|\frac{d^m\sigma}{dt^m}\right|(\tau) \right)$$

and

$$\begin{aligned}\tilde{\|}(Y,\sigma)\tilde{\|} = &\sup_{0\le\tau\le t} \left(\sum_{\substack{0\le l\le m,\\ 0\le m\le k}} \|\partial_t^l \partial_x^{m-l} Y(\tau,\cdot)\|_{L^2([x_0,L])} + \sum_{0\le l\le k} \|\partial_t^l \partial_x^{k+1-l} Y(\tau,\cdot)\|_{L^2([x_0,L])} \right)\\
&+ \sup_{0\le\tau\le t} \|\partial_t^{k+1} Y(\tau,\cdot) - \frac{d^{k+1}\sigma}{dt^{k+1}} q_1(\cdot,\sigma)\partial_x Y(\tau,\cdot)\|_{L^2([x_0,L])}\\
&+ \sum_{\substack{0\le l\le m,\\ 0\le m\le k+1}} \left(\|\partial_t^l \partial_x^{m-l} Y(\cdot,x_0)\|_{L^2[0,t]} + \|\partial_t^l \partial_x^{m-l} Y(\cdot,L)\|_{L^2[0,t]} \right)\\
&+ \sum_{0\le m\le k+1} \left\|\frac{d^m\sigma}{dt^m}\right\|_{L^2[0,t]},\end{aligned}$$

with λ will be defined in (3.46).

Furthermore, for any $l\in\mathbb{N}$ and given Y and σ such that $|||(Y,\sigma)|||<\infty$, we define

$$\begin{aligned}
\Phi(Z,t;Y,\sigma) &= a_{11}e_1\bar{u}_+(\partial_t Z)^2(t,x_0) + \int_{x_0}^L \{\bar{u}_+(\partial_t Z)^2 - a_{11}\bar{u}_+(\partial_x Z)^2 + g\bar{u}_+ Z^2\}(t,x)dx,\\
\mathcal{D}(Z,t;Y,\sigma) &= -2\int_0^t \bar{u}_+((a_{11}d_1 + a_{01}))(\partial_t^{l+1}Z)^2(\tau,x_0)d\tau + 2\int_0^t \bar{u}_+ a_{01}(\partial_t^{l+1}Z)^2(\tau,L)d\tau,\\
\Phi_l(Z,t;Y,\sigma) &= \sum_{m=0}^l \Phi(\partial_t^m Z,t;Y,\sigma),\quad \mathcal{D}_l(Z,t;Y,\sigma) = \sum_{m=0}^l \mathcal{D}(\partial_t^m Z,t;Y,\sigma),\\
\hat{\Phi}_l(Z,t;Y,\sigma) &= \Phi_{l-1}(Z,t;Y,\sigma) + \Phi_0(\partial_t^l Z - q_1(x,\sigma)Y_x\frac{d^l\sigma}{dt^l},t;Y,\sigma),\\
\hat{\mathcal{D}}_l(Z,t;Y,\sigma) &= \mathcal{D}_{l-1}(Z,t;Y,\sigma) + \mathcal{D}_0(\partial_t^l Z - q_1(x,\sigma)Y_x\frac{d^l\sigma}{dt^l},t;Y,\sigma).
\end{aligned}$$

It is easy to see that if $|||(Y,\sigma)|||\le\epsilon$ for some $\epsilon>0$, then

$$\Phi(Z,t;Y,\sigma)(t) \ge C\int_{x_0}^L (\partial_t Z)^2 + (\partial_x Z)^2 + Z^2)(t,x)dx,$$

for some constant $C>0$ independent of t.

PROPOSITION 3.7. *Assume (3.9) holds, then there exists an $\epsilon_0>0$ such that for any $\epsilon\in(0,\epsilon_0)$, if (Y,σ) is a smooth solution of the problem (3.22) and (3.28) with (h_1,h_2) and σ_0 satisfying*

$$|\sigma_0| + \|h_1\|_{H^{k+2}} + \|h_2\|_{H^{k+1}} \le \epsilon^2 \le \epsilon_0^2$$

and $|||(Y,\sigma)|||\le\epsilon$ for $t>T$, then

$$|||(Y,\sigma)||| \le \frac{\epsilon}{2}. \tag{3.29}$$

Proof. **Step 1: (Lower order energy estimates).** Taking the mth $(0\leq m\leq k-1)$ order derivative for the equation (3.22) with respect to t, then

$$\mathcal{L}(x,Y,\sigma)\partial_t^m Y=\mathcal{F}_m(x,Y,\sigma)+\tilde{\mathcal{F}}_m(x,Y,\sigma), \tag{3.30}$$

where

$$\mathcal{F}_m(x,Y,\sigma)=\sum_{1\leq l\leq m}C_m^l\left\{-\sum_{i,j=0}^{1}\partial_t^l a_{ij}\partial_{ij}\partial_t^{m-l}Y-\sum_{i=0}^{1}\partial_t^l b_i\partial_i\partial_t^{m-l}Y-\partial_t^l g\partial_t^{m-l}Y\right\},$$

and

$$\tilde{\mathcal{F}}_m(x,Y,\sigma)=\sum_{0\leq l\leq m}C_m^l\frac{d^{l+2}\sigma}{dt^{l+2}}\partial_t^{m-l}\left(q_1(x,\sigma)Y_x\right),$$

with the standard binomial coefficient $C_m^l=\begin{pmatrix}m\\ l\end{pmatrix}$.

Multiplying both sides of (3.30) by $\bar{u}_+\partial_t^{m+1}Y$ and integrating the result equation on $\Omega=:[0,t]\times[x_0,L]$ implies

$$\begin{aligned}
&\iint_\Omega \mathcal{L}(x,Z,\sigma)\partial_t^m Y\bar{u}_+\partial_t^{m+1}Y(\tau,x)d\tau dx\\
=&\int_{x_0}^{L}\frac{\bar{u}_+}{2}\left((\partial_t^{m+1}Y)^2-a_{11}(\partial_x\partial_t^m Y)^2+g(\partial_t^m Y)^2\right)(t,x)dx\\
&+\iint_\Omega\left(b_0\bar{u}_+-\partial_x\left(a_{01}\bar{u}_+\right)\right)(\partial_t^{m+1}Y)^2(\tau,x)d\tau dx\\
&+\iint_\Omega\left(b_1\bar{u}_+-\partial_x\left(a_{11}\bar{u}_+\right)\right)\partial_t^m\partial_x Y\partial_t^{m+1}Y(\tau,x)d\tau dx\\
&+\iint_\Omega \bar{u}_+\partial_t a_{11}\frac{(\partial_x\partial_t^m Y)^2}{2}(t,x)-\bar{u}_+\partial_t g\frac{(\partial_t^m Y)^2}{2}(\tau,x)d\tau dx\\
&+\int_0^t\left(a_{11}\bar{u}_+\partial_x\partial_t^m Y\partial_t^{m+1}Y+a_{01}\bar{u}_+(\partial_t^{m+1}Y)^2\right)(\tau,L)dt\\
&-\int_0^t\left(a_{11}\bar{u}_+\partial_x\partial_t^m Y\partial_t^{m+1}Y+a_{01}\bar{u}_+(\partial_t^{m+1}Y)^2\right)(\tau,x_0)d\tau\\
&-\int_{x_0}^{L}\frac{\bar{u}_+}{2}((\partial_t^{m+1}Y)^2-a_{11}(\partial_x\partial_t^m Y)^2+g(\partial_t^m Y)^2)(0,x)dx\\
=&\sum_{i=1}^{7}J_i.
\end{aligned}$$

Since the coefficients in the integrals J_2 and J_3 vanish at $(Y,\sigma)=(0,0)$, we have

$$\begin{aligned}
|J_2+J_3|\leq& C\int_0^t\left[\sum_{l=0}^{2}\left(\sum_{i=0}^{l}\|\partial_t^i\partial_x^{l-i}Y\|_{L^\infty([x_0,L])}+|\partial_t^l\sigma|(\tau)\right)\right]\\
&\times\sum_{l=1}^{m+1}\sum_{i=0}^{1}\|\partial_t^{l-i}\partial_x^i Y(\tau,\cdot)\|^2_{L^2[x_0,L]}d\tau
\end{aligned}$$

$$
\begin{aligned}
&\leq C \sup_{0\leq\tau\leq t}\left| e^{\frac{\lambda\tau}{64}} \left[\sum_{l=0}^{2}\left(\sum_{i=0}^{l}\|\partial_t^i\partial_x^{l-i}Y\|_{L^\infty([x_0,L])}+|\partial_t^l\sigma|(\tau)\right)\right]\right| \\
&\quad\times\int_0^t e^{-\frac{\lambda\tau}{64}}\tilde{\Phi}_m(Y,\tau;Y,\sigma)d\tau \\
&\leq C|||(Y,\sigma)|||\int_0^t e^{-\frac{\lambda\tau}{64}}\tilde{\Phi}_m(Y,\tau;Y,\sigma)d\tau. \qquad (3.31)
\end{aligned}
$$

Similarly,

$$
|J_4|\leq C|||(Y,\sigma)|||\int_0^t e^{-\frac{\lambda\tau}{64}}\tilde{\Phi}_m(Y,\tau;Y,\sigma)d\tau. \qquad (3.32)
$$

The boundary condition $Y_x(t,L)=0$ implies

$$
J_5=\int_0^t \frac{a_{01}}{\bar{\rho}_+}(\partial_t^{m+1}Y)^2(\tau,L)dt>0, \qquad (3.33)
$$

if $|||(Y,\sigma)|||$ is sufficiently small.

Differentiating the boundary condition

$$
\partial_x Y=d_1(Y_t,Y)Y_t+e_1(Y_t,Y)Y
$$

m times with respect to t, one has

$$
\partial_t^m Y_x=d_1\partial_t^{m+1}Y+e_1\partial_t^m Y+\mathcal{G}_m,\ \text{at } x=x_0,
$$

where $\mathcal{G}_m$ satisfies

$$
|\mathcal{G}_m|\leq C\left(\sum_{l=0}^{[\frac{m}{2}]+1}\sum_{i=0}^{l}(|\partial_t^l Y||\partial_t^{m+1-i}Y|+\ |\partial_t^l Y||\partial_t^{m-i}Y|)\right).
$$

Therefore,

$$
\begin{aligned}
J_6=&-\int_0^t \bar{u}_+(a_{11}d_1+a_{01})(\partial_t^{m+1}Y)^2(\tau,x_0)+\left(a_{11}e_1\bar{u}_+\partial_t^m Y\partial_t^{m+1}Y\right)(t,x_0)d\tau \\
&-\int_0^t \mathcal{G}_m a_{11}\bar{u}_+\partial_t^{m+1}Y(\tau,x_0)d\tau \\
\geq&-\int_0^t \bar{u}_+(a_{11}d_1+a_{01})(\partial_t^{m+1}Y)^2(\tau,x_0)(\tau,x_0)d\tau-\left(a_{11}e_1\bar{u}_+\frac{(\partial_t^m Y)^2}{2}\right)(t,x_0) \\
&+\left(a_{11}e_1\bar{u}_+\frac{(\partial_t^m Y)^2}{2}\right)(0,x_0)-C|||(Y,\sigma)|||\int_0^t\sum_{l=0}^{m+1}(\partial_t^l Y)^2(\tau,x_0)d\tau. \qquad (3.34)
\end{aligned}
$$

Summing up all the estimates in (3.31)–(3.34) yields

$$
\begin{aligned}
&\iint_\Omega \mathcal{L}(x,Y,\sigma)\partial_t^m Y\bar{u}_+\partial_t^{m+1}Y\,dtdx \\
\geq&\frac{1}{2}\Phi_m(Y,t;Y,\sigma)-\frac{1}{2}\Phi_m(Y,0;Y,\sigma)+\frac{1}{2}\mathcal{D}_m(Y,t;Y,\sigma)
\end{aligned}
$$

$$-C|||(Y,\sigma)|||\left[\int_0^t e^{-\frac{\lambda\tau}{64}}\tilde{\Phi}_m(Y,\tau;Y,\sigma)d\tau+\mathcal{D}_m(Y,t;Y,\sigma)\right]. \tag{3.35}$$

On the other hand, it holds that

$$\begin{aligned}&\left|\iint_\Omega \mathcal{F}_m\bar{u}_+\partial_t^{m+1}Y\,dtdx\right|\\ \le& C|||(Y,\sigma)|||\int_0^t e^{-\frac{\lambda\tau}{64}}\left[\tilde{\Phi}_m(Y,\tau;Y,\sigma)+|||(Y,\sigma)|||\sum_{l=0}^{m+1}\left|\frac{d^l\sigma}{dt^l}\right|^2\right]d\tau\end{aligned} \tag{3.36}$$

and

$$\left|\iint_\Omega \tilde{\mathcal{F}}_m\frac{\partial_t^{m+1}Y}{\bar{\rho}_+}dtdx\right|\le C|||(Y,\sigma)|||\int_0^t e^{-\frac{\lambda\tau}{64}}\left[\tilde{\Phi}_m(Y,\tau;Y,\sigma)+\sum_{l=0}^{m+2}\left|\frac{d^l\sigma}{dt^l}\right|^2\right]d\tau, \tag{3.37}$$

where the following estimate

$$\left\|\frac{d^{l+2}\sigma}{dt^{l+2}}\right\|_{L^2([x_0,L])}\le C\left|\frac{d^{l+2}\sigma}{dt^{l+2}}\right|$$

has been used. Combining the estimates in (3.35)–(3.37), one has

$$\begin{aligned}&\Phi_m(Y,t;Y,\sigma)+\mathcal{D}_m(Y,t;Y,\sigma)\\ \le&\Phi_m(Y,0;Y,\sigma)+C|||(Y,\sigma)|||\left[\int_0^t e^{-\frac{\lambda\tau}{64}}\left[\Phi_m(Y,\tau;Y,\sigma)+\sum_{l=0}^{m+2}\left|\frac{d^l\sigma}{dt^l}\right|^2\right]d\tau+\mathcal{D}_m(Y,t;Y,\sigma)\right],\end{aligned} \tag{3.38}$$

for $m=0,1,\dots,k-1$.

Step 2: (The highest order energy estimates). Take the kth order derivative for the Equation (3.22) with respect to t, then

$$\mathcal{L}(x,Y,\sigma)\partial_t^kY=\mathcal{F}_k(x,Y,\sigma)+\frac{d^{k+2}\sigma}{dt^{k+2}}q_1(x,\sigma)Y_x+\sum_{0\le l\le k-1}C_k^l\frac{d^{l+2}\sigma}{dt^{l+2}}\partial_t^{k-l}\left(q_1(x,\sigma)Y_x\right).$$

In order to handle the term $\frac{d^{k+2}\sigma}{dt^{k+2}}$, we consider the equation for $\check{Y}=\partial_t^kY-q_1(x,\sigma)Y_x\frac{d^k\sigma}{dt^k}$,

$$\mathcal{L}(x,Y,\sigma)\check{Y}=\mathcal{F}_k(x,Y,\sigma)+\check{\mathcal{F}}(x,Y,\sigma), \tag{3.39}$$

where

$$\begin{aligned}\check{\mathcal{F}}(x,Y,\sigma)=&-2\frac{d^{k+1}\sigma}{dt^{k+1}}\partial_t\left(q_1(x,\sigma)Y_x\right)-\frac{d^k\sigma}{dt^k}\partial_t^2\left(q_1(x,\sigma)Y_x\right)\\ &-\left(2a_{01}\partial_t\partial_x+a_{11}\partial_x^2+\sum_{i=0}^1b_i\partial_i+g\right)\left(\frac{d^k\sigma}{dt^k}q_1(x,\sigma)Y_x\right).\end{aligned}$$

Multiplying both sides of (3.39) by $\bar{u}_+\partial_t\check{Y}$ and integrating on Ω, one has

$$\Phi_0(\check{Y},t;Y,\sigma)+\mathcal{D}_0(\check{Y},t;Y,\sigma)$$

$$\leq \Phi_0(\check{Y},0;Y,\sigma)+C|||(Y,\sigma)|||\left[\int_0^t e^{-\frac{\lambda\tau}{64}}\left[\hat{\Phi}_k(Y,\tau;Y,\sigma)+\sum_{l=0}^{k+1}\left|\frac{d^l\sigma}{dt^l}\right|^2\right]d\tau+\hat{\mathcal{D}}_k(Y,t;Y,\sigma)\right]. \tag{3.40}$$

Then (3.40) and (3.38) imply

$$\begin{aligned}&\hat{\Phi}_k(Y,t;Y,\sigma)+\hat{\mathcal{D}}_k(Y,t;Y,\sigma)\\ \leq&\hat{\Phi}_k(Y,0;Y,\sigma)+C|||(Y,\sigma)|||\left\{\int_0^t e^{-\frac{\lambda\tau}{64}}\left[\hat{\Phi}_k(Y,\tau;Y,\sigma)+\sum_{l=0}^{k+1}\left|\frac{d^l\sigma}{dt^l}\right|^2\right]d\tau+\hat{\mathcal{D}}_k(Y,t;Y,\sigma)\right\}.\end{aligned} \tag{3.41}$$

Step 3: (The boundedness of the energy). Differentiating the equation for the shock front

$$\sigma(t)=\mathcal{A}_3(Y(t,x_0)) \tag{3.42}$$

with respect to t, we have

$$\sum_{l=0}^{k+1}\left|\frac{d^l\sigma}{dt^l}\right|^2(\tau)\leq C\left(|\sigma(\tau)|^2+\sum_{l=0}^{k+1}|\partial_t^l Y(\tau,x_0)|^2\right). \tag{3.43}$$

Therefore,

$$\sum_{l=0}^{k+1}\left|\frac{d^l\sigma}{dt^l}\right|^2\leq C\sum_{l=1}^{k}|\partial_t^l Y(\tau,x_0)|^2+|\partial_t\check{Y}|^2,$$

which together with (3.43) gives

$$\int_0^t e^{-\frac{\lambda\tau}{64}}\sum_{l=0}^{k+1}\left|\frac{d^l\sigma}{dt^l}\right|^2 d\tau\leq C\left(\int_0^t e^{-\frac{\lambda\tau}{64}}\Phi_0(Y,\tau;Y,\sigma)d\tau+\hat{\mathcal{D}}_k(Y,t;Y,\sigma)\right).$$

Thus, the energy estimate (3.41) is equivalent to

$$\begin{aligned}&\hat{\Phi}_k(Y,t;Y,\sigma)+\hat{\mathcal{D}}_k(Y,t;Y,\sigma)\\ \leq&\hat{\Phi}_k(Y,0;Y,\sigma)+C|||(Y,\sigma)|||\left(\hat{\mathcal{D}}_k(Y,\tau;Y,\sigma)+\int_0^t e^{-\frac{\lambda\tau}{64}}\hat{\Phi}_k(Y,\tau;Y,\sigma)d\tau\right).\end{aligned}$$

If $|||(Y,\sigma)|||\leq\epsilon$, then

$$\hat{\Phi}_k(Y,t;Y,\sigma)+\hat{\mathcal{D}}_k(Y,t;Y,\sigma)\leq C\hat{\Phi}(Y,0;Y,\sigma)\leq C\epsilon^4.$$

This yields

$$\tilde{\|}(Y,\sigma)\tilde{\|}\leq C\left(\sup_{0\leq\tau\leq t}\hat{\Phi}_k^{1/2}(Y,\tau;Y,\sigma)+\hat{\mathcal{D}}_k^{1/2}(Y,t;Y,\sigma)\right)\leq C\epsilon^2\leq\frac{\epsilon}{4}. \tag{3.44}$$

Step 4: (The decay of the lower energy and the shock position). The basic idea is to estimate the deviation of the solution Y to the nonlinear problem (3.22) from the solution $\bar{Y}$ to the linear problem (3.23).

At time $\tau=t_0$, choose $\bar{h}_1\in H^k$ and $\bar{h}_2\in H^{k-1}$ such that there exists a solution $\bar{Y}\in C^{k-1-i}([t_0,\infty);H^i([x_0,L]))$ of the linear problem (3.23) satisfying $\bar{Y}(t_0,\cdot)=\bar{h}_1$ and $\bar{Y}_t(t_0,\cdot)=\bar{h}_2$. Additionally $\bar{Y}$ satisfies

$$\sum_{l=0}^{k-1}\sum_{i=0}^{l}\|\partial_t^i\partial_x^{l-i}\bar{Y}(t_0,\cdot)\|_{L^2[x_0,L]}\le C|||(Y,\sigma)|||$$

for some uniform constant C, and

$$\hat{\Phi}_{k-4}(Y-\bar{Y},t_0;Y,\sigma)\le C|||(Y,\sigma)|||\hat{\Phi}_{k-4}(Y,t_0;Y,\sigma).$$

Note that $Y-\bar{Y}$ satisfies the equation

$$\begin{aligned}&\sum_{i,j=0}^{1}a_{ij}(x,Y,\sigma)\partial_{ij}(Y-\bar{Y})+\sum_{j=0}^{1}b_j(x,Y,\sigma)\partial_j(Y-\bar{Y})+g(x,Y,\sigma)(Y-\bar{Y})\\&=\sum_{i,j=0}^{1}(a_{ij}(x,Y,\sigma)-a_{ij}(x,0,0))\partial_{ij}\bar{Y}+\sum_{j=0}^{1}(b_j(x,Y,\sigma)-b_j(x,0,0))\partial_j\bar{Y}\\&\quad+(g(x,Y,\sigma)-g(x,0,0))\bar{Y}+\sigma''(t)q_1(x,\sigma)\partial_xY,\end{aligned}$$

and the boundary conditions

$$\begin{cases}\partial_x(Y-\bar{Y})=d_1(Y_t,Y)\partial_t(Y-\bar{Y})+e_1(Y_t,Y)(Y-\bar{Y})\\ \qquad\qquad+(d_1(Y_t,Y)-d_1(0,0))\partial_t\bar{Y}+(e_1(Y_t,Y)-e_1(0,0))\bar{Y}, \text{ at } x=x_0,\\ \partial_x(Y-\bar{Y})=0, \text{ at } x=L.\end{cases}$$

Then the energy estimate for $Y-\bar{Y}$ implies:

$$\begin{aligned}&\hat{\Phi}_{k-4}(Y-\bar{Y},t_0+T;Y,\sigma)+\hat{\mathcal{D}}_{k-4}(Y-\bar{Y},t_0+T;Y,\sigma)-\hat{\mathcal{D}}_{k-4}(Y-\bar{Y},t_0;Y,\sigma)\\&\le\hat{\Phi}_{k-4}(Y-\bar{Y},t_0;Y,\sigma)+C|||(Y,\sigma)|||\Bigg[\int_{t_0}^{t_0+T}\sum_{l=0}^{k-3}\left(\left|\frac{d^l\sigma}{dt^l}\right|^2+\|\partial_t^i\partial_x^{k-3-i}(Y-\bar{Y})\|_{L^2}^2\right)d\tau\\&\quad+\int_{t_0}^{t_0+T}\left(\hat{\Phi}_{k-4}^{1/2}(Y,\tau;Y,\sigma)\hat{\Phi}_{k-4}^{1/2}(Y-\bar{Y},\tau;Y,\sigma)+\hat{\Phi}_{k-4}(Y-\bar{Y},\tau;Y,\sigma)\right)d\tau\\&\quad+(\hat{\mathcal{D}}_{k-4}(Y-\bar{Y},t_0+T;Y,\sigma)-\hat{\mathcal{D}}_{k-4}(Y-\bar{Y},t_0;Y,\sigma))\\&\quad+(\hat{\mathcal{D}}_{k-4}(Y,t_0+T;Y,\sigma)-\hat{\mathcal{D}}_{k-4}(Y,t_0;Y,\sigma))\Bigg].\end{aligned}$$

Using the contraction of the energy for $\bar{Y}$ and noting that the derivation of Y and $\bar{Y}$ at t_0 is of higher order, one has

$$\frac{34+30\alpha_0}{64}\hat{\Phi}_{k-4}(Y,t_0+T;Y,\sigma)\le\frac{2+30\alpha_0}{32}\hat{\Phi}_{k-4}(Y,t_0;Y,\sigma) \tag{3.45}$$

if ϵ is sufficiently small. Using the method in Lemma 3.5, one obtains immediately from (3.45) and the shock front equation (3.42) that

$$\hat{\Phi}_{k-4}(Y_t,t;Y,\sigma)+\sigma^2(t)\le C(\tilde{\Phi}_{k-4}(Y,0;Y,\sigma)+\sigma^2(0))e^{-2\lambda t},$$

where

$$\lambda = -\frac{\ln \frac{1+\alpha}{2}}{2T}, \text{ and } \alpha = \frac{2+30\alpha_0}{17+15\alpha_0}. \tag{3.46}$$

Thus

$$\sum_{l=0}^{k-6} \|Y(t,\cdot)\|_{L^\infty[x_0,L]} \le C\epsilon^2 e^{-\lambda t}, \tag{3.47}$$

provided $\hat{\Phi}_{k-4}(Y,0;Y,\sigma) \le \epsilon^4$. This yields that

$$\sum_{l=0}^{k-6} \left| \frac{d^l\sigma}{dt^l} \right| \le C\epsilon^2 e^{-\lambda t}. \tag{3.48}$$

By combining (3.44), (3.47) and (3.48), (3.29) holds. This completes the proof. □

Proposition 3.7, together with the local existence, implies Theorem 3.2 by a continuation argument.

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