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TIME PERIODIC SOLUTIONS TO THE 3D COMPRESSIBLE FLUID MODELS OF KORTEWEG TYPE*

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Abstract. This paper is concerned with time periodic solutions to the three-dimensional compressible fluid models of Korteweg type under some smallness and structure conditions on a time periodic force. The proof is based on a regularized approximation scheme and the topological degree theory for time periodic solutions in a bounded domain. Furthermore, via a limiting process, the existence results can be obtained in the whole space.

Key words. 3D compressible Navier–Stokes–Korteweg system, time periodic solutions, uniform estimates, topological degree theory.

AMS subject classifications. 35M10, 35Q35, 35B10.

1. Introduction

In this paper, we investigate the existence of time periodic solutions to the following three-dimensional compressible fluid model of Korteweg type, which describes the motion of a general barotropic compressible viscous capillary fluid:

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}(\rho u \otimes u) - \mu \Delta u - \tilde{\mu} \nabla \mathrm{div} u + \nabla P(\rho) - \kappa \rho \nabla \Delta \rho = \rho f(x,t). \end{cases} \tag{1.1}$$

Here, $\rho > 0$, $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ denote the density and the velocity, respectively; the pressure P for isentropic flows is given by $P(\rho) = \rho^\gamma$, with $\gamma > 1$ being the specific heat ratio. Furthermore, the viscosity coefficients μ and $\tilde{\mu}$ satisfy the usual physical conditions $\mu > 0, \tilde{\mu} = \mu + \lambda$, where $\lambda + \frac{2}{3}\mu \geq 0$ and the constant $\kappa > 0$ is the Weber number and stands for the capillary coefficient. In addition, $f(x,t)$ is a given external time periodic force with a period $T > 0$ satisfying $f(-x,t) = -f(x,t)$.

Indeed, this system is known as the compressible Navier–Stokes–Korteweg (NSK) system (see the pioneering work of Dunn and Serrin [9] and also [1, 5]). So far, there are not many efforts made on the isentropic or non-isentropic compressible Navier–Stokes–Korteweg system, due to the difficulties induced by the nonlinearity and the dispersion coming from the capillary tensor. More precisely, Hattori and Li [13, 14, 15] proved the local existence and the global existence of strong/smooth solutions in Sobolev space for the small initial data. Recently, In [24, 22], the authors established the optimal decay rates of global smooth solutions on the isentropic case, and the non-isentropic case was discussed by Zhang and Tan [26]. Bresch, Desjardins, and Lin [2] considered the global existence of weak solutions in a periodic box or in strip domain provided the viscosity $\mu(\rho) = C\rho$ with $C > 0$ and $\lambda(\rho) = 0$, and Haspot improved their results in [12] in dimension one or two. When the external force, the given mass source, and/or the energy source were taken into consideration, Danchin and Desjardins [8] studied

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the existence of suitably smooth solutions in critical Besov space, Li [19] discussed the global existence and optimal L^2-decay rates of smooth solutions, the local existence of strong solutions was proven in [18], and the stationary solutions were investigated by Chen and Zhao [7] focusing on the non-isentropic case. For the problem of time periodic solutions, it seems to be discussed for the isentropic [6] and non-isentropic [4] compressible fluid models of Korteweg type only when the space dimension $N \geq 5$. And, under the same conditions as [4, 6], time periodic solutions to the compressible NSK system with friction were obtained in $\mathbb{R}^3$ by [3]. To the best of our knowledge, however, there are no results available on time periodic solutions of the problem (1.1) when the dimension $N \leq 4$.

We note that, when $\kappa = 0$, the system (1.1) becomes the compressible Navier–Stokes (NS) equations. The question of the existence of time periodic solutions to NS equations is largely settled. For the problem in a bounded domain, we can see [23, 20, 10, 11, 16] and the references therein. On the other hand, for the problem in an unbounded domain, we first mention the work by Ma, Ukai, and Yang [21]. The authors exploited linear decay analysis and the contraction mapping theorem to get time periodic solutions when the space dimension $N \geq 5$. Recently, Jin and Yang [17] applied the topological degree theory to improve their results in three-dimensional space.

The aim of this paper, inspired by the work [17], is to improve the existence result of [6], namely, we shall solve the existence of time periodic solutions to the problem (1.1) around a constant state $(\bar{\rho}, 0)$ in $\mathbb{R}^3$. In fact, compared to the work [17], we need to obtain the uniform estimates for higher-order quantities of the density and velocity, which guarantees that (3.62) holds.

Before stating our main result, we explain the notations and conventions used throughout this paper. Let us start by denoting $Q_T = \Omega \times (0,T)$ and defining t-anisotropic Sobolev spaces

$$W_p^{m,k}(Q_T) = \{u;\ D^\alpha u,\ D_t^\beta u \in L^p(Q_T),\ \text{for any } |\alpha| \leq m,\ |\beta| \leq k\}$$

with the norm

$$\|u\|_{W_p^{m,k}(Q_T)} = \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^p(Q_T)} + \sum_{|\beta| \leq k} \|D_t^\beta u\|_{L^p(Q_T)}.$$

For $0 < \alpha < 1$, we denote by $C^{\alpha,\alpha/2}(\bar{Q}_T)$ the set of all functions on Q_T such that the semi-norm

$$[u]_{\alpha,\alpha/2} = \sup_{(x,t) \neq (y,s)} \frac{|u(x,t) - u(y,s)|}{(|x-y|^2 + |t-s|)^{\alpha/2}} < +\infty,$$

endowed with the norm

$$|u|_{\alpha,\alpha/2} = [u]_{\alpha,\alpha/2} + \|u\|_{L^\infty}.$$

Moreover, $\deg(\cdot,\cdot,\cdot)$ stands for the Leray-Schauder degree.

Precisely, define the solution space in a bounded domain Ω and the whole space $\mathbb{R}^3$ by

$$X^\Omega = \left\{ (\rho,u)(x,t) \;\middle|\; \begin{array}{c} (\rho,u) \in L^\infty(0,T;L^6(\Omega)) \text{ satisfies } (a),(b),(c); \\ \rho_t \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)); \\ u_t \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)); \\ \nabla\rho \in L^\infty(0,T;H^2(\Omega)) \cap L^2(0,T;H^3(\Omega)); \\ \nabla u \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)). \end{array} \right\},$$

$$X=\left\{(\rho,u)(x,t)\left|\begin{array}{c}(\rho,u)\in L^{\infty}(0,T;L^{6}(\mathbb{R}^{3}))\text{ satisfies }(c);\\ \rho_{t}\in L^{\infty}(0,T;H^{1}(\mathbb{R}^{3}))\cap L^{2}(0,T;H^{2}(\mathbb{R}^{3}));\\ u_{t}\in L^{\infty}(0,T;L^{2}(\mathbb{R}^{3}))\cap L^{2}(0,T;H^{1}(\mathbb{R}^{3}));\\ \nabla\rho\in L^{\infty}(0,T;H^{2}(\mathbb{R}^{3}))\cap L^{2}(0,T;H^{3}(\mathbb{R}^{3}));\\ \nabla u\in L^{\infty}(0,T;H^{1}(\mathbb{R}^{3}))\cap L^{2}(0,T;H^{2}(\mathbb{R}^{3}))\end{array}\right.\right\},$$

respectively. Here, the conditions (a), (b), (c) can be presented by the following:

(a) (ρ,u) is time periodic function with periodic boundary condition.

(b) $\int_{\Omega}\rho dx=0$.

(c) $\rho(x,t)=\rho(-x,t),\ u(x,t)=-u(-x,t)$.

The norm $|||\cdot|||$ in X^{Ω} and X is given by

$$|||(\rho,u)|||^{2}=\sup_{0<t<T}\left\{\|(\rho,u)\|_{L^{6}}^{2}+\|(\rho_{t},\nabla\rho_{t},u_{t})\|_{L^{2}}^{2}+\|\nabla\rho\|_{H^{2}}^{2}+\|\nabla u\|_{H^{1}}^{2}\right\}$$
$$+\int_{0}^{T}\left(\|\rho_{t}\|_{H^{2}}^{2}+\|u_{t}\|_{H^{1}}^{2}+\|\nabla\rho\|_{H^{3}}^{2}+\|\nabla u\|_{H^{2}}^{2}\right)dt.$$

For some positive constant $\hbar$, set

$$X_{\hbar}^{\Omega}=\left\{(\rho,u)\in X^{\Omega};\ |||(\rho,u)|||<\hbar\right\}.$$

Our existence result of time periodic solutions for (1.1) is the following.

THEOREM 1.1. *Let the time periodic force $f(x,t)\in L^{2}(0,T;L^{\frac{6}{5}}(\mathbb{R}^{3}))\cap W_{2}^{1,1}((0,T)\times\mathbb{R}^{3})$ with $f(-x,t)=-f(x,t)$. Moreover, if*

$$\int_{0}^{T}\left(\|f\|_{L^{\frac{6}{5}}}^{2}+\|f\|_{H^{1}}^{4}\right)dt+\|f\|_{W^{1,1}}^{2}\leq\hbar^{*}$$

for some small constant $\hbar^{*}>0$, then problem (1.1) *has a time periodic solution in $X_{\hbar_0}$ with the same period as the external force f.*

To prove Theorem 1.1, we first consider the regularized problem of (1.1) in some functional space and construct an operator which can solve the regularized problem involving the parameter τ. Then, due to doing some energy estimates, we shall show that the operator is completely continuous. A matter worthy of note is that the process of parabolic regularized (that is, adding the term $\varepsilon\Delta\varrho$ to continuity equation) aims to prove the compactness of the operator. Second, we derive energy estimates and apply the topological degree theory to obtain the approximate solution for the regularized problem (2.1) in a bounded domain. Third, making use of t-anisotropic Sobolev imbedding theorem and some uniform bounds on the approximate solutions obtained, we shall establish a limit function, which is the desired time periodic solution.

The rest of the paper is arranged as follows. We are going to reformulate the problem in a bounded domain and give some preliminaries for later use in Section 2. In Section 3, we will do some uniform estimates and obtain the existence result in the bounded domain. The proof of Theorem 1.1 is given in the last section.

2. Preliminaries

In order to simplify the forthcoming calculation, set

$$\varrho=\rho-\bar{\rho},\quad v=u.$$

Then we can rewrite another equivalent form of (1.1), which is

$$\begin{cases}\varrho_t+\bar{\rho}\mathrm{div}v=-\mathrm{div}(\varrho v),\\(\bar{\rho}+\varrho)v_t+(\bar{\rho}+\varrho)(v\cdot\nabla)v-\mu\Delta v-\tilde{\mu}\nabla\mathrm{div}v+P'(\bar{\rho}+\varrho)\nabla\varrho\\\qquad-\kappa(\bar{\rho}+\varrho)\nabla\Delta\varrho=(\bar{\rho}+\varrho)f(x,t).\end{cases}\tag{2.1}$$

For the sake of simplicity, we restrict our attention to a particular class of spatial domains. Specifically, we assume $\Omega=(-L,L)^3\subset\mathbb{R}^3$ and the functions defined on Ω satisfy periodic boundary condition. Considering parabolic regularization, we regularize problem (2.1) in Ω, that is,

$$\begin{cases}\varrho_t-\varepsilon\Delta\varrho+\bar{\rho}\mathrm{div}v=-\mathrm{div}(\varrho v),\\(\bar{\rho}+\varrho)v_t+(\bar{\rho}+\varrho)(v\cdot\nabla)v-\mu\Delta v-\tilde{\mu}\nabla\mathrm{div}v+P'(\bar{\rho}+\varrho)\nabla\varrho\\\qquad-\kappa(\bar{\rho}+\varrho)\nabla\Delta\varrho=(\bar{\rho}+\varrho)f_\Omega(x,t),\end{cases}\tag{2.2}$$

where $f_\Omega(x,t)$ is a time periodic function and an odd function and is periodic with the space periodic 2L defined in the periodic domain, satisfying

$$f_\Omega\to f \text{ in } L^2(0,T;L^{\frac{6}{5}}(\mathbb{R}^3))\cap W_2^{1,1}((0,T)\times\mathbb{R}^3),\quad \text{as } L\to+\infty,$$
$$\int_0^T\|f_\Omega\|^2_{L^{\frac{6}{5}}(\Omega)}dt+\|f_\Omega\|^2_{W_2^{1,1}((0,T)\times\Omega)}\le 2\int_0^T\|f\|^2_{L^{\frac{6}{5}}(\mathbb{R}^3)}dt+2\|f\|^2_{W_2^{1,1}((0,T)\times\mathbb{R}^3)}.$$

Proposition 2.1. *Let f_Ω be as in the above and satisfy*

$$\int_0^T\left(\|f_\Omega\|^2_{L^{\frac{6}{5}}}+\|f_\Omega\|^4_{H^1}\right)dt+\|f_\Omega\|^2_{W_2^{1,1}}\le\hbar^*,$$

for some small constant $\hbar^*>0$. Then the problem (2.2) admits a solution (ϱ,v) in $X^\Omega_{\hbar_0}$. Here, $\hbar_0$ is a small constant independent of L and ε.

The proof of Proposition 2.1 shall be given in next section. Next, we recall some known elementary inequalities which will be used frequently later.

Lemma 2.2. *Assume that $\Omega_1\subset\mathbb{R}^N$ is a bounded domain and $\partial\Omega_1$ is locally Lipschitz continuous. If $u|_{\partial\Omega_1}=0$ (or $\int_{\Omega_1}udx=0$), then, for any $1\le p<N$, $1\le q\le p^*=\frac{Np}{N-p}$,*

$$\left(\int_{\Omega_1}|u|^qdx\right)^{1/q}\le C(N,p,q)|\Omega_1|^{1/q-1/p^*}\left(\int_{\Omega_1}|\nabla u|^qdx\right)^{1/p}.$$

In particular, if $q=p^*=\frac{Np}{N-p}$, then

$$\left(\int_{\Omega_1}|u|^{p^*}dx\right)^{1/p^*}\le C(N,p,q)\left(\int_{\Omega_1}|\nabla u|^pdx\right)^{1/p}.$$

LEMMA 2.3. *Assume that $\Omega_1\subset\mathbb{R}^3$ is a bounded domain and $\partial\Omega_1$ is locally Lipschitz continuous. If $u|_{\partial\Omega_1}=0$ (or $\int_{\Omega_1}udx=0$), then*

$$\begin{aligned}\|u\|_{L^3}&\le C\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}\le C\|u\|_{H^1},\\ \|u\|_{L^4}&\le C\|u\|_{L^2}^{1/4}\|\nabla u\|_{L^2}^{3/4},\\ \|u\|_{L^\infty}&\le C\|\nabla u\|_{H^1},\end{aligned}$$

where C is independent of Ω_1. Moreover, the above inequalities also hold in $\mathbb{R}^3$ if $u(x)\to 0$ as $|x|\to\infty$.

REMARK 2.4. It is worth mentioning, because we shall pass the limit of the approximate solutions in last section, that the constants C of the previous two lemmas independent of the domain Ω_1 play an important role in uniform estimates.

Furthermore, let us state a significant property of the Leray-Schauder degree from [25] as follows.

LEMMA 2.5. *Let $\Omega_1\subset\mathbb{R}^n$ be open and bounded, $g:\bar{\Omega}_1\to\mathbb{R}^n$ be a compact and continuous mapping, and I be a unit mapping. Assume that $p\notin(I-g)(\partial\Omega_1)$. If $p\notin(I-g)(\bar{\Omega}_1)$, then $\deg(I-g,\Omega_1,p)=0$. Thus, if $\deg(I-g,\Omega_1,p)\neq 0$, $(I-g)(x)=p$ admits at least one solution in Ω_1.*

3. Existence in bounded domain Ω

3.1. Introduction of an operator $\mathcal{H}$. To prove Proposition 2.1, we first work with the linear parabolic equations by introducing an operator

$$\begin{aligned}\mathcal{H}\colon X_\hbar^\Omega\times[0,1]&\to X^\Omega,\\ ((\rho,u),\tau)&\to(\varrho,v),\end{aligned}$$

with $\hbar$ being suitably small. Here, (ϱ,v) is the solution of the problem

$$\begin{cases}\varrho_t-\varepsilon\Delta\varrho+\bar{\rho}\mathrm{div}v=Q_1(\tau,\rho,u),\\ (\bar{\rho}+\tau\rho)v_t-\mu\Delta v-\tilde{\mu}\nabla\mathrm{div}v+\dfrac{P'(\bar{\rho})}{\bar{\rho}}(\bar{\rho}+\tau\rho)\nabla\varrho\\ \qquad\qquad-\kappa(\bar{\rho}+\tau\rho)\nabla\Delta\varrho=Q_2(\tau,\rho,u)+\tau(\bar{\rho}+\tau\rho)f_\Omega(x,t),\\ \int_\Omega\varrho dx=0,\end{cases}\tag{3.1}$$

where

$$\begin{aligned}&Q_1(\tau,\rho,u)=-\tau\mathrm{div}(\rho u),\\ &Q_2(\tau,\rho,u)=(\bar{\rho}+\tau\rho)\left(\frac{P'(\bar{\rho})}{\bar{\rho}}-\frac{P'(\bar{\rho}+\tau\rho)}{\bar{\rho}+\tau\rho}\right)\nabla\rho-\tau(\bar{\rho}+\tau\rho)(u\cdot\nabla)u.\end{aligned}$$

Set

$$U=(\varrho,v),\quad W=(\rho,u),\quad Q(W)=(Q_1,\frac{Q_2}{\bar{\rho}+\tau\rho}),\quad F=(0,\tau f_\Omega)$$

and define

$$\mathbb{A}=\begin{pmatrix}\varepsilon\Delta & -\bar{\rho}\mathrm{div}\\ -\dfrac{P'(\bar{\rho})}{\bar{\rho}}\nabla+\kappa\nabla\Delta & \dfrac{\mu}{\bar{\rho}+\tau\rho}\Delta+\dfrac{\tilde{\mu}}{\bar{\rho}+\tau\rho}\nabla\mathrm{div}\end{pmatrix}.$$

System (3.1) then takes the form

$$U_t = \mathbb{A}U + Q(W) + F.$$

REMARK 3.1. Because $\frac{d}{dt}\int_\Omega \varrho dx = 0$, it is obvious that $(\varrho + c, v)$ is a solution for any constant c if (ϱ, v) is a solution of the problem (2.2). Thus, the condition $\int_\Omega \varrho dx = 0$ seems to be necessary in order to grant the uniqueness of the solution.

After doing that, we proceed to show that the operator $\mathcal{H}$ is well defined.

LEMMA 3.2. *Let $\hbar$ suitably small. Then, for any $(\rho, u) \in X_\hbar^\Omega$, $\tau \in [0,1]$, the problem (3.1) admits a unique solution (ϱ, v) in X^Ω.*

Proof. First, we consider the corresponding homogeneous linear system $U_t = \mathbb{A}U$ of (3.1) in Ω

$$\begin{cases} \varrho_t - \varepsilon\Delta\varrho + \bar{\rho}\mathrm{div}v = 0, \\ v_t - \dfrac{\mu}{(\bar{\rho}+\tau\rho)}\Delta v - \dfrac{\tilde{\mu}}{(\bar{\rho}+\tau\rho)}\nabla\mathrm{div}v + \dfrac{P'(\bar{\rho})}{\bar{\rho}}\nabla\varrho - \kappa\nabla\Delta\varrho = 0, \end{cases} \tag{3.2}$$

with the initial value condition

$$\varrho(x,0) = \varrho_0(x), \quad v(x,0) = v_0(x).$$

Here, $\varrho_0(x)$ is an even function with $\int_\Omega \varrho_0(x)dx = 0$ and $v_0(x)$ is an odd function.

Multiplying $(3.2)_2$ by v and integrating it over Ω, we have from integrating by parts that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega v^2 dx + \int_\Omega \frac{\mu}{\bar{\rho}+\tau\rho}|\nabla v|^2 dx + \int_\Omega \frac{\tilde{\mu}}{\bar{\rho}+\tau\rho}|\mathrm{div}v|^2 dx \\ &\quad + \frac{P'(\bar{\rho})}{\bar{\rho}}\int_\Omega \nabla\varrho v dx + \kappa\int_\Omega \Delta\varrho \mathrm{div}v dx \\ &= \int_\Omega \frac{\tau\mu}{(\bar{\rho}+\tau\rho)^2}\nabla v\nabla\rho v dx + \int_\Omega \frac{\tau\tilde{\mu}}{(\bar{\rho}+\tau\rho)^2}\mathrm{div}v\nabla\rho v dx \\ &\le \frac{\tau(\mu+\tilde{\mu})}{(\bar{\rho}-\tau\|\rho\|_{L^\infty})^2}\|\nabla\rho\|_{L^3}\|\nabla v\|_{L^2}\|v\|_{L^6} \\ &\le C\frac{\tau(\mu+\tilde{\mu})}{(\bar{\rho}-\tau\|\rho\|_{L^\infty})^2}\|\nabla\rho\|_{H^1}\|\nabla v\|_{L^2}^2, \end{aligned} \tag{3.3}$$

where C is a constant independent of Ω from Lemma 2.3. In terms of the fact that $\|\rho\|_{L^\infty} \le C\|\nabla\rho\|_{H^1} \le C\hbar$, (3.3) implies

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v^2 dx + \int_\Omega \left(\frac{\mu}{3\bar{\rho}}|\nabla v|^2 + \frac{\tilde{\mu}}{3\bar{\rho}}|\mathrm{div}v|^2\right)dx + \frac{P'(\bar{\rho})}{\bar{\rho}}\int_\Omega \nabla\varrho v dx + \kappa\int_\Omega \Delta\varrho\mathrm{div}v dx \le 0. \tag{3.4}$$

Multiplying $(3.2)_1$ by ϱ and $\Delta\varrho$, respectively, and integrating them over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \varrho^2 dx + \varepsilon\int_\Omega |\nabla\varrho|^2 dx - \bar{\rho}\int_\Omega \nabla\varrho v dx = 0, \tag{3.5}$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\varrho|^2 dx + \varepsilon\int_\Omega |\Delta\varrho|^2 dx - \bar{\rho}\int_\Omega \Delta\varrho\mathrm{div}v dx = 0. \tag{3.6}$$

Putting (3.5)–(3.6) into (3.4), we arrive at

$$\frac{d}{dt}\int_\Omega\left(\frac{P'(\bar\rho)}{\bar\rho^2}\varrho^2+\frac{\kappa}{\bar\rho}|\nabla\varrho|^2+v^2\right)dx+\frac{2}{3\bar\rho}\int_\Omega\left(\mu|\nabla v|^2+\tilde\mu|\mathrm{div}v|^2\right)dx \\ +\frac{2\varepsilon}{\bar\rho}\int_\Omega\left(\frac{P'(\bar\rho)}{\bar\rho}|\nabla\varrho|^2+\kappa|\Delta\varrho|^2\right)dx\le 0. \tag{3.7}$$

Moreover, multiplying $(3.2)_2$ by $\mu\Delta v+\tilde\mu\nabla\mathrm{div}v$, we integrate to get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega\left(\mu|\nabla v|^2+\tilde\mu|\mathrm{div}v|^2\right)dx+\int_\Omega\frac{1}{\bar\rho+\tau\rho}(\mu\Delta v+\tilde\mu\nabla\mathrm{div}v)^2dx \\ -(\mu+\tilde\mu)\frac{P'(\bar\rho)}{\bar\rho}\int_\Omega\nabla\varrho\nabla\mathrm{div}v dx+\kappa(\mu+\tilde\mu)\int_\Omega\nabla\Delta\varrho\nabla\mathrm{div}v dx=0. \tag{3.8}$$

Applying ∇ to $(3.2)_1$ and then taking the L^2 inner product with $\nabla\varrho$ and $\nabla\Delta\varrho$,respectively, on the resultant equation we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla\varrho|^2dx+\varepsilon\int_\Omega|\Delta\varrho|^2dx+\bar\rho\int_\Omega\nabla\varrho\nabla\mathrm{div}v dx=0, \tag{3.9}$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega|\Delta\varrho|^2dx+\varepsilon\int_\Omega|\nabla\Delta\varrho|^2dx-\bar\rho\int_\Omega\nabla\Delta\varrho\nabla\mathrm{div}v dx=0. \tag{3.10}$$

Substituting (3.9)–(3.10) into (3.8) gives

$$\frac{1}{2}\frac{d}{dt}\int_\Omega\left(\mu|\nabla v|^2+\tilde\mu|\mathrm{div}v|^2+(\mu+\tilde\mu)\frac{P'(\bar\rho)}{\bar\rho^2}|\nabla\varrho|^2+(\mu+\tilde\mu)\frac{\kappa}{\bar\rho}|\Delta\varrho|^2\right)dx \\ +\varepsilon(\mu+\tilde\mu)\int_\Omega\left(\frac{P'(\bar\rho)}{\bar\rho^2}|\Delta\varrho|^2+\frac{\kappa}{\bar\rho}|\nabla\Delta\varrho|^2\right)dx+\int_\Omega\frac{1}{\bar\rho+\tau\rho}(\mu\Delta v+\tilde\mu\nabla\mathrm{div}v)^2dx=0. \tag{3.11}$$

From (3.7) and (3.11), by the Poincaré inequality, we have

$$\frac{d}{dt}(\|\varrho\|_{H^2}+\|v\|_{H^1})+C\varepsilon(\|\varrho\|_{H^2}+\|v\|_{H^1})\le 0,$$

Together with Grönwall's inequality, we obtain

$$\|\varrho\|_{H^2}+\|v\|_{H^1}\le(\|\varrho_0\|_{H^2}+\|v_0\|_{H^1})e^{-C\varepsilon t}. \tag{3.12}$$

By Duhamel's principle, the solution to the system (3.1) can be written in mild form as

$$U(t)=\int_{-\infty}^t e^{(t-s)\mathbb{A}}(Q(W(s))+F(s))ds,$$

where $e^{(t-s)\mathbb{A}}$ is the solution operator to the system (3.2). Then, (3.12) implies

$$\begin{aligned}\|\varrho\|_{H^2}+\|v\|_{H^1}&\le\int_{-\infty}^t\left(\left\|e^{(t-s)\mathbb{A}}Q_1(W(s))\right\|_{H^2}+\left\|e^{(t-s)\mathbb{A}}\left(\frac{Q_2}{\bar\rho+\tau\rho}+\tau f_\Omega\right)(s)\right\|_{H^1}\right)ds\\ &\le\int_{-\infty}^t e^{-C\varepsilon(t-s)}\left\{\|Q_1(W(s))\|_{H^2}+\left\|\left(\frac{Q_2}{\bar\rho+\tau\rho}+\tau f_\Omega\right)(s)\right\|_{H^1}\right\}ds\\ &\le\frac{1}{C\varepsilon}\sup_{t\in\mathbb{R}}\left\{\|Q_1(W(t))\|_{H^2}+\left\|\left(\frac{Q_2}{\bar\rho+\tau\rho}+\tau f_\Omega\right)(t)\right\|_{H^1}\right\}.\end{aligned}$$

Since the period of W and F is T, we have

$$\begin{aligned} U(t+T) &= \int_{-\infty}^{t+T} e^{(t+T-s)\mathbb{A}}(Q(W(s))+F(s))\,ds \\ &= \int_{-\infty}^{t+T} e^{(t-(s-T))\mathbb{A}}(Q(W(s-T))+F(s-T))\,ds \\ &= \int_{-\infty}^{t} e^{(t-s)\mathbb{A}}(Q(W(s))+F(s))\,ds = U(t). \end{aligned}$$

Hence, $(\varrho,v)\in\big(L^\infty(0,T;H^2(\Omega)),L^\infty(0,T;H^1(\Omega))\big)$ is a time periodic solution of (3.1).

Next, we are going to prove the uniqueness of the time periodic solution. Let us now assume that U_1 and U_2 are two solutions of (3.1). Then we have

$$(U_1-U_2)_t=\mathbb{A}(U_1-U_2).$$

We take change of variable by $U_1-U_2\to\tilde U=(\tilde\varrho,\tilde v)$, then $\tilde U=\mathbb{A}\tilde U$, and $\tilde U$ has the periodic property. Then we can do exactly as what we did with (3.2) in the first half of this proof. Thus, we easily check from (3.8)–(3.11) that

$$\begin{aligned} &\varepsilon(\mu+\tilde\mu)\int_0^T\int_\Omega\left(\frac{P'(\bar\rho)}{\bar\rho^2}|\Delta\tilde\varrho|^2+\frac{\kappa}{\bar\rho}|\nabla\Delta\tilde\varrho|^2\right)dxdt \\ &\quad+\int_0^T\int_\Omega\frac{1}{\bar\rho+\tau\rho}(\mu\Delta\tilde v+\tilde\mu\nabla\mathrm{div}\tilde v)^2dxdt=0. \end{aligned}$$

By by Poincaré inequality, it is easy to get $\tilde\varrho=\tilde v=0$. So we have $U_1=U_2$.

Furthermore, by the classical theory of parabolic equation, if $Q_1,Q_2+\tau(\bar\rho+\tau\rho)f_\Omega\in W_2^{1,1}((0,T)\times\Omega)$, then the problem (3.1) admits a unique solution $(\varrho,v)\in X^\Omega$. Similarly, when $(\rho,u)\in X_\hbar^\Omega$, (ϱ,v) is the unique solution of (3.1), which is in a sub-space of X^Ω.

In addition, if $(\varrho(x,t),v(x,t))$ is the periodic solution of (3.1), then $(\varrho(-x,t),-v(-x,t))$ is also the solution. By uniqueness, it follows that $(\varrho(x,t),v(x,t))=(\varrho(-x,t),-v(-x,t))$. This finishes the proof of Lemma 3.2. □

Next, we shall see that the operator $\mathcal{H}$ is completely continuous in the following lemma. For the sake of simplicity, set

$$m(\rho)=\frac{1}{\bar\rho+\tau\rho}-\frac{1}{\bar\rho},\qquad h(\rho)=\frac{P'(\bar\rho)}{\bar\rho}-\frac{P'(\bar\rho+\tau\rho)}{\bar\rho+\tau\rho},$$

the problem (3.1) takes the form

$$\begin{cases} \varrho_t-\varepsilon\Delta\varrho+\bar\rho\mathrm{div}v=-\tau\mathrm{div}(\rho u), \\ v_t-\dfrac{\mu}{\bar\rho}\Delta v-\dfrac{\tilde\mu}{\bar\rho}\nabla\mathrm{div}v+\dfrac{P'(\bar\rho)}{\bar\rho}\nabla\varrho-\kappa\nabla\Delta\varrho \\ \qquad\qquad=m(\rho)(\mu\Delta v+\tilde\mu\nabla\mathrm{div}v)+h(\rho)\nabla\rho-\tau(u\cdot\nabla)u+\tau f_\Omega(x,t), \\ \int_\Omega\varrho dx=0. \end{cases}\tag{3.13}$$

LEMMA 3.3. *If $\hbar$ is appropriately small, then the operator $\mathcal{H}$ is compact and continuous.*

Proof. To begin with, we prove the compactness of the operator $\mathcal{H}$.

Let (ϱ,v) be the solution of (3.13). For $0\le k\le 2$, applying ∇^k to $(3.13)_2$, and multiplying resultant equation by $\nabla^k v$, we integrate over Ω by parts to have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla^k v|^2dx+\frac{\mu}{\bar\rho}\int_\Omega|\nabla^{k+1}v|^2dx+\frac{\tilde\mu}{\bar\rho}\int_\Omega|\nabla^k\mathrm{div}v|^2dx\\
&-\frac{P'(\bar\rho)}{\bar\rho}\int_\Omega\nabla^k\varrho\nabla^k\mathrm{div}vdx+\kappa\int_\Omega\nabla^k\Delta\varrho\nabla^k\mathrm{div}vdx\\
=&\int_\Omega\nabla^k(m(\rho)(\mu\Delta v+\tilde\mu\nabla\mathrm{div}v))\nabla^k vdx+\int_\Omega\nabla^k(h(\rho)\nabla\rho)\nabla^k vdx\\
&-\tau\int_\Omega\nabla^k(u\nabla u)\nabla^k vdx+\tau\int_\Omega\nabla^k f_\Omega\nabla^k vdx. \qquad (3.14)
\end{aligned}$$

Applying ∇^k to $(3.13)_1$ and then taking the L^2 inner product with $\nabla^k\varrho$ and $\nabla^k\Delta\varrho$, respectively, on the resultant equation we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla^k\varrho|^2dx+\varepsilon\int_\Omega|\nabla^{k+1}\varrho|^2dx+\bar\rho\int_\Omega\nabla^k\varrho\nabla^k\mathrm{div}vdx\\
=&-\tau\int_\Omega\nabla^k\mathrm{div}(\rho u)\nabla^k\varrho dx, \qquad (3.15)
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla^{k+1}\varrho|^2dx+\varepsilon\int_\Omega|\nabla^k\Delta\varrho|^2dx-\bar\rho\int_\Omega\nabla^k\Delta\varrho\nabla^k\mathrm{div}vdx\\
=&\tau\int_\Omega\nabla^k\mathrm{div}(\rho u)\nabla^k\Delta\varrho dx. \qquad (3.16)
\end{aligned}$$

Combining (3.14)–(3.16) implies that, for $1\le k\le 2$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega\left(\frac{P'(\bar\rho)}{\bar\rho^2}|\nabla^k\varrho|^2+\frac{\kappa}{\bar\rho}|\nabla^{k+1}\varrho|^2+|\nabla^k v|^2\right)dx+\frac{\mu}{\bar\rho}\int_\Omega|\nabla^{k+1}v|^2dx\\
&+\frac{\tilde\mu}{\bar\rho}\int_\Omega|\nabla^k\mathrm{div}v|^2dx+\varepsilon\frac{P'(\bar\rho)}{\bar\rho^2}\int_\Omega|\nabla^{k+1}\varrho|^2dx+\varepsilon\frac{\kappa}{\bar\rho}\int_\Omega|\nabla^k\Delta\varrho|^2dx\\
\le&\varepsilon\frac{P'(\bar\rho)}{2\bar\rho^2}\|\nabla^{k+1}\varrho\|_{L^2}^2+\varepsilon\frac{\kappa}{2\bar\rho}\|\nabla^k\Delta\varrho\|_{L^2}^2+C\|\nabla^k(\rho u)\|_{L^2}^2+C\|\nabla^k\mathrm{div}(\rho u)\|_{L^2}^2\\
&+C\|\nabla^{k-1}(m(\rho)(\mu\Delta v+\tilde\mu\nabla\mathrm{div}v))\|_{L^2}^2+C\|\nabla^{k-1}(h(\rho)\nabla\rho)\|_{L^2}^2\\
&+C\|\nabla^{k-1}(u\nabla u)\|_{L^2}^2+C\|\nabla^{k-1}f_\Omega\|_{L^2}^2+\frac{\mu}{2\bar\rho}\|\nabla^{k+1}v\|_{L^2}^2\\
\le&\varepsilon\frac{P'(\bar\rho)}{2\bar\rho^2}\|\nabla^{k+1}\varrho\|_{L^2}^2+\varepsilon\frac{\kappa}{2\bar\rho}\|\nabla^k\Delta\varrho\|_{L^2}^2+\frac{\mu}{2\bar\rho}\|\nabla^{k+1}v\|_{L^2}^2+C\|\nabla\rho\|_{H^2}^2\|\nabla u\|_{H^2}^2\\
&+C\|\nabla\rho\|_{H^1}^2\|\nabla^{k+1}v\|_{L^2}^2+C\|\nabla\rho\|_{H^1}^4+C\|\nabla u\|_{H^1}^4+C\|\nabla^{k-1}f_\Omega\|_{L^2}^2, \qquad (3.17)
\end{aligned}$$

where we have used the fact $m(\rho)\sim\tau\rho$ and $h(\rho)\sim\tau\rho$ as well as Sobolev and Cauchy inequalities. For $k=0$, it holds that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega\left(\frac{P'(\bar\rho)}{\bar\rho^2}|\varrho|^2+\frac{\kappa}{\bar\rho}|\nabla\varrho|^2+|v|^2\right)dx+\frac{\mu}{\bar\rho}\int_\Omega|\nabla v|^2dx\\
&+\frac{\tilde\mu}{\bar\rho}\int_\Omega|\mathrm{div}v|^2dx+\varepsilon\frac{P'(\bar\rho)}{\bar\rho^2}\int_\Omega|\nabla\varrho|^2dx+\varepsilon\frac{\kappa}{\bar\rho}\int_\Omega|\Delta\varrho|^2dx
\end{aligned}$$

$$
\begin{aligned}
&\le \varepsilon\frac{P'(\bar\rho)}{2\bar\rho^2}\|\nabla\varrho\|_{L^2}^2+\varepsilon\frac{\kappa}{2\bar\rho}\|\Delta\varrho\|_{L^2}^2+\frac{\mu}{2\bar\rho}\|\nabla v\|_{L^2}^2+C\|\mathrm{div}(\rho u)\|_{L^{\frac65}}^2+C\|\mathrm{div}(\rho u)\|_{L^2}^2\\
&\quad+C\|\nabla(m(\rho)v)\|_{L^2}^2+C\|h(\rho)\nabla\rho\|_{L^{\frac65}}^2+C\|u\nabla u\|_{L^{\frac65}}^2+C\|f_\Omega\|_{L^{\frac65}}^2\\
&\le \varepsilon\frac{P'(\bar\rho)}{2\bar\rho^2}\|\nabla\varrho\|_{L^2}^2+\varepsilon\frac{\kappa}{2\bar\rho}\|\Delta\varrho\|_{L^2}^2+\frac{\mu}{2\bar\rho}\|\nabla v\|_{L^2}^2+C\|\nabla\rho\|_{L^2}^2\|\nabla u\|_{H^1}^2+C\|\nabla\rho\|_{L^2}^4\\
&\quad+C\|\nabla\rho\|_{H^1}^2\|\mathrm{div}u\|_{L^2}^2+C\|\nabla\rho\|_{H^1}^2\|\nabla v\|_{L^2}^2+C\|\nabla u\|_{L^2}^4+C\|f_\Omega\|_{L^{\frac65}}^2. \qquad (3.18)
\end{aligned}
$$

Since $\|\nabla\rho\|_{H^1}\le C\hbar$ for sufficiently small $\hbar$ and $0\le k\le 2$, it follows from (3.17)–(3.18) that

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{P'(\bar\rho)}{\bar\rho^2}\|\nabla^k\varrho\|_{L^2}^2+\frac{\kappa}{\bar\rho}\|\nabla^{k+1}\varrho\|_{L^2}^2+\|\nabla^k v\|_{L^2}^2\right)+\frac{\mu}{\bar\rho}\|\nabla^{k+1}v\|^2\\
&+\frac{\tilde\mu}{\bar\rho}\|\nabla^k\mathrm{div}v\|^2+\varepsilon\frac{P'(\bar\rho)}{\bar\rho^2}\|\nabla^{k+1}\varrho\|^2+\varepsilon\frac{\kappa}{\bar\rho}\|\nabla^k\Delta\varrho\|^2\\
&\le C\|\nabla\rho\|_{H^2}^2\|\nabla u\|_{H^2}^2+C\|\nabla\rho\|_{H^1}^4+C\|\nabla u\|_{H^1}^4+C\|\nabla^{k-1}f_\Omega\|_{L^2}^2+C\|f_\Omega\|_{L^{\frac65}}^2. \qquad (3.19)
\end{aligned}
$$

Integrating (3.19) over $[0,T]$ and summing up them from $k=0$ to 2 imply that

$$
\begin{aligned}
&\int_0^T\left(\frac{\mu}{\bar\rho}\|\nabla v\|_{H^2}^2+\varepsilon\frac{P'(\bar\rho)}{\bar\rho^2}\|\nabla\varrho\|_{H^2}^2+\varepsilon\frac{\kappa}{\bar\rho}\|\Delta\varrho\|_{H^2}^2\right)dt\\
&\le C\sup_{0<t<T}\left(\|\nabla\rho\|_{H^2}^2+\|\nabla u\|_{H^1}^2\right)\int_0^T\left(\|\nabla\rho\|_{H^1}^2+\|\nabla u\|_{H^2}^2\right)dt\\
&\quad+C\int_0^T\|f_\Omega\|_{H^1}^2dt+C\int_0^T\|f_\Omega\|_{L^{\frac65}}^2dt\\
&=Y^*. \qquad (3.20)
\end{aligned}
$$

Then there exists a time $t^*\in(0,T)$ such that

$$\mu T\|\nabla v(t^*)\|_{H^2}^2+\varepsilon T\|\nabla\varrho(t^*)\|_{H^3}^2\le CY^*.$$

By the Poincaré inequality, this implies that, for $k=1,2$,

$$\frac{P'(\bar\rho)}{\bar\rho^2}\|\nabla^k\varrho(t^*)\|_{L^2}^2+\frac{\kappa}{\bar\rho}\|\nabla^{k+1}\varrho(t^*)\|_{L^2}^2+\|\nabla^k v(t^*)\|_{L^2}^2\le CY^*,\quad t^*\in(0,T).$$

Furthermore, integrating (3.19) from t^* to t for any $t\in(t^*,T)$ leads to

$$\|\nabla\varrho(t)\|_{H^2}^2+\|\nabla v(t)\|_{H^1}^2\le CY^*.$$

By virtue of the time periodicity, one has

$$\|\nabla\varrho(0)\|_{H^2}^2+\|\nabla v(0)\|_{H^1}^2\le CY^*.$$

Repeating the above argument with $t\in(0,t^*)$, we obtain

$$\|\nabla\varrho(t)\|_{H^2}^2+\|\nabla v(t)\|_{H^1}^2\le CY^*,\quad t\in[0,T].$$

This further yields

$$\sup_{0<t<T}\left\{\|(\rho,u)(t)\|_{L^6}^2+\|\nabla\varrho(t)\|_{H^2}^2+\|\nabla v(t)\|_{H^1}^2\right\}\le CY^*. \qquad (3.21)$$

On the other hand, for $k=0,1$, applying ∇^k to $(3.13)_2$, and multiplying the resultant equation by $\nabla^k v_t$, and integrating over Ω, we have

$$\begin{aligned}
&\int_\Omega |\nabla^k v_t|^2 dx + \frac{1}{2\bar\rho}\frac{d}{dt}\int_\Omega \left(\mu|\nabla^{k+1}v|^2 + \tilde\mu|\nabla^k \mathrm{div} v|^2\right) dx \\
&+\frac{P'(\bar\rho)}{\bar\rho}\int_\Omega \nabla^k\nabla\varrho\nabla^k v_t dx - \kappa\int_\Omega \nabla^k\nabla\Delta\varrho\nabla^k v_t dx \\
=&\int_\Omega \nabla^k\left(m(\rho)(\mu\Delta v + \tilde\mu\nabla\mathrm{div} v)\right)\nabla^k v_t dx + \int_\Omega \nabla^k(h(\rho)\nabla\rho)\nabla^k v_t dx \\
&-\tau\int_\Omega \nabla^k(u\nabla u)\nabla^k v_t dx + \tau\int_\Omega \nabla^k f_\Omega \nabla^k v_t dx. \qquad (3.22)
\end{aligned}$$

Applying ∇^k to $(3.13)_1$ and then taking the L^2 inner product with $\nabla^k\varrho_t$ and $\nabla^k\Delta\varrho_t$, respectively, on the resultant equation, we obtain

$$\begin{aligned}
&\int_\Omega |\nabla^k\varrho_t|^2 dx + \frac{\varepsilon}{2}\frac{d}{dt}\int_\Omega |\nabla^{k+1}\varrho|^2 dx + \bar\rho\int_\Omega \nabla^k\varrho_t\nabla^k \mathrm{div} v dx \\
=&-\tau\int_\Omega \nabla^k \mathrm{div}(\rho u)\nabla^k\varrho_t dx, \qquad (3.23)
\end{aligned}$$

$$\begin{aligned}
&\int_\Omega |\nabla^{k+1}\varrho_t|^2 dx + \frac{\varepsilon}{2}\frac{d}{dt}\int_\Omega |\nabla^k\Delta\varrho|^2 dx - \bar\rho\int_\Omega \nabla^k\Delta\varrho_t\nabla^k \mathrm{div} v dx \\
=&\tau\int_\Omega \nabla^k \mathrm{div}(\rho u)\nabla^k\Delta\varrho_t dx. \qquad (3.24)
\end{aligned}$$

Summing up (3.22)–(3.24), we integrate from 0 to T to have

$$\begin{aligned}
&\int_0^T \left(\|\varrho_t\|_{H^2}^2 + \|v_t\|_{H^1}^2\right) dt \\
\le& C\int_0^T \left(\|\nabla\varrho\|_{H^3}^2 + \|\nabla v\|_{H^2}^2\right) dt + C\sup_{0<t<T}\left\{\|\nabla\rho\|_{H^1}^2\right\}\int_0^T \|\nabla v\|_{H^2}^2 dt \\
&+C\sup_{0<t<T}\left(\|\nabla\rho\|_{H^2}^2 + \|\nabla u\|_{H^1}^2\right)\int_0^T \left(\|\nabla\rho\|_{H^1}^2 + \|\nabla u\|_{H^2}^2\right) dt + C\int_0^T \|f_\Omega\|_{H^1}^2 dt. \qquad (3.25)
\end{aligned}$$

Then there exists $t^*\in(0,T)$ such that

$$\begin{aligned}
&\|\varrho_t(t^*)\|_{H^2}^2 + \|v_t(t^*)\|_{H^1}^2 \\
\le& C\int_0^T \left(\|\nabla\varrho\|_{H^3}^2 + \|\nabla v\|_{H^2}^2\right) dt + C\sup_{0<t<T}\left\{\|\nabla\rho\|_{H^1}^2\right\}\int_0^T \|\nabla v\|_{H^2}^2 dt \\
&+C\sup_{0<t<T}\left(\|\nabla\rho\|_{H^2}^2 + \|\nabla u\|_{H^1}^2\right)\int_0^T \left(\|\nabla\rho\|_{H^1}^2 + \|\nabla u\|_{H^2}^2\right) dt + C\int_0^T \|f_\Omega\|_{H^1}^2 dt.
\end{aligned}$$

Applying ∂_t to $(3.13)_2$ and then taking the L^2 inner product with v_t on the resulting identity, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |v_t|^2 dx + \frac{\mu}{\bar\rho}\int_\Omega |\nabla v_t|^2 dx + \frac{\tilde\mu}{\bar\rho}\int_\Omega |\mathrm{div} v_t|^2 dx \\
&-\frac{P'(\bar\rho)}{\bar\rho}\int_\Omega \varrho_t \mathrm{div} v_t dx + \kappa\int_\Omega \Delta\varrho_t \mathrm{div} v_t dx
\end{aligned}$$

$$
\begin{aligned}
&=\int_\Omega \partial_t(m(\rho)(\mu\Delta v+\tilde{\mu}\nabla \mathrm{div} v))v_t dx+\int_\Omega \partial_t(h(\rho)\nabla\rho)v_t dx\\
&\quad -\tau\int_\Omega \partial_t(u\nabla u)v_t dx+\tau\int_\Omega f_{\Omega t}\cdot v_t dx. \qquad (3.26)
\end{aligned}
$$

We apply ∂_t to $(3.13)_1$ and take the L^2 inner product with ϱ_t and $\Delta\varrho_t$, respectively, on the resulting identity to obtain

$$
\frac{1}{2}\frac{d}{dt}\int_\Omega |\varrho_t|^2 dx+\varepsilon\int_\Omega |\nabla\varrho_t|^2 dx+\bar{\rho}\int_\Omega \varrho_t \mathrm{div} v_t dx=-\tau\int_\Omega \partial_t \mathrm{div}(\rho u)\varrho_t dx, \qquad (3.27)
$$

$$
\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\varrho_t|^2 dx+\varepsilon\int_\Omega |\Delta\varrho_t|^2 dx-\bar{\rho}\int_\Omega \Delta\varrho_t \mathrm{div} v_t dx=\tau\int_\Omega \partial_t \mathrm{div}(\rho u)\Delta\varrho_t dx. \qquad (3.28)
$$

Substituting (3.27)–(3.28) into (3.26) and integrating from t^* to t for any $t^*\le t\le t^*+T$ yield

$$
\begin{aligned}
&\sup_{0<t<T}\left\{\frac{P'(\bar{\rho})}{\bar{\rho}^2}\|\varrho_t\|_{L^2}^2+\frac{\kappa}{\bar{\rho}}\|\nabla\varrho_t\|_{L^2}^2+\|v_t\|_{L^2}^2\right\}\\
&\le C\left(\|\varrho_t(t^*)\|_{H^1}^2)+\|v_t(t^*)\|_{L^2}^2\right)+C\int_0^T \|v_t\|_{L^2}^2 dt+C\sup_{0<t<T}\left\{\|\nabla u\|_{H^1}^2\right\}\int_0^T \|\rho_t\|_{H^1}^2 dt\\
&\quad +C\sup_{0<t<T}\left\{\|u_t\|_{L^2}^2\right\}\int_0^T \left(\|\nabla\rho\|_{H^2}^2+\|\nabla u\|_{H^2}^2\right)dt+C\sup_{0<t<T}\left\{\|\rho_t\|_{H^1}^2\right\}\int_0^T \|\nabla v\|_{H^2}^2 dt\\
&\quad +C\sup_{0<t<T}\left\{\|\nabla\rho\|_{H^2}^2\right\}\int_0^T \|\nabla v_t\|_{L^2}^2 dt+C\sup_{0<t<T}\left\{\|\nabla u\|_{H^1}^2\right\}\int_0^T \|\nabla u_t\|_{L^2}^2 dt\\
&\quad +C\sup_{0<t<T}\left\{\|\nabla\rho\|_{H^1}^2\right\}\int_0^T \left(\|\nabla u_t\|_{L^2}^2+\|\nabla\rho_t\|_{L^2}^2\right)dt+C\int_0^T \|f_{\Omega t}\|_{L^2}^2 dt\\
&\le C\int_0^T \|v_t\|_{L^2}^2 dt+C\sup_{0<t<T}\left\{\|\nabla u\|_{H^1}^2\right\}\int_0^T \left(\|\rho_t\|_{H^1}^2+\|\nabla u_t\|_{L^2}^2\right)dt\\
&\quad +C\sup_{0<t<T}\left\{\|\rho_t\|_{H^1}^2\right\}\int_0^T \|\nabla v\|_{H^2}^2 dt+C\sup_{0<t<T}\left\{\|\nabla\rho\|_{H^2}^2\right\}\int_0^T \|\nabla v_t\|_{L^2}^2 dt\\
&\quad +C\sup_{0<t<T}\left\{\|\nabla\rho\|_{H^1}^2\right\}\int_0^T \left(\|\nabla u_t\|_{L^2}^2+\|\nabla\rho_t\|_{L^2}^2+\|\nabla v\|_{H^2}^2\right)dt\\
&\quad +C\sup_{0<t<T}\left(\|\nabla\rho\|_{H^2}^2+\|\nabla u\|_{H^1}^2\right)\int_0^T \left(\|\nabla\rho\|_{H^1}^2+\|\nabla u\|_{H^2}^2\right)dt\\
&\quad +C\sup_{0<t<T}\left\{\|u_t\|_{L^2}^2\right\}\int_0^T \left(\|\nabla\rho\|_{H^2}^2+\|\nabla u\|_{H^2}^2\right)dt+C\|f_\Omega\|_{W_2^{1,1}}^2. \qquad (3.29)
\end{aligned}
$$

In addition, integrating (3.19) and (3.26)–(3.28) from t to $t+h$ and then integrating them form 0 to T gives

$$
\begin{aligned}
&\int_0^T \Big(\left(\|\nabla\varrho(t+h)\|_{H^2}^2+\|\nabla v(t+h)\|_{H^1}^2+\|\varrho_t(t+h)\|_{H^1}^2+\|v_t(t+h)\|_{L^2}^2\right)\\
&\qquad -\left(\|\nabla\varrho(t)\|_{H^2}^2+\|\nabla^k v(t)\|_{H^1}^2\|\varrho_t(t)\|_{H^1}^2+\|v_t(t)\|_{L^2}^2\right)\Big)dt\\
&\le C\int_0^T \|v_t\|_{L^2}^2 dt+C\sup_{0<t<T}\left\{\|\nabla u\|_{H^1}^2\right\}\int_0^T \left(\|\rho_t\|_{H^1}^2+\|\nabla u_t\|_{L^2}^2\right)dt
\end{aligned}
$$

$$
\begin{aligned}
&+C\sup_{0<t<T}\left\{\|\rho_t\|_{H^1}^2\right\}\int_0^T\|\nabla v\|_{H^2}^2dt+C\sup_{0<t<T}\left\{\|\nabla\rho\|_{H^2}^2\right\}\int_0^T\|\nabla v_t\|_{L^2}^2dt\\
&+C\sup_{0<t<T}\left\{\|\nabla\rho\|_{H^1}^2\right\}\int_0^T\left(\|\nabla u_t\|_{L^2}^2+\|\nabla\rho_t\|_{L^2}^2+\|\nabla v\|_{H^2}^2\right)dt\\
&+C\sup_{0<t<T}\left(\|\nabla\rho\|_{H^2}^2+\|\nabla u\|_{H^1}^2\right)\int_0^T\left(\|\nabla\rho\|_{H^1}^2+\|\nabla u\|_{H^2}^2\right)dt+C\|f_\Omega\|_{W_2^{1,1}}^2\\
&+C\sup_{0<t<T}\left\{\|u_t\|_{L^2}^2\right\}\int_0^T\left(\|\nabla\rho\|_{H^2}^2+\|\nabla u\|_{H^2}^2\right)dt+C\int_0^T\|f_\Omega\|_{L^{\frac{6}{5}}}^2dt. \qquad (3.30)
\end{aligned}
$$

Hence, (3.20)–(3.21), (3.25), and (3.29)–(3.30) as well as the strong compactness of L^p space yield that $\mathcal{H}$ is a compact operator.

In the end, we show that the operator $\mathcal{H}$ is continuous. Assume that $(\rho_n,u_n)\subset X_\hbar^\Omega$, $\tau_n\in[0,1]$, $(\rho,u)\subset X_\hbar^\Omega$, $\tau\in[0,1]$, and, when $\lim_{n\to\infty}\tau_n=\tau$,

$$
\begin{aligned}
\lim_{n\to\infty}\sup_{0<t<T}&\Big\{\|(\rho_n-\rho,u_n-u)\|_{L^6}^2+\|(\rho_{nt}-\rho_t,\nabla\rho_{nt}-\nabla\rho_t,u_{nt}-u_t)\|_{L^2}^2\\
&+\|\nabla\rho_n-\nabla\rho\|_{H^2}^2+\|\nabla u_n-\nabla u\|_{H^1}^2\Big\}+\int_0^T\Big(\|\rho_{nt}-\rho_t\|_{H^2}^2+\|u_{nt}-u_t\|_{H^1}^2\\
&+\|\nabla\rho_n-\nabla\rho\|_{H^3}^2+\|\nabla u_n-\nabla u\|_{H^2}^2\Big)dt=0.
\end{aligned}
$$

Let $(\varrho_n,v_n)=\mathcal{H}((\rho_n,u_n),\tau_n)$, $(\varrho,v)=\mathcal{H}((\rho,u),\tau)$. Then $(\varrho_n-\varrho,v_n-v)$ is a periodic solution of the following equations with periodic boundary condition:

$$
\begin{cases}
\tilde{\varrho}_t-\varepsilon\Delta\tilde{\varrho}+\bar{\rho}\mathrm{div}\tilde{v}=(\tau-\tau_n)\mathrm{div}(\rho u)-\tau_n\mathrm{div}\left((\rho_n-\rho)u+\rho_n(u_n-u)\right),\\
\tilde{v}_t-\dfrac{1}{\bar{\rho}+\tau\rho}(\mu\Delta\tilde{v}+\tilde{\mu}\nabla\mathrm{div}\tilde{v})+\dfrac{P'(\bar{\rho}+\tau\rho)}{\bar{\rho}+\tau\rho}\nabla\tilde{\varrho}-\kappa\nabla\Delta\tilde{\varrho}\\
\quad=\left(\dfrac{1}{\bar{\rho}+\tau_n\rho_n}-\dfrac{1}{\bar{\rho}+\tau\rho}\right)(\mu\Delta v_n+\tilde{\mu}\nabla\mathrm{div}v_n)-\left(\dfrac{P'(\bar{\rho}+\tau_n\rho_n)}{\bar{\rho}+\tau_n\rho_n}-\dfrac{P'(\bar{\rho}+\tau\rho)}{\bar{\rho}+\tau\rho}\right)\nabla\rho_n\\
\qquad+(\tau-\tau_n)u_n\nabla u_n-\tau(u_n-u)\nabla u_n-\tau u\nabla(u_n-u)+(\tau_n-\tau)f(x,t).
\end{cases}
$$

Similar to the proof of the compactness of the operator $\mathcal{H}$, we have

$$
\begin{aligned}
\lim_{n\to\infty}\sup_{0<t<T}&\Big\{\|(\varrho_n-\varrho,v_n-v)\|_{L^6}^2+\|(\varrho_{nt}-\varrho_t,\nabla\varrho_{nt}-\nabla\varrho_t,v_{nt}-v_t)\|_{L^2}^2\\
&+\|\nabla\varrho_n-\nabla\varrho\|_{H^2}^2+\|\nabla v_n-\nabla v\|_{H^1}^2\Big\}+\int_0^T\Big(\|\varrho_{nt}-\varrho_t\|_{H^2}^2+\|v_{nt}-v_t\|_{H^1}^2\\
&+\|\nabla\varrho_n-\nabla\varrho\|_{H^3}^2+\|\nabla v_n-\nabla v\|_{H^2}^2\Big)dt=0.
\end{aligned}
$$

This implies the continuity of the operator $\mathcal{H}$. □

3.2. Uniform estimates. This subsection is devoted to giving a series of uniform estimates. Firstly, in the case of $\tau=0$ of (2.2), it is easy to prove $\mathcal{H}((\rho,u),0)\equiv 0$ by the same procedure as that used for the proof of uniqueness in Lemma 3.2 (see also Section 3 in [17]). Thus, we only need to consider the case $\tau\in(0,1]$.

In what follows, consider the system in X^Ω

$$\begin{cases} \varrho_t - \varepsilon\Delta\varrho + \bar\rho \mathrm{div} v = -\tau \mathrm{div}(\varrho v), \\ (\bar\rho+\tau\varrho)v_t - \mu\Delta v - \tilde\mu\nabla\mathrm{div} v + P'(\bar\rho+\tau\varrho)\nabla\varrho \\ \qquad -\kappa(\bar\rho+\tau\varrho)\nabla\Delta\varrho + \tau(\bar\rho+\tau\varrho)(v\cdot\nabla)v = \tau(\bar\rho+\tau\varrho)f_\Omega(x,t), \\ \int_\Omega \varrho dx = 0. \end{cases} \tag{3.31}$$

We shall get Lemma 3.4–3.9 in the following by the elaborate calculation.

LEMMA 3.4. *Let $\tau\in(0,1]$, $|\varrho|\le\frac{\bar\rho}{2}$. If (ϱ,v) is a solution of (3.31), then it holds that*

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega \left((\bar\rho+\tau\varrho)v^2 + \frac{2}{\tau^2(\gamma-1)}P(\bar\rho+\tau\varrho) + \kappa|\nabla\varrho|^2\right)dx \\ &\quad + \frac{\varepsilon}{\gamma-1}\int_\Omega P''(\bar\rho+\tau\varrho)|\nabla\varrho|^2dx + \kappa\varepsilon\int_\Omega|\Delta\varrho|^2dx + \int_\Omega\left(\mu|\nabla v|^2+\tilde\mu|\mathrm{div} v|^2\right)dx \\ &\le C\tau\varepsilon\|\nabla v\|_{L^2}^3\|\Delta v\|_{L^2} + C\tau\|f_\Omega\|^2_{L^{\frac{6}{5}}} \end{aligned} \tag{3.32}$$

where C is a constant independent of L and ε.

Proof. Multiplying $(3.31)_2$ by v and integrating over Ω, we have with the periodic boundary condition

$$\begin{aligned} &\frac{1}{2}\int_\Omega(\bar\rho+\tau\varrho)\frac{d}{dt}v^2dx + \int_\Omega\left(\mu|\nabla v|^2+\tilde\mu|\mathrm{div} v|^2\right)dx + \int_\Omega P'(\bar\rho+\tau\varrho)\nabla\varrho\cdot v dx \\ &\quad + \kappa\int_\Omega \mathrm{div}[(\bar\rho+\tau\varrho)v]\Delta\varrho dx + \int_\Omega\tau(\bar\rho+\tau\varrho)v\nabla v\cdot v dx = \int_\Omega\tau(\bar\rho+\tau\varrho)f_\Omega v dx. \end{aligned} \tag{3.33}$$

From $(3.31)_1$, we arrive at

$$\begin{aligned} \int_\Omega \mathrm{div}[(\bar\rho+\tau\varrho)v]\Delta\varrho dx &= \int_\Omega \tau\nabla\varrho\cdot v\Delta\varrho dx + \int_\Omega(\bar\rho+\tau\varrho)\mathrm{div} v\Delta\varrho dx \\ &= \int_\Omega\left(\bar\rho\mathrm{div} v + \tau\mathrm{div}(\varrho v)\right)\Delta\varrho dx \\ &= \int_\Omega\left(-\varrho_t+\varepsilon\Delta\varrho\right)\Delta\varrho dx \\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla\varrho|^2dx + \varepsilon\int_\Omega|\Delta\varrho|^2dx \end{aligned} \tag{3.34}$$

and

$$\begin{aligned} \int_\Omega\tau(\bar\rho+\tau\varrho)v\nabla v\cdot v dx &= \int_\Omega\tau(\bar\rho+\tau\varrho)v\cdot\nabla\left(\frac{v^2}{2}\right)dx \\ &= -\int_\Omega\frac{\tau^2}{2}\nabla\varrho v\cdot v^2dx - \int_\Omega\frac{\tau}{2}(\bar\rho+\tau\varrho)\mathrm{div} v\cdot v^2dx \\ &= \int_\Omega\frac{\tau}{2}\varrho_t v^2dx - \int_\Omega\frac{\tau\varepsilon}{2}\Delta\varrho v^2dx. \end{aligned} \tag{3.35}$$

We multiply $(3.31)_1$ by $P'(\bar\rho+\tau\varrho)$ to get

$$\frac{1}{\tau}\frac{d}{dt}\int_\Omega P(\bar\rho+\tau\varrho)dx + \varepsilon\tau\int_\Omega P''(\bar\rho+\tau\varrho)|\nabla\varrho|^2dx$$

$$
\begin{aligned}
&=\tau\bar{\rho}\int_\Omega P''(\bar{\rho}+\tau\varrho)\nabla\varrho\cdot vdx+\tau^2\int_\Omega P''(\bar{\rho}+\tau\varrho)\nabla\varrho\cdot\varrho vdx\\
&=\tau\int_\Omega(\bar{\rho}+\tau\varrho)P''(\bar{\rho}+\tau\varrho)\nabla\varrho\cdot vdx\\
&=\tau(\gamma-1)\int_\Omega P'(\bar{\rho}+\tau\varrho)\nabla\varrho\cdot vdx. \qquad (3.36)
\end{aligned}
$$

Inserting (3.34)–(3.36) into (3.33) yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega\left((\bar{\rho}+\tau\varrho)v^2+\frac{2}{\tau^2(\gamma-1)}P(\bar{\rho}+\tau\varrho)+\kappa|\nabla\varrho|^2\right)dx+\int_\Omega\left(\mu|\nabla v|^2+\tilde{\mu}|\mathrm{div}v|^2\right)dx\\
&\quad+\frac{\varepsilon}{\gamma-1}\int_\Omega P''(\bar{\rho}+\tau\varrho)|\nabla\varrho|^2dx+\kappa\varepsilon\int_\Omega|\Delta\varrho|^2dx\\
&=-\tau\varepsilon\int_\Omega\nabla\varrho\cdot v\nabla vdx+\tau\int_\Omega(\bar{\rho}+\tau\varrho)f_\Omega vdx\\
&\le\frac{\varepsilon}{2(\gamma-1)}\int_\Omega P''(\bar{\rho}+\tau\varrho)|\nabla\varrho|^2dx+C\tau\varepsilon\|v\|_{L^6}^2\|\nabla v\|_{L^3}^2+C\tau\|f_\Omega\|_{L^{\frac{6}{5}}}\|v\|_{L^6}\\
&\le\frac{\varepsilon}{2(\gamma-1)}\int_\Omega P''(\bar{\rho}+\tau\varrho)|\nabla\varrho|^2dx+C\tau\varepsilon\|\nabla v\|_{L^2}^3\|\Delta v\|_{L^2}+C\tau\|f_\Omega\|_{L^{\frac{6}{5}}}^2+\frac{\mu}{2}\|\nabla v\|_{L^2}^2.
\end{aligned}
$$

Here, we have used Lemma 2.3 and Hölder and Young's inequalities. Therefore, (3.32) holds obviously. ∎

LEMMA 3.5. *Under the assumptions in Lemma 3.4, it holds that*

$$
\begin{aligned}
&\int_\Omega(\bar{\rho}+\tau\varrho)v_t^2dx+\frac{1}{2}\frac{d}{dt}\int_\Omega\left(\mu|\nabla v|^2+\tilde{\mu}|\mathrm{div}v|^2\right)dx\\
&\quad-\frac{d}{dt}\int_\Omega\left(\frac{1}{\tau}P(\bar{\rho}+\tau\varrho)\mathrm{div}v+\kappa\bar{\rho}\nabla\varrho\nabla\mathrm{div}v\right)dx\\
&\le C\|\nabla v\|_{H^1}^2+\varepsilon^2\|\Delta\varrho\|_{H^1}^2+C\tau\|\nabla\varrho\|_{H^1}^4\|\nabla v\|_{L^2}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|\nabla\Delta\varrho\|_{L^2}^2\\
&\quad+C\tau\|\Delta\varrho\|_{L^2}^2\|\nabla\mathrm{div}v\|_{L^2}^2+C\tau\|\nabla^2v\|_{L^2}\|\nabla v\|_{L^2}^3+C\tau\|f_\Omega\|_{L^2}^2, \qquad (3.37)
\end{aligned}
$$

where C is a constant independent of L and ε.

Proof. Multiplying $(3.31)_2$ by v_t and integrating over Ω, we have

$$
\begin{aligned}
&\int_\Omega(\bar{\rho}+\tau\varrho)v_t^2dx+\frac{1}{2}\frac{d}{dt}\int_\Omega\left(\mu|\nabla v|^2+\tilde{\mu}|\mathrm{div}v|^2\right)dx-\int_\Omega\frac{1}{\tau}P(\bar{\rho}+\tau\varrho)\mathrm{div}v_tdx\\
&-\kappa\int_\Omega(\bar{\rho}+\tau\varrho)\nabla\Delta\varrho\cdot v_tdx+\int_\Omega\tau(\bar{\rho}+\tau\varrho)v\nabla v\cdot v_tdx=\tau\int_\Omega(\bar{\rho}+\tau\varrho)f_\Omega v_tdx. \qquad (3.38)
\end{aligned}
$$

We multiply $(3.31)_1$ by $P'(\bar{\rho}+\tau\varrho)\mathrm{div}v$ and integrate over Ω to obtain

$$
\begin{aligned}
&\int_\Omega P'(\bar{\rho}+\tau\varrho)\varrho_t\mathrm{div}vdx+\int_\Omega(\bar{\rho}+\tau\varrho)P'(\bar{\rho}+\tau\varrho)|\mathrm{div}v|^2dx-\varepsilon\int_\Omega P'(\bar{\rho}+\tau\varrho)\Delta\varrho\mathrm{div}vdx\\
&=-\tau\int_\Omega P'(\bar{\rho}+\tau\varrho)\nabla\varrho\cdot v\mathrm{div}vdx. \qquad (3.39)
\end{aligned}
$$

Applying ∇ to $(3.31)_1$ and taking the L^2 inner product with $\nabla\mathrm{div}v$ on the resulting identity, we get

$$\begin{aligned}&\int_\Omega \nabla\varrho_t\nabla\mathrm{div}v dx+\int_\Omega(\bar\rho+\tau\varrho)|\nabla\mathrm{div}v|^2dx-\varepsilon\int_\Omega\nabla\Delta\varrho\nabla\mathrm{div}v dx\\&=-\tau\int_\Omega\big(\nabla\varrho\mathrm{div}v\nabla\mathrm{div}v+\nabla\varrho\nabla v\nabla\mathrm{div}v+v\nabla^2\varrho\nabla\mathrm{div}v\big)\,dx.\end{aligned}\tag{3.40}$$

Substituting (3.39)–(3.40) into (3.38), we conclude that

$$\begin{aligned}&\int_\Omega(\bar\rho+\tau\varrho)v_t^2dx+\frac12\frac{d}{dt}\int_\Omega\left(\mu|\nabla v|^2+\tilde\mu|\mathrm{div}v|^2-\frac{2}{\tau}P(\bar\rho+\tau\varrho)\mathrm{div}v-2\kappa\bar\rho\nabla\varrho\nabla\mathrm{div}v\right)dx\\
=&\gamma\int_\Omega P(\bar\rho+\tau\varrho)|\mathrm{div}v|^2dx-\varepsilon\int_\Omega P'(\bar\rho+\tau\varrho)\Delta\varrho\mathrm{div}v dx+\tau\int_\Omega P'(\bar\rho+\tau\varrho)\nabla\varrho\cdot v\mathrm{div}v dx\\
&+\kappa\int_\Omega\bar\rho(\bar\rho+\tau\varrho)|\nabla\mathrm{div}v|^2dx-\varepsilon\kappa\int_\Omega\bar\rho\nabla\Delta\varrho\nabla\mathrm{div}v dx+\kappa\tau\int_\Omega\varrho\nabla\Delta\varrho v_t dx\\
&+\kappa\bar\rho\tau\int_\Omega\big(\nabla\varrho\mathrm{div}v\nabla\mathrm{div}v+\nabla\varrho\nabla v\nabla\mathrm{div}v+v\nabla^2\varrho\nabla\mathrm{div}v\big)\,dx\\
&-\int_\Omega\tau(\bar\rho+\tau\varrho)v\nabla v\cdot v_t dx+\tau\int_\Omega(\bar\rho+\tau\varrho)f_\Omega v_t dx\\
\le& C\|\mathrm{div}v\|_{L^2}^2+C\varepsilon\|\Delta\varrho\|_{L^2}\|\mathrm{div}v\|_{L^2}+C\tau\|\nabla\varrho\|_{L^2}\|\nabla v\|_{L^2}\|\mathrm{div}v\|_{L^3}+C\|\nabla\mathrm{div}v\|_{L^2}^2\\
&+C\varepsilon\|\nabla\Delta\varrho\|_{L^2}\|\nabla\mathrm{div}v\|_{L^2}+C\tau\|\nabla^2\varrho\|_{L^2}\|\nabla v\|_{L^3}\|\nabla\mathrm{div}v\|_{L^2}+\frac12\int_\Omega(\bar\rho+\tau\varrho)v_t^2dx\\
&+C\tau\|\nabla v\|_{H^1}\|\Delta\varrho\|_{L^2}\|\nabla\mathrm{div}v\|_{L^2}+C\tau\|\nabla\varrho\|_{H^1}^2\|\nabla\Delta\varrho\|_{L^2}^2\\
&+C\tau\|\nabla^2v\|_{L^2}\|\nabla v\|_{L^2}^3+C\tau\|f_\Omega\|_{L^2}^2\\
\le&\frac12\int_\Omega(\bar\rho+\tau\varrho)v_t^2dx+C\|\nabla v\|_{H^1}^2+\varepsilon^2\|\Delta\varrho\|_{H^1}^2+C\tau\|\nabla\varrho\|_{H^1}^4\|\nabla v\|_{L^2}^2\\
&+C\tau\|\nabla\varrho\|_{H^1}^2\|\nabla\Delta\varrho\|_{L^2}^2+C\tau\|\Delta\varrho\|_{L^2}^2\|\nabla\mathrm{div}v\|_{L^2}^2+C\tau\|\nabla^2v\|_{L^2}\|\nabla v\|_{L^2}^3+C\tau\|f_\Omega\|_{L^2}^2.\end{aligned}$$

This in turn gives (3.37). ∎

LEMMA 3.6. *Under the assumptions in Lemma 3.4, it holds that*

$$\begin{aligned}&\frac12\frac{d}{dt}\int_\Omega\left(\bar\rho|\nabla v|^2+\frac{P'(\bar\rho)}{\bar\rho}|\nabla\varrho|^2+\kappa|\Delta\varrho|^2\right)dx\\
&+\varepsilon\int_\Omega\left(\frac{P'(\bar\rho)}{\bar\rho}|\Delta\varrho|^2+\kappa|\nabla\Delta\varrho|^2\right)dx+\int_\Omega\big(\mu|\Delta v|^2+\tilde\mu|\nabla\mathrm{div}v|^2\big)\,dx\\
\le& C\tau\|\nabla\varrho\|_{L^2}^3\|\nabla^2\varrho\|_{L^2}+C\tau\|\nabla\varrho\|_{H^1}^2\|v_t\|_{L^2}^2+C\tau\|\Delta\varrho\|_{L^2}^3\|\nabla\Delta\varrho\|_{L^2}\\
&+C\tau\|\nabla\varrho\|_{H^1}^2\|\nabla\Delta\varrho\|_{L^2}^2+C\tau\|\nabla v\|_{H^1}^2\|\nabla v\|_{L^2}^2+C\tau\|f_\Omega\|_{L^2}^2,\end{aligned}\tag{3.41}$$

where C is a constant independent of L and ε.

Proof. Multiplying $(3.31)_2$ by Δv and integrating over Ω, we arrive at

$$\begin{aligned}&\frac12\frac{d}{dt}\int_\Omega\bar\rho|\nabla v|^2dx+\int_\Omega\big(\mu|\Delta v|^2+\tilde\mu|\nabla\mathrm{div}v|^2\big)\,dx\\
=&P'(\bar\rho)\int_\Omega\nabla\varrho\Delta v dx+\tau\int_\Omega\varrho v_t\Delta v dx+\int_\Omega(P'(\bar\rho+\tau\varrho)-P'(\bar\rho))\nabla\varrho\Delta v dx\\
&-\kappa\int_\Omega(\bar\rho+\tau\varrho)\nabla\Delta\varrho\Delta v dx+\tau\int_\Omega(\bar\rho+\tau\varrho)v\nabla v\cdot\Delta v dx-\tau\int_\Omega(\bar\rho+\tau\varrho)f_\Omega\Delta v dx.\end{aligned}\tag{3.42}$$

Notice that $\mathrm{div}\Delta v = \Delta\mathrm{div} v$ leads to

$$\int_\Omega \nabla\varrho\Delta v dx = -\int_\Omega \varrho\mathrm{div}\Delta v dx = \int_\Omega \nabla\varrho\nabla\mathrm{div} v dx,$$

$$\int_\Omega \nabla\Delta\varrho\Delta v dx = -\int_\Omega \Delta\varrho\mathrm{div}\Delta v dx = \int_\Omega \nabla\Delta\varrho\nabla\mathrm{div} v dx.$$

Applying ∇ to $(3.31)_1$ and taking the L^2 inner product with $\nabla\varrho$ and $\nabla\Delta\varrho$, respectively, on the resulting identity yield

$$\begin{aligned}&-\int_\Omega \bar\rho\nabla\mathrm{div} v\nabla\varrho dx - \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\varrho|^2 dx = \varepsilon\int_\Omega |\Delta\varrho|^2 dx + \tau\int_\Omega \nabla\mathrm{div}(\varrho v)\nabla\varrho dx\\ =&\varepsilon\int_\Omega |\Delta\varrho|^2 dx + \frac{\tau}{2}\int_\Omega |\nabla\varrho|^2\mathrm{div} v dx + \tau\int_\Omega \nabla\varrho\nabla v\nabla\varrho dx + \tau\int_\Omega \varrho\nabla\varrho\nabla\mathrm{div} v dx,\end{aligned} \tag{3.43}$$

$$\begin{aligned}&\int_\Omega \bar\rho\nabla\mathrm{div} v\nabla\Delta\varrho dx - \frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta\varrho|^2 dx\\ =&\varepsilon\int_\Omega |\nabla\Delta\varrho|^2 dx - \frac{\tau}{2}\int_\Omega |\Delta\varrho|^2\mathrm{div} v dx + \tau\int_\Omega \nabla^2\varrho\nabla v\Delta\varrho dx - \tau\int_\Omega \nabla\varrho\nabla v\nabla\Delta\varrho dx\\ &-\tau\int_\Omega \nabla\varrho\mathrm{div} v\nabla\Delta\varrho dx - \tau\int_\Omega \varrho\nabla\mathrm{div} v\nabla\Delta\varrho dx.\end{aligned} \tag{3.44}$$

Adding (3.42)–(3.44) together and bearing in mind $(P'(\bar\rho+\tau\varrho)-P'(\bar\rho))\sim\tau\varrho$, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega \left(\bar\rho|\nabla v|^2 + \frac{P'(\bar\rho)}{\bar\rho}|\nabla\varrho|^2 + \kappa|\Delta\varrho|^2\right)dx\\ &+\varepsilon\int_\Omega \left(\frac{P'(\bar\rho)}{\bar\rho}|\Delta\varrho|^2 + \kappa|\nabla\Delta\varrho|^2\right)dx + \int_\Omega \left(\mu|\Delta v|^2 + \tilde\mu|\nabla\mathrm{div} v|^2\right)dx\\ =&-\frac{P'(\bar\rho)}{\bar\rho}\tau\int_\Omega \left(\frac{1}{2}|\nabla\varrho|^2\mathrm{div} v + \nabla\varrho\nabla v\nabla\varrho + \varrho\nabla\varrho\nabla\mathrm{div} v\right)dx + \tau\int_\Omega \varrho v_t\Delta v dx\\ &+\int_\Omega (P'(\bar\rho+\tau\varrho)-P'(\bar\rho))\nabla\varrho\Delta v dx + \frac{\kappa\tau}{2}\int_\Omega |\Delta\varrho|^2\mathrm{div} v dx - \kappa\tau\int_\Omega \nabla^2\varrho\nabla v\Delta\varrho dx\\ &+\kappa\tau\int_\Omega \nabla\varrho\nabla v\nabla\Delta\varrho dx + \kappa\tau\int_\Omega \nabla\varrho\mathrm{div} v\nabla\Delta\varrho dx + \kappa\tau\int_\Omega \varrho\nabla\mathrm{div} v\nabla\Delta\varrho dx\\ &-\kappa\tau\int_\Omega \varrho\Delta v\nabla\Delta\varrho dx + \tau\int_\Omega (\bar\rho+\tau\varrho)v\nabla v\cdot\Delta v dx - \tau\int_\Omega (\bar\rho+\tau\varrho)f_\Omega\Delta v dx\\ \le& C\tau\|\nabla\varrho\|_{L^2}\|\nabla\varrho\|_{L^3}\|\Delta v\|_{L^2} + C\tau\|\nabla\varrho\|_{H^1}\|v_t\|_{L^2}\|\Delta v\|_{L^2} + C\tau\|\Delta\varrho\|_{L^2}\|\Delta\varrho\|_{L^3}\|\Delta v\|_{L^2}\\ &+C\tau\|\nabla\varrho\|_{H^1}\|\Delta v\|_{L^2}\|\nabla\Delta\varrho\|_{L^2} + C\tau\|\nabla v\|_{H^1}\|\nabla v\|_{L^2}\|\Delta v\|_{L^2} + C\tau\|f_\Omega\|_{L^2}\|\Delta v\|_{L^2}\\ \le& C\tau\|\nabla\varrho\|_{L^2}^3\|\nabla^2\varrho\|_{L^2} + C\tau\|\nabla\varrho\|_{H^1}^2\|v_t\|_{L^2}^2 + C\tau\|\Delta\varrho\|_{L^2}^3\|\nabla\Delta\varrho\|_{L^2}\\ &+C\tau\|\nabla\varrho\|_{H^1}^2\|\nabla\Delta\varrho\|_{L^2}^2 + C\tau\|\nabla v\|_{H^1}^2\|\nabla v\|_{L^2}^2 + C\tau\|f_\Omega\|_{L^2}^2 + \frac{\mu}{2}\|\Delta v\|_{L^2}^2.\end{aligned}$$

Thus, (3.41) follows immediately from the above estimates. □

LEMMA 3.7. *Under the assumptions in Lemma 3.4, it holds that*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega\Big(D(\bar\rho+\tau\varrho)v_t^2+\frac{DP'(\bar\rho)}{\bar\rho}\varrho_t^2+D\kappa|\nabla\varrho_t|^2+\bar\rho|\nabla\mathrm{div}v|^2\\
&\quad+\frac{P'(\bar\rho)}{\bar\rho}|\Delta\varrho|^2+\kappa|\nabla\Delta\varrho|^2\Big)dx+\frac{\mu}{4}D\int_\Omega|\nabla v_t|^dx\\
&\quad+\frac{\tilde\mu}{2}D\int_\Omega|\mathrm{div}v_t|^dx+\tilde\mu\int_\Omega|\Delta\mathrm{div}v|^2dx+\mu\int_\Omega|\mathrm{curl}\Delta v|^2dx\\
&\quad+\varepsilon D\int_\Omega\left(\frac{P'(\bar\rho)}{\bar\rho}|\nabla\varrho_t|^2+\kappa|\Delta\varrho_t|^2\right)dx+\varepsilon\int_\Omega\left(\frac{P'(\bar\rho)}{\bar\rho}|\nabla\Delta\varrho|^2+\kappa|\Delta\Delta\varrho|^2\right)dx\\
&\le C\tau\|\varrho_t\|_{L^2}^2\|\nabla\varrho\|_{H^1}^2+C\tau\|\varrho_t\|_{L^2}^4\|v_t\|_{L^2}^2+C\tau\|\varrho_t\|_{H^1}^2\|\nabla\Delta\varrho\|_{L^2}^2\\
&\quad+C\tau\|\varrho_t\|_{L^2}^2\|\nabla v\|_{L^2}^2\|\nabla v\|_{H^1}^2+C\tau\|v_t\|_{H^1}^2\|\nabla v\|_{H^1}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|\nabla v_t\|_{L^2}^2\\
&\quad+C\tau\|\nabla\varrho\|_{H^1}^2\|\nabla v\|_{L^2}^2\|\Delta v\|_{L^2}^2+C\tau\|\nabla v\|_{H^1}^2\|\Delta v\|_{L^2}^2+C\tau\|\nabla\varrho\|_{H^1}^4\|v_t\|_{L^2}^2\\
&\quad+C\tau\|\nabla\Delta\varrho\|_{H^1}^2\|\nabla v\|_{H^1}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|\Delta\Delta\varrho\|_{L^2}^2+C_{\eta_1}\tau\|\varrho_t\|_{H^1}^4\\
&\quad+C_{\eta_1}\tau\|\nabla\varrho\|_{H^1}^4+C_{\eta_1}\tau\|\nabla\Delta\varrho\|_{L^2}^2\|\Delta\Delta\varrho\|_{L^2}^2+\eta_1\|\nabla v\|_{H^1}^2\\
&\quad+\eta_2\|v_t\|_{L^2}^2+C_{\eta_2}\tau\|f_{\Omega t}\|_{L^2}^2+C\tau\|f_\Omega\|_{L^3}^4+C\tau\|\nabla f_\Omega\|_{L^2}^2. \qquad (3.45)
\end{aligned}$$

Here C, D, η_1, η_2, C_{η_1}, and C_{η_2} are constants independent of L and ε. Moreover, D can be fixed to be suitably large, η_1 and η_2 can be chosen to be arbitrarily small, and C_{η_1}, C_{η_2} are constants depending on η_1 and η_2, respectively.

Proof. Applying ∂_t to $(3.31)_2$ and then taking the L^2 inner product with v_t on the resulting identity, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega(\bar\rho+\tau\varrho)v_t^2dx+\int_\Omega\left(\mu|\nabla v_t|^2+\tilde\mu|\mathrm{div}v_t|^2\right)dx\\
&=P'(\bar\rho)\int_\Omega\varrho_t\mathrm{div}v_tdx+\int_\Omega\left(P'(\bar\rho+\tau\varrho)-P'(\bar\rho)\right)\varrho_t\mathrm{div}v_tdx-\frac{\tau}{2}\int_\Omega\varrho_tv_t^2dx\\
&\quad+\kappa\tau\int_\Omega\varrho_t\nabla\Delta\varrho v_tdx-\kappa\tau\int_\Omega\nabla\varrho\Delta\varrho_tv_tdx-\kappa\int_\Omega(\bar\rho+\tau\varrho)\Delta\varrho_t\mathrm{div}v_tdx\\
&\quad-\tau^2\int_\Omega\varrho_tv\nabla v\cdot v_tdx-\tau\int_\Omega(\bar\rho+\tau\varrho)v_t\nabla v\cdot v_tdx-\tau\int_\Omega(\bar\rho+\tau\varrho)v\nabla v_t\cdot v_tdx\\
&\quad+\tau\int_\Omega(\bar\rho+\tau\varrho)f_{\Omega t}\cdot v_tdx+\tau^2\int_\Omega\varrho_tf_\Omega v_tdx. \qquad (3.46)
\end{aligned}$$

We apply ∂_t to $(3.31)_1$ and take the L^2 inner product with ϱ_t and $\Delta\varrho_t$, respectively, on the resulting identity to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega\varrho_t^2dx+\bar\rho\int_\Omega\varrho_t\mathrm{div}v_tdx+\varepsilon\int_\Omega|\nabla\varrho_t|^2dx\\
&=-\frac{\tau}{2}\int_\Omega|\varrho_t|^2\mathrm{div}vdx-\tau\int_\Omega\nabla\varrho\cdot\varrho_tv_tdx-\tau\int_\Omega\varrho\varrho_t\mathrm{div}v_tdx, \qquad (3.47)
\end{aligned}$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla\varrho_t|^2dx-\bar\rho\int_\Omega\Delta\varrho_t\mathrm{div}v_tdx+\varepsilon\int_\Omega|\Delta\varrho_t|^2dx$$

$$
\begin{aligned}
&= -\frac{\tau}{2}\int_\Omega \nabla\varrho_t \mathrm{div} v \nabla\varrho_t dx + \tau\int_\Omega \nabla\varrho\cdot v_t\Delta\varrho_t dx \\
&\quad + \tau\int_\Omega \varrho_t \mathrm{div} v\Delta\varrho_t dx + \tau\int_\Omega \varrho \mathrm{div} v_t\Delta\varrho_t dx. \qquad (3.48)
\end{aligned}
$$

Substituting (3.47)–(3.48) into (3.46) yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega\left((\bar\rho+\tau\varrho)v_t^2+\frac{P'(\bar\rho)}{\bar\rho}\varrho_t^2+\kappa|\nabla\varrho_t|^2\right)dx+\int_\Omega\left(\mu|\nabla v_t|^2+\tilde\mu|\mathrm{div} v_t|^2\right)dx \\
&\quad+\varepsilon\int_\Omega\left(\frac{P'(\bar\rho)}{\bar\rho}|\nabla\varrho_t|^2+\kappa|\Delta\varrho_t|^2\right)dx \\
&= -\tau\frac{P'(\bar\rho)}{\bar\rho}\int_\Omega\left(\frac{1}{2}|\varrho_t|^2\mathrm{div} v+\nabla\varrho\cdot\varrho_t v_t+\varrho\varrho_t\mathrm{div} v_t\right)dx+\kappa\tau\int_\Omega\varrho_t\nabla\Delta\varrho\cdot v_t dx \\
&\quad+\tau\int_\Omega\left(P'(\bar\rho+\tau\varrho)-P'(\bar\rho)\right)\varrho_t\mathrm{div} v_t dx-\frac{\tau}{2}\int_\Omega\varrho_t v_t^2 dx-\frac{\kappa\tau}{2}\int_\Omega\nabla\varrho_t\mathrm{div} v\nabla\varrho_t dx \\
&\quad-\kappa\tau\int_\Omega\nabla\varrho_t\mathrm{div} v\nabla\varrho_t dx+\frac{\kappa\tau}{2}\int_\Omega\varrho_t^2\Delta\mathrm{div} v dx-\tau\int_\Omega(\bar\rho+\tau\varrho)v_t\nabla v\cdot v_t dx \\
&-\tau\int_\Omega(\bar\rho+\tau\varrho)v\nabla v_t v_t dx-\tau^2\int_\Omega\varrho_t v\nabla v\cdot v_t dx+\tau\int_\Omega(\bar\rho+\tau\varrho)f_{\Omega t}\cdot v_t dx+\tau^2\int_\Omega\varrho_t f_\Omega v_t dx \\
&\le C\tau\|\varrho_t\|_{L^2}^2\|\mathrm{div} v\|_{H^2}+C\tau\|\varrho_t\|_{L^2}\|\nabla\varrho\|_{H^1}\|\nabla v_t\|_{L^2}+C\tau\|\varrho_t\|_{L^2}\|\nabla\varrho\|_{H^1}\|\mathrm{div} v_t\|_{L^2} \\
&\quad+C\tau\|\varrho_t\|_{L^2}\|v_t\|_{L^3}\|\nabla v_t\|_{L^2}+C\tau\|\varrho_t\|_{H^1}\|\nabla\Delta\varrho\|_{L^2}\|\nabla v_t\|_{L^2}+C\tau\|\varrho_t\|_{H^1}^2\|\mathrm{div} v\|_{H^2} \\
&\quad+C\tau\|\varrho_t\|_{L^2}\|\nabla v\|_{L^2}\|\nabla^2 v\|_{L^2}\|\nabla v_t\|_{L^2}+C\tau\|v_t\|_{L^2}\|\nabla v\|_{H^1}\|\nabla v_t\|_{L^2} \\
&\quad+C\tau\|f_{\Omega t}\|_{L^2}\|v_t\|_{L^2}+C\tau\|\varrho_t\|_{L^2}\|f_\Omega\|_{L^3}\|\nabla v_t\|_{L^2} \\
&\le C\tau\|\varrho_t\|_{H^1}^2\|\mathrm{div} v\|_{H^2}+C\tau\|\varrho_t\|_{L^2}^2\|\nabla\varrho\|_{H^1}^2+C\tau\|\varrho_t\|_{L^2}^4\|v_t\|_{L^2}^2+C\tau\|\varrho_t\|_{H^1}^2\|\nabla\Delta\varrho\|_{L^2}^2 \\
&\quad+C\tau\|\varrho_t\|_{L^2}^2\|\nabla v\|_{L^2}^2\|\nabla^2 v\|_{L^2}^2+C\tau\|v_t\|_{L^2}^2\|\nabla v\|_{H^1}^2+C\tau\|f_{\Omega t}\|_{L^2}\|v_t\|_{L^2} \\
&\quad+C\tau\|\varrho_t\|_{L^2}^2\|f_\Omega\|_{L^3}^2+\frac{\mu}{2}\|\nabla v_t\|_{L^2}^2+\frac{\tilde\mu}{2}\|\mathrm{div} v_t\|_{L^2}^2.
\end{aligned}
$$

This in turn gives

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega\left((\bar\rho+\tau\varrho)v_t^2+\frac{P'(\bar\rho)}{\bar\rho}\varrho_t^2+\kappa|\nabla\varrho_t|^2\right)dx+\int_\Omega\left(\mu|\nabla v_t|^2+\tilde\mu|\mathrm{div} v_t|^2\right)dx \\
&\quad+\varepsilon\int_\Omega\left(\frac{P'(\bar\rho)}{\bar\rho}|\nabla\varrho_t|^2+\kappa|\Delta\varrho_t|^2\right)dx \\
&\le C\tau\|\varrho_t\|_{H^1}^2\|\mathrm{div} v\|_{H^2}+C\tau\|\varrho_t\|_{L^2}^2\|\nabla\varrho\|_{H^1}^2+C\tau\|\varrho_t\|_{L^2}^4\|v_t\|_{L^2}^2 \\
&\quad+C\tau\|\varrho_t\|_{H^1}^2\|\nabla\Delta\varrho\|_{L^2}^2+C\tau\|\varrho_t\|_{L^2}^2\|\nabla v\|_{L^2}^2\|\nabla^2 v\|_{L^2}^2 \\
&\quad+C\tau\|v_t\|_{L^2}^2\|\nabla v\|_{H^1}^2+C\tau\|f_{\Omega t}\|_{L^2}\|v_t\|_{L^2}+C\tau\|\varrho_t\|_{L^2}^2\|f_\Omega\|_{L^3}^2. \qquad (3.49)
\end{aligned}
$$

Next, we multiply $(3.31)_2$ by $\nabla\Delta\mathrm{div} v$ and integrate it over Ω to get

$$
\frac{\bar\rho}{2}\frac{d}{dt}\int_\Omega|\nabla\mathrm{div} v|^2 dx+(\mu+\tilde\mu)\int_\Omega|\Delta\mathrm{div} v|^2 dx
$$

$$
\begin{aligned}
=&\int_\Omega P'(\bar\rho)\Delta\varrho\Delta\mathrm{div}v dx+\tau\int_\Omega \varrho\mathrm{div}v_t\Delta\mathrm{div}v dx+\tau\int_\Omega\nabla\varrho v_t\Delta\mathrm{div}v dx\\
&+\tau\int_\Omega P''(\bar\rho+\tau\varrho)|\nabla\varrho|^2\Delta\mathrm{div}v dx+\int_\Omega\left(P'(\bar\rho+\tau\varrho)-P'(\bar\rho)\right)\Delta\varrho\Delta\mathrm{div}v dx\\
&-\kappa\int_\Omega(\bar\rho+\tau\varrho)\Delta\Delta\varrho\Delta\mathrm{div}v dx-\kappa\tau\int_\Omega\nabla\varrho\nabla\Delta\varrho\Delta\mathrm{div}v dx\\
&+\tau\int_\Omega\mathrm{div}\left((\bar\rho+\tau\varrho)v\nabla v\right)\Delta\mathrm{div}v dx-\tau\int_\Omega\mathrm{div}\left((\bar\rho+\tau\varrho)f_\Omega\right)\Delta\mathrm{div}v dx. \qquad (3.50)
\end{aligned}
$$

Applying Δ to $(3.13)_1$ and then taking the L^2 inner product with $\Delta\varrho$ and $\Delta\Delta\varrho$, respectively, on the resulting equation, we arrive at

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega|\Delta\varrho|^2dx+\bar\rho\int_\Omega\Delta\mathrm{div}v\Delta\varrho dx+\varepsilon\int_\Omega|\nabla\Delta\varrho|^2dx\\
=&-\tau\int_\Omega\Big(\frac{1}{2}|\Delta\varrho|^2\mathrm{div}v+\nabla\varrho\Delta v\Delta\varrho+2\nabla^2\varrho\nabla v\Delta\varrho+\varrho\Delta\mathrm{div}v\Delta\varrho+2\nabla\varrho\nabla\mathrm{div}v\Delta\varrho\Big)dx, \qquad (3.51)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla\Delta\varrho|^2dx-\bar\rho\int_\Omega\Delta\mathrm{div}v\Delta\Delta\varrho dx+\varepsilon\int_\Omega|\Delta\Delta\varrho|^2dx\\
=&\tau\int_\Omega\Big(-\frac{1}{2}\nabla\Delta\varrho\nabla v\nabla\Delta\varrho+\nabla\varrho\Delta v\Delta\Delta\varrho+\Delta\varrho\mathrm{div}v\Delta\Delta\varrho+2\nabla^2\varrho\nabla v\Delta\Delta\varrho\\
&+\varrho\Delta\mathrm{div}v\Delta\Delta\varrho+2\nabla\varrho\nabla\mathrm{div}v\Delta\Delta\varrho\Big)dx. \qquad (3.52)
\end{aligned}
$$

Adding (3.51)–(3.52) to (3.50) yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega\left(\bar\rho|\nabla\mathrm{div}v|^2+\frac{P'(\bar\rho)}{\bar\rho}|\Delta\varrho|^2+\kappa|\nabla\Delta\varrho|^2\right)dx\\
&+(\mu+\tilde\mu)\int_\Omega|\Delta\mathrm{div}v|^2dx+\varepsilon\int_\Omega\left(\frac{P'(\bar\rho)}{\bar\rho}|\nabla\Delta\varrho|^2+\kappa|\Delta\Delta\varrho|^2\right)dx\\
=&-\frac{\tau P'(\bar\rho)}{\bar\rho}\int_\Omega\Big(\frac{1}{2}|\Delta\varrho|^2\mathrm{div}v+\nabla\varrho\Delta v\Delta\varrho+2\nabla^2\varrho\nabla v\Delta\varrho+\varrho\Delta\mathrm{div}v\Delta\varrho+2\nabla\varrho\nabla\mathrm{div}v\Delta\varrho\Big)dx\\
&+\tau\int_\Omega\varrho\mathrm{div}v_t\Delta\mathrm{div}v dx+\tau\int_\Omega\nabla\varrho\cdot v_t\Delta\mathrm{div}v dx+\tau\int_\Omega P''(\bar\rho+\tau\varrho)|\nabla\varrho|^2\Delta\mathrm{div}v dx\\
&+\int_\Omega\left(P'(\bar\rho+\tau\varrho)-P'(\bar\rho)\right)\Delta\varrho\Delta\mathrm{div}v dx+\kappa\tau\int_\Omega\Big(-\frac{1}{2}\nabla\Delta\varrho\nabla v\nabla\Delta\varrho+2\nabla\varrho\nabla\mathrm{div}v\Delta\Delta\varrho\\
&+\Delta\varrho\mathrm{div}v\Delta\Delta\varrho+2\nabla^2\varrho\nabla v\Delta\Delta\varrho+\varrho\Delta\mathrm{div}v\Delta\Delta\varrho+\nabla\varrho\Delta v\Delta\Delta\varrho\Big)dx-\kappa\tau\int_\Omega\varrho\Delta\Delta\varrho\Delta\mathrm{div}v dx\\
&-\kappa\tau\int_\Omega\nabla\varrho\nabla\Delta\varrho\Delta\mathrm{div}v dx+\tau^2\int_\Omega\nabla\varrho\cdot v\nabla v\Delta\mathrm{div}v dx+\tau\int_\Omega(\bar\rho+\tau\varrho)|\nabla v|^2\Delta\mathrm{div}v dx\\
&+\tau\int_\Omega(\bar\rho+\tau\varrho)v\Delta v\Delta\mathrm{div}v dx-\tau^2\int_\Omega\nabla\varrho f_\Omega\Delta\mathrm{div}v dx-\int_\Omega(\bar\rho+\tau\varrho)\mathrm{div}f_\Omega\Delta\mathrm{div}v dx\\
\le&C\tau\|\Delta\varrho\|_{L^2}^2\|\nabla v\|_{H^2}+C\tau\|\nabla\varrho\|_{H^1}\|\Delta\varrho\|_{L^2}\|\Delta\mathrm{div}v\|_{L^2}+C\tau\|\nabla\varrho\|_{H^1}\|\mathrm{div}v_t\|_{L^2}\|\Delta\mathrm{div}v\|_{L^2}\\
&+C\tau\|\nabla\varrho\|_{H^1}\|\nabla v_t\|_{L^2}\|\Delta\mathrm{div}v\|_{L^2}+C\tau\|\nabla\varrho\|_{L^4}^2\|\Delta\mathrm{div}v\|_{L^2}+C\tau\|\nabla v\|_{L^4}^2\|\Delta\mathrm{div}v\|_{L^2}\\
&+C\tau\|\nabla\Delta\varrho\|_{L^2}\|\Delta\Delta\varrho\|_{L^2}\|\nabla v\|_{H^1}+C\tau\|\nabla\varrho\|_{H^1}\|\Delta\Delta\varrho\|_{L^2}\|\Delta\mathrm{div}v\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&+C\tau\|\nabla\varrho\|_{H^1}\|\nabla v\|_{L^2}\|\Delta v\|_{L^2}\|\Delta\mathrm{div}v\|_{L^2}+C\tau\|\nabla v\|_{H^1}\|\Delta v\|_{L^2}\|\Delta\mathrm{div}v\|_{L^2}\\
&+C\tau\|\nabla\varrho\|_{H^1}\|f_\Omega\|_{L^3}\|\Delta\mathrm{div}v\|_{L^2}+C\tau\|\mathrm{div}f_\Omega\|_{L^2}\|\Delta\mathrm{div}v\|_{L^2}\\
\le&\mu\|\Delta\mathrm{div}v\|_{L^2}^2+C\tau\|\Delta\varrho\|_{L^2}^2\|\nabla v\|_{H^2}+C\tau\|\nabla\Delta\varrho\|_{L^2}\|\Delta\Delta\varrho\|_{L^2}\|\nabla v\|_{H^1}+C\tau\|\nabla\varrho\|_{H^1}^2\|\Delta\varrho\|_{L^2}^2\\
&+C\tau\|\nabla\varrho\|_{H^1}^2\|\nabla v_t\|_{L^2}^2+C\tau\|\nabla\varrho\|_{L^2}\|\nabla^2\varrho\|_{L^2}^3+C\tau\|\nabla\varrho\|_{H^1}^2\|\Delta\Delta\varrho\|_{L^2}^2+C\tau\|\nabla v\|_{L^2}\|\Delta v\|_{L^2}^3\\
&+C\tau\|\nabla v\|_{H^1}^2\|\Delta v\|_{L^2}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|\nabla v\|_{L^2}^2\|\Delta v\|_{L^2}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|f_\Omega\|_{L^3}^2+C\tau\|\mathrm{div}f_\Omega\|_{L^2}^2,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega\left(\bar\rho|\nabla\mathrm{div}v|^2+\frac{P'(\bar\rho)}{\bar\rho}|\Delta\varrho|^2+\kappa|\nabla\Delta\varrho|^2\right)dx\\
&\quad+\tilde\mu\int_\Omega|\Delta\mathrm{div}v|^2dx+\varepsilon\int_\Omega\left(\frac{P'(\bar\rho)}{\bar\rho}|\nabla\Delta\varrho|^2+\kappa|\Delta\Delta\varrho|^2\right)dx\\
\le&C\tau\|\Delta\varrho\|_{L^2}^2\|\nabla v\|_{H^2}+C\tau\|\nabla\Delta\varrho\|_{L^2}\|\Delta\Delta\varrho\|_{L^2}\|\nabla v\|_{H^1}+C\tau\|\nabla\varrho\|_{H^1}^2\|\Delta\varrho\|_{L^2}^2\\
&+C\tau\|\nabla\varrho\|_{H^1}^2\|\nabla v_t\|_{L^2}^2+C\tau\|\nabla\varrho\|_{L^2}\|\nabla^2\varrho\|_{L^2}^3+C\tau\|\nabla\varrho\|_{H^1}^2\|\Delta\Delta\varrho\|_{L^2}^2\\
&+C\tau\|\nabla v\|_{L^2}\|\Delta v\|_{L^2}^3+C\tau\|\nabla v\|_{H^1}^2\|\Delta v\|_{L^2}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|\nabla v\|_{L^2}^2\|\Delta v\|_{L^2}^2\\
&+C\tau\|\nabla\varrho\|_{H^1}^2\|f_\Omega\|_{L^3}^2+C\tau\|\mathrm{div}f_\Omega\|_{L^2}^2.
\end{aligned}
\tag{3.53}
$$

Moreover, applying curl to $(3.31)_2$ and employing the fact $\mathrm{curl}\nabla\cdot=0$, we find

$$
\mathrm{curl}((\bar\rho+\tau\varrho)v_t)-\mu\mathrm{curl}\Delta v-\kappa\tau\mathrm{curl}(\varrho\nabla\Delta\varrho)+\tau\mathrm{curl}((\bar\rho+\tau\varrho)v\nabla v)=\tau\mathrm{curl}((\bar\rho+\tau\varrho)f_\Omega).
$$

We multiply the above equation by $\mathrm{curl}\Delta v$ and integrate over Ω to conclude that

$$
\begin{aligned}
\mu\int_\Omega|\mathrm{curl}\Delta v|^2dx\le&C\|\nabla v_t\|_{L^2}^2+C\tau\|\nabla\varrho\|_{H^1}^4\|v_t\|_{L^2}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|\Delta\Delta\varrho\|_{L^2}^2\\
&+C\tau\|\nabla v\|_{L^2}\|\Delta v\|_{L^2}^3+C\tau\|\nabla v\|_{H^1}^2\|\Delta v\|_{L^2}^2\\
&+C\tau\|\nabla\varrho\|_{H^1}^2\|\nabla v\|_{L^2}^2\|\Delta v\|_{L^2}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|f_\Omega\|_{L^3}^2+C\tau\|\nabla f_\Omega\|_{L^2}^2.
\end{aligned}
\tag{3.54}
$$

Notice that

$$
\|\nabla\Delta v\|_{L^2}^2\le C\left(\|\mathrm{div}\Delta v\|_{L^2}^2+\|\mathrm{curl}\Delta v\|_{L^2}^2\right).
$$

Combining (3.49) and (3.53)–(3.54) yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega\Big(D(\bar\rho+\tau\varrho)v_t^2+\frac{DP'(\bar\rho)}{\bar\rho}\varrho_t^2+D\kappa|\nabla\varrho_t|^2+\bar\rho|\nabla\mathrm{div}v|^2+\frac{P'(\bar\rho)}{\bar\rho}|\Delta\varrho|^2\\
&\quad+\kappa|\nabla\Delta\varrho|^2\Big)dx+\frac{\mu}{4}D\int_\Omega|\nabla v_t|^2dx+\frac{\tilde\mu}{2}D\int_\Omega|\mathrm{div}v_t|^2dx+\tilde\mu\int_\Omega|\Delta\mathrm{div}v|^2dx\\
&\quad+\mu\int_\Omega|\mathrm{curl}\Delta v|^2dx+\varepsilon D\int_\Omega\left(\frac{P'(\bar\rho)}{\bar\rho}|\nabla\varrho_t|^2+\kappa|\Delta\varrho_t|^2\right)dx\\
&\quad+\varepsilon\int_\Omega\left(\frac{P'(\bar\rho)}{\bar\rho}|\nabla\Delta\varrho|^2+\kappa|\Delta\Delta\varrho|^2\right)dx
\end{aligned}
$$

$$
\begin{aligned}
&\le C\tau\|\varrho_t\|_{L^2}^2\|\nabla\varrho\|_{H^1}^2+C\tau\|\varrho_t\|_{L^2}^4\|v_t\|_{L^2}^2+C\tau\|v_t\|_{L^2}^2\|\nabla v\|_{H^1}^2\\
&\quad+C\tau\|\varrho_t\|_{H^1}^2\|\nabla\Delta\varrho\|_{L^2}^2+C\tau\|\varrho_t\|_{L^2}^2\|\nabla v\|_{L^2}^2\|\nabla v\|_{H^1}^2+C\tau\|v_t\|_{H^1}^2\|\nabla v\|_{H^1}^2\\
&\quad+C\tau\|\nabla\varrho\|_{H^1}^2\|\nabla v\|_{L^2}^2\|\Delta v\|_{L^2}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|\nabla v_t\|_{L^2}^2+C\tau\|\nabla v\|_{H^1}^2\|\Delta v\|_{L^2}^2\\
&\quad+C\tau\|\nabla\varrho\|_{H^1}^4\|v_t\|_{L^2}^2+C\tau\|\nabla\Delta\varrho\|_{H^1}^2\|\nabla v\|_{H^1}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|\Delta\Delta\varrho\|_{L^2}^2\\
&\quad+C_{\eta_1}\tau\|\varrho_t\|_{H^1}^4+C_{\eta_1}\tau\|\nabla\varrho\|_{H^1}^4+C_{\eta_1}\tau\|\nabla\Delta\varrho\|_{L^2}^2\|\Delta\Delta\varrho\|_{L^2}^2\\
&\quad+\eta_1\|\nabla v\|_{H^1}^2+\eta_2\|v_t\|_{L^2}^2+C_{\eta_2}\tau\|f_{\Omega t}\|_{L^2}^2+C\tau\|f_\Omega\|_{L^3}^4+C\tau\|\nabla f_\Omega\|_{L^2}^2, \qquad (3.55)
\end{aligned}
$$

provided $D>0$ is chosen to be suitably large. Hence, (3.45) holds. □

LEMMA 3.8. *Under the assumptions in Lemma 3.4, it holds that*

$$
\begin{aligned}
&\int_\Omega \varrho_t^2dx+\varepsilon\frac{d}{dt}\int_\Omega|\nabla\varrho|^2dx\\
&\le C\|\mathrm{div}v\|_{L^2}^2+C\tau\|\nabla\varrho\|_{L^2}^2\|\nabla v\|_{H^1}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|\nabla v\|_{L^2}^2, \qquad (3.56)
\end{aligned}
$$

$$
\int_\Omega|\nabla\varrho_t|^2dx+\varepsilon\frac{d}{dt}\int_\Omega|\Delta\varrho|^2dx\le C\|\nabla\mathrm{div}v\|_{L^2}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|\nabla v\|_{H^1}^2, \qquad (3.57)
$$

$$
\begin{aligned}
&\int_\Omega|\Delta\varrho_t|^2dx+\varepsilon\frac{d}{dt}\int_\Omega|\nabla\Delta\varrho|^2dx\\
&\le C\|\Delta\mathrm{div}v\|_{L^2}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|\Delta\mathrm{div}v\|_{L^2}^2+C\tau\|\nabla\Delta\varrho\|_{L^2}^2\|\nabla v\|_{H^1}^2, \qquad (3.58)
\end{aligned}
$$

$$
\begin{aligned}
\int_\Omega\left(|\nabla\varrho|^2+|\Delta\varrho|^2\right)dx\le{}& C\left(\|v_t\|_{L^2}^2+\|\Delta v\|_{L^2}^2+\|\nabla\mathrm{div}v\|_{L^2}^2\right)+C\tau\|\nabla\varrho\|_{H^1}^2\|\nabla\Delta\varrho\|_{L^2}^2\\
&+C\tau\|\nabla v\|_{H^1}^2\|\nabla v\|_{L^2}^2+C\tau\|f_\Omega\|_{L^2}^2, \qquad (3.59)
\end{aligned}
$$

$$
\begin{aligned}
\int_\Omega\left(|\nabla\Delta\varrho|^2+|\Delta\Delta\varrho|^2\right)dx\le{}& C\left(\|\nabla v_t\|_{L^2}^2+\|\Delta\mathrm{div}v\|_{L^2}^2\right)+C\tau\|\nabla\varrho\|_{H^1}^2\|\nabla v_t\|_{L^2}^2\\
&+C\tau\|\nabla v\|_{H^1}^2\|\Delta v\|_{L^2}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|\Delta\Delta\varrho\|_{L^2}^2\\
&+C\tau\|\nabla^2\varrho\|_{L^2}^2\|\nabla v\|_{L^2}^2\|\Delta v\|_{L^2}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|\Delta\varrho\|_{L^2}^2\\
&+C\tau\|\nabla^2\varrho\|_{L^2}^2\|f_\Omega\|_{L^3}^2+C\tau\|\mathrm{div}f_\Omega\|_{L^2}^2, \qquad (3.60)
\end{aligned}
$$

where C is a constant independent of L and ε.

Proof. Firstly, multiply $(3.31)_1$ by ϱ_t and $\Delta\varrho_t$, respectively, and then integrate over Ω to find that

$$
\begin{aligned}
&\int_\Omega\varrho_t^2dx+\frac{\varepsilon}{2}\frac{d}{dt}\int_\Omega|\nabla\varrho|^2dx=-\bar\rho\int_\Omega\varrho_t\mathrm{div}v\,dx-\tau\int_\Omega\mathrm{div}(\varrho v)\varrho_tdx\\
&\le\frac12\int_\Omega\varrho_t^2dx+C\|\mathrm{div}v\|_{L^2}^2+C\tau\|\nabla\varrho\|_{L^2}^2\|\nabla v\|_{H^1}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|\nabla v\|_{L^2}^2,
\end{aligned}
$$

$$\int_\Omega |\nabla\varrho_t|^2dx+\frac{\varepsilon}{2}\frac{d}{dt}\int_\Omega |\Delta\varrho|^2dx=-\bar{\rho}\int_\Omega \nabla\varrho_t\nabla\mathrm{div}v dx-\tau\int_\Omega \nabla\mathrm{div}(\varrho v)\nabla\varrho_t dx$$
$$\le\frac{1}{2}\int_\Omega |\nabla\varrho_t|^2dx+C\|\nabla\mathrm{div}v\|_{L^2}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|\nabla v\|_{H^1}^2.$$

These imply (3.56) and (3.57).

Similarly, applying Δ to $(3.31)_1$ and taking the L^2 inner product with $\Delta\varrho_t$ on the resulting identity, we have

$$\int_\Omega |\Delta\varrho_t|^2dx+\frac{\varepsilon}{2}\frac{d}{dt}\int_\Omega |\nabla\Delta\varrho|^2dx=-\bar{\rho}\int_\Omega \Delta\varrho_t\Delta\mathrm{div}v dx-\tau\int_\Omega \Delta\mathrm{div}(\varrho v)\Delta\varrho_t dx$$
$$\le\frac{1}{2}\int_\Omega |\Delta\varrho_t|^2dx+C\|\Delta\mathrm{div}v\|_{L^2}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|\Delta\mathrm{div}v\|_{L^2}^2+C\tau\|\nabla\Delta\varrho\|_{L^2}^2\|\nabla v\|_{H^1}^2.$$

This in turn gives (3.58).

Next, we multiply $(3.31)_2$ by $\nabla\varrho$ and integrate over Ω to obtain

$$\begin{aligned}
&\int_\Omega P'(\bar{\rho}+\tau\varrho)|\nabla\varrho|^2dx+\kappa\bar{\rho}\int_\Omega |\Delta\varrho|^2dx\\
=&-\int_\Omega (\bar{\rho}+\tau\varrho)v_t\nabla\varrho dx+\mu\int_\Omega \Delta v\nabla\varrho dx+\tilde{\mu}\int_\Omega \nabla\mathrm{div}v\nabla\varrho dx\\
&+\kappa\tau\int_\Omega \varrho\nabla\Delta\varrho\nabla\varrho dx-\tau\int_\Omega (\bar{\rho}+\tau\varrho)v\nabla v\nabla\varrho dx+\tau\int_\Omega (\bar{\rho}+\tau\varrho)f_\Omega\nabla\varrho dx\\
\le&\frac{1}{2}P'(\bar{\rho}+\tau\varrho)\int_\Omega |\nabla\varrho|^2dx+C\left(\|v_t\|_{L^2}^2+\|\Delta v\|_{L^2}^2+\|\nabla\mathrm{div}v\|_{L^2}^2\right)\\
&+C\tau\|\nabla\varrho\|_{H^1}^2\|\nabla\Delta\varrho\|_{L^2}^2+C\tau\|\nabla v\|_{H^1}^2\|\nabla v\|_{L^2}^2+C\tau\|f_\Omega\|_{L^2}^2,
\end{aligned}$$

which implies (3.59).

Finally, we apply Δ to $(3.31)_2$ and take the L^2 inner product with $\nabla\Delta\varrho$ on the resulting equation to arrive at

$$\begin{aligned}
&\int_\Omega P'(\bar{\rho}+\tau\varrho)|\nabla\Delta\varrho|^2dx+\kappa\bar{\rho}\int_\Omega |\Delta\Delta\varrho|^2dx\\
=&\tau\int_\Omega \nabla\varrho v_t\Delta\Delta\varrho dx+\int_\Omega (\bar{\rho}+\tau\varrho)\mathrm{div}v_t\Delta\Delta\varrho dx-(\mu+\tilde{\mu})\int_\Omega \Delta\mathrm{div}v\Delta\Delta\varrho dx\\
&+\tau\int_\Omega P''(\bar{\rho}+\tau\varrho)|\nabla\varrho|^2\Delta\Delta\varrho dx-\tau\int_\Omega P''(\bar{\rho}+\tau\varrho)\nabla\varrho\Delta\varrho\nabla\Delta\varrho dx-\kappa\tau\int_\Omega \varrho|\Delta\Delta\varrho|^2dx\\
&-\kappa\tau\int_\Omega \nabla\varrho\nabla\Delta\varrho\Delta\Delta\varrho dx+\tau^2\int_\Omega \nabla\varrho\cdot v\nabla v\Delta\Delta\varrho dx+\tau\int_\Omega (\bar{\rho}+\tau\varrho)|\nabla v|^2\Delta\Delta\varrho dx\\
&+\tau\int_\Omega (\bar{\rho}+\tau\varrho)v\Delta v\Delta\Delta\varrho dx-\tau^2\int_\Omega \nabla\varrho f_\Omega\Delta\Delta\varrho dx-\tau\int_\Omega (\bar{\rho}+\tau\varrho)\mathrm{div}f_\Omega\Delta\Delta\varrho dx\\
\le&\frac{\kappa\bar{\rho}}{2}\int_\Omega |\Delta\Delta\varrho|^2dx+C\left(\|\nabla v_t\|_{L^2}^2+\|\Delta\mathrm{div}v\|_{L^2}^2\right)+C\tau\|\nabla\varrho\|_{H^1}^2\|\nabla v_t\|_{L^2}^2\\
&+C\tau\|\nabla v\|_{H^1}^2\|\Delta v\|_{L^2}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|\Delta\Delta\varrho\|_{L^2}^2+C\tau\|\nabla^2\varrho\|_{L^2}^2\|\nabla v\|_{L^2}^2\|\Delta v\|_{L^2}^2\\
&+C\tau\|\nabla\varrho\|_{H^1}^2\|\Delta\varrho\|_{L^2}^2+C\tau\|\nabla^2\varrho\|_{L^2}^2\|f_\Omega\|_{L^3}^2+C\tau\|\mathrm{div}f_\Omega\|_{L^2}^2,
\end{aligned}$$

which leads to (3.60). ∎

LEMMA 3.9. *Under the assumptions in Lemma 3.4, it holds that*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \Big(\tilde{D}\bar{D}\bar{\rho}|\nabla \mathrm{div} v|^2+\tilde{D}\bar{D}\frac{P'(\bar\rho)}{\bar\rho}|\Delta\varrho|^2+\tilde{D}\bar{D}\kappa|\nabla\Delta\varrho|^2+\mu|\Delta v|^2+\tilde\mu|\nabla \mathrm{div} v|^2\\
&\quad+2P'(\bar\rho)\Delta\varrho \mathrm{div} v+2\kappa\bar\rho\nabla\Delta\varrho\Delta v\Big)dx++\tilde{D}\varepsilon\frac{d}{dt}\int_\Omega\left(|\nabla\varrho|^2+|\nabla\Delta\varrho|^2\right)dx\\
&+\frac{\bar\rho}{2}\int_\Omega|\nabla v_t|^2dx+\tilde{D}\int_\Omega\left(\varrho_t^2+|\Delta\varrho_t|^2\right)dx+\tilde{D}\bar{D}\int_\Omega\tilde\mu|\Delta \mathrm{div} v|^2dx\\
&+\tilde{D}\bar{D}\varepsilon\int_\Omega\left(\frac{P'(\bar\rho)}{\bar\rho}|\nabla\Delta\varrho|^2+\kappa|\Delta\Delta\varrho|^2\right)dx\\
\le& C\|\mathrm{div} v\|_{L^2}^2+C\tau\|\Delta\varrho\|_{L^2}^2\|\nabla v\|_{H^2}+C\tau\|\nabla\varrho\|_{H^1}^2\|\Delta \mathrm{div} v\|_{L^2}^2+C\tau\|\nabla\Delta\varrho\|_{L^2}^2\|\nabla v\|_{H^1}^2\\
&+C\tau\|\nabla\Delta\varrho\|_{L^2}\|\Delta\Delta\varrho\|_{L^2}\|\nabla v\|_{H^1}+C\tau\|\Delta\varrho\|_{L^2}^4\|v_t\|_{L^2}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|\nabla v\|_{L^2}^2\|\Delta v\|_{L^2}^2\\
&+C\tau\|\nabla\varrho\|_{H^1}^2\|\Delta\Delta\varrho\|_{L^2}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|\Delta\varrho\|_{L^2}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|\nabla v_t\|_{L^2}^2\\
&+C\tau\|\nabla v\|_{H^1}^2\|\Delta v\|_{L^2}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|\nabla v\|_{H^1}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|f_\Omega\|_{L^3}^2+C\tau\|\nabla f_\Omega\|_{L^2}^2,
\end{aligned}\tag{3.61}$$

where C, $\tilde{D}$, and $\bar{D}$ are constants independent of L and ε, and $\tilde{D}$, $\bar{D}$ can be chosen to be appropriately large.

Proof. Multiply $(3.31)_2$ by Δv_t to find that

$$\begin{aligned}
&\int_\Omega(\bar\rho+\tau\varrho)|\nabla v_t|^2dx+\frac{1}{2}\frac{d}{dt}\int_\Omega\left(\mu|\Delta v|^2+\tilde\mu|\nabla \mathrm{div} v|^2\right)dx\\
&\quad+P'(\bar\rho)\frac{d}{dt}\int_\Omega\Delta\varrho \mathrm{div} v dx+\kappa\bar\rho\frac{d}{dt}\int_\Omega\nabla\Delta\varrho\Delta v dx\\
=&-\tau\int_\Omega\nabla\varrho v_t\nabla v_t dx-\tau\int_\Omega P''(\bar\rho+\tau\varrho)|\nabla\varrho|^2\mathrm{div} v_t dx+P'(\bar\rho)\int_\Omega\varrho_t\Delta \mathrm{div} v dx\\
&-\int_\Omega\left(P'(\bar\rho+\tau\varrho)-P'(\bar\rho)\right)\Delta\varrho \mathrm{div} v_t dx+\kappa\tau\int_\Omega\nabla\varrho\nabla\Delta\varrho \mathrm{div} v_t dx+\kappa\tau\int_\Omega\varrho\Delta\Delta\varrho \mathrm{div} v_t dx\\
&-\kappa\bar\rho\int_\Omega\Delta\varrho_t\Delta \mathrm{div} v dx-\tau\int_\Omega\nabla\left((\bar\rho+\tau\varrho)v\nabla v\right)\nabla v_t dx+\tau\int_\Omega\nabla\left((\bar\rho+\tau\varrho)f_\Omega\right)\nabla v_t dx\\
\le&\frac{\bar\rho}{4}\|\nabla v_t\|_{L^2}^2+C\|\varrho_t\|_{L^2}\|\Delta \mathrm{div} v\|_{L^2}+C\|\Delta\varrho_t\|_{L^2}\|\Delta \mathrm{div} v\|_{L^2}+C\tau\|\Delta\varrho\|_{L^2}^4\|v_t\|_{L^2}^2\\
&+C\tau\|\nabla\varrho\|_{H^1}^2\|\Delta\varrho\|_{L^2}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|\Delta\Delta\varrho\|_{L^2}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|\nabla v\|_{L^2}^2\|\Delta v\|_{L^2}^2\\
&+C\tau\|\nabla v\|_{H^1}^2\|\Delta v\|_{L^2}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|f_\Omega\|_{L^3}^2+C\tau\|\nabla f_\Omega\|_{L^2}^2,
\end{aligned}$$

Together with (3.53), (3.56), and (3.58), by choosing $\tilde{D}$ and $\bar{D}$ suitably large. We get (3.61). □

3.3. Proof of Proposition 2.1. Note that solving problem (2.2) is equivalent to solving the equation

$$U-\mathcal{H}(U,1)=0,\quad U=(\varrho,v)\in X^\Omega.$$

To this end, we apply the topological degree theory. The proof is divided into two steps.

Step 1. Throughout this step, in order to apply topological degree theory, we have to show that there exists $\hbar_0>0$ such that

$$(I-\mathcal{H}(\cdot,\tau))(\partial B_{\hbar_0}(0))\neq 0, \qquad \text{for any } \tau\in[0,1], \tag{3.62}$$

where $B_{\hbar_0}(0)$ is the ball of radius $\hbar_0$ centered at the origin in X^{Ω}.

For suitably large $\hat{D}$ and D^*, consider $\hat{D}D^*\times(3.32)+D^*(3.37)+\hat{D}D^*\times(3.41)+D^*\times(3.45)+D^*\times(3.56)+(3.57)+(3.58)+(3.59)+(3.60)$, we integrate it from 0 to T to deduce that

$$\begin{aligned}
&\frac{\hat{D}D^*}{4}\int_0^T\int_\Omega\left(\mu|\nabla v|^2+\tilde{\mu}|\mathrm{div}v|^2+\mu|\Delta v|^2+\tilde{\mu}|\nabla\mathrm{div}v|^2\right)dxdt\\
&\quad+\frac{\bar{\rho}D^*}{4}\int_0^T\int_\Omega v_t dxdt+\frac{DD^*}{4}\int_0^T\int_\Omega\left(\mu|\nabla v_t|^2+2\tilde{\mu}|\mathrm{div}v_t|^2\right)dxdt\\
&\quad+D^*\int_0^T\int_\Omega\left(\tilde{\mu}|\Delta\mathrm{div}v|^2+\mu|\mathrm{curl}\Delta v|^2\right)dxdt+\int_0^T\int_\Omega\left(D^*\varrho_t^2+|\nabla\varrho_t|^2+|\Delta\varrho_t|^2\right)dxdt\\
&\quad+\int_0^T\int_\Omega\left(|\nabla\varrho|^2+|\Delta\varrho|^2+|\nabla\Delta\varrho|^2+|\Delta\Delta\varrho|^2\right)dxdt+\varepsilon DD^*\int_0^T\int_\Omega\Big(\frac{P'(\bar{\rho})}{\bar{\rho}}|\nabla\varrho_t|^2\\
&\quad+\kappa|\Delta\varrho_t|^2\Big)dxdt+\varepsilon D^*\int_0^T\int_\Omega\left(\frac{P'(\bar{\rho})}{\bar{\rho}}|\nabla\Delta\varrho|^2+\kappa|\Delta\Delta\varrho|^2\right)dxdt\\
&\le C\tau\sup_{0<t<T}\left\{\|\nabla\varrho\|_{H^1}^2+\|\nabla v\|_{H^1}^2\right\}\int_0^T\left(\|v_t\|_{H^1}^2+\|\nabla v\|_{H^1}^2\right)dt\\
&\quad+C\tau\sup_{0<t<T}\left\{\|\nabla\varrho\|_{H^1}^4\right\}\int_0^T\|\nabla v\|_{L^2}^2dt+C\tau\sup_{0<t<T}\left\{\|\varrho_t\|_{L^2}^4+\|\Delta\varrho\|_{H^1}^2\right\}\int_0^T\|v_t\|_{L^2}^2dt\\
&\quad+C\tau\sup_{0<t<T}\left\{\|\nabla\varrho\|_{H^1}^2+\|\nabla v\|_{H^1}^4\right\}\int_0^T\|\varrho_t\|_{L^2}^2dt+C\tau\sup_{0<t<T}\left\{\|\nabla\varrho\|_{H^2}^2\right\}\int_0^T\|\nabla\varrho\|_{H^3}^2dt\\
&\quad+C\tau\sup_{0<t<T}\left\{\|\nabla\varrho\|_{H^1}^2\right\}\int_0^T\|\Delta\mathrm{div}v\|_{L^2}^2dt+C\tau\sup_{0<t<T}\left\{\|\nabla\varrho\|_{H^1}^4\right\}\int_0^T\|v_t\|_{L^2}^2dt\\
&\quad+C\tau\sup_{0<t<T}\left\{\|\varrho_t\|_{H^1}^2\right\}\int_0^T\|\varrho_t\|_{H^1}^2dt+C\tau\sup_{0<t<T}\left\{\|\nabla v\|_{H^1}^2+\|\nabla v\|_{H^1}^4\right\}\int_0^T\|\nabla\varrho\|_{H^1}^2dt\\
&\quad+C\tau\sup_{0<t<T}\left\{\|\nabla v\|_{H^1}^2+\|\varrho_t\|_{H^1}^2\right\}\int_0^T\|\nabla\Delta\varrho\|_{H^1}^2dt+C\tau\int_0^T\|f_{\Omega t}\|_{L^2}^2dt\\
&\quad+C\tau\int_0^T\|f_\Omega\|_{L^{\frac{6}{5}}}^2dt+C\tau\int_0^T\|f_\Omega\|_{L^3}^4dt+C\tau\int_0^T\|f_\Omega\|_{H^1}^2dt\\
&\le C_1\tau\hbar_0^4+C_2\tau\hbar_0^6+C_3\tau\left(\int_0^T\left(\|f_\Omega\|_{L^{\frac{6}{5}}}^2+\|f_\Omega\|_{H^1}^4\right)dt+\|f_\Omega\|_{W_2^{1,1}}^2\right).
\end{aligned}\tag{3.63}$$

Then there exits $t^*\in(0,T)$ such that

$$\begin{aligned}
&\int_\Omega\left(v_t^2+|\nabla v_t|^2+|\mathrm{div}v_t|^2+\varrho_t^2+|\nabla\varrho_t|^2+|\Delta\varrho_t|^2\right)(x,t^*)dx\\
&\quad+\int_\Omega\left(|\nabla v|^2+|\mathrm{div}v|^2+|\Delta v|^2+|\nabla\mathrm{div}v|^2+|\Delta\mathrm{div}v|^2+|\mathrm{curl}\Delta v|^2\right)(x,t^*)dx\\
&\quad+\int_\Omega\left(|\nabla\varrho|^2+|\Delta\varrho|^2+|\nabla\Delta\varrho|^2+|\Delta\Delta\varrho|^2\right)(x,t^*)dx
\end{aligned}$$

$$+\varepsilon\int_\Omega\left(|\nabla\varrho_t|^2+|\Delta\varrho_t|^2+|\nabla\Delta\varrho|^2+|\Delta\Delta\varrho|^2\right)(x,t^*)dx$$
$$\leq\tilde{C}_1\tau\hbar_0^4+\tilde{C}_2\tau\hbar_0^6+\tilde{C}_3\tau\left(\int_0^T\left(\|f_\Omega\|^2_{L^{\frac{6}{5}}}+\|f_\Omega\|^4_{H^1}\right)dt+\|f_\Omega\|^2_{W_2^{1,1}}\right).$$

Adding up (3.41), (3.49), and (3.61) yields

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\left(\bar\rho|\nabla v|^2+\frac{P'(\bar\rho)}{\bar\rho}|\nabla\varrho|^2+\kappa|\Delta\varrho|^2+(\bar\rho+\tau\varrho)v_t^2+\frac{P'(\bar\rho)}{\bar\rho}\varrho_t^2+\kappa|\nabla\varrho_t|^2\right)dx\\
&+\frac{d}{dt}\int_\Omega\left(\tilde{D}\bar{D}\left(\bar\rho|\nabla\mathrm{div}v|^2+\frac{P'(\bar\rho)}{\bar\rho}|\Delta\varrho|^2+\kappa|\nabla\Delta\varrho|^2\right)+\mu|\Delta v|^2+\tilde\mu|\nabla\mathrm{div}v|^2\right)dx\\
&+\frac{d}{dt}\int_\Omega(2P'(\bar\rho)\Delta\varrho\mathrm{div}v+2\kappa\bar\rho\nabla\Delta\varrho\Delta v)\,dx+\tilde{D}\varepsilon\frac{d}{dt}\int_\Omega\left(|\nabla\varrho|^2+|\nabla\Delta\varrho|^2\right)dx\\
\leq&C\left(\|\nabla v\|^2_{H^1}+\|v_t\|^2_{L^2}\right)+C\tau\|\varrho_t\|^2_{H^1}\|\mathrm{div}v\|_{H^2}+C\tau\|\Delta\varrho\|^2_{L^2}\|\nabla v\|_{H^2}+C\tau\|\nabla\varrho\|^4_{H^2}\\
&+C\tau\|\nabla v\|^4_{H^1}+C\tau\|\varrho_t\|^4_{H^1}+C\tau\|v_t\|^4_{L^2}+C\tau\|\varrho_t\|^4_{L^2}\|v_t\|^2_{L^2}+C\tau\|\varrho_t\|^2_{L^2}\|\nabla v\|^4_{H^1}\\
&+C\tau\|\nabla\varrho\|^2_{H^1}\|\nabla v\|^4_{H^1}+C\tau\|\nabla\varrho\|^2_{H^1}\|\nabla v_t\|^2_{L^2}+C\tau\|\nabla\Delta\varrho\|^2_{L^2}\|\Delta\Delta\varrho\|^2_{L^2}\\
&+C\tau\|\nabla\varrho\|^2_{H^1}\|\Delta\Delta\varrho\|^2_{L^2}+C\tau\|\Delta\varrho\|^4_{L^2}\|v_t\|^2_{L^2}+C\tau\|\nabla\varrho\|^2_{H^1}\|\Delta\mathrm{div}v\|^2_{L^2}\\
&+C\tau\|f_\Omega\|^2_{H^1}++C\tau\|f_{\Omega t}\|^2_{L^2}+C\tau\|f_\Omega\|^4_{H^1}.\end{aligned}\tag{3.64}$$

Notice that

$$\int_\Omega 2P'(\bar\rho)\Delta\varrho\mathrm{div}v dx\leq\frac{\bar\rho}{2}\int_\Omega|\nabla v|^2dx+\frac{\tilde{D}\bar{D}P'(\bar\rho)}{4\bar\rho}\int_\Omega|\Delta\varrho|^2dx,\tag{3.65}$$

$$\int_\Omega 2\kappa\bar\rho\nabla\Delta\varrho\Delta v dx\leq\bar\rho\int_\Omega|\nabla\mathrm{div}v|^2dx+\frac{\tilde{D}\bar{D}P'(\bar\rho)}{4\bar\rho}\int_\Omega|\Delta\varrho|^2dx,\tag{3.66}$$

when $\tilde{D}$ and $\bar{D}$ are appropriately large. Integrating (3.64) from t^* to t for any $t^*<t<t^*+T$ gives

$$\begin{aligned}&\sup_{0<t<T}\int_\Omega\big(|\nabla v|^2+|\nabla\varrho|^2+|\Delta\varrho|^2+v_t^2+\varrho_t^2+|\nabla\varrho_t|^2\\
&\quad+|\nabla\mathrm{div}v|^2+|\nabla\Delta\varrho|^2+|\Delta v|^2\big)(x,t)dx\\
\leq&\int_\Omega\left(|\nabla v|^2+|\nabla\varrho|^2+|\Delta\varrho|^2+v_t^2+\varrho_t^2+|\nabla\varrho_t|^2+|\nabla\mathrm{div}v|^2+|\nabla\Delta\varrho|^2+|\Delta v|^2\right)(x,t^*)dx\\
&+C\int_0^T\left(\|\nabla v\|^2_{H^1}+\|v_t\|^2_{L^2}+\|\varrho_t\|^2_{H^1}+\|\Delta\varrho\|^2_{L^2}\right)dt+C\tau\sup_{0<t<T}\left\{\|v_t\|^2_{L^2}\right\}\int_0^T\|v_t\|^2_{L^2}dt\\
&+C\tau\sup_{0<t<T}\left\{\|\Delta\varrho\|^2_{L^2}\right\}\int_0^T\|\nabla v\|^2_{H^2}dt+C\tau\sup_{0<t<T}\left\{\|\nabla\varrho\|^2_{H^2}\right\}\int_0^T\|\Delta\Delta\varrho\|^2_{L^2}dt\\
&+C\tau\sup_{0<t<T}\left\{\|\nabla v\|^2_{H^1}\right\}\int_0^T\|\nabla v\|^2_{H^1}dt+C\tau\sup_{0<t<T}\left\{\|\nabla\varrho\|^2_{H^2}\right\}\int_0^T\|\nabla\varrho\|^2_{H^2}dt\\
&+C\tau\sup_{0<t<T}\left\{\|\nabla v\|^4_{H^1}\right\}\int_0^T\left(\|\nabla\varrho\|^2_{H^1}+\|\varrho_t\|^2_{L^2}\right)dt+C\tau\sup_{0<t<T}\left\{\|\varrho_t\|^4_{L^2}\right\}\int_0^T\|v_t\|^2_{L^2}dt\\
&+C\tau\sup_{0<t<T}\left\{\|\varrho_t\|^2_{H^1}\right\}\int_0^T\left(\|\varrho_t\|^2_{H^1}+\|\nabla v\|^2_{H^2}\right)dt+C\tau\sup_{0<t<T}\left\{\|\nabla\varrho\|^4_{L^2}\right\}\int_0^T\|v_t\|^2_{L^2}dt\end{aligned}$$

$$
\begin{aligned}
&+C\tau\sup_{0<t<T}\left\{\|\nabla\varrho\|_{H^1}^2\right\}\int_0^T\left(\|\Delta\mathrm{div}v\|_{L^2}^2+\|\nabla v_t\|_{L^2}^2\right)dt\\
&+C\tau\|f_\Omega\|_{W_2^{1,1}}^2+C\tau\int_0^T\|f_\Omega\|_{H^1}^4dt\\
\le& C_4\tau\hbar_0^4+C_5\tau\hbar_0^6+C_6\tau\left(\int_0^T\|f_\Omega\|_{H^1}^4dt+\|f_\Omega\|_{W_2^{1,1}}^2\right). \qquad (3.67)
\end{aligned}
$$

In view of (3.63) and (3.67), it is easy to get

$$
\begin{aligned}
&\sup_{0<t<T}\int_\Omega\left(|\nabla v|^2+|\nabla\varrho|^2+|\Delta\varrho|^2+v_t^2+\varrho_t^2+|\nabla\varrho_t|^2\right.\\
&\quad\left.+|\nabla\mathrm{div}v|^2+|\nabla\Delta\varrho|^2+|\Delta v|^2\right)(x,t)dx\\
&+\int_0^T\int_\Omega\left(v_t^2+|\nabla v_t|^2+|\mathrm{div}v_t|^2+\varrho_t^2+|\nabla\varrho_t|^2+|\Delta\varrho_t|^2\right)(x,t)dxdt\\
&+\int_0^T\int_\Omega\left(|\nabla v|^2+|\mathrm{div}v|^2+|\Delta v|^2+|\nabla\mathrm{div}v|^2+|\Delta\mathrm{div}v|^2+|\mathrm{curl}\Delta v|^2\right)(x,t)dxdt\\
&+\int_0^T\int_\Omega\left(|\nabla\varrho|^2+|\Delta\varrho|^2+|\nabla\Delta\varrho|^2+|\Delta\Delta\varrho|^2\right)(x,t)dxdt\\
&+\varepsilon\int_0^T\int_\Omega\left(|\nabla\varrho_t|^2+|\Delta\varrho_t|^2+|\nabla\Delta\varrho|^2+|\Delta\Delta\varrho|^2\right)(x,t)dxdt\\
\le&\hat{C}_1\tau\hbar_0^4+\hat{C}_2\tau\hbar_0^6+\hat{C}_3\tau\left(\int_0^T\left(\|f_\Omega\|_{L^{\frac{6}{5}}}^2+\|f_\Omega\|_{H^1}^4\right)dt+\|f_\Omega\|_{W_2^{1,1}}^2\right).
\end{aligned}
$$

Note that $\|(\varrho,v)\|_{L^6}\le C\|\nabla(\varrho,v)\|_{L^2}$. Then, when $\hbar_0$ and $\int_0^T\left(\|f_\Omega\|_{L^{\frac{6}{5}}}^2+\|f_\Omega\|_{H^1}^4\right)dt+\|f_\Omega\|_{W_2^{1,1}}^2$ are suitably small, we eventually obtain

$$
\begin{aligned}
&|||(\varrho,v)|||^2+\varepsilon\int_0^T\left(\|\nabla\varrho_t\|_{H^1}^2+\|\nabla\Delta\varrho\|_{H^1}^2\right)dt\\
\le&\hat{C}_1\tau\hbar_0^4+\hat{C}_2\tau\hbar_0^6+\hat{C}_3\tau\left(\int_0^T\left(\|f_\Omega\|_{L^{\frac{6}{5}}}^2+\|f_\Omega\|_{H^1}^4\right)dt+\|f_\Omega\|_{W_2^{1,1}}^2\right)\le\frac{1}{2}\hbar_0^2,
\end{aligned}
$$

which implies that (3.62) holds.

Step 2. Since $\mathcal{H}((\rho,u),0)\equiv 0$, then

$$
\deg(I-\mathcal{H}(\cdot,1),B_{\hbar_0}(0),0)=\deg(I-\mathcal{H}(\cdot,0),B_{\hbar_0}(0),0)=\deg(I,B_{\hbar_0}(0),0)=1.
$$

Thus, by Lemma 2.5 and Lemma 3.3, we conclude that the problem (2.2) has a solution (ϱ,v) with $|||(\varrho,v)|||\le\hbar_0$. This finishes the proof of Proposition 2.1.

4. Existence in $\mathbb{R}^3$

This last section is devoted to proving Theorem 1.1 by passing to the limit in the regularized problem (2.2).

Proof. (Proof of Theorem 1.1.) Let $(\varrho_\Omega,v_\Omega)$ be the time periodic solution constructed in previous section. Due to t-anisotropic Sobolev imbedding theorem (cf. Theorem 1.4.1 in [25]), we have $\varrho_\Omega,v_\Omega\in C^{\alpha,\alpha/2}(\bar{Q}_T)$ and

$$
[\varrho_\Omega,v_\Omega]_{\alpha,\alpha/2}\le C\hbar_0.
$$

Note that $\hbar_0$ is independent of L and ε. Let $\varepsilon\to 0$, and then let $L\to\infty$, for any fixed $\Omega_L=(-L,L)$. There exists a subsequence $\{(\varrho_n,v_n)\}_{n=1}^{\infty}$ and $(\varrho,v)\in X_{\hbar_0}^{\Omega_L}$ such that

$$
\begin{gathered}
(\varrho_n,v_n)\to(\varrho,v), \text{ uniformly in } \Omega_L;\\
(\varrho_n,v_n)\to(\varrho,v), \text{ strongly in } L^2(0,T;L^6(\Omega_L));\\
(\nabla\varrho_n,\nabla v_n)\to(\nabla\varrho,\nabla v), \text{ weakly-}* \text{ in } \big(L^\infty(0,T;H^2(\Omega_L)),L^\infty(0,T;H^1(\Omega_L))\big);\\
(\varrho_{nt},v_{nt})\to(\varrho_t,v_t), \text{ weakly-}* \text{ in } \big(L^\infty(0,T;H^1(\Omega_L)),L^\infty(0,T;L^2(\Omega_L))\big);\\
(\nabla\varrho_n,\nabla v_n)\to(\nabla\varrho,\nabla v), \text{ weakly in } \big(L^2(0,T;H^3(\Omega_L)),L^2(0,T;H^2(\Omega_L))\big);\\
(\varrho_{nt},v_{nt})\to(\varrho_t,v_t), \text{ weakly in } \big(L^2(0,T;H^2(\Omega_L)),L^2(0,T;H^1(\Omega_L))\big).
\end{gathered}
$$

Furthermore, coming back to (3.64)–(3.66), for any small constant ξ, we have

$$
\begin{aligned}
\int_0^T\Big|\int_\Omega&\big(|\nabla v|^2+|\nabla\varrho|^2+|\Delta\varrho|^2+v_t^2+\varrho_t^2+|\nabla\varrho_t|^2+|\nabla\mathrm{div}v|^2+|\nabla\Delta\varrho|^2\\
&+|\Delta v|^2\big)(x,t+\xi)dx-\int_\Omega\big(|\nabla v|^2+|\nabla\varrho|^2+|\Delta\varrho|^2+v_t^2+\varrho_t^2+|\nabla\varrho_t|^2\\
&+|\nabla\mathrm{div}v|^2+|\nabla\Delta\varrho|^2+|\Delta v|^2\big)(x,t)dx\Big|dt\le C|\xi|,
\end{aligned}
$$

where C is independent of L. Therefore, we conclude that

$$
\begin{gathered}
(\varrho_{nt},v_{nt})\to(\varrho_t,v_t), \text{ strongly in } L^2(0,T;L^2(\Omega_L));\\
(\nabla\varrho_n,\nabla v_n)\to(\nabla\varrho,\nabla v), \text{ strongly in } \big(L^2(0,T;H^2(\Omega_L)),L^2(0,T;H^1(\Omega_L))\big).
\end{gathered}
$$

Hence, taking a sequence L_n with $L_n\to+\infty$ as $n\to\infty$, let $\{(\varrho_n^k,v_n^k)\}$ be the convergent function sequence in Ω_{L_k} given in the above sense. And let $\{(\varrho_n^{k+1},v_n^{k+1})\}$ be a subsequence of $\{(\varrho_n^k,v_n^k)\}$, which converges in $\Omega_{L_{k+1}}$ $(k=1,2,\dots,n,\dots)$. We repeat the argument as follows:

$$
\begin{array}{llll}
(\varrho_1^1,v_1^1) & (\varrho_2^1,v_2^1) & \cdots & (\varrho_n^1,v_n^1) \quad \text{converges in } \Omega_{L_1}\\
(\varrho_1^2,v_1^2) & (\varrho_2^2,v_2^2) & \cdots & (\varrho_n^2,v_n^2) \quad \text{converges in } \Omega_{L_2}\\
\multicolumn{4}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots}\\
(\varrho_1^n,v_1^n) & (\varrho_2^n,v_2^n) & \cdots & (\varrho_n^n,v_n^n) \quad \text{converges in } \Omega_{L_n}\\
\multicolumn{4}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots}
\end{array}
$$

We obtain a Cantor diagonal subsequence $\{(\varrho_n^n,v_n^n)\}$, which converges to (ϱ,v) in Ω_L for any $L>0$. Since $L>0$ is arbitrary, we see that the limit function $(\varrho,v)\in X_{\hbar_0}$ is indeed a time periodic solution of (2.1) in $\mathbb{R}^3$. The proof of the theorem is thus completed. □

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