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GLOBALLY SMOOTH SOLUTION AND BLOW-UP PHENOMENON FOR A NONLINEARLY COUPLED SCHRÖDINGER SYSTEM IN ATOMIC BOSE–EINSTEIN CONDENSATES*

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Abstract. In this paper, we study the nonlinearly coupled Schrödinger equations for atomic Bose–Einstein condensates. By the Galërkin method and a priori estimates, the global existence of a smooth solution is obtained. Under some assumptions of the coefficients and p, the blow-up theorem is established.

Key words. Galërkin method, locally smooth solution, globally smooth solution, a priori estimates, blow-up solution.

AMS subject classifications. 35E15, 35Q53.

1. Introduction

In this paper we consider the following nonlinearly coupled Schrödinger system

$$\begin{cases} i\hbar u_t = \left(-\frac{\hbar^2}{2M}\Delta+\lambda_u|u|^2+\lambda|v|^2+g_{11}|u|^{2p}+g|u|^{p-1}|v|^{p+1}\right)u+\sqrt{2}\alpha\bar{u}v, \\ i\hbar v_t = \left(-\frac{\hbar^2}{4M}\Delta+\varepsilon+\lambda_v|v|^2+\lambda|u|^2+g|u|^{p+1}|v|^{p-1}+g_{22}|v|^{2p}\right)v+\frac{\alpha}{\sqrt{2}}u^2, \end{cases} \tag{1.1}$$

with the initial condition and periodic boundary conditions

$$u(x,0)=u_0(x), \qquad v(x,0)=v_0(x), \qquad x\in\Omega, \tag{1.2}$$

$$u(x+2L,t)=u(x,t), \qquad v(x+2L,t)=v(x,t), \quad x\in\Omega, \quad t\geq 0, \tag{1.3}$$

where $\Delta=\frac{\partial^2}{\partial x^2}$; $i=\sqrt{-1}$; $p\geq 1$; $L>0$, $\Omega=(-L,L)$; $\hbar$ is Planck constant; $M>0$ is the mass of a single atom; λ_u, λ_v, λ represent the strengths of the atom-atom, molecule-molecule, and atom-molecule interactions, respectively; ε, α are real constants.

It was an eagerly anticipated event when the Ketterle's group found the Feschbach resonances in the inter-particle interactions of a dilute Bose–Einstein condensate of Na-atoms at MIT [22]. Since all quantities of interest in the atomic BECs crucially depend on the scattering length, a tunable interaction suggests very interesting studies of the many-body behavior of condensate systems. Then, for the time evolution of the dilute single-condensate system, the corresponding equation becomes the Gross–Pitaevskii equation, which is the Dirac time-dependent variational scheme.

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When $p=\hbar=M=1$, system (1.1) is similar to the following coupled Gross–Pitaevskii equations

$$\begin{cases} iu_t=\left(-\frac{\nabla^2}{2}+V(x)+(\lambda_u|u|^2+\lambda|v|^2)\right)u+\alpha v, \\ iv_t=\left(-\frac{\nabla^2}{2}+V(x)+\varepsilon+(\lambda_v|v|^2+\lambda|u|^2)\right)v+\alpha u. \end{cases} \tag{1.4}$$

Several authors are interested in the existence of a solution for this problem. In 2011, Weizhu Bao and Yongyong Cai [4] proved the existence and uniqueness results for the ground states of the above coupled Gross–Pitaevskii equations and obtained the limiting behavior of the ground states with large parameters. Tingchun Wang and Xiaofei Zhao [23] also studied this problem. They proposed and analyzed the finite difference methods for solving (1.4) in two dimensions.

The single Gross–Pitaevskii equation has been considered by many authors. In 2003, Weizhu Bao, Dieter Jaksch, and Peter A. Markowich [6] obtained the numerical solution of the time-dependent Gross–Pitaevskii equation (TGPE). In [5], Weizhu Bao and Weijun Tang proposed a new numerical method to compute the ground-state solution of trapped interacting Bose–Einstein condensation at zero or very low temperature by directly minimizing the energy functional via finite element approximation. Further discussion can be found in [2, 3, 18, 19].

However, to our knowledge, the TGPE has not yet been fully studied. When $p=1$, system (1.1) reduces to

$$\begin{cases} i\hbar u_t=\left(-\frac{\hbar^2}{2M}\Delta+a|u|^2+b|v|^2\right)u+\sqrt{2}\alpha\bar{u}v, \\ i\hbar v_t=\left(-\frac{\hbar^2}{4M}\Delta+\varepsilon+b|u|^2+c|v|^2\right)v+\frac{\alpha}{\sqrt{2}}u^2, \end{cases}. \tag{1.5}$$

These coupled nonlinear equations replace the usual Gross–Pitaevskii equation that describes the time evolution of the dilute single-condensate system [12, 17, 22]. In this paper, we consider system (1.1), which is a more general problem than (1.5), as a mathematical model in nonlinear partial differential equations. Our aim is to obtain the globally smooth solution of system (1.1) and, under some conditions, to establish the blow-up theorem for the case of $p\geq 2$. The main difficulty is to establish certain delicate a priori estimates that govern our strategy to prove the existence of the smooth solution.

In [13–16], the initial value problem and the periodic boundary value problem were studied by Boling Guo for a class of systems of standard nonlinear Schrödinger equations. In this paper, we prove the existence and uniqueness of the global solution to the periodic boundary value problem for the nonlinearly coupled Schrödinger system (1.1) by using the Faedo–Galëkin method.

Since the a priori estimates of the solution to system (1.1)–(1.3) are unconcerned with the period L, we can derive the global smooth solution as $L\to\infty$, a.e. $x\in R$. The global smooth solution to the periodic boundary value problem for the system (1.1) is proved in Theorem 1.2 and for the case of $x\in R$ it is established in Theorem 1.3

Before starting the main results, we review the notation and the calculus inequalities used in this paper.

To simplify the notation in this paper we shall denote by $\int U(x)dx$ the integration $\int_\Omega U(x)dx$ and let C be a generic constant, which may assume different values in different formulas.

Let $L^m(\Omega), 1\le m\le\infty$ be the classical Lebesgue space with the norm

$$\|u\|_m=\left(\int_\Omega |u|^m dx\right)^{\frac{1}{m}}, \qquad (1\le m<\infty),$$

$$\|u\|_\infty = ess.sup.\{|u(x)|: x\in\Omega\}, \qquad (m=\infty).$$

The usual L^2 inner product is $(u,v)=\int_\Omega u\overline{v}dx$, where $\overline{v}$ denotes the complex conjugate of v and the norm of L^2 is $\|u\|_2=\sqrt{(u,u)}$.

Denote by $H^m(\Omega), m=1,2,\ldots$ the Sobolev space of complex-valued functions with the norm

$$\|u\|_{H^m}=\left(\int_\Omega \sum_{|\alpha|\le m}|D^\alpha u|^2 dx\right)^{\frac{1}{2}}.$$

Define $\Lambda=\{u\in H^1(R): |x|u\in L^2(R)\}$ with norm $\|u\|_\Lambda^2 dx=\int_R(|\nabla u|^2+|x|^2|u|^2)dx$ and $\Lambda\times\Lambda$ is simply denoted by Λ^2.

The following auxiliary lemmas will be needed.

Lemma 1.1 (The Gagliardo–Nirenberg inequality). *Assume $u\in L^q(R)$, $\partial_x^m u\in L^r(R)$, $1\le q,\ r\le\infty$. Let p and α satisfy*

$$\frac{1}{p}=j+\alpha(\frac{1}{r}-m)+(1-\alpha)\frac{1}{q}; \quad \frac{j}{m}\le\alpha\le 1.$$

Then,

$$\|\partial_x^j u\|_p\le C(p,m,j,q,r)\|\partial_x^m u\|_r^\alpha\|u\|_q^{1-\alpha}. \tag{1.6}$$

In particular, when $m=1$, $j=0$, $p=4$, $r=2$, $q=2$, we have

$$\|u\|_4^4\le C\|u_x\|_2^1\|u\|_2^3, \tag{1.7}$$

$$\|u\|_{2p+2}^{2p+2}\le C\|u_x\|_2^p\|u\|_2^{p+2}. \tag{1.8}$$

Lemma 1.2 (The Gronwall inequality). *Let c be a constant, and $b(t),u(t)$ be nonnegative continuous functions in the interval [0,T] satisfying*

$$u(t)\le c+\int_0^t b(\tau)u(\tau)d\tau, \qquad t\in[0,T].$$

Then, $u(t)$ satisfies the estimate

$$u(t)\le c\exp\left(\int_0^t b(\tau)d\tau\right), \qquad \textit{for} \quad t\in[0,T]. \tag{1.9}$$

The main result of this paper is stated in the following theorem.

Theorem 1.1. *Let the initial data $(u_0(x),v_0(x))\in H^m_{per}(\Omega)\times H^m_{per}(\Omega)$, and $m>\frac{1}{2}$. Then there exists a $T_0>0$ such that system (1.1)–(1.3) has a unique solution (u,v), which satisfies*

$$(u,v)\in L^\infty([0,T_0];H^m_{per}(\Omega))^2. \tag{1.10}$$

THEOREM 1.2. *Let the initial data* $(u_0(x),v_0(x))\in H^m_{per}(\Omega)\times H^m_{per}(\Omega)$ *and* $m>\frac{1}{2}$. *Suppose one of the following conditions holds:*

(i) $p\geq 2,\quad g_{11},\quad g_{22}\geq 0,\ \begin{pmatrix} g_{11} & g \\ g & g_{22}\end{pmatrix}$ *is positive definite,*

(ii) $1\leq p<2$.

Then, for all $T>0$, *system (1.1)–(1.3) has a uniquely global solution*

$$(u,v)\in L^\infty\left([0,T);H^m_{per}(\Omega)\right)^2. \tag{1.11}$$

THEOREM 1.3. *Let the initial data* $(u_0(x),v_0(x))\in H^m(R)\times H^m(R)$ *and* $m>\frac{1}{2}$. *If one of the following conditions holds*

(i) $p\geq 2,\quad g_{11},\quad g_{22}\geq 0,\ \begin{pmatrix} g_{11} & g \\ g & g_{22}\end{pmatrix}$ *is positive definite,*

(ii) $1\leq p<2$,

then the system (1.1)–(1.2) has a uniquely global solution

$$(u,v)\in L^\infty_{loc}([0,\infty);H^m(R))^2. \tag{1.12}$$

THEOREM 1.4. *Let* $p\geq 2$ *and* $(u,v)\in\Lambda^2$. *Let* $g_{11},\ g_{22}\geq 0,\ \begin{pmatrix} g_{11} & g \\ g & g_{22}\end{pmatrix}$ *be negative definite. If* $\epsilon>|\frac{3\sqrt{2}}{8}\alpha|$, $\lambda_u>|\frac{3\sqrt{2}}{2}\alpha|$, $\hbar,\lambda_v,\lambda>0$ *and one of the following conditions holds*

(i) $E_0=E(u_0,v_0)<0$,

(ii) $E_0=0$ *and* $Im\int(x\overline{u}_0v_{x0}+x\overline{v}_{x0}u_0)<0$,

(iii) $E_0>0$ *and* $Im\int(x\overline{u}_0v_{x0}+x\overline{v}_0u_{x0})<-\frac{2}{\hbar}\sqrt{h(0)E_0}$,

then solution (u,v) *of system (1.1) blows up in a finite time, i.e. there exists* $T_*>0$ *such that*

$$\lim_{t\to T_*^-}\int(|u_x|^2+|v_x|^2)dx=+\infty. \tag{1.13}$$

Theorem 1.1 can be easily proved by the Galërkin method (see, e.g., [11]). The detailed proof is omitted here.

2. The global existence of smooth solution

In this section, we give the demonstration of *a priori* estimates that guarantee the existence of the global smooth solution of system (1.1)–(1.3). Also, we get the uniqueness of the solution.

LEMMA 2.1. *Let* $u_0(x)\in L^2(\Omega),v_0(x)\in L^2(\Omega)$ *and* (u,v) *be a locally smooth solution of system (1.1) with initial data* (u_0,v_0). *Then, we have the identity*

$$\|u(x,t)\|_2^2+2\|v(x,t)\|_2^2\equiv\|u_0(x)\|_2^2+2\|v_0(x)\|_2^2. \tag{2.1}$$

Proof. Taking the inner product for the first equation of system (1.1) with u and the second equation with v, respectively, and integrating the resulting equations with respect to x on Ω, and then taking the imaginary part of the resulting equations, we obtain

$$\begin{cases} \frac{\hbar}{2}\frac{d}{dt}\|u\|_2^2 = \sqrt{2}\alpha \mathrm{Im}\int (\overline{u})^2 v dx, \\ \frac{\hbar}{2}\frac{d}{dt}\|v\|_2^2 = \frac{\alpha}{\sqrt{2}}\mathrm{Im}\int u^2 \overline{v} dx. \end{cases} \tag{2.2}$$

Multiplying the second equation of system (2.2) by 2 and then summing up the first equation, it follows that

$$\frac{\hbar}{2}\frac{d}{dt}\|u\|_2^2 + \hbar\frac{d}{dt}\|v\|_2^2 = 0,$$

which implies identity (2.1). ∎

Lemma 2.2. *Under the conditions of Lemma 2.1, if* $(u_0,v_0)\in H^1_{per}(\Omega)\times H^1_{per}(\Omega)$, $M>0$, *and one of the following conditions holds*

(*i*) $p\geq 2$, $g_{11}, g_{22}\geq 0$, $\begin{pmatrix} g_{11} & g \\ g & g_{22}\end{pmatrix}$ *is positive definite,*

(*ii*) $1\leq p<2$,

then we can get

$$\sup_{0\leq t\leq T}(2\|u(\cdot,t)\|_{H^1} + \|v(\cdot,t)\|_{H^1}) \leq C, \qquad \forall T>0, \tag{2.3}$$

where C is a constant depending only on $\|u_0\|_{H^1_{per}}, \|v_0\|_{H^1_{per}}$.

Proof. We take the inner product of the first equation of system (1.1) with u_t and the second equation with v_t. Integrating and taking the real part of the resulting equations, we get

$$\begin{cases} 0 = \frac{\hbar^2}{4M}\frac{d}{dt}\int |u_x|^2 dx + \frac{\lambda_u}{4}\frac{d}{dt}\int |u|^4 dx + \lambda Re\int |v|^2 u\overline{u}_t dx + \sqrt{2}\alpha Re\int \overline{u}v\overline{u}_t dx \\ \qquad + g_{11}\int |u|^{2p}u\overline{u}_t dx + g\int |u|^{p-1}|v|^{p+1}u\overline{u}_t dx, \\ 0 = \frac{\hbar^2}{8M}\frac{d}{dt}\int |v_x|^2 dx + \frac{\varepsilon}{2}\frac{d}{dt}\int |v|^2 dx + \frac{\lambda_v}{4}\frac{d}{dt}\int |v|^4 dx + \lambda Re\int |u|^2 v\overline{v}_t dx \\ \qquad + \frac{\alpha}{\sqrt{2}}Re\int u^2\overline{v}_t dx + g\int |u|^{p+1}|v|^{p-1}v\overline{v}_t dx + g_{22}\int |v|^{2p}v\overline{v}_t dx. \end{cases} \tag{2.4}$$

Summing up the two equations of the system (2.4), we have

$$\begin{aligned} &\frac{\hbar^2}{4M}\frac{d}{dt}\left(\int |u_x|^2 dx + \frac{1}{2}\int |v_x|^2 dx\right) + \frac{1}{4}\frac{d}{dt}\left(\lambda_u\int |u|^4 dx + \lambda_v\int |v|^4 dx\right) \\ &+ \frac{\varepsilon}{2}\frac{d}{dt}\int |v|^2 dx + \frac{\lambda}{2}\frac{d}{dt}\int |u|^2|v|^2 dx + \frac{\alpha}{\sqrt{2}}Re\frac{d}{dt}\int u^2\overline{v}dx \\ &+ \frac{1}{2p+2}\frac{d}{dt}\left(g_{11}\int |u|^{2p+2}dx + 2g\int |u|^{p+1}|v|^{p+1}dx + g_{22}\int |v|^{2p+2}dx\right) = 0. \end{aligned}$$

Let

$$\mathrm{I} := \frac{\hbar^2}{4M}\left(\int |u_x|^2 dx + \frac{1}{2}\int |v_x|^2 dx\right), \quad \mathrm{II} := \frac{1}{4}\left(\lambda_u\int |u|^4 dx + \lambda_v\int |v|^4 dx\right),$$

$$\mathrm{III} := \frac{\lambda}{2}\int |u|^2|v|^2 dx, \qquad \mathrm{IV} := \frac{\varepsilon}{2}\int |v|^2 dx, \qquad \mathrm{V} := \frac{\alpha}{\sqrt{2}}Re\int u^2\overline{v}dx,$$

$$\mathrm{VI} := \frac{1}{2p+2}\left(g_{11}\int |u|^{2p+2}dx + 2g\int |u|^{p+1}|v|^{p+1}dx + g_{22}\int |v|^{2p+2}dx\right).$$

Then

$$E(t) = \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}+\mathrm{VI} \equiv E(0). \tag{2.5}$$

Applying Lemma 1.1 and the Young inequality, we have

$$\|u\|_4^4 \le C\|u\|_2^3\|u_x\|_2 \le \frac{C}{2}\left(\frac{1}{\delta^2}\|u\|_2^6 + \delta^2\|u_x\|_2^2\right), \tag{2.6}$$

$$\|v\|_4^4 \le C\|v\|_2^3\|v_x\|_2 \le \frac{C}{2}\left(\frac{1}{\delta^2}\|v\|_2^6 + \delta^2\|v_x\|_2^2\right). \tag{2.7}$$

Then, we can bound term II by

$$|\mathrm{II}| \le \frac{C}{8}\left(\frac{1}{\delta^2}(\lambda_u\|u\|_2^6 + \lambda_v\|v\|_2^6) + \delta^2(\lambda_u\|u_x\|_2^2 + \lambda_v\|v_x\|_2^2)\right). \tag{2.8}$$

For term III, using the Hölder's inequality

$$\frac{\lambda}{2}\int |u|^2|v|^2dx \le \frac{\lambda}{4}\left(\|u\|_4^4 + \|v\|_4^4\right). \tag{2.9}$$

Combining inequalities (2.6) and (2.7), term III can be bounded by

$$|\mathrm{III}| \le \frac{C}{8}\left(\frac{1}{\delta^2}(\|u\|_2^6 + \|v\|_2^6) + \delta^2(\|u_x\|_2^2 + \|v_x\|_2^2)\right). \tag{2.10}$$

The term

$$\mathrm{V} = \frac{\alpha}{\sqrt{2}} Re\int u^2\overline{v}dx \le \frac{\alpha}{\sqrt{2}}\int |u|^2|v|dx \le C\|v\|_2\|u\|_4^2. \tag{2.11}$$

Applying inequality (2.7) and Lemma 2.1 yields

$$|\mathrm{V}| \le C\left(\frac{1}{\delta^2}\|u\|_2^6 + \delta^2\|u_x\|_2^2\right). \tag{2.12}$$

Using the estimates of terms II, III, and V, we deduce

$$|\mathrm{II}| + |\mathrm{III}| + |\mathrm{V}| \le C\left(\frac{1}{\delta^2}(\|u\|_2^6 + \|v\|_2^6) + \delta^2(\|u_x\|_2^2 + \|v_x\|_2^2)\right). \tag{2.13}$$

In view of Lemma 2.1, it follows that

$$\mathrm{IV} = \frac{\varepsilon}{2}\|v\|_2^2 \le C. \tag{2.14}$$

Next, we need to estimate term VI:

$$\|u\|_{2p+2}^{2p+2} \le C\|u_x\|_2^p\|u\|_2^{p+2}, \qquad \|v\|_{2p+2}^{2p+2} \le C\|v_x\|_2^p\|v\|_2^{p+2}.$$

Using Hölder's inequality, we have

$$\begin{aligned}\int |u|^{p+1}|v|^{p+1} &\leq C(\|u\|_{2p+2}^{2p+2}+\|v\|_{2p+2}^{2p+2})\\ &\leq C\left(\frac{1}{\delta}(\|u\|_2^{(p+2)\frac{2}{2-p}}+\|v\|_2^{(p+2)\frac{2}{2-p}})+\delta(\|u_x\|_2^2+\|v_x\|_2^2)\right).\end{aligned} \tag{2.15}$$

Therefore, term VI can be bounded by

$$|\mathrm{VI}|\leq C\left(\frac{1}{\delta}(\|u\|_2^{(p+2)\frac{2}{2-p}}+\|v\|_2^{(p+2)\frac{2}{2-p}})+\delta(\|u_x\|_2^2+\|v_x\|_2^2)\right). \tag{2.16}$$

For $1\leq p<2$, combining the estimates in (2.13), (2.14), and (2.16) yields

$$(\frac{\hbar^2}{4M}-\delta)(\|u_x\|_2^2+\frac{1}{2}\|v_x\|_2^2)\leq C,$$

almost all

$$2\|u_x\|_2^2+\|v_x\|_2^2\leq C.$$

For $p\geq 2$

$$E:=\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}+\mathrm{VI}\equiv C.$$

Note that the matrix $\begin{pmatrix} g_{11} & g\\ g & g_{22}\end{pmatrix}\geq 0$. Consequently,

$$\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}\leq C.$$

Combining estimates (2.13) and (2.14) yields

$$\mathrm{I}-\delta(\|u_x\|_2^2+\frac{1}{2}\|v_x\|_2^2)\leq C,$$

a.e.

$$2\|u_x\|_2^2+\|v_x\|_2^2\leq C.$$

This completes the proof of Lemma 2.2. □

LEMMA 2.3. *Let T be any positive number, $u_0\in H^2_{per}(\Omega)$, $v_0\in H^2_{per}(\Omega)$. Under the conditions of Lemma 2.1, we have*

$$\sup_{0\leq t\leq T}(2\|u(\cdot,t)\|_{H^2}+\|v(\cdot,t)\|_{H^2})\leq C,\qquad \forall T>0, \tag{2.17}$$

where the constant C depends only on T and $\|u_0\|_{H^2_{per}},\|v_0\|_{H^2_{per}}$.

Proof. Taking the inner product of u_{xxxx} with the first equation of system (1.1) and v_{xxxx} with the second equation and integrating the resulting equations with respect to x on Ω, and then taking the imaginary part of the resulting equations, we obtain

$$\frac{\hbar}{2}\frac{d}{dt}\|u_{xx}\|_2^2=\lambda_u\mathrm{Im}\int\left(u^2(\overline{u}_{xx})^2+2|u_x|^2u\overline{u}_{xx}+2|u|_x^2u_x\overline{u}_{xx}\right)dx$$

$$
\begin{aligned}
&+\lambda \mathrm{Im}\int \left(v_{xx}\overline{v}u\overline{u}_{xx}+\overline{v}_{xx}vu\overline{u}_{xx}+2|v_x|^2u\overline{u}_{xx}+2|v|_x^2u_x\overline{u}_{xx}\right)dx\\
&+g_{11}\mathrm{Im}\int (|u|^{2p}u)_{xx}\overline{u}_{xx}dx+g\mathrm{Im}\int (|u|^{p-1}|v|^{p+1}u)_{xx}\overline{u}_{xx}dx\\
&+\sqrt{2}\alpha\mathrm{Im}\int \left(|u_{xx}|^2\overline{v}+\overline{v}_{xx}u\overline{u}_{xx}+2(u_x\overline{v}_x)\overline{u}_{xx}\right)dx,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\hbar}{2}\frac{d}{dt}\|v_{xx}\|_2^2=&\lambda_v\mathrm{Im}\int \left(v^2(\overline{v}_{xx})^2+2|v_x|^2v\overline{v}_{xx}+2|v|_x^2v_x\overline{v}_{xx}\right)dx\\
&+\lambda \mathrm{Im}\int \left(u_{xx}\overline{u}v\overline{v}_{xx}+\overline{u}_{xx}uv\overline{v}_{xx}+2|u_x|^2v\overline{v}_{xx}+2(|u|_x^2v_x)\overline{v}_{xx}\right)dx\\
&+g_{22}\mathrm{Im}\int (|v|^{2p}v)_{xx}\overline{v}_{xx}dx+g\mathrm{Im}\int (|v|^{p-1}|u|^{p+1}v)_{xx}\overline{v}_{xx}dx\\
&+\frac{2\alpha}{\sqrt{2}}\mathrm{Im}\int (u_{xx}u\overline{v}_{xx}+2u_xu_x)\overline{v}_{xx}dx.
\end{aligned}
$$

Denote $\frac{\hbar}{2}\frac{d}{dt}\|u_{xx}\|_2^2=\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}$, where

$$
\begin{aligned}
\mathrm{I}&\equiv\lambda_u\mathrm{Im}\int \left(u^2(\overline{u}_{xx})^2+2|u_x|^2u\overline{u}_{xx}+2|u|_x^2u_x\overline{u}_{xx}\right)dx,\\
\mathrm{II}&\equiv\lambda\mathrm{Im}\int \left(v_{xx}\overline{v}u\overline{u}_{xx}+\overline{v}_{xx}vu\overline{u}_{xx}+2|v_x|^2u\overline{u}_{xx}+2|v|_x^2u_x\overline{u}_{xx}\right)dx,\\
\mathrm{III}&\equiv g_{11}\mathrm{Im}\int (|u|^{2p}u)_{xx}\overline{u}_{xx}dx+g\mathrm{Im}\int (|u|^{p-1}|v|^{p+1}u)_{xx}\overline{u}_{xx}dx,\\
\mathrm{IV}&\equiv\sqrt{2}\alpha\mathrm{Im}\int \left(|u_{xx}|^2\overline{v}+\overline{v}_{xx}u\overline{u}_{xx}+2u_x\overline{v}_x\overline{u}_{xx}\right)dx.
\end{aligned}
$$

We denote $\frac{\hbar}{2}\frac{d}{dt}\|v_{xx}\|_2^2=\mathrm{V}+\mathrm{VI}+\mathrm{VII}+\mathrm{VIII}$, where

$$
\begin{aligned}
\mathrm{V}&\equiv\lambda_v\mathrm{Im}\int \left(v^2(\overline{v}_{xx})^2+2|v_x|^2v\overline{v}_{xx}+2|v|_x^2v\overline{v}_{xx}\right)dx,\\
\mathrm{VI}&\equiv\lambda\mathrm{Im}\int \left(u_{xx}\overline{u}v\overline{v}_{xx}+\overline{u}_{xx}uv\overline{v}_{xx}+2|u_x|^2v\overline{v}_{xx}+2|u|_x^2v\overline{v}_{xx}\right)dx,\\
\mathrm{VII}&\equiv g_{22}\mathrm{Im}\int (|v|^{2p}v)_{xx}\overline{v}_{xx}dx+g\mathrm{Im}\int (|v|^{p-1}|u|^{p+1}v)_{xx}\overline{v}_{xx}dx,\\
\mathrm{VIII}&\equiv\frac{2\alpha}{\sqrt{2}}\mathrm{Im}\int (u_{xx}u\overline{v}_{xx}+2u_xu_x\overline{v}_{xx})dx.
\end{aligned}
$$

Firstly, we estimate term I.

By using the Sobolev embedding theorem and Hölder's inequality, we have

$$|\mathrm{I}|\le C\|u_{xx}\|_2^2+C_1\|u_x\|_4^2\|u_{xx}\|_2.$$

Applying inequality (1.7) and Lemma 2.2 yields

$$C_1\|u_x\|_4^2\|u_{xx}\|_2\le C_2\|u_{xx}\|_2^{\frac{1}{2}}\|u_x\|_2^{\frac{3}{2}}\|u_{xx}\|_2\le C_3\|u_{xx}\|_2^{\frac{3}{2}}\le C_4\|u_{xx}\|_2^2+C_5.$$

So

$$|\mathrm{I}|\le C_1\|u_{xx}\|_2^2+C_2. \tag{2.18}$$

Next, we will deal with term II.

Applying the Sobolev embedding theorem and the Hölder inequality

$$|\mathrm{II}|\le C_1(\|u_{xx}\|_2^2+\|v_{xx}\|_2^2)+C_2\|u_{xx}\|_2\|v_x\|_4^2+C_3\|v_x\|_4\|u_x\|_4\|u_{xx}\|_2$$

Applying the inequality (1.7)and Lemma 2.2

$$\begin{aligned}C_2\|u_{xx}\|_2\|v_x\|_4^2+C_3\|v_x\|_4\|u_x\|_4\|u_{xx}\|_2 &\le C_4\|u_{xx}\|_2\|v_{xx}\|_2^{\frac{1}{2}}\|v_x\|_2^{\frac{3}{2}}+C_5\|v_{xx}\|_2^{\frac{1}{4}}\|u_{xx}\|_2^{\frac{5}{4}}\\ &\le C(\|u_{xx}\|_2^2+\|v_{xx}\|_2^2+1).\end{aligned}$$

Therefore,

$$|\mathrm{II}|\le C(\|v_{xx}\|_2^2+\|u_{xx}\|_2^2+1). \tag{2.19}$$

For term III,

$$\begin{aligned}g_{11}\mathrm{Im}\int(|u|^{2p}u)_{xx}\overline{u}_{xx}dx &= g_{11}\mathrm{Im}\int(|u|^{2p})_{xx}u\overline{u}_{xx}dx+2g_{11}\mathrm{Im}\int|u|_x^{2p}u_x\overline{u}_{xx}dx\\ &= g_{11}\mathrm{Im}\{\int 2p|u|^{2p-2}|u_x|^2u\overline{u}_{xx}+p|u|^{2p-2}(\overline{u}_{xx})^2u^2\\ &\quad +p(p-1)|u|^{2p-4}(|u|^2)_x^2u\overline{u}_{xx}dx\}+2g_{11}\mathrm{Im}\int|u|_x^{2p}u_x\overline{u}_{xx}dx.\end{aligned}$$

Using the Sobolev embedding theorem and Hölder's inequality, we have

$$\left|g_{11}\mathrm{Im}\int(|u|^{2p}u)_{xx}\overline{u}_{xx}dx\right|\le C(1+\|u_{xx}\|_2^2).$$

Similarly,

$$\begin{aligned}&g\mathrm{Im}\int(|u|^{p-1}|v|^{p+1}u)_{xx}\overline{u}_{xx}dx\\ &=g\mathrm{Im}\left(\int|u|_{xx}^{p-1}|v|^{p+1}u\overline{u}_{xx}dx+\int|v|_{xx}^{p+1}|u|^{p-1}u\overline{u}_{xx}dx\right)\\ &\quad+2g\mathrm{Im}\left(\int|u|_x^{p-1}|v|_x^{p+1}udx+\int|u|^{p-1}|v|_x^{p+1}u_xdx+\int|u|_x^{p-1}|v|^{p+1}u_xdx\right)\\ &=g\mathrm{Im}\left(\int(p-1)|u|^{p-3}|u_{xx}|^2|v|^{p+1}u\overline{u}_{xx}dx+\int\frac{p-1}{2}|u|^{p-3}(\overline{u}_{xx})^2u^2|v|^{p+1}dx\right)\\ &\quad+g\mathrm{Im}\left(\int\frac{p-1}{2}\frac{p-3}{2}|u|^{p-5}(|u|_x)^2|v|^{p+1}u\overline{u}_{xx}dx+\int(|v|^{p+1})_{xx}|u|^{p-1}u\overline{u}_{xx}dx\right)\\ &\quad+2g\mathrm{Im}\left(\int|u|_x^{p-1}|v|_x^{p+1}udx+\int|u|^{p-1}|v|_x^{p+1}u_xdx+\int|u|_x^{p-1}|v|^{p+1}u_xdx\right)\\ &\le C(1+\|v_{xx}\|_2^2+\|u_{xx}\|_2^2).\end{aligned}$$

Therefore

$$|\mathrm{III}|\le C(1+\|v_{xx}\|_2^2+\|u_{xx}\|_2^2). \tag{2.20}$$

For term IV, we can immediately get the estimate

$$|\mathrm{IV}|\le C(\|v_{xx}\|_2^2+\|u_{xx}\|_2^2+1). \tag{2.21}$$

Comparing term I with term V and term II with term VI, we get

$$|\mathrm{V}|\leq C(\|v_{xx}\|_2^2+1),\qquad |\mathrm{VI}|\leq C(\|v_{xx}\|_2^2+\|u_{xx}\|_2^2+1). \tag{2.22}$$

Comparing III and VII, we get

$$|\mathrm{VII}|\leq C(1+\|v_{xx}\|_2^2+\|u_{xx}\|_2^2). \tag{2.23}$$

For term VIII, we have

$$|\mathrm{VIII}|\leq C(\|u_{xx}\|_2^2+\|v_{xx}\|_2^2+1). \tag{2.24}$$

Applying (2.18)–(2.24), we can obtain the following estimate

$$\frac{d}{dt}(\|u_{xx}\|_2^2+\|v_{xx}\|_2^2)\leq C(\|u_{xx}\|_2^2+\|v_{xx}\|_2^2+1).$$

Using the Gronwall inequality

$$\|u_{xx}\|_2^2+\|v_{xx}\|_2^2\leq C.$$

This completes the proof of Lemma 2.3. □

LEMMA 2.4. *Let $m\geq 0$ be an integer. Under the conditions of Lemma 2.1, we have*

$$\sup_{0\leq t\leq T}(2\|u(\cdot,t)\|_{H^m}+\|v(\cdot,t)\|_{H^m})\leq C,\qquad \forall T>0, \tag{2.25}$$

where the constant C depends only on T and $\|u_0\|_{H^m},\|v_0\|_{H^m}$.

Proof. This lemma is proved by mathematical induction as follows. When $m=0,1,2$, according to lemmas 2.1, 2.2, and 2.3, the inequality (2.25) holds.

Suppose that (2.25) is valid for $m\leq k$. We will prove that (2.25) holds for $m=k+1$.

Taking the inner product of $D^{2(k+1)}u$ with the first equation of system (1.1) and $D^{2(k+1)}v$ with the second equation, and then integrating and taking the imaginary part of the resulting equations, we get

$$\begin{aligned}
&\frac{\hbar}{2}\frac{d}{dt}\|D^{k+1}u\|_2^2\\
&=\mathrm{Im}\left(\lambda_u\int D^{k+1}(|u|^2u)D^{k+1}\overline{u}dx+\lambda\int D^{k+1}(|v|^2u)D^{k+1}\overline{u}dx\right)\\
&\quad+\mathrm{Im}\left(g_{11}\int D^{k+1}(|u|^{2p}u)D^{k+1}\overline{u}dx+g\int D^{k+1}(|u|^{p-1}|v|^{p+1}u)D^{k+1}\overline{u}dx\right)\\
&\quad\quad+\sqrt{2}\alpha\mathrm{Im}\int D^{k+1}(\overline{u}v)D^{k+1}\overline{u}dx:=\mathrm{I}+\mathrm{II}+\mathrm{III},\\
&\frac{\hbar}{2}\frac{d}{dt}\|D^{k+1}v\|_2^2\\
&=\mathrm{Im}\left(\lambda_v\int D^{k+1}(|v|^2v)D^{k+1}\overline{v}dx+\lambda\int D^{k+1}(|u|^2v)D^{k+1}\overline{v}dx\right)\\
&\quad+\mathrm{Im}\left(g_{22}\int D^{k+1}(|v|^{2p}v)D^{k+1}\overline{v}dx+g\int D^{k+1}(|v|^{p-1}|u|^{p+1}v)D^{k+1}\overline{v}dx\right)\\
&\quad\quad+\frac{\alpha}{\sqrt{2}}\mathrm{Im}\int D^{k+1}(u^2)D^{k+1}\overline{v}dx:=\mathrm{IV}+\mathrm{V}+\mathrm{VI}.
\end{aligned}$$

By the normal computation, we can obtain

$$\int D^{k+1}(|u|^2u)D^{k+1}\overline{u}dx=\int (D^{k+1}|u|^2)uD^{k+1}\overline{u}+C_1\int (D^k|u|^2)DuD^{k+1}\overline{u}dx+\cdots$$
$$+C_k\int (D|u|^2)D^kuD^{k+1}\overline{u}dx+\int |u|^2D^{k+1}uD^{k+1}\overline{u}dx. \tag{2.26}$$

Using the induction assumption for $m\le k$, $\|u\|_{H^m}+\|v\|_{H^m}\le C$, so when $k=3$, we can get $\|u\|_{H^3}\le C$, $\|v\|_{H^3}\le C$. Using the Sobolev embedding theorem, $\|D^2u\|_\infty\le C$, $\|D^2v\|_\infty\le C$. Therefore,

$$\lambda_u\mathrm{Im}\int D^{k+1}(|u|^2u)D^{k+1}\overline{u}dx\le C_1\|D^{k+1}u\|_2^2+C_2, \tag{2.27}$$

Applying the same computation yields

$$\lambda\mathrm{Im}\int D^{k+1}(|v|^2u)D^{k+1}\overline{u}dx\le C_1\|D^{k+1}v\|_2^2+C_2\|D^{k+1}u\|_2^2+C_3. \tag{2.28}$$

Therefore,

$$|\mathrm{I}|\le C_1\|D^{k+1}v\|_2^2+C_2\|D^{k+1}u\|_2^2+C_3. \tag{2.29}$$

Also by the normal computation, we can obtain

$$\int D^{k+1}(|u|^{2p}u)D^{k+1}\overline{u}dx=\int (D^{k+1}|u|^{2p})uD^{k+1}\overline{u}+C_1\int (D^k|u|^{2p})DuD^{k+1}\overline{u}dx+\cdots$$
$$+C_k\int (D|u|^{2p})D^kuD^{k+1}\overline{u}dx+\int |u|^{2p}D^{k+1}uD^{k+1}\overline{u}dx,$$

so

$$g_{11}\mathrm{Im}\int D^{k+1}(|u|^{2p}u)D^{k+1}\overline{u}dx\le C_1\|D^{k+1}u\|_2^2+C_2.$$

Using the same computation yields

$$g\mathrm{Im}\int D^{k+1}(|u|^{p-1}|v|^{p+1}u)D^{k+1}\overline{u}dx\le C_1\|D^{k+1}v\|_2^2+C_2\|D^{k+1}u\|_2^2+C_3.$$

Hence

$$|\mathrm{II}|\le C_1\|D^{k+1}v\|_2^2+C_2\|D^{k+1}u\|_2^2+C_3. \tag{2.30}$$

Applying the induction computation, we have

$$\int (D^{k+1}\overline{u}v)D^{k+1}\overline{u}dx=\int D^{k+1}\overline{u}vD^{k+1}\overline{u}dx+C_1\int D^m\overline{u}DvD^{k+1}\overline{u}dx+\cdots$$
$$+C_k\int D\overline{u}D^kvD^{k+1}\overline{u}dx+\int \overline{u}D^{k+1}vD^{k+1}\overline{u}dx.$$

So, we have the estimate of term III

$$|\mathrm{III}|\le C_1\|D^{k+1}u\|_2^2+C_2\|D^{k+1}v\|_2^2+C_3. \tag{2.31}$$

Comparing term I with term IV, and term II with term V, we get

$$|\mathrm{IV}| \leq C_1\|D^{k+1}v\|_2^2 + C_2\|D^{k+1}u\|_2^2 + C_3, \tag{2.32}$$

$$|\mathrm{V}| \leq C_1\|D^{k+1}v\|_2^2 + C_2\|D^{k+1}u\|_2^2 + C_3. \tag{2.33}$$

For term VI, by direct computation, we have

$$\begin{aligned}\int D^{k+1}u^2 D^{k+1}\overline{u}dx = &\int D^{k+1}uuD^{k+1}\overline{u}dx + C_1\int D^k uDuD^{k+1}\overline{u}dx + \cdots \\ &+ C_k\int DuD^k uD^{k+1}\overline{u}dx + \int uD^{k+1}uD^{k+1}\overline{u}dx,\end{aligned}$$

Using the induction assumption and the Sobolev embedding theorem yields

$$|\mathrm{VI}| \leq C_1\|D^{k+1}u\|_2^2 + C_2. \tag{2.34}$$

Comparing (2.29)–(2.34), we have

$$\frac{d}{dt}(\|D^{k+1}u\|_2^2 + \|D^{k+1}v\|_2^2) \leq C(\|D^{k+1}u\|_2^2 + \|D^{k+1}v\|_2^2 + 1).$$

Using the Gronwall inequality yields

$$\|D^{k+1}u\|_2^2 + \|D^{k+1}v\|_2^2 \leq C.$$

This completes the proof of Lemma 2.4. □

Finally, we prove the uniqueness of the solution to system (1.1)–(1.3) in the following.

Let $(u_1, v_1), (u_2, v_2)$ be two solutions which satisfy system (1.1)–(1.3). Then $(s = u_1 - u_2, m = v_1 - v_2)$ satisfies

$$\begin{cases} i\hbar s_t = -\frac{\hbar^2}{2M}\Delta s + \lambda_u(|u_1|^2 u_1 - |u_2|^2 u_2) + \lambda(|v_1|^2 u_1 - |v_2|^2 u_2) + g_{11}(|u_1|^{2p}u_1 \\ \qquad - |u_2|^{2p}u_2) + g(|v_1|^{p+1}|u_1|^{p-1}u_1 - |v_2|^{p+1}|u_2|^{p-1}u_2) + \sqrt{2}\alpha(\overline{u}_1 v_1 - \overline{u}_2 v_2), \\ i\hbar m_t = -\frac{\hbar^2}{4M}\Delta m + \varepsilon m + \lambda_v(|v_1|^2 v_1 - |v_2|^2 v_2) + \lambda(|u_1|^2 v_1 - |u_2|^2 v_2) + g_{22}(|v_1|^{2p}v_1 \\ \qquad - |v_2|^{2p}v_2) + g(|u_1|^{p+1}|v_1|^{p-1}v_1 - |u_2|^{p+1}|v_2|^{p-1}v_2) + \frac{\alpha}{\sqrt{2}}(u_1^2 - u_2^2), \end{cases} \tag{2.35}$$

$$s(0) = 0, \qquad m(0) = 0.$$

Taking the inner product of the first equation of system (2.35) with s and the second equation with m, considering the imaginary part of the resulting equations, we obtain

$$\begin{aligned}\frac{\hbar}{2}\frac{d}{dt}\|s\|_2^2 = &\lambda_u \mathrm{Im}\int(|u_1|^2 u_1 - |u_2|^2 u_2)\overline{s}dx + \lambda\mathrm{Im}\int(|v_1|^2 u_1 - |v_2|^2 u_2)\overline{s}dx \\ &+ g_{11}\mathrm{Im}\int(|u_1|^{2p}u_1 - |u_2|^{2p}u_2)\overline{s}dx + \sqrt{2}\alpha\mathrm{Im}\int(\overline{u}_1 v_1 - \overline{u}_2 v_2)\overline{s}dx \\ &+ g\mathrm{Im}\int(|v_1|^{p+1}|u_1|^{p-1}u_1 - |v_2|^{p+1}|u_2|^{p-1}u_2)\overline{s}dx,\end{aligned}$$

$$\frac{\hbar}{2}\frac{d}{dt}\|m\|_2^2 = \lambda_v \mathrm{Im}\int(|v_1|^2v_1 - |v_2|^2v_2)\overline{m}dx + \lambda\mathrm{Im}\int(|u_1|^2v_1 - |u_2|^2v_2)\overline{m}dx$$
$$+ g_{22}\mathrm{Im}\int(|v_1|^{2p}v_1 - |v_2|^{2p}v_2)\overline{m}dx + \frac{\alpha}{\sqrt{2}}\mathrm{Im}\int(u_1^2 - u_2^2)\overline{m}dx$$
$$+ g\mathrm{Im}\int(|u_1|^{p+1}|v_1|^{p-1}v_1 - |u_2|^{p+1}|v_2|^{p-1}v_2)\overline{m}dx.$$

However,

$$\lambda_u \mathrm{Im}\int\left(|u_1|^2u_1 - |u_2|^2u_2\right)\overline{s}dx \le C\int\left|(|u_1|^2 s\overline{s} + (|u_1|^2 - |u_2|^2)u_2\overline{s})\right|dx \le C_1\|s\|_2^2.$$

Similarly,

$$g_{11}\mathrm{Im}\int(|u_1|^{2p}u_1 - |u_2|^{2p}u_2)\overline{s}dx \le C\|s\|_2^2,$$
$$\lambda_v \mathrm{Im}\int(|v_1|^2v_1 - |v_2|^2v_2)\overline{m}dx \le C\|m\|_2^2,$$
$$g_{22}\mathrm{Im}\int(|v_1|^{2p}v_1 - |v_2|^{2p}v_2)\overline{m}dx \le C\|m\|_2^2.$$

Also,

$$\lambda\mathrm{Im}\int\left(|v_1|^2u_1 - |v_2|^2u_2\right)\overline{s}dx$$
$$\le C\int\left|(|v_1|^2 s\overline{s} + (|v_1|^2 - |v_2|^2)u_2\overline{s})\right|dx \le C(\|s\|_2^2 + \|m\|_2^2)$$
$$g\mathrm{Im}\int(|v_1|^{p+1}|u_1|^{p-1}u_1 - |v_2|^{p+1}|u_2|^{p-1}u_2)\overline{s}dx \le C(\|s\|_2^2 + \|m\|_2^2),$$
$$g\mathrm{Im}\int(|u_1|^{p+1}|v_1|^{p-1}v_1 - |u_2|^{p+1}|v_2|^{p-1}v_2)\overline{m}dx \le C(\|s\|_2^2 + \|m\|_2^2),$$
$$\int(\overline{u}_1v_1 - \overline{u}_2v_2)\overline{s}dx$$
$$= \int(\overline{u}_1v_1 - \overline{u}_1v_2 + \overline{u}_1v_2 - \overline{u}_2v_2)\overline{s}dx \le C_1\int(m\overline{s} + |s|^2)dx \le C_2(\|m\|_2^2 + \|s\|_2^2),$$
$$\int(u_1^2 - u_2^2)\overline{m}dx = \int\left((u_1^2 - u_1u_2) + (u_1u_2 - u_2^2)\right)\overline{m}dx \le C_1\int s\overline{m}dx \le C_2(\|s\|_2^2 + \|m\|_2^2).$$

By the above inequalities, one can easily check that

$$\frac{d}{dt}(\|s\|_2^2 + \|m\|_2^2) \le C(\|s\|_2^2 + \|m\|_2^2).$$

Applying the Gronwall inequality, we get $s = 0, m = 0$. Thus the uniqueness is obtained.

REMARK 2.1. By virtue of the local smooth solution, the a priori estimates, and the continuous extension theorem, we obtain the global smooth solution of the period initial value problem (1.1)–(1.3). Thus, Theorem 1.2 is obtained.

REMARK 2.2. All the above estimates are unconcerned with the period L and only depend on the norm of initial data. Therefore, by using the a priori estimates of the solution to the system (1.1)–(1.3) for L, as in [25], we derive the global smooth solution as $L \to \infty$. Thus, Theorem 1.3 is obtained.

3. Blow-up phenomenon of the solution

In this section, we give some conditions on the existence of blow-up solutions of system (1.1).

THEOREM 3.1. *Let $p\geq 2,(u_0,v_0)\in\Lambda^2$ and $(u(t),v(t))\in C([0,T_0),\Lambda^2)$ be the solution of system (1.1). Define $h(t)=\int_R|x|^2(|u(t)|^2+2|v(t)|^2)dx$. Then, $h(t)$ is well defined for $t\in[0,T_0)$. Moreover,*

$$h'(t)=\frac{2\hbar}{M}Im\int_R(\overline{u}xu_x+\overline{v}xv_x)dx, \tag{3.1}$$

$$\begin{aligned}h''(t)=&\frac{2\hbar^2}{M^2}\int_R(|u_x|^2+\frac{1}{2}|v_x|^2)dx+\frac{1}{M}\left(\lambda_u\int_R|u|^4dx+\lambda_v\int_R|v|^4dx+2\lambda\int_R|u|^2|v|^2dx\right)\\&+\frac{\sqrt{2}\alpha}{M}Re\int_R u^2\overline{v}dx\\&+\frac{2p}{M(p+1)}\left(g_{11}\int_R|u|^{2p+2}dx+g_{22}\int_R|v|^{2p+2}dx+2g\int_R|u|^{p+1}|v|^{p+1}dx\right).\end{aligned} \tag{3.2}$$

Proof. We only prove (3.1) and (3.2) formally.

$$\begin{aligned}h'(t)=&2Re\int_R|x|^2(\overline{u}u_t+2\overline{v}v_t)dx=-\frac{\hbar}{M}\mathrm{Im}\int_R|x|^2(\overline{u}u_{xx}+\overline{v}v_{xx})dx\\=&\frac{2\hbar}{M}\mathrm{Im}\int_R(\overline{u}xu_x+\overline{v}xv_x)dx,\end{aligned}$$

$$\begin{aligned}h''(t)=&-\frac{2\hbar}{M}\mathrm{Im}\int_R(u_t(\overline{u}+2x\overline{u}_x)+v_t(\overline{v}+2x\overline{v}_x))dx\\=&-\frac{2}{M}Re\int_R\left((\frac{\hbar^2\partial^2}{2M\partial_{xx}}-\lambda_u|u|^2-\lambda|v|^2-g_{11}|u|^{2p}-g|u|^{p-1}|v|^{p+1})u-\sqrt{2}\alpha\overline{u}v\right)(\overline{u}+2x\overline{u}_x)dx\\&+\int_R\left((\frac{\hbar^2\partial^2}{4M\partial_{xx}}-\varepsilon-\lambda_v|v|^2-\lambda|u|^2-g|u|^{p+1}|v|^{p-1}-g_{22}|v|^{2p})v-\frac{\alpha}{\sqrt{2}}u^2\right)(\overline{v}+2x\overline{v}_x)dx.\end{aligned}$$

Note that

$$\begin{aligned}\frac{\hbar^2}{2M}Re\int_R\overline{u}u_{xx}dx=&-\frac{\hbar^2}{2M}\int_R|u_x|^2dx,\\\frac{\hbar^2}{2M}Re\int_R 2u_{xx}x\overline{u}_xdx=&-\frac{\hbar^2}{2M}\int_R|u_x|^2dx.\end{aligned}$$

Then,

$$\frac{\hbar^2}{2M}Re\int_R u_{xx}(\overline{u}+2x\overline{u}_x)dx=-\frac{\hbar^2}{M}\int_R|u_x|^2dx.$$

Similarly,

$$\begin{aligned}\frac{\hbar^2}{4M}Re\int_R v_{xx}(\overline{v}+2x\overline{v}_x)dx=&-\frac{\hbar^2}{2M}\int_R|v_x|^2dx,\\\lambda_uRe\int_R|u|^2u(\overline{u}+2x\overline{u}_x)dx=&\frac{\lambda_u}{2}\int_R|u|^4dx,\end{aligned}$$

$$\lambda Re\int_R |v|^2 u(\overline{u}+2x\overline{u}_x)dx = \lambda\int_R \left(|u|^2|v|^2+|v|^2x|u|_x^2\right)dx,$$

$$g_{11}Re\int_R |u|^{2p}u(\overline{u}+2x\overline{u}_x)dx = g_{11}\int_R\left(|u|^{2p+2}-\frac{1}{p+1}|u|^{2p+2}\right)dx,$$

$$gRe\int_R |u|^{p-1}|v|^{p+1}u(\overline{u}+2x\overline{u}_x)dx = g\int_R\left(|u|^{p+1}|v|^{p+1}+\frac{2}{p+1}|v|^{p+1}x|u|_x^{p+1}\right)dx,$$

$$\lambda_v Re\int_R |v|^2 v(\overline{v}+2x\overline{v}_x)dx = \frac{\lambda_v}{2}\int_R |v|^4dx,$$

$$\lambda Re\int_R |u|^2 v(\overline{v}+2x\overline{v}_x)dx = \lambda\int_R \left(|v|^2|u|^2+|u|^2x|v|_x^2\right)dx,$$

$$Re\int_R \varepsilon v(\overline{v}+2x\overline{v}_x)dx = \varepsilon\int_R \left(|v|^2-|v|^2\right)dx = 0,$$

$$gRe\int_R |u|^{p+1}|v|^{p-1}v(\overline{v}+2x\overline{v}_x)dx = g\int_R\left(|v|^{p+1}|u|^{p+1}+\frac{2}{p+1}|u|^{p+1}x|v|_x^{p+1}\right)dx,$$

$$g_{22}Re\int_R |v|^{2p}v(\overline{v}+2x\overline{v}_x)dx = g_{22}\int_R\left(|v|^{2p+2}-\frac{1}{p+1}|v|^{2p+2}\right)dx.$$

Therefore,

$$\begin{aligned} h''(t) = {} & \frac{2\hbar^2}{M^2}\int_R (|u_x|^2+\frac{1}{2}|v_x|^2)dx + \frac{1}{M}\left(\lambda_u\int_R |u|^4dx+\lambda_v\int_R |v|^4dx+2\lambda\int_R |u|^2|v|^2dx\right) \\ & + \frac{\sqrt{2}\alpha}{M}Re\int_R u^2\overline{v}dx \\ & \quad + \frac{2p}{M(p+1)}\left(g_{11}\int_R |u|^{2p+2}dx+g_{22}\int_R |v|^{2p+2}dx+2g\int_R |u|^{p+1}|v|^{p+1}dx\right). \end{aligned}$$

This completes the proof of Theorem 3.1. □

Proof. (Proof of Theorem 1.4.) Assuming that the solutions exist globally in time, we obtain from Theorem 3.1 that

$$\begin{aligned} h''(t) = {} & \frac{8}{M}E(t) - \frac{4\varepsilon}{M}\int_R |v|^2dx - \frac{1}{M}\left(\lambda_u\int_R |u|^4dx+\lambda_v\int_R |v|^4dx+2\lambda\int_R |u|^2|v|^2dx\right) \\ & - \frac{3\sqrt{2}\alpha}{M}Re\int_R u^2\overline{v}dx \\ & \quad - \frac{2(2-p)}{(p+1)M}\left(g_{11}\int_R |u|^{2p+2}dx+g_{22}\int_R |v|^{2p+2}dx+2g\int_R |u|^{p+1}|v|^{p+1}dx\right). \end{aligned}$$

Using Hölder's inequality, we have

$$-\frac{3\sqrt{2}\alpha}{M}Re\int_R u^2\overline{v}dx \le \left|\frac{3\sqrt{2}\alpha}{2M}\right|\left(\int_R |u|^4dx+\int_R |v|^2dx\right).$$

So,

$$\begin{aligned} h''(t) \le {} & \frac{8}{M}E(t) - \left(\frac{4\varepsilon}{M}-\left|\frac{3\sqrt{2}\alpha}{2M}\right|\right)\int_R |v|^2dx - \frac{1}{M}\left(\lambda_u-\left|\frac{3\sqrt{2}\alpha}{2M}\right|\right)\int_R |u|^4dx \\ & - \frac{\lambda_v}{M}\int_R |v|^4dx - \frac{2\lambda}{M}\int_R |u|^2|v|^2dx \end{aligned}$$

$$-\frac{2(2-p)}{(p+1)M}\left(\int_R (g_{11}|u|^{2p+2}+g_{22}|v|^{2p+2}+2g|u|^{p+1}|v|^{p+1})dx\right).$$

Because $M>0$, $(4\epsilon-|\frac{3\sqrt{2}\alpha}{2}|)>0$, $(\lambda_u-|\frac{3\sqrt{2}\alpha}{2M}|)>0$, $\lambda_v,\lambda>0$, we deduce

$$h''(t)\le\frac{8}{M}E(t)-\frac{2(2-p)}{(p+1)M}\left(g_{11}\int_R|u|^{2p+2}dx+g_{22}\int_R|v|^{2p+2}dx+2g\int_R|u|^{p+1}|v|^{p+1}dx\right).$$

Since $p\ge 2$, $g_{11}>0$, and $\begin{pmatrix} g_{11} & g \\ g & g_{22}\end{pmatrix}$ is negative definite, it follows that

$$h''(t)\le\frac{8}{M}E(t)=\frac{8}{M}E(0). \tag{3.3}$$

By a classical analysis, we have that

$$h(t)=h(0)+h'(0)t+\int_0^t(t-\tau)h''(\tau)d\tau,\qquad 0\le t<+\infty. \tag{3.4}$$

It follows that

$$h(t)\le h(0)+h'(0)t+\frac{4}{M}E_0t^2,\qquad 0\le t<+\infty.$$

Moreover, $h(t)$ is a nonnegative function,

$$h(0)=\int|x|^2(|u_0|^2+2|v_0|^2)dx$$

and

$$h'(0)=\frac{2\hbar}{M}\mathrm{Im}\int(x\overline{u}_0u_{0x}+x\overline{v}_0v_{0x})dx. \tag{3.5}$$

In the following, we discuss this theorem through three cases.

(a) If (i) holds, from (3.3) we have $h''(t)\le\frac{8}{M}E(0)<0$. Then, $h(t)$ is a concave function of t which implies that there exists $T^*<\infty$ such that $\lim_{t\to T^*}h(t)=0$.

(b) If (ii) holds, from (3.3) and (3.5) we have $h''(t)\le\frac{8}{M}E(0)=0$ and $h'(0)<0$. Thus there exists $T^*<\infty$ such that $\lim_{t\to T^*}h(t)=0$.

(c) If (iii) holds, assuming

$$f(t)=h(0)+h'(0)t+\frac{4}{M}E_0t^2,$$

we have $\left(h'(0)^2-\frac{16}{M}h(0)E_0\right)\ge 0$. Thus there exists at least t_1 such that

$$f(t_1)=h(0)+h'(0)t_1+\frac{4}{M}E_0t_1^2=0. \tag{3.6}$$

From (3.4) and (3.6), there exists $T_*<\infty$ such that $\lim_{t\to T_*}h(t)=0$.

By (a), (b), and (c), we can get $\lim_{t\to T_*}h(t)=0$, which together with (2.1) leads to a contradiction. Thus, the maximal existence time T of the solution (u,v) to the system (1.1)–(1.3) is finite.

This completes the proof of Theorem 1.4. ∎

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