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EMERGENCE OF FLOCKING FOR A MULTI-AGENT SYSTEM MOVING WITH CONSTANT SPEED*

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Abstract. We present a Cucker–Smale-type flocking model for interacting multi-agents(or particles) moving with constant speed in arbitrary dimensions, and derive a sufficient condition for the asymptotic flocking in terms of spatial and velocity diameters, coupling strength and a communication weight. In literature, several Vicsek-type models with a unit speed constraint have been proposed in the modeling of self-organization and planar models were extensively studied via the dynamics of the heading angle. Our proposed model has a velocity coupling that is orthogonal to the velocity of the test agent to ensure the constancy of speed of the test agent along the dynamic process. For a flocking estimate, we derive a system of dissipative differential inequalities for spatial and velocity diameters, and we also employ a robust Lyapunov functional approach.

Key words. Cucker–Smale model, flocking, Lyapunov functional, unit speed constraint, Vicsek model.

AMS subject classifications. Primary: 70K20; Secondary: 34D05.

1. Introduction

Emergent collective motions such as flocking and synchronization have often been observed in biological complex systems [4, 6–8, 21, 22, 25, 29–32], and has received considerable attention in biology, engineering, and physics because of their applications in the controlling of man-made systems and networks [13–15, 18, 23, 24, 26–28]. The systematic studies of such emergent phenomena were initially addressed by two innovators, Winfree [32] and Kuramoto [16, 17] several decades ago. Why does a complex biological system exhibit flocking and synchronous behavior? Such a question has long concerned ecologists. One possible answer is that the collective behaviors are a better strategy than individual behaviors from the viewpoint of survival. Indeed, the mechanism that leads to flocking and synchronization is not that simple. In fact, in order to understand this mechanism, we need to identify environmental information and agent's individual motion. In this paper, we do not attempt to answer this intriguing question. However, we instead try to model such ordered phenomena using a simple dynamical system for possible applications in the control theory of multi-agent systems. More precisely, we propose a second-order multi-agent model with constant speed constraint in any dimensions. As aforementioned, there are some previous works [10, 14, 15, 23, 24, 28, 31] on the flocking dynamics of two-dimensional models for multi-agent systems with the unit-speed constraint. In this case, since agents have unit speeds, the velocity lies on the $\mathbb{S}^1$ so that the dynamics of velocity is completely determined by that of the heading angle. This is the point where the two-dimensionality of the underlying dynamics crucially surfaces up. In contrast, for high dimensions, we have to employ more angles, which make the dynamical systems will be so complicated if we use the

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generalized spherical coordinate. Recently, several Vicsek-type particle models with a unit speed constraint were proposed in [3, 5, 9] for the study of alignment (or flocking). In this paper, motivated by the works [19, 20] on the quantum synchronization, we present a new Cucker–Smale-type flocking model with a constant speed constraint so that our proposed model generalizes the planar model introduced in [10], where the two-dimensionality of the underlying physical space is crucially used. Before we move on, we briefly discuss why constant speed constraint matters in the flocking modeling. As far as the authors know, this is mainly due to the historical development of flocking in physics literature. The modeling of flocking phenomena were first introduced by Vicsek's group [31] in the physics community and the unit speed constraint was employed in relation with the phase models for synchronization, e.g., the Kuramoto model [16, 17], and Vicsek's work advanced further research [7, 14, 15, 18, 22–24, 28] on the collective dynamics of interacting multi-agent systems in engineering and physics. Unlike the Cucker–Smale model, where the constant speed constraint was not imposed, the constant speed constraint will result in the shrinking of the set of admissible initial configurations leading to the asymptotic flocking (see Theorem 3.1).

Let (x_i, v_i) denote the phase space coordinate of the ith agent. Then, the generalized CS model reads as follows:
For $i = 1, \ldots, N$,

$$\frac{dx_i}{dt} = v_i, \qquad \frac{dv_i}{dt} = K \sum_{j=1}^{N} \psi_{ij} \Gamma(v_i, v_j),$$

where K is the coupling strength and $\Gamma(v_i, v_j)$ is the velocity coupling between v_i and v_j. For a symmetric case in which

$$\psi_{ij} = \psi_{ji}, \quad \Gamma(v_i, v_j) = \Gamma(v_j, v_i), \qquad 1 \le i, j \le N, \tag{1.1}$$

the total momentum $\sum_{i=1}^{N} v_i$ is a conserved quantity, and therefore, if flocking occurs, then the velocities of individual agents will tend to the average initial velocity such that the speeds of agents are not invariant along the CS flow. Therefore, to incorporate the constant speed constraint and the Cucker–Smale flocking mechanism, we need to disregard one symmetry in (1.1). Here we disregard the latter case, i.e., we do not require the velocity coupling to be symmetric in the exchange of v_i and v_j in Γ. Recently, Ha, Jeong and Kang [10] attempted to combine these two mechanisms and heuristically derived a simple planar model similar to the Vicsek [31] and the Justh–Krishnaprasad models [14,15]. This derivation crucially uses the polar coordinate.

There are two main results of this paper. First, we derive a Cucker–Smale-type flocking model by incorporating the CS flocking mechanism and constant speed assumption together in *arbitrary dimensions*. As aforementioned, we take a velocity coupling $\Gamma(v_i, v_j)$ satisfying

$$\Gamma(v_i, v_j) \ne \Gamma(v_j, v_i) \qquad \Gamma(v_i, v_j) \cdot v_i = 0.$$

More precisely, let x_i and v_i be the spatial position and velocity of the ith particles, respectively. For the Cucker–Smale flocking [4], the velocity coupling $\Gamma(v_i, v_j)$ is simply defined as the relative velocity $v_j - v_i$, and as mentioned before, this coupling does not preserve the speed. Therefore, we instead employ the following velocity coupling motivated by [19, 20] (see Section 2.1 for details):

$$\Gamma(v_i,v_j) := v_j - \frac{\langle v_j,v_i\rangle}{\langle v_i,v_i\rangle} v_i.$$

Second, we introduce the position and velocity diameters $D(x)$ and $D(v)$ (see Notation at the end of Introduction), respectively, and show that these functionals satisfy a system of dissipative differential inequalities for the metric dependent communication weights $\psi_{ij} = \psi(||x_i - x_j||)$:

$$\left|\frac{dD(x)}{dt}\right| \le D(v), \quad \frac{dD(v)}{dt} \le -KC_0\psi(D(x))\mathcal{A}(v_0)D(v), \quad \text{a.e. } t>0.$$

Then we employ the Lyapunov functional approach for the above SDDI system to derive the desired asymptotic flocking estimate (see Section 3.2 for more details).

The rest of the paper is organized as follows: In Section 2, we discuss our model and present a key a priori estimates for the proposed model. In Section 3, we derive the SDDI for the spatial and velocity diameters, and provide a flocking estimate using the Lyapunov functional approach. Finally, Section 4 is devoted to the summary of our main results. In Appendix A, we present an example leading to the nonexistence of a global flocking in the absence of the geometric condition $\mathcal{A}(v_0) > 0$.

Notation: Throughout the paper, we set

$$x := (x_1,\dots,x_N), \quad v := (v_1,\dots,v_N),$$

and for configuration (x,v), we introduce the position and velocity diameters as follows.

$$D(x) := \max_{1\le i,j\le N} ||x_i - x_j||, \quad D(v) := \max_{1\le i,j\le N} ||v_i - v_j||.$$

2. Preliminaries

In this section, we briefly discuss our model and present an a priori estimate regarding the variation of angle between two distinct agent velocities.

2.1. Discussion of our model.

First, we recall the Cucker–Smale flocking model, before we discuss our proposed model. Let (x_i,v_i) be the position and velocity pair of the ith agent and assume that the internal forcing from the neighboring field agent j to the ith agent is given by the weighted relative velocity $\psi_{ij}(v_j - v_i)$. In this setting, the Cucker–Smale (CS) model on the Euclidean configuration space $\mathbb{R}^d \times \mathbb{R}^d$ reads as follows:

$$\begin{aligned} &\frac{dx_i}{dt} = v_i, \quad t>0, \\ &\frac{dv_i}{dt} = K\sum_{k=1}^N \psi_{ik}(v_k - v_i). \end{aligned} \tag{2.1}$$

Presently, we do not assume any specific ansatz for ψ_{ij}. For the symmetric weight $\psi_{ij} = \psi_{ji}$, the total momentum $\sum_{i=1}^N v_i$ is a conserved quantity, and therefore, if flocking occurs, the agent's individual velocity should converge to the averaged velocity $\frac{1}{N}\sum_{i=1}^N v_i$. Thus, the speed of agents is not preserved along the dynamics (2.1) (see [2, 4, 11, 12]), unless the agent speeds are that of averaged one. To guarantee constant speed, we employ Lohe's idea as follows. In [19, 20], Lohe introduced the following first-order model on the sphere from the quantum synchronization model:

$$\dot{x}_i = \Omega_i \frac{x_i}{||x_i||^2} + \underbrace{\frac{K}{N}\sum_{k=1}^{N}\Big(x_k - \frac{\langle x_i, x_k\rangle x_i}{||x_i||^2}\Big)}_{\diamond}, \quad i = 1, \dots, N,$$

where x_i is the position of the ith particle. Note that the term $\diamond$ represents an aggregation mechanism and make particle stay on the sphere. Following Lohe's idea, we employ the velocity coupling term $\Gamma(v_k, v_i)$:

$$\Gamma(v_k, v_i) := v_k - \frac{\langle v_k, v_i\rangle}{\langle v_i, v_i\rangle} v_i$$

which has the property:

$$\langle v_i, \Gamma(v_k, v_i)\rangle = 0.$$

This leads to the conservation of modulus for v_i in the time-evolution of (2.2) (see Lemma 2.1). To obtain Cucker–Smale-type system, we insert a communication weight ψ_{ij} in front of each coupling term to incorporate the degree of interactions between i and jth agents. Based on the above discussion, we introduce the following modified CS model:

$$\begin{aligned} \frac{dx_i}{dt} &= v_i, \quad t > 0, \\ \frac{dv_i}{dt} &= K\sum_{k=1}^{N}\psi_{ik}\Big(v_k - \frac{\langle v_k, v_i\rangle}{\langle v_i, v_i\rangle} v_i\Big), \end{aligned} \tag{2.2}$$

where ψ_{ik} is a non-negative metric dependent communication weight.

Note that even for the symmetric communication $\psi_{ik} = \psi_{ki}$, the velocity coupling is not symmetric in i and k, i.e.,

$$v_k - \frac{\langle v_k, v_i\rangle}{\langle v_i, v_i\rangle} v_i \neq v_i - \frac{\langle v_i, v_k\rangle}{\langle v_k, v_k\rangle} v_k.$$

Thus, the total momentum may not be conserved, i.e.,

$$\sum_{i=1}^{N} v_i(t) \neq \sum_{i=1}^{N} v_i(0), \quad t \geq 0,$$

so that we may achieve a constancy of the speeds. Then, the following lemma guarantees the constancy of speed.

LEMMA 2.1. *Let (x_i, v_i) be a solution to system (2.2). Then, the speed of agents is constant along the flow (2.2):*

$$||v_i(t)|| = ||v_{i0}||, \quad t > 0.$$

Proof. We take an inner product $(2.2)_2$ with v_i to determine

$$\frac{1}{2}\frac{d||v_i||^2}{dt} = K\sum_{k=1}^N \psi_{ik}\Big(\langle v_i, v_k\rangle - \frac{\langle v_k, v_i\rangle}{\langle v_i, v_i\rangle}\langle v_i, v_i\rangle\Big) = 0.$$

This implies that

$$\frac{d}{dt}||v_i||^2 = 0, \quad \text{i.e.,} \quad ||v_i(t)|| = ||v_{i0}||, \quad t > 0.$$

□

Next, we discuss how the system (2.2) can be reduced to the model introduced in [10] for the special case:

$$d = 2 \quad \text{and} \quad ||v_{i0}|| = 1, \ 1 \le i \le N.$$

In this case, it follows from Lemma 2.1 that

$$||v_i(t)|| = 1 \quad \text{for all } t > 0 \text{ and } i = 1, \ldots, N.$$

Thus, we can rewrite v_i as a polar form:

$$v_i = (\cos\theta_i, \sin\theta_i). \tag{2.3}$$

Then, we substitute the ansatz (2.3) into the equation $(2.2)_2$ to obtain

$$\begin{aligned}&(-\sin\theta_i, \cos\theta_i)\dot{\theta}_i\\ &= K\sum_{k=1}^N \psi_{ij}\Big[(\cos\theta_k, \sin\theta_k) - (\cos\theta_i\cos\theta_k + \sin\theta_i\sin\theta_k)(\cos\theta_i, \sin\theta_i)\Big].\end{aligned}$$

We take an inner product with $(-\sin\theta_i, \cos\theta_i)$ to obtain

$$\frac{d\theta_i}{dt} = K\sum_{k=1}^N \psi_{ik}\sin(\theta_k - \theta_i).$$

Thus, our proposed model (2.2) becomes the model in [10]:

$$\begin{aligned}\frac{dx_i}{dt} &= e^{\sqrt{-1}\theta_i},\\ \frac{d\theta_i}{dt} &= K\sum_{k=1}^N \psi_{ki}\sin(\theta_k - \theta_i).\end{aligned}$$

2.2. A priori estimates. In this part, we present a key a priori estimate for the model (2.2). For later flocking estimates, we next introduce a functional $\mathcal{A}(v)$ to measure the maximal angle between v_is

$$\mathcal{A}(v) := \min_{i\neq j}\langle v_i, v_j\rangle.$$

Note that $\mathcal{A}(v(t))$ measures a maximum angle between the velocities of distinct agents in the sense that

$$\langle v_i, v_j\rangle = r^2\cos\theta_{ij}, \quad \text{where } \theta_{ij} \text{ is the angle between } i\text{th and } j\text{th particles.}$$

Lemma 2.2. *Let (x,v) be a solution to system (2.2) with initial configuration (x_0,v_0) satisfying*

$$||v_{i0}||=r, \quad 1\leq i\leq N, \qquad \mathcal{A}(v_0)>0.$$

Then, the functional $\mathcal{A}(v(t))$ is non-decreasing along the flow (2.2):

$$\mathcal{A}(v(t))\geq \mathcal{A}(v_0), \quad t>0.$$

Proof. First, we note that Lemma 2.1 yields

$$||v_i(t)||=r, \quad t\geq 0,\ 1\leq i\leq N.$$

We take an inner product with v_j and $\dot{v}_i$ to obtain

$$\langle \dot{v}_i,v_j\rangle = K\sum_{k=1}^{N}\psi_{ik}\Big(\langle v_k,v_j\rangle - \frac{\langle v_k,v_i\rangle}{r^2}\langle v_i,v_j\rangle\Big). \tag{2.4}$$

On the other hand, we change the role of i and j to obtain

$$\langle v_i,\dot{v}_j\rangle = K\sum_{k=1}^{N}\psi_{jk}\Big(\langle v_i,v_k\rangle - \frac{\langle v_k,v_j\rangle}{r^2}\langle v_i,v_j\rangle\Big). \tag{2.5}$$

We combine (2.4) and (2.5) to obtain

$$\frac{d\langle v_i,v_j\rangle}{dt} = K\sum_{k=1}^{N}\psi_{ik}\Big(\langle v_k,v_j\rangle - \frac{\langle v_k,v_i\rangle}{r^2}\langle v_i,v_j\rangle\Big) + K\sum_{k=1}^{N}\psi_{jk}\Big(\langle v_i,v_k\rangle - \frac{\langle v_k,v_j\rangle}{r^2}\langle v_i,v_j\rangle\Big).$$

We define a set $\mathcal{T}$ as follows:

$$\mathcal{T}:=\Big\{t\in\mathbb{R}_+ : \mathcal{A}(v(t))>0\Big\}.$$

Since $\mathcal{A}(v_0)>0$, there exists some $\delta>0$ such that

$$\mathcal{A}(v(t))>0, \quad t\in(0,\delta).$$

Thus, $\delta\in\mathcal{T}$, i.e., $\mathcal{T}$ is non-empty. We now claim

$$T_* := \sup\mathcal{T}=\infty.$$

Suppose not, i.e., $T_*<\infty$. Then it follows from the continuity of $\mathcal{A}(\cdot)$ that

$$\lim_{t\to T_*-}\mathcal{A}(v(t))=0. \tag{2.6}$$

For a fixed t, we take indices i_t and j_t such that

$$\langle v_{i_t},v_{j_t}\rangle = \min_{i\neq j}\langle v_i(t),v_j(t)\rangle. \tag{2.7}$$

For such i_t and j_t, we have

$$\frac{d\langle v_{i_t},v_{j_t}\rangle}{dt} = K\sum_{k=1}^{N}\psi_{i_tk}\Big(\langle v_{j_t},v_k\rangle - \frac{\langle v_{i_t},v_k\rangle}{r^2}\langle v_{j_t},v_{i_t}\rangle\Big)$$

$$
\begin{aligned}
&+K\sum_{k=1}^{N}\psi_{j_tk}\Big(\langle v_{i_t},v_k\rangle-\frac{\langle v_{j_t},v_k\rangle}{r^2}\langle v_{j_t},v_{i_t}\rangle\Big)\\
&=\mathcal{I}_{11}+\mathcal{I}_{12},\quad t\in[0,T_*).
\end{aligned}
$$

We use the minimality (2.7) of $\langle v_{i_t},v_{j_t}\rangle$:

$$
\frac{\langle v_{i_t},v_k\rangle}{r^2}\langle v_{j_t},v_{i_t}\rangle\le\frac{\langle v_{i_t},v_k\rangle}{r^2}\langle v_{j_t},v_k\rangle
$$

to obtain

$$
\begin{aligned}
\mathcal{I}_{11}&=K\sum_{k=1}^{N}\psi_{i_tk}\Big(\langle v_{j_t},v_k\rangle-\frac{\langle v_{i_t},v_k\rangle}{r^2}\langle v_{j_t},v_{i_t}\rangle\Big)\\
&\ge K\sum_{k=1}^{N}\psi_{i_tk}\Big(1-\frac{\langle v_{i_t},v_k\rangle}{r^2}\Big)\langle v_{j_t},v_k\rangle\\
&\ge 0.
\end{aligned}
$$

By the same argument, we have

$$
\mathcal{I}_{12}\ge 0.
$$

Thus, $\mathcal{A}(v(t))$ is a non-decreasing function on $[0,T_*)$ such that

$$
\mathcal{A}(v(t))\ge\mathcal{A}(v_0)>0.
$$

This gives a contradiction to (2.6), and therefore, we have

$$
T_*=\infty,\quad \mathcal{A}(v(t))\ge\mathcal{A}(v_0),\quad t\in[0,\infty).
$$

□

3. Asymptotic flocking estimates

In this section, we derive a system of dissipative differential inequalities (SDDI) for diameters $D(x)$ and $D(v)$, and for the metric dependent communication weight $\psi_{ik}\ge 0$, we show that the flocking occurs asymptotically. Indeed, for the linear time dependent weight $\psi(t)$, the flocking estimate will be directly followed by the SDDI(see Remark 3.2). First, we recall the definition of asymptotic flocking as follows.

Definition 3.1. *Let (x,v) be the solution to system (2.2). Then, the system $(x(t),v(t))$ exhibits asymptotic flocking if and only if the following conditions hold.*

$$
\sup_{0\le t<\infty}||x_i(t)-x_j(t)||<\infty,\quad \lim_{t\to\infty}||v_i(t)-v_j(t)||=0,\qquad 1\le i,j\le N.
$$

Remark 3.1. Note that the asymptotic flocking is equivalent to

$$
\sup_{t\ge 0}D(x(t))<\infty,\qquad \lim_{t\to\infty}D(v(t))=0.
$$

3.1. Derivation of SDDI. In this part, we derive the SDDI for $D(v)$ and $D(x)$.

LEMMA 3.2. *For a given $T\in(0,\infty]$, let $(x(t),v(t))$ be a solution to system (2.2) satisfying*

$$||v_i(t)||=r \quad \text{and} \quad D(x(t))D(v(t))\mathcal{A}((t))>0, \quad t\in(0,T).$$

Then, $D(x)$ and $D(v)$ satisfy

$$\left|\frac{dD(x)}{dt}\right|\le D(v), \qquad \frac{dD(v)}{dt}\le -\frac{KN\psi_m}{2r^2}\mathcal{A}(v)D(v), \qquad a.e.\ t\in(0,T), \tag{3.1}$$

where $\psi_m=\min_{i,j}\psi_{ij}$.

Proof.

• (Derivation of the first inequality): It follows from $(2.2)_2$ that we have

$$\frac{d}{dt}(x_i-x_j)=v_i-v_j. \tag{3.2}$$

We take an inner product (3.2) with (x_i-x_j) to obtain

$$\frac{1}{2}\frac{d}{dt}||x_i-x_j||^2=\langle v_i-v_j,x_i-x_j\rangle.$$

Now, we use Young's inequality to obtain

$$\left|||x_i-x_j||\frac{d}{dt}||x_i-x_j||\right|=|\langle v_i-v_j,x_i-x_j\rangle|\le ||v_i-v_j||\ ||x_i-x_j||,$$

Thus, we have

$$\left|\frac{d}{dt}||x_i-x_j||\right|\le ||v_i-v_j||\le D(v), \quad \text{for all } i \text{ and } j.$$

• (Derivation of the second inequality): By assumption, we have

$$||v_i(t)||=r, \quad \langle v_i(t),v_j(t)\rangle>0, \quad t\in(0,T),\ 1\le i,j\le N.$$

These yield

$$r^2-\langle v_i,v_j\rangle=\frac{1}{2}||v_i-v_j||^2. \tag{3.3}$$

Now, we use the relation (3.3) to obtain

$$r^2-\mathcal{A}(v)=r^2-\min_{i\neq j}\langle v_i,v_j\rangle=\frac{1}{2}\max_{i\neq j}||v_i-v_j||^2=\frac{1}{2}D(v)^2. \tag{3.4}$$

On the other hand, note that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||v_i-v_j||^2&=-\frac{d}{dt}\langle v_i,v_j\rangle=-\langle \dot v_i,v_j\rangle-\langle v_i,\dot v_j\rangle\\&=-K\sum_{k=1}^N\psi_{ik}\Big(\langle v_j,v_k\rangle-\frac{\langle v_i,v_k\rangle}{r^2}\langle v_j,v_i\rangle\Big)\end{aligned}$$

$$-K\sum_{k=1}^{N}\psi_{jk}\Big(\langle v_i,v_k\rangle-\frac{\langle v_j,v_k\rangle}{r^2}\langle v_j,v_i\rangle\Big). \tag{3.5}$$

For a given t, we take indices i_t and j_t such that

$$D(v(t))=||v_{i_t}-v_{j_t}||.$$

Then for such i_t and j_t, we use (3.4) and (3.5) to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}D(v(t))^2=&-K\sum_{k=1}^{N}\psi_{i_tk}\Big(\langle v_{j_t},v_k\rangle-\frac{\langle v_{i_t},v_k\rangle}{r^2}\langle v_{j_t},v_{i_t}\rangle\Big)\\&-K\sum_{k=1}^{N}\psi_{j_tk}\Big(\langle v_{i_t},v_k\rangle-\frac{\langle v_{j_t},v_k\rangle}{r^2}\langle v_{j_t},v_{i_t}\rangle\Big).\end{aligned} \tag{3.6}$$

We use the minimality of $\langle v_{i_t},v_{j_t}\rangle$ and

$$r^2\geq\langle v_i,v_j\rangle>0,\quad 1\leq i,j\leq N$$

to obtain

$$\langle v_{j_t},v_k\rangle-\frac{\langle v_{i_t},v_k\rangle}{r^2}\langle v_{j_t},v_{i_t}\rangle>0,\quad \langle v_{i_t},v_k\rangle-\frac{\langle v_{j_t},v_k\rangle}{r^2}\langle v_{j_t},v_{i_t}\rangle>0. \tag{3.7}$$

Now, we use (3.6) and (3.7) to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}D(v(t))^2\leq&-K\psi_m(t)\sum_{k=1}^{N}\Big(\langle v_{j_t},v_k\rangle-\frac{\langle v_{i_t},v_k\rangle}{r^2}\langle v_{j_t},v_{i_t}\rangle\Big)\\&-K\psi_m(t)\sum_{k=1}^{N}\Big(\langle v_{i_t},v_k\rangle-\frac{\langle v_{j_t},v_k\rangle}{r^2}\langle v_{j_t},v_{i_t}\rangle\Big),\end{aligned} \tag{3.8}$$

where we used the notation $\psi_m(t):=\min_{i\neq j}\psi_{ij}$.

To obtain the desired estimate, we rearrange the terms in R.H.S. of (3.8) as follows:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}D(v(t))^2\leq&-K\psi_m(t)\sum_{k=1}^{N}\Big(\langle v_{j_t},v_k\rangle-\frac{\langle v_{j_t},v_k\rangle}{r^2}\langle v_{j_t},v_{i_t}\rangle\Big)\\&-K\psi_m(t)\sum_{k=1}^{N}\Big(\langle v_{i_t},v_k\rangle-\frac{\langle v_{i_t},v_k\rangle}{r^2}\langle v_{j_t},v_{i_t}\rangle\Big)\\:=&-\Big(\mathcal{I}_{21}+\mathcal{I}_{22}\Big).\end{aligned}$$

Next, we estimate $\mathcal{I}_{21}$ and $\mathcal{I}_{22}$ separately.

We use

$$\langle v_{j_t},v_k\rangle\geq\langle v_{j_t},v_{i_t}\rangle=\mathcal{A}(v(t)),\quad 1\leq k\leq N,$$
$$1-\frac{\langle v_{i_t},v_{j_t}\rangle}{r^2}=\frac{1}{2}||v_{i_t}-v_{j_t}||^2$$

to obtain

$$\mathcal{I}_{21}=K\psi_m(t)\sum_{k=1}^{N}\Big(\langle v_{j_t},v_k\rangle-\frac{\langle v_{j_t},v_k\rangle}{r^2}\langle v_{j_t},v_{i_t}\rangle\Big)$$

$$
\begin{aligned}
&= K\psi_m(t)\sum_{k=1}^{N}\Big(1-\frac{\langle v_{j_t},v_{i_t}\rangle}{r^2}\Big)\langle v_{j_t},v_k\rangle \\
&= \frac{K\psi_m(t)}{2r^2}||v_{i_t}-v_{j_t}||^2\sum_{k=1}^{N}\langle v_{j_t},v_k\rangle \\
&\ge \frac{KN\mathcal{A}(v)\psi_m(t)}{2r^2}||v_{i_t}-v_{j_t}||^2
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{I}_{22} &= K\psi_m(t)\sum_{k=1}^{N}\Big(\langle v_{i_t},v_k\rangle-\frac{\langle v_{i_t},v_k\rangle}{r^2}\langle v_{j_t},v_{i_t}\rangle\Big) \\
&= K\psi_m(t)\sum_{k=1}^{N}\Big(1-\frac{\langle v_{j_t},v_k\rangle}{r^2}\Big)\langle v_{i_t},v_k\rangle \\
&= \frac{K\psi_m(t)}{2r^2}||v_{i_t}-v_{j_t}||^2\sum_{k=1}^{N}\langle v_{i_t},v_k\rangle \\
&\ge \frac{KN\mathcal{A}(v)\psi_m(t)}{2r^2}||v_{i_t}-v_{j_t}||^2.
\end{aligned}
$$

Thus, we have

$$
\frac{d}{dt}(D(v))^2 \le -\frac{KN\mathcal{A}(v)\psi_m(t)}{r^2}||v_{i_t}-v_{j_t}||^2 = -\frac{KN\mathcal{A}(v)\psi_m(t)}{r^2}D(v)^2.
$$

This yields the desired result. □

REMARK 3.2. For a linear communication weight with the property

$$
\min_{i,j}\psi_{ij}(t) \ge \frac{C}{(1+t)^\alpha}, \quad 0\le\alpha<1,
$$

it follows from the second inequality in (3.1) and Lemma 2.2 that

$$
D(x(t))<\infty, \quad \lim_{t\to\infty} D(x(t))=0.
$$

3.2. Asymptotic flocking estimate. In this part, we present an asymptotic flocking estimate for system (2.2) with metric dependent ψ introduced in [4,11,21]. For this, we set ψ to be non-increasing in the argument and bounded:

$$
\begin{aligned}
&\psi(r)>0, \qquad (\psi(r_1)-\psi(r_2))(r_1-r_2)\le 0, \qquad \sup_{r\ge 0}\psi(r)\le\psi_M<\infty, \\
&\psi_{ij} =: \text{Either } \frac{\psi(||x_i-x_j||)}{N}, \quad \text{or} \quad \frac{\psi(||x_j-x_i||)}{\sum_{k=1}^{N}\psi(||x_k-x_i||)}.
\end{aligned} \tag{3.9}
$$

Then, it is easy to see that

$$
\min_{i,j}\psi_{ij} \ge \frac{\min\left\{\frac{1}{\psi_M},1\right\}}{N}\psi(D(x)). \tag{3.10}
$$

It follows from lemmas 3.2 and (3.10) that

$$\begin{aligned} \left|\frac{dD(x)}{dt}\right| &\le D(v), \quad \text{a.e. } t \in (0,T), \\ \frac{dD(v)}{dt} &\le -KC_0\psi(D(x))D(v), \end{aligned} \tag{3.11}$$

where the constant C_0 is a positive constant defined by

$$C_0(K,r,\psi_M,v_0) := \frac{1}{2r^2}\min\left\{\frac{1}{\psi_M},1\right\}\mathcal{A}(v_0). \tag{3.12}$$

Following [11], we introduce the Lyapunov-type functionals:

$$\mathcal{H}_\pm(t) := D(v(t)) \pm KC_0 \int_0^{D(x(t))} \psi(s)ds, \quad t \in (0,T). \tag{3.13}$$

Note that the Lyapunov functional (3.13) looks like the same as the functionals [11,12,21] for the CS models with symmetric and non-symmetric communication weights [4,21]. However, in our case, the constant C_0 in (3.12) has a geometric factor $\mathcal{A}(v_0)$ measuring the maximal angle between agent's initial velocities. This is simply due to the nature of nonlinear velocity couplings to make the unit modulus of velocities.

Lemma 3.3. *For a given $T \in (0,\infty]$, let $(x(t),v(t))$ be a solution to system (2.2) and (3.9) satisfying*

$$||v_i(t)|| = r \quad \text{and} \quad D(x(t))D(v(t))\mathcal{A}(v(t)) > 0, \quad t \in (0,T).$$

Then, for any solution to system (2.2) with communication weight (3.9), we have

$$D(v(t)) + KC_0\left|\int_{D(x_0)}^{D(x(t))} \psi(s)ds\right| \le D(v_0), \quad t \in (0,T).$$

Proof. It follows from (3.11) that

$$\begin{aligned} \frac{d\mathcal{H}_\pm}{dt} &= \frac{dD(v)}{dt} \pm C_0\psi(D(x))\frac{dD(x)}{dt} \\ &\le -KC_0\psi(D(x))D(v) \pm C_0K\psi(D(x))\frac{dD(x)}{dt} \\ &= -KC_0\psi(D(x))\Big(D(v) \mp \frac{dD(x)}{dt}\Big) \\ &\le 0. \end{aligned}$$

Thus, we have

$$\mathcal{H}_\pm(t) \le \mathcal{H}_\pm(0), \quad t \ge 0.$$

This and (3.13) yield the desired estimate. □

We are now ready to present our main result for system (2.2).

Theorem 3.4. *Suppose that the communication weight ψ and initial configuration (x_0,v_0) satisfy (3.9) and*

$$(i)\ ||v_i(0)|| = r, \quad 1 \le i \le N, \qquad \mathcal{A}(v_0) > 0,$$

$$(ii)\ 0 < D(v_0) < KC_0 \min\Big\{ \int_{D(x_0)}^{\infty} \psi(s)ds,\ \int_0^{D(x_0)} \psi(s)ds \Big\}.$$

Then, there exist a unique solution $(x(t), v(t))$ *to system (2.2) satisfying the asymptotic flocking conditions.*

$$\begin{aligned} &(i)\ \sup_{t\geq 0} D(x(t)) < D^\infty, \\ &(ii)\ D(v(t)) \leq D(v_0)\exp\Big(-KC_0\psi(D^\infty)t\Big), \quad t\geq 0, \end{aligned} \tag{3.14}$$

where D^∞ *is a positive constant implicitly defined by the following relation:*

$$D(v_0) := KC_0 \int_{D(x_0)}^{D^\infty} \psi(s)ds. \tag{3.15}$$

Proof. Since R.H.S. of (2.2) is linear and Lipschitz continuous, the standard Cauchy-Lipschitz theory yields the global solution $(x(t), v(t))$ to system (2.2). Hence, we focus on the flocking estimate (3.14). In order to use Lemma 3.3, we have to show that

$$||x_i(t)|| = r, \qquad D(x(t))D(v(t))\mathcal{A}(v(t)) > 0, \quad t \in (0,\infty).$$

It follows from Lemma 2.1 and 2.2 that

$$||x_i(t)|| = r, \quad \mathcal{A}(v(t)) \geq \mathcal{A}(v_0) > 0. \tag{3.16}$$

Thus, it suffices to verify that the diameters $D(x(t))$ and $D(v(t))$ cannot be zero in a finite-time. First, we show that the velocity diameter $D(v(t))$ cannot be zero in finite-time. Suppose not, i.e., there exists the smallest time $T_* \in (0,\infty)$ such that

$$D(v(T_*)) = 0, \quad v_i(T_*) = v_j(T_*), \qquad 1 \leq i,j \leq N.$$

Then, it follows from (2.2) that

$$\frac{dv_i(T_*)}{dt} = 0, \quad 1 \leq i \leq N.$$

On the other hand, by differentiating the equation (2.2), we also have

$$\frac{d^k v_i(T_*)}{dt^k} = 0, \quad k \geq 2.$$

Then, there exists $\delta > 0$ such that

$$v_i(t) = v_j(t), \quad T_* - \delta < t < T_* + \delta.$$

This contradicts to the minimality of T_* and we obtain

$$D(v(t)) > 0, \quad t \in (0,\infty). \tag{3.17}$$

Since ψ is strictly decreasing and non-negative in its argument,

$$F(\delta) := KC_0 \int_\delta^{D(x_0)} \psi(s)ds \text{ is a non-increasing continuous function in } \delta.$$

Thus, there exists a unique $D_m > 0$ such that

$$D(v_0) = KC_0 \int_{D_m}^{D(x_0)} \psi(s) ds, \quad D_m < D(x_0),$$

since

$$0 < D(v_0) < KC_0 \int_0^{D(x_0)} \psi(s) ds = F(0), \quad F(D(x_0)) = 0,$$

We now claim

$$D(x(t)) \geq D_m > 0, \quad t \in (0, \infty).$$

Suppose not, i.e., there exists $T_1 \in (0,\infty)$ such that

$$D(x(T_1)) < D_m.$$

Then, by the continuity of $D(x(\cdot))$, there exists $\delta_1 > 0$ such that

$$D(x(t)) < D_m, \quad t \in (T_1 - \delta, T_1 + \delta).$$

This yields

$$D(v_0) < KC_0 \int_{D(x(T_1))}^{D(x_0)} \psi(s) ds. \tag{3.18}$$

On the other hand, it follows from Lemma 3.3 that

$$KC_0 \left| \int_{D(x_0)}^{D(x(t))} \psi(s) ds \right| \leq D(v_0), \quad t \geq 0.$$

which is contradictory to (3.18). Therefore, we have

$$D(x(t)) \geq D_m, \quad t \geq 0. \tag{3.19}$$

Finally, we combine (3.16), (3.17), and (3.19) to obtain

$$||x_i(t)|| = r, \quad D(x(t)) D(v(t)) \mathcal{A}(v(t)) > 0, \quad t \in [0,\infty).$$

Therefore, we can use Lemma 3.3 for all $t > 0$.

• Part A (Uniform boundedness of $D(x(t))$): Since ψ is strictly decreasing in its argument, we can choose a unique D^∞ satisfying the implicit relation (3.15). We now claim:

$$D(x(t)) \leq D^\infty, \quad t \geq 0. \tag{3.20}$$

Suppose not, i.e., there exists a $T_* \in (0,\infty)$ such that

$$D(x(T_*)) > D^\infty.$$

Then, it follows from Lemma 3.3 that

$$D(v_0) \geq KC_0 \int_{D(x_0)}^{D(x(T_*))} \psi(s) ds > KC_0 \int_{D(x_0)}^{D^\infty} \psi(s) ds = D(v_0).$$

This is a contradiction. Thus, we have the desired claim (3.20).

• Part B (Exponential decay of $D(v(t))$): It follows from the second equation in (3.11) that

$$\frac{dD(v)}{dt} \le -KC_0\psi(D^\infty)D(v).$$

This yields the desired exponential decay. □

REMARK 3.3. In [1, 11, 21], similar Lyapunov functional approach has been applied to the Cucker–Smale-type models without constant speed constraint. Due to the constant speed constraint, we have to restrict admissible initial configurations to satisfy a geometric condition:

$$\mathcal{A}(v_0) > 0, \tag{3.21}$$

which is not present the classical Cucker–Smale-type models. This geometric constraint (3.21) is crucial in the flocking estimate. Once the condition (3.21) is violated, we can find an explicit example for the nonexistence of a global flocking, see Appendix A.

4. Conclusion

In this paper, we introduced a Cucker–Smale-type flocking model with a unit speed constraint in any arbitrary dimension. Since the original CS flocking model is not compatible with the constant speed constraint, we introduced a new non-symmetric velocity coupling leading to the constancy of speed. Previously, several Vicsek-type flocking models with the unit speed constraint have been proposed in [3,5,9,10,14,15,31] and a flocking estimate has been studied for the planar model in [10]. Our new velocity coupling is motivated by the study of the quantum synchronization [19, 20]. For the flocking estimate, we introduced the position and velocity diameters and derived a system of dissipative differential inequalities. Using the Lyapunov functional approach, we showed that the flocking occurs exponentially fast for some restricted class of initial configurations.

Appendix A. Nonexistence of global flocking. In this appendix, we provide an explicit three particle configuration leading to the nonexistence of a global flocking.

Consider a symmetric initial configuration $\{(x_i,v_i)\}_{i=1}^3$ on the planar domain $\mathbb{R}^2$ for system (2.2) for three particles

$$\begin{aligned} &x_1(0) = (0,M), \quad x_2(0) = (0,0), \quad x_3(0) = (0,0),\\ &v_1(0) = (0,1), \quad v_2(0) = (1-\varepsilon, -\sqrt{1-(1-\varepsilon)^2}),\\ &v_3(0) = (-(1-\varepsilon), -\sqrt{1-(1-\varepsilon)^2}), \quad 0<\varepsilon<1. \end{aligned} \tag{A.1}$$

In the sequel, we assume the Cucker–Smale communication weight:

$$\psi(s) = \frac{1}{(1+s^2)^{\frac{\beta}{2}}}, \quad 0<\beta<1.$$

Below, we use the notation v_i^j to denote the jth component of $v_i \in \mathbb{R}^2$, i.e., we set $v_i = (v_i^1, v_i^2)$.

LEMMA A.1. *For a fixed $\varepsilon \in (0,1)$, there exists a large positive constant $M = M(\varepsilon)$ such that system (2.2) admits a solution (x,v) with initial data (A.1) satisfying*

$$\sup_{t\ge 0} |v_2^1(t) - v_3^1(t)| \le |v_2^1(0) - v_3^1(0)|.$$

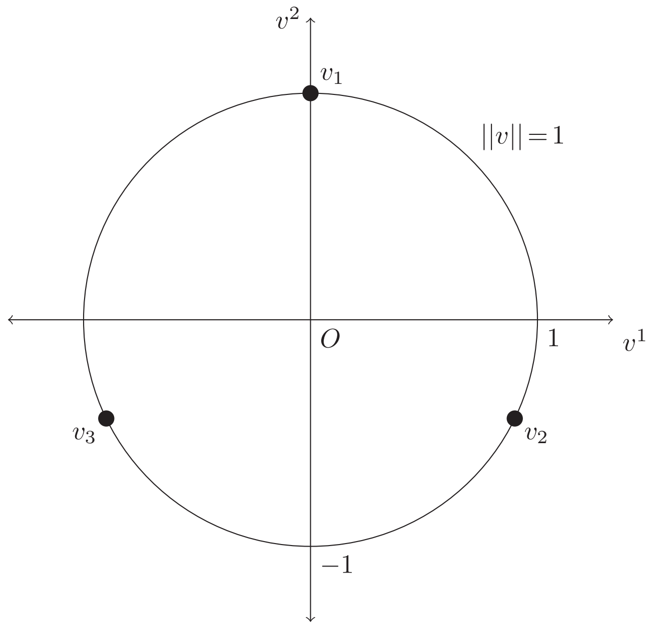

FIG. A.1. *Configuration for particles of velocity.*

Proof. Note that by symmetry, the solution (x,v) satisfies the following relations:

$$\begin{aligned}&v_1(t)=(0,1), \quad v_2^1(t)=-v_3^1(t), \quad v_2^2(t)=v_3^2(t),\\ &x_1(t)=(0,M+t), \quad x_2(t)=\int_0^t v_2(s)ds, \quad x_3(t)=\int_0^t v_3(s)ds, \quad t\geq 0.\end{aligned}$$

This implies

$$\begin{aligned}&\psi(|x_2-x_1|)=\psi(|x_3-x_1|), \quad \langle v_2,v_1\rangle=\langle v_3,v_1\rangle,\\ &\langle v_2,v_3\rangle=v_2^1v_3^1+v_2^2v_3^2=-v_2^1v_2^1+v_2^2v_2^2.\end{aligned} \tag{A.2}$$

We next claim:

$$\frac{d}{dt}|v_2^1-v_3^1|\leq 0, \quad \text{a.e., } t\in[0,\infty).$$

Proof of claim: It follows from system (2.2) that

$$\begin{aligned}\frac{dv_2}{dt}&=\frac{K}{3}(\psi(|x_2-x_1|)(v_1-\langle v_2,v_1\rangle v_2)+\psi(|x_2-x_3|)(v_3-\langle v_2,v_3\rangle v_2)),\\ \frac{dv_3}{dt}&=\frac{K}{3}(\psi(|x_3-x_1|)(v_1-\langle v_3,v_1\rangle v_3)+\psi(|x_3-x_2|)(v_2-\langle v_2,v_3\rangle v_3)).\end{aligned} \tag{A.3}$$

Then, we use (A.2) and (A.3) to obtain

$$
\begin{aligned}
&\frac{d}{dt}(v_2-v_3)\\
&\quad=\frac{K}{3}\Big(\psi(|x_2-x_1|)(v_1-\langle v_2,v_1\rangle v_2)+\psi(|x_2-x_3|)(v_3-\langle v_2,v_3\rangle v_2)\Big)\\
&\qquad-\frac{K}{3}\Big(\psi(|x_3-x_1|)(v_1-\langle v_3,v_1\rangle v_3)+\psi(|x_3-x_2|)(v_2-\langle v_2,v_3\rangle v_3)\Big)\\
&\quad=\frac{K}{3}\Big(\psi(|x_2-x_1|)(-\langle v_2,v_1\rangle v_2+\langle v_3,v_1\rangle v_3)\\
&\qquad+\psi(|x_2-x_3|)(v_3-\langle v_2,v_3\rangle v_2-v_2+\langle v_2,v_3\rangle v_3)\Big)\\
&\quad=\frac{K}{3}\Big(-\psi(|x_2-x_1|)\langle v_2,v_1\rangle(v_2-v_3)+(1+\langle v_2,v_3\rangle)\psi(|x_2-x_3|)(v_3-v_2)\Big). \qquad \text{(A.4)}
\end{aligned}
$$

We now take the first component of the above relation (A.4) to obtain

$$
\begin{aligned}
&\frac{d}{dt}(v_2^1-v_3^1)\\
&=\frac{K}{3}\Big(-\psi(|x_2-x_1|)\langle v_2,v_1\rangle(v_2^1-v_3^1)+(1+\langle v_2,v_3\rangle)\psi(|x_2-x_3|)(v_3^1-v_2^1)\Big). \qquad \text{(A.5)}
\end{aligned}
$$

If we multiply both sides of (A.5) by $(v_2^1-v_3^1)$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|v_2^1-v_3^1|^2\\
&=\frac{K}{3}\Big(-\psi(|x_2-x_1|)\langle v_2,v_1\rangle|v_2^1-v_3^1|^2-(1+\langle v_2,v_3\rangle)\psi(|x_2-x_3|)|v_2^1-v_3^1|^2\Big).
\end{aligned}
$$

We define a set $\mathcal{T}_c$:

$$
\mathcal{T}_c:=\Big\{T\leq(0,\infty]:\ \frac{d}{dt}|v_2^1(t)-v_3^1(t)|\leq 0,\quad \text{a.e., } t\in[0,T)\Big\},\qquad T^*:=\sup\mathcal{T}.
$$

Since

$$
\begin{aligned}
&\psi(|x_2(0)-x_1(0)|)=\psi(M)\to 0\quad \text{as } M\to\infty,\\
&\langle v_2(0),v_1(0)\rangle=-\sqrt{1-(1-\varepsilon)^2},\quad |v_2^1(0)-v_3^1(0)|=2(1-\varepsilon),
\end{aligned}
$$

we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|v_2^1-v_3^1|^2\Big|_{t=0+}\\
&=\frac{K}{3}\Big[4\psi(M)\sqrt{1-(1-\varepsilon)^2}(1-\varepsilon)^2-8\psi(|x_2(0)-x_3(0)|)(1-\varepsilon)^2\Big]\\
&\leq 0,\quad \text{by choosing } M \text{ sufficiently large.}
\end{aligned}
$$

Thus, there exists $\delta>0$ such that

$$
\frac{d}{dt}|v_2^1(t)-v_3^1(t)|\leq 0,\quad \text{a.e., } t\in[0,\delta),\quad \text{i.e.,}\quad \delta\in\mathcal{T}.
$$

Hence, the set $\mathcal{T}_c$ is not empty and T^* is well defined. Next, we show that

$$T^* = \infty.$$

Suppose not, i.e., $T^* < \infty$. Then for $t \in [0, T^*)$, we have

$$\frac{d}{dt}|v_2^1 - v_3^1| \le 0, \quad v_2^2 = v_3^2 < 0, \quad t \in [0, T^*). \tag{A.6}$$

The relation (A.6), $||v_i|| = 1$ and $v_1(t) = (0,1)$ imply

$$|x_2 - x_1| > M + t, \quad |x_2 - x_3| < 2t.$$

and

$$\begin{aligned}\langle v_2, v_3\rangle &= v_2^1 v_3^1 + v_2^2 v_3^2 = -v_2^1 v_2^1 + v_2^2 v_2^2 \\ &\ge -v_2^1(0) v_2^1(0) + v_2^2(0) v_2^2(0) \\ &= 1 - 2(1-\varepsilon)^2 = 1 - 2(1 - 2\varepsilon + \varepsilon^2).\end{aligned}$$

Thus, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|v_2^1 - v_3^1|^2 \\ &\le \frac{K}{3}\Big(\psi(|x_2 - x_1|)|v_2^1 - v_3^1|^2 - (1 + \langle v_2, v_3\rangle)\psi(|x_2 - x_3|)|v_2^1 - v_3^1|^2\Big) \\ &\le \frac{K}{3}\Big(\psi(M+t)|v_2^1 - v_3^1|^2 - (1 + \langle v_2, v_3\rangle)\psi(2t)|v_2^1 - v_3^1|^2\Big) \\ &\le \frac{K}{3}\Big(\psi(M+t)|v_2^1 - v_3^1|^2 - 2\varepsilon\psi(2t)|v_2^1 - v_3^1|^2\Big) \\ &= \frac{K}{3}\Big(\psi(M+t) - 2\varepsilon\psi(2t)\Big)|v_2^1 - v_3^1|^2.\end{aligned}$$

Note that follows the relation

$$\psi(M+t) - \varepsilon\psi(2t) = \left(\frac{1}{1 + (M+t)^2}\right)^{\frac{\beta}{2}} - \varepsilon\left(\frac{1}{1+4t^2}\right)^{\frac{\beta}{2}} < 0$$

is equivalent to

$$1 + (M+t)^2 - \frac{1+4t^2}{\varepsilon^{\frac{2}{\beta}}} = 1 - \frac{1}{\varepsilon^{\frac{2}{\beta}}} + M^2 + 2Mt + \left(1 - \frac{4}{\varepsilon^{\frac{2}{\beta}}}\right)t^2 > 0.$$

Thus, if we assume

$$0 < t < t_+ := \frac{-M - \sqrt{M^2 - (1 - \frac{1}{\varepsilon^{\frac{2}{\beta}}})(1 - \frac{1}{\varepsilon^{\frac{2}{\beta}}} + M^2)}}{(1 - \frac{4}{\varepsilon^{\frac{2}{\beta}}})},$$

then we have

$$\psi(M+t) - \varepsilon\psi(2t) = \left(\frac{1}{1 + (M+t)^2}\right)^{\frac{\beta}{2}} - \varepsilon\left(\frac{1}{1+4t^2}\right)^{\frac{\beta}{2}} < 0.$$

In conclusion, for $t\in[0,\min\{T^*,t_+\})$, the following inequality holds:

$$\frac{d}{dt}|v_2^1-v_3^1|\le -\frac{K\varepsilon}{3}\psi(2t)|v_2^1-v_3^1|.$$

This yields

$$|v_2^1(t)-v_3^1(t)|\le e^{-\frac{K\varepsilon}{3}\int_0^t\psi(2s)ds}|v_2^1(0)-v_3^1(0)|,\quad t<\min\{T^*,t_+\}.$$

Note that

$$\begin{aligned}\lim_{M\to\infty}\frac{t_+}{M}&=\lim_{M\to\infty}\frac{-M-\sqrt{M^2-\left(1-\frac{1}{\varepsilon^{\frac{2}{\beta}}}\right)\left(1-\frac{1}{\varepsilon^{\frac{2}{\beta}}}+M^2\right)}}{M\left(1-\frac{4}{\varepsilon^{\frac{2}{\beta}}}\right)}\\&=\frac{\frac{1}{\varepsilon^{\frac{1}{\beta}}}+1}{\frac{4}{\varepsilon^{\frac{2}{\beta}}}-1}=\frac{\varepsilon^{\frac{1}{\beta}}+\varepsilon^{\frac{2}{\beta}}}{4-\varepsilon^{\frac{2}{\beta}}}>0.\end{aligned}$$

If we take $M>0$ sufficiently large, then t_+ satisfies

$$t_+>\frac{M}{2}\frac{\varepsilon^{\frac{1}{\beta}}+\varepsilon^{\frac{2}{\beta}}}{4-\varepsilon^{\frac{2}{\beta}}}>T^*.$$

This leads to

$$\frac{d}{dt}|v_2^1-v_3^1|\le -\frac{K\varepsilon}{3}\psi(2t)|v_2^1-v_3^1|,\qquad t\in[0,T^*).$$

Since we assumed $|v_2^x(t)-v_3^x(t)|\ge 0$ for $t\in\mathbb{R}_+$, by continuity we have

$$\frac{d}{dt}|v_2^1(T^*)-v_3^1(T^*)|\le 0.$$

This gives a contradiction. Thus, we have $T^*=\infty$. ∎

PROPOSITION A.2. *For a fixed $\varepsilon\in(0,1)$, there exists a large positive constant M such that system (2.2) admits a solution (x,v) with initial data (A.1) satisfying*

$$v_1(t)=(0,1),\quad \textit{for all } t\ge 0,\qquad \lim_{t\to\infty}v_2(t)\ne(0,1),\quad \lim_{t\to\infty}v_3(t)\ne(0,1),$$

i.e., there is no flocking behavior.

Proof. To prove the nonexistence of flocking, it suffices to show that v_2 and v_3 cannot converge to v_1 as $t\to\infty$. To do so, the relative difference $|v_2^1-v_3^1|$ should increase from the initial distance for some time. However, it follows from Lemma A.1 that we have

$$\sup_{t\ge 0}|v_2^1(t)-v_3^1(t)|\le|v_2^1(0)-v_3^1(0)|.$$

Thus, we cannot have a flocking. ∎

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