POPULATION STABILIZATION IN BRANCHING BROWNIAN MOTION WITH ABSORPTION AND DRIFT*

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Abstract. We consider, through PDE methods, branching Brownian motion with drift and absorption. It is well known that there exists a critical drift which separates those processes which die out almost surely from those which survive with positive probability. In this work, we consider lower-order corrections to the critical drift which ensures a nonnegative, bounded expected number of particles and convergence of this expectation to a limiting nonnegative number which is positive for some initial data. In particular, we show that the average number of particles stabilizes at the convergence rate $O(\log(t)/t)$ if and only if the multiplicative factor of the $O(t^{-1/2})$ correction term is $3\sqrt{\pi}t^{-1/2}$. Otherwise, the convergence rate is $O(1/\sqrt{t})$. We point out some connections between this work and recent work investigating the expansion of the front location for the initial value problem in Fisher–KPP.

Key words. Branching Brownian motion, absorption, population dynamics, selection models, Fisher–KPP, delay, Bramson correction.

AMS subject classifications. 92D25, 35K57.

1. Introduction

In this paper, we consider a probabilistic model, branching Brownian motion, through analytic means. In this process, each particle currently alive moves as an independent Brownian motion and independently splits at a constant rate into two particles, each of which then evolves as the parent. In addition, we are interested in this model where the particles are pushed by a given deterministic drift $-\dot{X}$ and where particles are killed upon reaching the origin.

This connection to the Fisher–KPP equation and its use as a model for populations undergoing selection have made branching Brownian motion the subject of intense interest in recent years; see [1,2,4,5,9,11,15,16,18–21,25,26,31]. Early work by McKean, [27], focused on understanding the statistics of the rightmost particle of branching Brownian motion with neither drift nor absorption in order to understand solutions to the Fisher–KPP equation. We describe in more detail the connection between this work and Fisher–KPP at the conclusion of this section. In 1978, Kesten introduced drift and absorption at the origin into the model in [21]. In this work, he showed that $\dot{X}=2$ is the critical drift, separating systems which die out with probability one from systems with positive survival probability. Recently, more precise results have been obtained regarding the distribution of particles at or near the critical drift and the convergence of the statistics of the rightmost particle; see [1, 4, 5, 9, 18–20, 25]. The preceding is an incomplete bibliography, and the interested reader should investigate the references within the works referenced above.

Our perspective is slightly different. We wish to understand the effect of lower-order terms in expansion of the drift on the average number of surviving particles. To be more explicit, let N_t be the number of particles at time t with the position of the ith particle given by Y_t^i, and define

$$v(t,x) = \mathbb{E}^x\left[\sum_{i=1}^{N_t} v_0\left(Y_t^i\right)\right],$$

for some compactly supported function v_0. Then, for $x \geq X(t)$, v solves the equation

$$\begin{cases} v_t = v_{xx} + v, \\ v(t, X(t)) = 0, \\ v(0, x) = v_0(x); \end{cases} \qquad (1.1)$$

see the "many-to-one" lemma of [17, 19]. In particular, if v_0 is the indicator function of $[a, b]$, this is simply the expected number of particles in $[a, b]$. In [15, 16], the authors, using (1.1) as an approximation for Fisher–KPP, show that, letting

$$\dot{X} = 2 + \frac{r}{t},$$

the only choice for r which yields non-trivial long-time behavior is $-3/2$. We obtain a correction of order $t^{-3/2}$, which gives us more precise information on the average number of particles by a refinement of their methods and using a connection between the mass of the solution of (1.1), v, with the normal derivative at the origin, $v_x(0)$. This allows us to find an expansion for $\dot{X}$, independent of initial data, which yields faster convergence to the limiting mass. We now state this precisely.

Statement of results. Before we state the main theorem, we give a bit of notation and recast our problem. For ease of exposition, we use the following notation in order to omit tracking multiplicative constants that arise.

NOTATION 1.1. *For two values a and b which may depend on time and various other data, we write*

$$a \lesssim b$$

if there is some multiplicative constant, $C > 0$, that is independent of time such that

$$a \leq Cb.$$

Such a constant may depend on initial data or various other constants.

We note that, despite this notation, we occasionally need to introduce an arbitrary constant into our equations. Whenever we do, we denote by C such an arbitrary constant which may change line by line but which is independent of time.

In addition, in order to avoid the complications inherent in a moving boundary, we shift to a moving frame to obtain the equation

$$\begin{cases} v_t - \dot{X}(t) v_x = v_{xx} + v, & \text{for } x \geq 0 \\ v(t, 0) = 0, \\ v(0, x) = v_0(x). \end{cases} \qquad (1.2)$$

THEOREM 1.2.   *Let v satisfy (1.2) with $X(t)$ given by*

$$X(t) = 2(t+1) - \frac{3}{2} \log(t+1) - \frac{\overline{c}}{\sqrt{t+1}} \qquad (1.3)$$

and where $v_0(x)$ is a compactly supported function on $[0, \infty]$. Then there exists $\alpha_0 \geq 0$ such that, for any $p \in [1, \infty]$, we have

$$\lim_{t \to \infty} \|v(t, x) - \alpha_0 x e^{-x}\|_{L^p([0, \infty])} = 0. \qquad (1.4)$$

In addition, we have that, if $\bar{c} = 3\sqrt{\pi}$, then

$$\left| \alpha_0 - \int_0^\infty v(t,x)dx \right| \lesssim \frac{\log(t)}{t}.$$ (1.5)

Otherwise, we have that

$$\frac{1}{\sqrt{t}} \lesssim \left| \alpha_0 - \int_0^\infty v(t,x)dx \right| \lesssim \frac{1}{\sqrt{t}}.$$ (1.6)

Finally, for certain initial data, we have $\alpha_0 > 0$.

The main focus of this paper is in obtaining the mass stabilization rates in (1.5) and (1.6). We state the convergence in L^p and the positivity of α_0 in order to reassure the skeptical reader that these rates have meaning. The L^p convergence and the positivity of α_0 arise naturally in our work, and we make note when they become apparent.

Our proof of (1.5) in Theorem 1.2 involves no direct estimates on the mass of the solution. In systems such as (1.2), there is a close relationship between the derivative of the solution at $x = 0$ and its total mass. Hence, we focus on obtaining estimates on the derivative of the solution at the origin. To this end, the main ingredient of the proof of Theorem 1.2 is the following lemma.

LEMMA 1.3. *Let v satisfy (1.2) with $X(t)$ and $\bar{c}$ given by (1.3) and where the initial data v_0 is compactly supported on $[0, \infty]$. Then there exists a nonnegative constant α_0 such that $v_x(t, 0)$ converges to α_0. If $\bar{c} = 3\sqrt{\pi}$, then we have*

$$|v_x(t, 0) - \alpha_0| \lesssim \frac{\log(t)}{t}.$$

On the other hand, if $\bar{c} \neq 3\sqrt{\pi}$, then we have that, for sufficiently large t,

$$\frac{1}{\sqrt{t}} \lesssim |v_x(t, 0) - \alpha_0| \lesssim \frac{1}{\sqrt{t}}.$$

Connection with front speeds in Fisher–KPP. Much of the renewed interest in understanding system (1.1) and its probabilistic counterpart, branching Brownian motion with drift and absorption, lies in its connection to the Fisher–KPP equation

$$\begin{cases} u_t = u_{xx} + u(1 - u), \\ u(0, x) = u_0(x), \end{cases}$$ (1.7)

where u_0 is some localized smooth function on $\mathbb{R}$ taking values in $[0, 1]$. This equation usually arises as a model for population dynamics; see [12, 13, 28, 32].

Equation (1.7) was originally studied in the early twentieth century in [13, 22], and it was observed that traveling wave solutions to the system exist. To be more explicit, there are global-in-time solutions of the form $u(t, x) = \phi_c(x - ct)$ for profiles ϕ_c and speeds $c \geq 2$. Later, in [3], Aronson and Weinberger showed that solutions with more general initial data spread at the speed of the critical traveling wave and admit the same behavior, in that the steady state $u \equiv 1$ invades the unsteady state $u \equiv 0$. Namely, given a solution to (1.7) where u_0 is compactly supported, nonnegative, and nonzero, we have

$$\min_{|x| \leq ct} u(x) = 1, \quad \text{for all } c < 2,$$

and

$$\max_{|x| \geq ct} u(x) = 0, \quad \text{for all } c > 2.$$

In this case, as in Kesten's paper [21], the critical speed is 2.

In the celebrated papers [7, 8], Bramson obtained more precise asymptotics of the front location with probabilistic methods. More specifically, he showed that, for any $m \in (0, 1)$, there exists a shift x_m, depending on the initial conditions and m, such that

$$\{x \in \mathbb{R} : u(t, x) = m\} \subset \left[2t - \frac{3}{2}\log(t) - x_m - o(1), 2t - \frac{3}{2}\log(t) - x_m + o(1) \right].$$

Recently, Roberts simplified the proofs of these results in [31]. Ebert and van Saarloos, in [10], obtained, through non-rigorous methods, the next term in the expansion. Namely, using matched asymptotics, they argue that the front speed is given by

$$2t - \frac{3}{2}\log(t) - x_m - \frac{3\sqrt{\pi}}{\sqrt{t}} + O(t^{-1}). \tag{1.8}$$

Interestingly, though the constant term has dependence on the initial data, the lower-order term is believed to be universal. We point out that the expansion obtained by Ebert and van Saarloos is the same as the expansion we obtain in (1.3) through Theorem 1.2. In fact, part of the motivation of our work has been to provide some understanding of the $3\sqrt{\pi}$ term, as the Ebert and van Saarloos paper does not provide any interpretation beyond the matched asymptotics in the formal derivation. The most recent works regarding the front location in Fisher–KPP are [15, 16] by Hamel et al. In these papers, the authors used PDE methods, which we have borrowed and expanded on here, in order to obtain results similar to those of Bramson at the expense of the precision in the constant term. In addition, we mention a work in preparation, [6], in which the authors, through probabilistic methods, investigate the value of $\bar{c}$ given in (1.8) in the Fisher–KPP context.

That our expansion for the critical drift in (1.1) is the same as Ebert and van Saarloos's expansion for the front location for (1.7) is not a surprise. Solutions to (1.1) provide a convenient family of sub- and supersolutions to (1.7) which are exceptionally faithful approximations of the tail of solutions to (1.7) provided that $X(t)$ is chosen carefully. As we mentioned above, the authors in [15, 16, 29], use equations such as (1.1) in order to obtain precise results about the front position in Fisher–KPP. In addition, we remind the reader that solutions of (1.1) and (1.7) are connected through their interpretation as statistics of branching Brownian motion; see [27]. The close relationship between these two probabilistic models has been leveraged to transfer understanding from one system to the other; see [11, 15, 26] and many of the papers mentioned above.

Outline of the paper. This paper is organized as follows. In Section 2, we show how the mass stabilization rates given in Theorem 1.2 follow from Lemma 1.3. This boils down to leveraging a connection between the total mass of a solution to (1.2) and its derivative as $x = 0$ to obtain an ODE governing the total mass. Then, we apply the estimates provided by Lemma 1.3 to conclude.

In Section 3, we switch to self-similar variables in order to reduce the equation to a simple parabolic PDE with a decaying forcing term. This change of variables is also used in [15, 16] and is convenient because it changes (1.2) into a PDE that has a discrete spectrum which we know explicitly. This allows us to decompose our solution into three

parts: a steady state, a slowly decaying function with vanishing normal derivative, and a rapidly decaying remainder. This reduces Lemma 1.3 to proving this decomposition. We note that the convergence of v to $\alpha_0 x e^{-x}$ is an easy consequence of this, and, as such, we omit the proof of it.

Finally, in Section 4, we give the proof of this decomposition by explicitly solving for the steady state and the slowly decaying function and by obtaining estimates for the quickly decaying remainder. At some point in the analysis, the positivity of α_0 becomes obvious; we share when this happens.

2. Deducing population stabilization rates from Lemma 1.3

Proof. (Proof of (1.5) and (1.6):) We first cover the case where $\bar{c} = 3\sqrt{\pi}$. Let α_0 be the constant from Lemma 1.3. Integrating equation (1.2) and defining

$$P(t) = \int_0^\infty v(t,x)dx - \alpha_0,$$

we obtain

$$P'(t) - P = -v_x(t,0) + \alpha_0.$$

To see this, notice that parabolic regularity theory gives us that v is smooth in time and space and that v and v' will decay exponentially as x tends to infinity. Hence, the boundary terms at infinity vanish, and we are justified in pulling the time derivative outside of the integral that defines P.

We point out that $P(t)$ is bounded above uniformly in time. We may see this by, for instance, re-proving the bound (20) from [15] in our setting[1]. We note that this bound follows directly from the analysis in this paper.

Define $E(t) = e^{-t}M(t)$. Notice that $E'(t) = (M' - M)e^{-t}$ and that

$$\lim_{t \to \infty} E(t) = 0$$

since $M(t)$ is bounded above. Hence, we have that

$$E'(t) = (\alpha_0 - v_x(t,0))e^{-t}.$$

Integrating both sides of this and applying Lemma 1.3, we get

$$|M(t)|e^{-t} = \left| \int_t^\infty E'(s)ds \right| = \left| \int_t^\infty (\alpha_0 - v_x(s,0))e^{-s}ds \right|$$

$$\lesssim \int_t^\infty \frac{\log(s)}{s}e^{-s}ds \le \frac{\log(t)}{t}\int_t^\infty e^{-s}ds = \frac{\log(t)}{t}e^{-t}.$$

Multiplying both sides by the exponential finishes the upper bound for this choice of $\bar{c}$.

If $\bar{c} \ne 3\sqrt{\pi}$, we may apply Lemma 1.3 in the same manner to get the desired upper bound in the statement of the proof. In order to obtain the lower bound, let M and E

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[1] Actually, their bound may be used out-of-the-box in our setting by creating a super-solution with well-chosen initial data. Since this is a minor point, we do not go into detail.

be as above. For sufficiently large t, we may apply Lemma 1.3 to obtain that

$$|M(t)|e^{-t} = \left| \int_t^\infty E'(s)ds \right| \gtrsim \int_t^\infty \frac{e^{-s}}{\sqrt{s}} ds$$

$$\geq \frac{1}{\sqrt{2t}} \int_t^{2t} e^{-s} ds \gtrsim \frac{e^{-t} - e^{-2t}}{\sqrt{t}}.$$

Multiplying both sides by e^t and taking sufficiently large t, we obtain the desired lower bound. This finishes the proof. □

3. Self-similar variables

In order to understand the solutions to (1.2) where $X(t)$ is given by (1.3), we change variables a number of times following the development in [15, 16]. First, we remove an exponential to obtain the function $\bar{v} = e^x v$ satisfying

$$\bar{v}_t + \left(\frac{3}{2(t+1)} - \frac{\bar{c}}{2(t+1)^{3/2}} \right) \bar{v}_x = \bar{v}_{xx} + \left(\frac{3}{2(t+1)} - \frac{\bar{c}}{2(t+1)^{3/2}} \right) \bar{v}. \tag{3.1}$$

Changing to self-similar variables $\tau = \log(1+t)$ and $y = x(1+t)^{-1/2}$, we obtain

$$w_\tau - \frac{y}{2} w_y - w_{yy} - \frac{3}{2} w = \left(\frac{\bar{c}}{2e^\tau} - \frac{3}{2e^{\tau/2}} \right) w_y - \frac{\bar{c}}{2e^{\tau/2}} w, \tag{3.2}$$

where we have defined $w(\tau, y) = \bar{v}(e^\tau - 1, ye^{-\tau/2})$. Let $W(\tau, y) = e^{\tau/2} e^{y^2/8} w(\tau, y)$, and we get

$$W_\tau + MW = \left(\frac{\bar{c}}{2e^\tau} - \frac{3}{2e^{\tau/2}} \right) \left(W_y - \frac{y}{4} W \right) - \frac{\bar{c}}{2e^{\tau/2}} W, \tag{3.3}$$

where

$$M = -\partial_y^2 + \left(\frac{y^2}{16} - \frac{3}{4} \right).$$

The initial data and the boundary conditions for W are given by

$$\begin{cases} W(\tau, 0) = 0, \\ W(0, y) = e^{y^2/8} e^y v_0(y). \end{cases}$$

We work mainly with W in the sequel. Notice that the operator on the left-hand side of (3.3) is the equation of a simple harmonic oscillator with a decaying forcing term. We use the fact that we understand the eigenvalues and eigenfunctions of this operator in the analysis that follows.

In these coordinates, Lemma 1.3 is easily reduced to proving the following decomposition, as the following gives complete information about $v_x(t, 0)$.

LEMMA 3.1. *Let W satisfy* (3.3) *with smooth, compactly supported initial data. Then there exist α, g, and R such that*

$$W(\tau, y) = \alpha y e^{-y^2/8} + e^{-\tau/2} g(y) + R(\tau, y). \tag{3.4}$$

In addition, $|R_y(0)| \lesssim (1+\tau)e^{-\tau}$. Finally, $g_y(0) = 0$ if and only if $\bar{c} = 3\sqrt{\pi}$

We prove this lemma in the sequel by decomposing W into three functions: one part is the steady solution of the equation, one is a slowly decaying function with zero derivative at $y = 0$, and one is a quickly decaying function. First, we show that W is bounded in L^2 and converges to the steady state at the rate $e^{-\tau/2}$. This also shows the existence of α above. Then, we prove the existence of g and use the fact that we may solve for it explicitly. Finally, we leverage these facts to prove the existence of R.

4. Proof of Lemma 3.1

Before we begin the proof, notice that M defines a nonnegative, definite, symmetric quadratic form on the space

$$X := \{\phi \in L^2 : y\phi \in L^2, \phi_y \in L^2\}, \tag{4.1}$$

which we call Q. Namely, for all $\phi \in X$, we define

$$Q(\phi) := \int \phi(M\phi)dy. \tag{4.2}$$

Note that Q satisfies the following inequality:

$$\int \frac{y}{4}\phi^2 dy \leq Q(\phi) + \|\phi\|_2^2. \tag{4.3}$$

We use this inequality often in the sequel.

Let $e_0, e_1, \ldots$ denote the eigenfunctions of M. We know by [16] that the first two eigenvalues are 0 and 1. Hence, Q is nonnegative on X and $Q(\phi) \geq \|\phi\|_2^2$ on $\mathrm{Span}\{e_1, e_2, \ldots\}$. Moreover, we know that

$$e_0(y) = \frac{1}{\sqrt{2\sqrt{\pi}}} y e^{-y^2/8}.$$

First, we show that W converges in L^2 to the steady state at the rate $e^{-\tau/2}$. In addition, this shows the existence of α in Lemma 3.1, since e_0 is the steady state of (3.3). We remark that the potential positivity of the total mass in Theorem 1.2 follows from the work below. We comment further on this following the proof.

LEMMA 4.1. *Suppose that W satisfies* (3.3). *Then there exists α such that*

$$\|W - \alpha y e^{-y^2/8}\|_2 \lesssim e^{-\tau/2}.$$

Proof. First, we show that W is bounded in L^2. To this end, multiplying (3.3) by W and integrating by parts, we get

$$\frac{1}{2}\frac{d}{d\tau}\int |W|^2 dy + Q(W) = \int \left[\left(\frac{\bar{c}}{2e^\tau} - \frac{3}{2e^{\tau/2}}\right)yW^2 - \frac{\bar{c}}{2e^{\tau/2}}W^2\right]dy$$

$$\leq 4e^{-\tau/2}\left(\frac{\bar{c}}{2e^{\tau/2}} - \frac{3}{2}\right)\left[Q(W) + \|W\|_2^2\right] + e^{-\tau/2}\frac{\bar{c}}{2}\|W\|_2^2$$

$$\lesssim e^{-\tau/2}\left[Q(W) + \|W\|_2^2\right],$$

where we used (4.3) to obtain the second inequality. We may then rewrite this as

$$\frac{1}{2}\frac{d}{d\tau}\|W\|_2^2 + \left(1 - Ce^{-\tau/2}\right)Q(W) \lesssim e^{-\tau/2}\|W\|_2^2.$$

We may choose τ_0 such that, for $\tau \geq \tau_0$, we have $1 - Ce^{-\tau/2} \geq 0$. We note that, by working in the original Euclidean coordinates (t, x), we may easily check that W remains in L^2 for $\tau \in [0, \tau_0]$. Since Q is nonnegative, we have that

$$\frac{d}{d\tau}\|W\|_2^2 \lesssim e^{-\tau/2}\|W\|_2^2.$$

Solving this differential inequality gives us that W is uniformly bounded in L^2.

Now, we finish the proof in two steps. First, we look at the projection of W onto e_0. This gives us the steady state. Then, we look at the component of W orthogonal to e_0. To be explicit, we decompose W as

$$W(\tau,y) = W_1(\tau)e_0(y) + \tilde{W}(\tau,y), \tag{4.4}$$

where $\tilde{W}$ is an element of $\mathrm{Span}\{e_1, e_2, \dots\}$.

In order to understand W_1, we multiply (3.3) by e_0 and integrate by parts to obtain

$$
\begin{aligned}
(W_1)_\tau &= \int W_\tau(\tau,y)e_0(y)dy + \int MW(\tau,y)e_0(y)dy \\
&= \int \left(\frac{\bar{c}}{2e^\tau} - \frac{3}{2e^{\tau/2}}\right)\left(W_y(\tau,y) - \frac{y}{4}W(\tau,y)\right)e_0(y)dy - \int \frac{\bar{c}}{2e^{\tau/2}}W(\tau,y)e_0(y)dy \\
&\lesssim e^{-\tau/2}\int W(\tau,y)\left[|(e_0)_y| + ye_0(y)\right]dy + e^{-\tau/2}|W_1(\tau)| \\
&\lesssim e^{-\tau/2}\|W\|_2^2.
\end{aligned}
$$

We used here that M is symmetric and that $Me_0 = 0$. We may obtain the same inequality for $-(W_1)_\tau$ by multiplying by $-e_0$ instead. Hence, we obtain

$$|(W_1)_\tau| \lesssim e^{-\tau/2}\|W\|_2 \lesssim e^{-\tau/2}. \tag{4.5}$$

Hence, there exists α' such that W_1 tends to α' as τ tends to infinity. Moreover, we have that $|\alpha' - W_1| \lesssim e^{-\tau/2}$. Hence, we need only show that $\tilde{W}$ decays fast enough in order to finish the proof.

To obtain the decay of $\tilde{W}$, we use (4.4) in (3.3) to obtain

$$
\begin{aligned}
\tilde{W}_\tau + M\tilde{W} &= e^{-\tau/2}\left[\left(\frac{\bar{c}}{2e^{\tau/2}} - \frac{3}{2}\right)\left(\tilde{W}_y - \frac{y}{4}\tilde{W}\right) - \frac{\bar{c}}{2}\tilde{W}\right] \\
&\quad + e^{-\tau/2}\left[\left(\frac{\bar{c}}{2e^\tau} - \frac{3}{2e^{\tau/2}}\right)\left(W_1(e_0)_y - \frac{y}{4}W_1e_0\right) - \frac{\bar{c}}{2e^{\tau/2}}W_1e_0\right] - (W_1)_\tau e_0.
\end{aligned}
$$

Noting that $\tilde{W}$ lives in the span of $e_1, e_2, \dots$, we have that $Q(\tilde{W}) \geq \|\tilde{W}\|_2^2$. Hence, when we multiply the equation above by $\tilde{W}$, integrate by parts, and use our inequality on Q, we obtain

$$\frac{1}{2}\frac{d}{d\tau}\|\tilde{W}\|_2^2 + \left(1 - Ce^{-\tau/2}\right)\|\tilde{W}\|_2^2 \lesssim e^{-\tau/2}\|\tilde{W}\|_2. \tag{4.6}$$

Solving this differential inequality yields

$$\|\tilde{W}\|_2 \lesssim e^{-\tau/2},$$

finishing the proof. $\qquad\square$

REMARK 4.2.   By changing coordinates, we see that

$$W_0(y) = e^{y^2/8}e^y v_0(y),$$

which gives us that

$$\langle W_0, ye^{-y^2/8}\rangle = \int_0^\infty ye^y v_0(y)dy.$$

Hence, if $\int \xi v_0(\xi)d\xi$ is large enough, then α_0 must be positive. Indeed, (4.6) implies that, if $\tilde{W}$ is small initially, it stays small, and then (4.5) implies that, if W_1 is large initially ,it remains large. When equation (1.2) is used to approximate Fisher–KPP, this may be overcome by either choosing a larger initial condition or running the system for sufficiently long that this integral is large, depending on whether one is looking for a supersolution or a subsolution.

Now, we investigate g and R in (3.4). The ansatz implicit in (3.4) gives us the following equation

$$
\begin{aligned}
&-\frac{e^{-\tau/2}}{2}g+e^{-\tau/2}Mg+R_\tau+MR\\
&=e^{-\tau/2}\left[\frac{3\alpha}{4}y^2e^{-y^2/8}-\frac{\alpha\bar{c}}{2}ye^{-y^2/8}-\frac{3\alpha}{2}e^{-y^2/8}\right]\\
&\quad+e^{-\tau}\left[\frac{\alpha\bar{c}}{2}e^{-y^2/8}-\frac{\alpha\bar{c}}{4}y^2e^{-y^2/8}-\frac{3}{2}g_y+\frac{3y}{8}g-\frac{\bar{c}}{2}g+\frac{\bar{c}e^{-\tau/2}}{2}\left(g_y-\frac{y}{4}g\right)\right]\\
&\quad+e^{-\tau/2}\left[\left(\frac{\bar{c}e^{-\tau/2}}{2}-\frac{3}{2}\right)\left(R_y-\frac{y}{4}R\right)-\frac{\bar{c}}{2}R\right],
\end{aligned}
$$

where α is as in Lemma 4.1. We separate this into equations for g and R by associating the terms on the right of order $e^{-\tau/2}$ with g and the rest with R. This yields

$$
Mg-\frac{g}{2}=\alpha e^{-y^2/8}\left[\frac{3}{4}y^2-\frac{\bar{c}}{2}y-\frac{3}{2}\right]
\tag{4.7}
$$

and

$$
R_\tau+MR=e^{-\tau}f(y,\tau)+e^{-\tau/2}\left[\left(\frac{\bar{c}e^{-\tau/2}}{2}-\frac{3}{2}\right)\left(R_y-\frac{y}{4}R\right)-\frac{\bar{c}}{2}R\right],
\tag{4.8}
$$

where f is given by

$$
f(\tau,y)=\left[\frac{\alpha\bar{c}}{2}e^{-y^2/8}-\frac{\alpha\bar{c}}{4}y^2e^{-y^2/8}-\frac{3}{2}g_y+\frac{3y}{8}g-\frac{\bar{c}}{2}g+\frac{\bar{c}e^{-\tau/2}}{2}\left(g_y-\frac{y}{4}g\right)\right].
\tag{4.9}
$$

We first show that (4.7) is well-defined and that g has the property that $g_y(0)=0$ if and only if $\bar{c}=3\sqrt{\pi}$. Then, we show (4.8) is well-defined as well.

LEMMA 4.3.   *There exists a smooth solution $g\in X$ of (4.7) which is locally bounded in C^1. In addition, $g_y(0)=0$ if and only if $\bar{c}=3\sqrt{\pi}$.*

Proof.   First we show, abstractly, that such a solution exists. Then, we write down an explicit solution to the equation. This explicit solution allow us to understand $g_y(0)$.

In order to show the existence of a solution $g\in X$, we proceed as in Lemma 4.1. Namely, we write

$$
g=g_1e_0+\tilde{g},
\tag{4.10}
$$

where $\tilde{g}$ is in the span of $e_1,e_2,\ldots$. To bound g_1, we multiply (4.7) by e_0 and integrate. This gives an explicit formula for g_1 independent of $\tilde{g}$. Namely,

$$
g_1=-2\alpha\int\left[\frac{3}{4}y^2-\frac{\bar{c}}{2}-\frac{3}{2}\right]e_0(0)e^{-y^2/8}dy.
\tag{4.11}
$$

Then, fixing g_1 as this value, we simply write down the equation for $\tilde{g}$ given by

$$M\tilde{g} - \frac{\tilde{g}}{2} = \alpha \left[\frac{3}{4}y^2 - \frac{\bar{c}}{2}y - \frac{3}{2} \right] + \frac{g_1}{2}e_0. \tag{4.12}$$

On Span$\{e_1, e_2, \dots\}$ with the norm of X, the operator $M - 1/2$ is coercive, i.e. its spectrum is bounded below by a positive constant. Here, we are using that, for $\phi \in X \cap \text{Span}\{e_1, e_2, \dots\}$, we have $Q(\phi) \geq \|\phi\|_2^2$. Hence, the Lax–Milgram theorem implies that (4.12) is uniquely solvable in Span$\{e_1, e_2, \dots\}$. This, along with (4.11), gives us that (4.7) is uniquely solvable in X. The standard elliptic theory, as in [14], then, implies that g is in fact in H_{loc}^k for every k. This, in addition, implies that g is smooth and locally bounded in C^1.

We now solve (4.7) explicitly. By changing variables to $z = y^2/4$ and letting $G = e^{z/2}g$, we obtain the equation:

$$-zG_{zz} - \left(\frac{1}{2} - z \right) G_z - G = \alpha \left[3z - \bar{c}\sqrt{z} - \frac{3}{2} \right].$$

Following the work of Ebert and van Saarloos in [10] shows us that the explicit solution is of the form

$$G(z) = \alpha \left[\frac{3}{2} + 2\bar{c}\sqrt{z} - \frac{3}{2}F_2(z) + a_1(1 - 2z) + a_2 H(z) \right],$$

where F_2 and H are given by

$$F_2(z) = \sqrt{\pi} \sum_{n=2}^{\infty} \frac{z^n}{n(n-1)\Gamma(1/2+n)} \quad \text{and} \quad H(z) = -\frac{\sqrt{z}}{4} \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{\Gamma(-1/2+n)}{\Gamma(3/2+n)}. \tag{4.13}$$

As in [10], we may check that

$$\lim_{z \to \infty} \frac{H(z)}{z^{-3/2}e^z} = -\frac{1}{4}, \quad \text{and} \quad \lim_{z \to \infty} \frac{F_2(z)}{z^{-3/2}e^z} = \sqrt{\pi}.$$

Hence, H and F_2 are clearly not in L^2 in our original variables, even with the additional $e^{-z/2}$ factor. Thus, we must choose a_2 such that these terms cancel at $z = \infty$. In addition, we must choose a_1 such that $G(0) = 0$. Hence, we obtain

$$G(z) = \alpha \left[\frac{3}{2} + 2\bar{c}\sqrt{z} - \frac{3}{2}G_2(z) + \frac{-3}{2}(1 - 2z) - 6\sqrt{z}H(z) \right]$$

$$= \alpha \left[2\bar{c}\sqrt{z} - \frac{3}{2}G_2(z) + 3z - 6\sqrt{\pi}H(z) \right].$$

By uniqueness, we may return to the original variables to obtain an explicit formula for $g_y(0)$. Namely, we have that

$$g(y) = \alpha e^{-y^2/8} \left[\bar{c}y - \frac{3}{2}G_2(y^2/4) - \frac{3y^2}{4} - 6\sqrt{\pi}H(y^2/4) \right].$$

With this formula and with the first term in the sequence for H, we can easily see that $g_y(0) = 0$ if and only if $\bar{c} = 3\sqrt{\pi}$, finishing the proof. $\qquad\square$

Lemma 4.1 and Lemma 4.3 tell us that R must decay to zero in L^2 as τ tends to infinity. This is a key fact that we use in the following lemma, which finishes the proof of Lemma 3.1.

LEMMA 4.4.   *Let R satisfy* (4.8) *with α and g given above. Then we have the following bounds:*

$$\|R\|_2 \lesssim \tau e^{-\tau} \quad and \quad |R_y(0)| \lesssim \tau e^{-\tau}.$$

Proof.   We proceed by decomposing R as we did W and g in the proofs of lemmas 4.1 and 4.3. Namely, let

$$R(\tau,y) = R_1(\tau)e_0(y) + \tilde{R}(\tau,y), \tag{4.14}$$

where $\tilde{R}$ is orthogonal to e_0. To obtain a bound on R_1, we first note that lemmas 4.1 and 4.3 imply that R decays to zero in L^2 at least at the rate $e^{-\tau/2}$. Then, we multiply (4.8) by e_0 and integrate to obtain

$$(R_1)_\tau = e^{-\tau}\langle f, e_0\rangle - e^{-\tau/2}\langle R, \left(\frac{\bar{c}e^{-\tau/2}}{2} - \frac{3}{2}\right)\left((e_0)_y + \frac{y}{4}e_0\right) + \frac{\bar{c}}{2}e_0\rangle.$$

Using the formula for f given in (4.9), we see that $\langle f, e_0\rangle$ is bounded in X uniformly in time. In addition, we know that $e_0 \in X$ does not depend on time. Hence, applying Cauchy–Schwarz, we obtain

$$|(R_1)_\tau| \lesssim e^{-\tau} + e^{-\tau/2}\|R\|. \tag{4.15}$$

Since we know that $\|R\|_2 \lesssim e^{-\tau/2}$, it follows that

$$|(R_1)_\tau| \lesssim e^{-\tau}.$$

This gives us that

$$|R_1(\tau)| = |R_1(\infty) - R_1(\tau)| = \left|\int_\tau^\infty (R_1)_\tau(s)ds\right| \lesssim \int_\tau^\infty e^{-s}ds = e^{-\tau}.$$

This is the desired bound on R_1.

In order to finish the proof of the L^2 bound on R, we need to bound $\tilde{R}$ in L^2. To this end, we use the decomposition (4.14) along with (4.8) to note that $\tilde{R}$ satisfies

$$\tilde{R}_\tau + M\tilde{R} = e^{-\tau}f(y,\tau) + e^{-\tau/2}\left[\left(\frac{\bar{c}e^{-\tau/2}}{2} - \frac{3}{2}\right)\left(\tilde{R}_y - \frac{y}{4}\tilde{R}\right) - \frac{\bar{c}}{2}\tilde{R}\right]$$

$$+ R_1 e^{-\tau/2}\left[\left(\frac{\bar{c}e^{-\tau/2}}{2} - \frac{3}{2}\right)\left((e_0)_y - \frac{y}{4}e_0\right) - \frac{\bar{c}}{2}e_0\right] - (R_1)_\tau e_0.$$

Multiplying this by $\tilde{R}$ and integrating by parts yields

$$\frac{1}{2}\frac{d}{d\tau}\|\tilde{R}\|_2^2 + Q(\tilde{R})$$

$$\lesssim e^{-\tau}\|\tilde{R}\|_2 + e^{-\tau/2}\left(\int (y+1)\tilde{R}^2 dy\right) + e^{-\tau/2}R_1\|\tilde{R}\|_2 + |(R_1)_\tau|\|\tilde{R}\|_2.$$

Again using the inequality (4.3) along with the inequality on R_1 and $(R_1)_\tau$ that we just obtained, we note that

$$\frac{d}{d\tau}\|\tilde{R}\|_2^2 + (2 - Ce^{-\tau/2})\|\tilde{R}\|_2^2 \lesssim e^{-\tau}\|\tilde{R}\|_2, \tag{4.16}$$

where C is some universal constant.

We will now solve this differential inequality. We do this in two steps. First, we obtain a rough bound on the decay of $\tilde{R}$, and then we leverage this to get the correct decay. To start, we may set τ_0 to be the time when $2 - Ce^{-\tau/2} \geq 7/4$ for all $\tau \geq \tau_0$. Then, we see that, for $\tau \geq \tau_0$, we have

$$\frac{d}{d\tau}\|\tilde{R}\|_2^2 + \frac{7}{4}\|\tilde{R}\|_2^2 \lesssim e^{-\tau}\|\tilde{R}\|_2.$$

This gives us that

$$\frac{d}{d\tau}\left(e^{\frac{7\tau}{4}}\|\tilde{R}\|_2^2\right) \lesssim e^{-\tau/8}\left(e^{\frac{7\tau}{4}}\|\tilde{R}\|_2^2\right)^{1/2}.$$

Integrating this in τ implies that $\|\tilde{R}\|_2 \lesssim e^{-7\tau/8}$.

Call $\psi(\tau) = e^{\tau}\|\tilde{R}\|_2$, and we have that

$$\psi_\tau - Ce^{-\tau/2}\psi \lesssim 1.$$

From above, we have that $\psi(\tau) \lesssim e^{\tau/8}$. The combination of these implies that

$$\psi_\tau \lesssim 1,$$

which in turn implies that $\psi(\tau) \lesssim 1 + \tau$. This gives us the desired inequality

$$\|\tilde{R}\|_2 \lesssim (\tau + 1)e^{-\tau},$$

which finishes the proof of the initial bound of R.

To finish the proof, we need to bound $R_y(0)$. This, however, is a simple consequence of our bounds on the L^2 norm of R and the right-hand side of (4.8). Indeed, with these the standard parabolic regularity theory, which may be found in [23, 24], give us the desired bound on $R_y(0)$, finishing the proof. □

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