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DERIVATION OF THE BURGERS' EQUATION FROM THE GAS DYNAMICS*

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Abstract. We establish in this paper that under long-wavelength small-amplitude approximation, the solution to the gas dynamics system converges globally in time to the solution of the Burgers' equation for well prepared initial data.

Key words. Burgers' equation, long wavelength limit, gas dynamics system.

AMS subject classifications. 35Q53; 35Q35.

1. Introduction

In this paper, we study the long-wavelength, small amplitude limit of the gas dynamics system to the Burgers' equation. Consider the following gas dynamics system

$$\begin{cases} n_t+(nu)_x=0 \\ (nu)_t+(nu^2+p)_x=\mu u_{xx}, \end{cases} \tag{1.1}$$

where $n(t,x)$, $u(t,x)$ are the density and velocity of the gas at time $t\geq 0$ and position $x\in\mathbb{R}$, respectively. The pressure $p=p(n)=An^\gamma$ for $\gamma\geq 1$ is the ratio of specific heats. When $\gamma=1$, it describes an isothermal process, while when $\gamma>1$, it describes an adiabatic process. The constant $\mu\geq 0$ is the viscosity coefficient. The system (1.1) is also known as the compressible Navier–Stokes equations.

The Burgers' equation [1]

$$u_t+uu_x=\mu u_{xx}$$

is one of the simplest yet most important nonlinear PDEs, and has received a great deal of attention due to the fact that it models a number of physically important phenomena such as shock waves and acoustic transmissions. Due to its important features, it is also frequently used as a test equation in numerical schemes.

It was shown formally that under the weak nonlinearity and long-wavelength approximation, the Burgers' equation can be derived under the Gardner–Morikawa transformation (see [6]). However, to the best knowledge of the authors there are no rigorous mathematical justifications. In this paper, we justify this limit with mathematical rigor.

In the next section, we give the formal derivation and the main results (Theorem 2.2). In the third section, we prove Theorem 2.2 by uniform energy estimates for the remainder terms in Sobolev spaces. An appendix is given to derive the key remainder equation (2.15).

2. Formal expansion and the main result

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2.1. Formal expansion. To study the dynamics of the small but finite amplitude of the gas dynamics, we transform the independent space-time coordinates to the stretching coordinate (Gardner–Morikawa transformation, see [6])

$$\xi = \varepsilon(x - c_0 t), \qquad \tau = \varepsilon^2 t, \tag{2.1}$$

where ε denotes the amplitude of the initial perturbation and is assumed to be small compared with unity. The parameter c_0 is the wave phase parameter to be determined to balance the time variation of a state variable and dissipative effects. Using (2.1) in (1.1), we obtain (after denoting (ξ,τ) as (x,t))

$$\begin{cases} \varepsilon n_t - c_0 n_x + (nu)_x = 0, & \text{(2.2a)} \\ \varepsilon u_t - c_0 u_x + u u_x + A\gamma n^{\gamma-2} n_x = \varepsilon \dfrac{\mu}{n} u_{xx}, & \text{(2.2b)} \end{cases}$$

where $p = An^\gamma$ is explicitly used.

We consider the formal expansion as series in powers of ε about an equilibrium state $(n,u) = (1,0)$

$$\begin{aligned} n &= 1 + \varepsilon n^{(1)} + \varepsilon^2 n^{(2)} + \cdots \\ u &= \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \cdots . \end{aligned} \tag{2.3}$$

To determine the coefficients of $(n^{(i)}, u^{(i)})$, we plug the formal expansion (2.3) into the rescaled equation (2.2). Now, we equate the coefficients on both sides of the resulting equation in front of different powers of the parameter ε.

At the first order, we have

$$(\mathcal{S}_0) \quad \begin{cases} -c_0 n_x^{(1)} + u_x^{(1)} = 0, & \text{(2.4a)} \\ A\gamma n_x^{(1)} - c_0 u_x^{(1)} = 0. & \text{(2.4b)} \end{cases}$$

To get a nontrivial solution of $n^{(1)}$ and $u^{(1)}$, we require the determinant of the matrix of coefficients to vanish to obtain

$$c_0^2 = A\gamma, \tag{2.5}$$

which will be assumed throughout this paper. There are two solutions of (2.5), i.e., $c_0 = \pm\sqrt{A\gamma}$, representing the right-going and the left-going waves respectively. From (2.4), we assume

$$u^{(1)} = c_0 n^{(1)}, \tag{2.6}$$

which make (2.4) valid with $n^{(1)}$ to be determined.

At the second order, we have

$$(\mathcal{S}_1) \quad \begin{cases} \partial_t n^{(1)} - c_0 \partial_x n^{(2)} + \partial_x u^{(2)} + \partial_x (n^{(1)} u^{(1)}) = 0; & \text{(2.7a)} \\ \partial_t u^{(1)} - c_0 \partial_x u^{(2)} + u^{(1)} \partial_x u^{(1)} & \\ \qquad + A\gamma \partial_x n^{(2)} + A\gamma(\gamma-2) n^{(1)} \partial_x n^{(1)} = \mu \partial_{xx} u^{(1)}. & \text{(2.7b)} \end{cases}$$

Multiplying (2.7a) with c_0 and adding the resultant to (2.7b), after rearranging we obtain the *Burgers' equation*

$$\partial_t n^{(1)} + \frac{A\gamma(\gamma+1)}{2c_0} n^{(1)} \partial_x n^{(1)} = \frac{\mu}{2} \partial_{xx} n^{(1)}, \tag{2.8}$$

where we have used (2.6) explicitly. Here we have used (2.5) explicitly to cancel the coefficients in front of $n^{(2)}$ and $u^{(2)}$. From (2.7a), we can write

$$u^{(2)}=c_0n^{(2)}+r^{(1)}=c_0n^{(2)}-\int^x \partial_t n^{(1)}+\partial_x(n^{(1)}u^{(1)})d\xi, \tag{2.9}$$

where $r^{(1)}$ depends only on $(n^{(1)},u^{(1)})$. Proceeding as above, we have *the coefficients of* ε^3*:*

$$(\mathcal{S}_2)\quad \begin{cases} \partial_t n^{(2)}-c_0\partial_x n^{(3)}+\partial_x u^{(3)}+\partial_x(n^{(1)}u^{(2)})+\partial_x(n^{(2)}u^{(1)})=0; & \text{(2.10a)}\\ \partial_t u^{(2)}-c_0\partial_x u^{(3)}+A\gamma\partial_x n^{(3)}+\partial_x(u^{(1)}u^{(2)})+A\gamma(\gamma-2)\partial_x(n^{(1)}n^{(2)}) & \\ \quad +\dfrac{A\gamma(\gamma-2)(\gamma-3)}{2}(n^{(1)})^2\partial_x n^{(1)}=\mu\partial_{xx}u^{(2)}-\mu n^{(1)}\partial_{xx}u^{(1)}. & \text{(2.10b)} \end{cases}$$

Multiplying (2.10a) with c_0, and adding the resultant to (2.10b), we obtain a linearized Burgers' equation for $n^{(2)}$:

$$\partial_t n^{(2)}+\frac{A\gamma(\gamma+1)}{2c_0}\partial_x(n^{(1)}n^{(2)})=\frac{\mu}{2}\partial_{xx}n^{(1)}+G^{(1)}, \tag{2.11}$$

where the nonhomogeneous term $G^{(1)}$ involves only $n^{(1)}$ and $u^{(1)}$. Since explicit form of $G^{(1)}$ plays no role in this paper, we don't work it out explicitly.

Inductively, we can formally derive the equation to any order of ε, we are interested in. Let $k\geq 2$ be an integer. From the system $(\mathcal{S}_{k-1})$ for the coefficients of ε^k, we obtain

$$u^{(k)}=c_0n^{(k)}+r^{(k-1)}, \tag{2.12}$$

where r^{k-1} depends only on $(n^{(j)},u^{(j)})$ for $1\leq j\leq k-1$. Then in the evolution system $(\mathcal{S}_k)$ for the coefficients of ε^{k+1}, by the same procedure that leads to (2.8), we obtain the linearized Burgers' equation for $n^{(k)}$:

$$\partial_t n^{(k)}+\frac{A\gamma(\gamma+1)}{2c_0}\partial_x\left(n^{(1)}n^{(k)}\right)=\frac{\mu}{2}\partial_{xx}n^{(k)}+G^{(k-1)}, \tag{2.13}$$

where $G^{(k-1)}$ only depends on $(n^{(j)},u^{(j)})$ for $1\leq j\leq k-1$, which are "known" from the $(k-1)^{th}$ step. If we define $G^{(0)}=0$, the Burgers' equation (2.8) can be unified in the form of (2.13). Note that the system (2.12) and (2.13) is self contained, which does not depend on $(n^{(j)},u^{(j)})$ for $j\leq k+1$.

For the solvability of $(n^{(k)},u^{(k)})$, we recall the following classical result for the existence of sufficiently smooth solutions in a small time interval.

Proposition 2.1. *Let $\mu\geq 0$ and $s\geq 2$. Then for any $(n_0^{(i)},u_0^{(i)})\in H^s$ satisfying* (2.12) *at time $t=0$, there exists a maximal time of existence $T>0$ such that the initial value problem* (2.13) *has a unique solution $(n^{(i)}(t),u^{(i)}(t))$ on $0\leq t<T$ with initial data $(n_0^{(i)},n_0^{(i)})$, such that $u\in L^\infty([0,T'];H^s(\mathbb{R}))$ for every $T'<T$.*

The proof can be modified from classical results of Kato [3].

2.2. Remainder equation. However, to show that as $\varepsilon\to 0$, $n^{(1)}$ converges to a solution of the Burgers' equation, we must make the above procedure rigorous. We need to cut off and consider the remainder terms. To do so, we consider the following expansions with remainder terms $(n_R^\varepsilon, u_R^\varepsilon)$:

$$\begin{cases} n &= 1+\varepsilon n^{(1)}+\varepsilon^2 n^{(2)}+\varepsilon^3 n^{(3)}+\varepsilon^2 n_R^\varepsilon \\ u &= \varepsilon u^{(1)}+\varepsilon^2 u^{(2)}+\varepsilon^3 u^{(3)}+\varepsilon^2 u_R^\varepsilon, \end{cases} \tag{2.14}$$

where $(n_R^\varepsilon, u_R^\varepsilon)$ are the remainder terms that may depend on ε. Here $(n^{(i)}, u^{(i)})$ for $1 \le i \le 3$ satisfy (2.6), (2.8), (2.9), (2.11), (2.12), and (2.13) with $k=3$, whose existence and regularity is guaranteed by Proposition 2.1. To further simplify the notations in what follows, we denote

$$\tilde{n} = n^{(1)}+\varepsilon n^{(2)}+\varepsilon^2 n^{(3)}, \quad \tilde{u} = u^{(1)}+\varepsilon u^{(2)}+\varepsilon^2 u^{(3)}.$$

Plugging the expansion (2.14) into the gas dynamics equation (1.1), and then subtracting $\varepsilon^2 \times$(2.7) and $\varepsilon^3 \times$(2.10), we have the remainder equation for $(n_R^\varepsilon, u_R^\varepsilon)$:

$$\begin{cases} \partial_t n_R^\varepsilon - \dfrac{c_0}{\varepsilon}\partial_x n_R^\varepsilon + \dfrac{1}{\varepsilon}\partial_x u_R^\varepsilon + \partial_x[(\tilde{n}+\varepsilon n_R^\varepsilon)u_R^\varepsilon] + \partial_x[\tilde{u} u_R^\varepsilon] + \varepsilon \mathcal{R}_1 = 0, & \text{(2.15a)} \\ \partial_t u_R^\varepsilon - \dfrac{c_0}{\varepsilon}\partial_x u_R^\varepsilon + \dfrac{A\gamma}{\varepsilon}\partial_x n_R^\varepsilon + (\tilde{u}+\varepsilon u_R^\varepsilon)\partial_x u_R^\varepsilon + a_{21}(\varepsilon n_R^\varepsilon)\partial_x n_R^\varepsilon & \text{(2.15b)} \\ \qquad + a_{22}(\varepsilon n_R^\varepsilon) n_R^\varepsilon + \partial_x \tilde{u} u_R^\varepsilon = \dfrac{\mu}{n}\partial_{xx} u_R^\varepsilon - \varepsilon \mu b n_R^\varepsilon - \varepsilon \mathcal{R}_2 & \text{(2.15c)} \end{cases}$$

where a_{21}, a_{22} depend on $n^{(1)}$, $n^{(2)}$, $n^{(3)}$, their spatial derivatives and $\varepsilon n_R^\varepsilon$, and

$$\begin{aligned} b &= \frac{1}{n}(\partial_{xx}u^{(1)} + \varepsilon \partial_{xx} u^{(2)} - \varepsilon \partial_{xx} u^{(1)} n^{(1)}), \\ \mathcal{R}_1 &= \partial_t n^{(3)} + \partial_x \Big(\sum_{1\le i,j\le 3; i+j\ge 4} \varepsilon^{i+j-4} n^{(i)} u^{(j)} \Big), \\ \mathcal{R}_2 &= \partial_t u^{(3)} + \sum_{1\le i,j\le 3; i+j\ge 4} \varepsilon^{i+j-4} u^{(i)} \partial_x u^{(j)} + a_{23}(n^{(1)}, n^{(2)}, n^{(3)}) + a_{24}(\varepsilon n_R^\varepsilon). \end{aligned} \tag{2.16}$$

The derivation of such a remainder system is detailed in Appendix A.

This system is an hyperbolic system with singular perturbation, whose uniform (in ε) estimate is not straightforward. To rigorously justify the limit from gas dynamics to the Burgers' equation, we need uniform in ε estimate for this system.

2.3. Main results. We state the main result of this paper in the following

THEOREM 2.2. *Let $\mu \ge 0$, $\gamma \ge 1$ and $s \ge 2$ in Proposition 2.1 be sufficiently large. Let $(n^{(i)}, u^{(i)}) \in H^s$ be a solution on $[0,T)$ constructed in Proposition 2.1 for* (2.13) *with initial data $(n_0^{(i)}, u_0^{(i)}) \in H^s$ satisfying* (2.12). *Assume $(n_{R0}^\varepsilon, u_{R0}^\varepsilon) \in H^2$ and has the expansion*

$$\begin{aligned} n_0 &= 1+\varepsilon^1 n_0^{(1)}+\varepsilon^2 n_0^{(2)}+\varepsilon^3 n_0^{(3)}+\varepsilon^2 n_{R0}^\varepsilon, \\ u_0 &= \varepsilon^1 u_0^{(1)}+\varepsilon^2 u_0^{(2)}+\varepsilon^3 u_0^{(3)}+\varepsilon^2 u_{R0}^\varepsilon, \end{aligned}$$

then for any $T' < T$, there exists $\varepsilon_{T'} > 0$ such that if $0 < \varepsilon < \varepsilon_{T'}$, the solution of the system (2.2) *with initial data (n_0, u_0) can be expressed as*

$$\begin{aligned} n &= 1+\varepsilon^1 n^{(1)}+\varepsilon^2 n^{(2)}+\varepsilon^3 n^{(3)}+\varepsilon^2 n_R^\varepsilon, \\ u &= \varepsilon^1 u^{(1)}+\varepsilon^2 u^{(2)}+\varepsilon^3 u^{(3)}+\varepsilon^2 u_R^\varepsilon, \end{aligned}$$

such that

$$\sup_{t\in[0,T']}\|n_R^\varepsilon(t)\|_{H^2}^2+\|u_R^\varepsilon(t)\|_{H^2}^2\le C_{T'}$$

for all $0<\varepsilon<\varepsilon_{T'}$.

We see that if by some chance $T=\infty$, then T' can be made arbitrary large.

This theorem applies when $\mu\ge 0$. In the long-wavelength limit, when $\mu>0$, the Burgers' equation governs the dynamics, while when $\mu=0$, the inviscid Burgers' equation does. The result also works in the spatial periodic case $x\in\mathbb{T}$, which is not developed here. To prove Theorem (2.2), we have to prove that $(n_R^\varepsilon,u_R^\varepsilon)$ is bounded in H^2 uniformly in ε. This is given in Section 2 by delicate energy estimate. In Appendix A, we give a detailed derivation of the remainder system (2.15).

We also remark that under a different Gardner–Morikawa transformation

$$\xi=\varepsilon(x-c_0t),\qquad \tau=\varepsilon^3t,$$

KdV equation can be derived [6] and justified by the method in this paper. See also the derivation of the KdV equation and the two dimensional Kadomtsev–Petviashvili II (KP-II) equation and the three dimensional Zakharov–Kuznetsov equation in higher dimensions from the Euler–Poisson systems in [2, 5] and the references therein.

3. Uniform estimates

Since the viscous term $\mu\partial_{xx}u_R^\varepsilon$ plays a beneficial role in the estimate, we only consider the inviscid case of $\mu=0$ in the following. The case $\mu>0$ is easier and can be proved in the same way. Furthermore, we renormalize the equation by setting the coefficients $A\gamma=c_0=1$. Then the remainder equations can be written in the following form ($\mu=0$)

$$\begin{cases}\partial_t n_R^\varepsilon-\dfrac{1-u}{\varepsilon}\partial_x n_R^\varepsilon+\dfrac{n}{\varepsilon}\partial_x u_R^\varepsilon+\partial_x\tilde{n}u_R^\varepsilon+\partial_x\tilde{u}n_R^\varepsilon+\varepsilon\mathcal{R}_1=0, & \text{(3.1a)}\\ \partial_t u_R^\varepsilon-\dfrac{1-u}{\varepsilon}\partial_x u_R^\varepsilon+\dfrac{1+\varepsilon a_{21}(\varepsilon n_R^\varepsilon)}{\varepsilon}\partial_x n_R^\varepsilon+a_{22}(\varepsilon n_R^\varepsilon)n_R^\varepsilon+\partial_x\tilde{u}u_R^\varepsilon=-\varepsilon\mathcal{R}_2, & \text{(3.1b)}\end{cases}$$

where $\mathcal{R}_2$, a_{21}, and a_{22} are defined in (2.16). The system is hyperbolic and symmetrizable [4]. The local existence of smooth solutions is then standard from Kato's theory [3], which is stated in the following

Proposition 3.1. *Let $s>\frac{d}{2}+1$, $\varepsilon>0$ be fixed, and the initial data $(n_{R0}^\varepsilon,n_{R0}^\varepsilon)\in H^s$. Then there exists a maximal existence time T_ε and a solution $(n_R^\varepsilon,n_R^\varepsilon)$ of (3.1) such that*

$$(n_R^\varepsilon,n_R^\varepsilon)\in C([0,T];H^s)\cap C^1([0,T];H^{s-1}),$$

for every $T<T_\varepsilon$.

We want to prove that $T_\varepsilon\ge T$ as $\varepsilon\to 0$ for some $T>0$. To slightly simplify the presentation, we let $\tilde{C}$ be constant, which will be fixed later, much larger than the bound of the initial data $(n_R^\varepsilon,u_R^\varepsilon)|_{t=0}$ in H^2, i.e.,

$$\|(n_R^\varepsilon,u_R^\varepsilon)(t)\|_{H^2}\le\tilde{C},\quad \forall t\in[0,T_\varepsilon]. \tag{3.2}$$

As a consequence of the expansion (2.14), there exists $\varepsilon_0>0$ such that

$$1/2<n<3/2,\quad |u|\le 1/2,\qquad \forall\varepsilon<\varepsilon_0. \tag{3.3}$$

Furthermore, from Lemma A.1 in Appendix A, when $\varepsilon<\varepsilon_0$, we have

$$\|a_{21}\|_{H^2},\|a_{22}\|_{H^2}\le C_1(\varepsilon\tilde{C}).$$

LEMMA 3.2. *Let $\alpha=0,1$ and $(n_R^\varepsilon,u_R^\varepsilon)$ be a solution to the equation* (3.1), *then there exist some constant C such that*

$$\|\varepsilon\partial_t n_R^\varepsilon\|_{H^\alpha}^2\le C(1+\|n_R^\varepsilon\|_{H^{\alpha+1}}^2+\|u_R^\varepsilon\|_{H^{\alpha+1}}^2).$$

Proof. We first consider $\alpha=0$. Multiplying (3.1a) by ε and taking L^2-norm yield

$$\begin{aligned}\|\varepsilon\partial_t n_R^\varepsilon\|^2\le&\|(1-u)\partial_x n_R^\varepsilon\|^2+\|n\partial_x u_R^\varepsilon\|^2+\varepsilon^2\|\partial_x\tilde{u}n_R^\varepsilon\|^2+\varepsilon^2\|\partial_x\tilde{n}u_R^\varepsilon\|^2+\varepsilon^4\|\mathcal{R}_1\|^2\\ \le&C(\|\partial_x n_R^\varepsilon\|^2+\|\partial_x u_R^\varepsilon\|^2)+C\varepsilon^2(1+\|n_R^\varepsilon\|^2+\|u_R^\varepsilon\|^2)\\ \le&C(1+\|n_R^\varepsilon\|_{H^1}^2+\|u_R^\varepsilon\|_{H^1}^2),\end{aligned}\tag{3.4}$$

thanks to (3.3).

When $\alpha=1$, we take ∂_x of (3.1a), and then take L^2-norm to yield

$$\|\varepsilon\partial_{tx}n_R^\varepsilon\|^2\le C(1+\|n_R^\varepsilon\|_{H^2}^2+\|u_R^\varepsilon\|_{H^2}^2).\tag{3.5}$$

Adding (3.4) and (3.5) together, we complete the proof. ∎

PROPOSITION 3.3. *Let $\sigma=0,1,2$ and $(n_R^\varepsilon,u_R^\varepsilon)$ be a solution to the equation* (3.1). *Then the following inequality holds*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int|\partial_x^\sigma u_R^\varepsilon|^2dx+\frac{1}{2}\frac{d}{dt}\int\frac{1+\varepsilon a_{21}(\varepsilon n_R^\varepsilon)}{n}|\partial_x^\sigma n_R^\varepsilon|^2dx\\ \le&C_1(1+\varepsilon\|n_R^\varepsilon\|_{H^2}+\varepsilon\|u_R^\varepsilon\|_{H^2})(\|n_R^\varepsilon\|_{H^\sigma}^2+\|u_R^\varepsilon\|_{H^\sigma}^2).\end{aligned}\tag{3.6}$$

Proof. Taking ∂_x^σ of (3.1b), and then taking inner product of the resultant and $\partial_x^\sigma u_R^\varepsilon$, we have by integrating by parts

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int|\partial_x^\sigma u_R^\varepsilon|^2=&\int\partial_x^\sigma(\frac{1-u}{\varepsilon}\partial_x u_R^\varepsilon)\partial_x^\sigma u_R^\varepsilon-\int\partial_x^\sigma(\frac{1+\varepsilon a_{21}(\varepsilon n_R^\varepsilon)}{\varepsilon}\partial_x n_R^\varepsilon)\partial_x^\sigma u_R^\varepsilon\\ &-\int\partial_x^\sigma(a_{22}(\varepsilon n_R^\varepsilon)n_R^\varepsilon)\partial_x^\sigma u_R^\varepsilon-\int\partial_x^\sigma(\partial_x\tilde{u}u_R^\varepsilon)\partial_x^\sigma u_R^\varepsilon-\varepsilon\int\partial_x^\sigma\mathcal{R}_2\partial_x^\sigma u_R^\varepsilon.\end{aligned}\tag{3.7}$$

We denote the first to the fifth terms on the LHS as I_i, for $i=1,\dots,5$.

For the term I_1, we have by integrating by parts

$$\begin{aligned}I_1=&\int\frac{1}{\varepsilon}\partial_x^{\sigma+1}u_R^\varepsilon\partial_x^\sigma u_R^\varepsilon-\int\frac{u}{\varepsilon}\partial_x^{\sigma+1}u_R^\varepsilon\partial_x^\sigma u_R^\varepsilon-\sum_{1\le\beta\le\sigma}C_\sigma^\beta\int\partial_x^\beta(\tilde{u}+\varepsilon u_R^\varepsilon)\partial_x^{\sigma-\beta+1}u_R^\varepsilon\partial_x^\sigma u_R^\varepsilon\\ =&\frac{1}{2}\int\partial_x(\tilde{u}+\varepsilon u_R^\varepsilon)|\partial_x^\sigma u_R^\varepsilon|^2-\sum_{1\le\beta\le\sigma}C_\sigma^\beta\int\partial_x^\beta(\tilde{u}+\varepsilon u_R^\varepsilon)\partial_x^{\sigma-\beta+1}u_R^\varepsilon\partial_x^\sigma u_R^\varepsilon\\ \le&C\|u_R^\varepsilon\|_{H^\sigma}^2+C\varepsilon\|\partial_x u_R^\varepsilon\|_{L^\infty}\|u_R^\varepsilon\|_{H^\sigma}^2.\end{aligned}$$

When $\sigma=0$, there is no such summation term.

For the second term I_2, we have by chain rule

$$I_2=-\int\frac{1+\varepsilon a_{21}(\varepsilon n_R^\varepsilon)}{\varepsilon}\partial_x^{\sigma+1}n_R^\varepsilon\partial_x^\sigma u_R^\varepsilon-\sum_{1\le\beta\le\sigma}\int\partial_x^\beta a_{21}(\varepsilon n_R^\varepsilon)\partial_x^{\sigma-\beta+1}n_R^\varepsilon\partial_x^\sigma u_R^\varepsilon$$

$$=: \mathcal{B} + I_{21}. \tag{3.8}$$

When $\sigma = 0$, $I_{21} = 0$. When $\beta = 1$, by Sobolev embedding and Lemma A.1, we have

$$\begin{aligned} I_{21} &\leq C \|\partial_x a_{21}\|_{L^\infty} \|\partial_x^\sigma n_R^\varepsilon\| \|\partial_x^\sigma u_R^\varepsilon\| \\ &\leq C_1(\varepsilon \|n_R^\varepsilon\|_{H^2})(\|n_R^\varepsilon\|_{H^\sigma}^2 + \|u_R^\varepsilon\|_{H^\sigma}^2). \end{aligned}$$

Similarly, when $\beta = 2$ and $\sigma = 2$, we have

$$\begin{aligned} I_{21} &\leq C \|\partial_{xx} a_{21}\|_{L^2} \|\partial_x n_R^\varepsilon\|_{L^\infty} \|\partial_x^\sigma u_R^\varepsilon\| \\ &\leq C_1(\varepsilon \|n_R^\varepsilon\|_{H^2})(\|n_R^\varepsilon\|_{H^\sigma}^2 + \|u_R^\varepsilon\|_{H^\sigma}^2). \end{aligned}$$

Therefore, we have

$$I_2 \leq C_1(\varepsilon \|n_R^\varepsilon\|_{H^2})(\|n_R^\varepsilon\|_{H^\sigma}^2 + \|u_R^\varepsilon\|_{H^\sigma}^2) + \mathcal{B}, \tag{3.9}$$

where $\mathcal{B}$ defined in (3.8) will be estimated together with J_3 in (3.10) in the following.

The estimate of I_3 is similar to that of I_{21} in (3.8). We have

$$I_3 \leq C(\varepsilon \|\partial_x n_R^\varepsilon\|_{L^\infty}) \|n_R^\varepsilon\|_{H^\sigma} \|u_R^\varepsilon\|_{H^\sigma}.$$

The term I_4 is bounded similarly,

$$I_4 \leq C \|u_R^\varepsilon\|_{H^\sigma}^2.$$

For I_5, by Lemma A.2, we have

$$\begin{aligned} I_5 &\leq C(\varepsilon \partial_x n_R^\varepsilon\|_{H^2}) \|u_R^\varepsilon\|_{H^\sigma} \\ &\leq C(\varepsilon \partial_x n_R^\varepsilon\|_{H^2})(1 + \|u_R^\varepsilon\|_{H^\sigma}^2). \end{aligned}$$

To cope with the term $\mathcal{B}$ and to get an estimate for n_R^ε, we resort to the equation (3.1a). By taking ∂_x^σ of (3.1a), and then taking inner product with $n^{-1}(1 + \varepsilon a_{21}(\varepsilon n_R^\varepsilon)) \partial_x^\sigma n_R^\varepsilon$, we obtain

$$\begin{aligned} &\int \frac{1 + \varepsilon a_{21}(\varepsilon n_R^\varepsilon)}{n} \partial_t \partial_x^\sigma n_R^\varepsilon \partial_x^\sigma n_R^\varepsilon - \int \partial_x^\sigma \left(\frac{(1-u)}{\varepsilon} \partial_x n_R^\varepsilon \right) \frac{1 + \varepsilon a_{21}(\varepsilon n_R^\varepsilon)}{n} \partial_x^\sigma n_R^\varepsilon \\ &+ \int \frac{1 + \varepsilon a_{21}(\varepsilon n_R^\varepsilon)}{\varepsilon} \partial_x^{\sigma+1} u_R^\varepsilon \partial_x^\sigma n_R^\varepsilon + \sum_{1 \leq \beta \leq \sigma} \int \partial_x^\beta (\frac{n}{\varepsilon}) \partial_x^{\sigma - \beta + 1} u_R^\varepsilon \frac{1 + \varepsilon a_{21}(\varepsilon n_R^\varepsilon)}{n} \partial_x^\sigma n_R^\varepsilon dx \\ &+ \int \frac{1 + \varepsilon a_{21}(\varepsilon n_R^\varepsilon)}{n} \partial_x^\sigma (\partial_x \tilde{n} u_R^\varepsilon) \partial_x^\sigma n_R^\varepsilon + \int \frac{1 + \varepsilon a_{21}(\varepsilon n_R^\varepsilon)}{n} \partial_x^\sigma (\partial_x \tilde{u} n_R^\varepsilon) \partial_x^\sigma n_R^\varepsilon \\ &+ \int \frac{1 + \varepsilon a_{21}(\varepsilon n_R^\varepsilon)}{n} \varepsilon \partial_x^\sigma \mathcal{R}_1 \partial_x^\sigma n_R^\varepsilon =: \sum_{i=1}^{7} J_i = 0. \end{aligned} \tag{3.10}$$

In the following, we estimate them term by term.

For the term J_1, we have by integrating by parts w.r.t. time t

$$J_1 = \frac{1}{2} \frac{d}{dt} \int \frac{1 + \varepsilon a_{21}(\varepsilon n_R^\varepsilon)}{n} |\partial_x^\sigma n_R^\varepsilon|^2 dx - \frac{1}{2} \int \partial_t (\frac{1 + \varepsilon a_{21}(\varepsilon n_R^\varepsilon)}{n}) |\partial_x^\sigma n_R^\varepsilon|^2 dx. \tag{3.11}$$

By direct computation,

$$\partial_t(\frac{1+\varepsilon a_{21}(\varepsilon n_R^\varepsilon)}{n}) = \frac{\varepsilon(\partial_t n^{(i)}\partial_{n^{(i)}}a_{21}+\varepsilon\partial_t n_R^\varepsilon\partial_{n_R^\varepsilon}a_{21})}{n} - \frac{(1+\varepsilon a_{21}(\varepsilon n_R^\varepsilon))(\varepsilon\partial_t\tilde{n}+\varepsilon^2\partial_t n_R^\varepsilon)}{n^2},$$

which yields from Remark A.1 that

$$\|\partial_t(\frac{1+\varepsilon a_{21}(\varepsilon n_R^\varepsilon)}{n})\|_{L^\infty} \le \varepsilon C(\varepsilon\|n_R^\varepsilon\|_{L^\infty})(1+\varepsilon\|\partial_t n_R^\varepsilon\|_{L^\infty}).$$

By Lemma 3.2, Hölder's inequality, and the Sobolev embedding theorem, the second term on the RHS of (3.11) can be bounded by

$$\begin{aligned}\left|\frac{1}{2}\int\partial_t(\frac{1+\varepsilon a_{21}(\varepsilon n_R^\varepsilon)}{n})|\partial_x^\sigma n_R^\varepsilon|^2dx\right| &\le C(\varepsilon\|\partial_{n_R^\varepsilon}\|_{L^\infty})(1+\varepsilon\|\varepsilon\partial_t n_R^\varepsilon\|_{L^\infty})\|n_R^\varepsilon\|_{H^\sigma}^2\\ &\le C_1(1+\varepsilon\|n_R^\varepsilon\|_{H^2}+\varepsilon\|u_R^\varepsilon\|_{H^2})\|n_R^\varepsilon\|_{H^\sigma}^2,\end{aligned}$$

where $C_1 = C_1(\varepsilon\|\partial_{n_R^\varepsilon}\|_{H^2})$. We therefore have from (3.11) that

$$J_1 \ge \frac{1}{2}\frac{d}{dt}\int\frac{1+\varepsilon a_{21}(\varepsilon n_R^\varepsilon)}{n}|\partial_x^\sigma n_R^\varepsilon|^2dx - C_1(1+\varepsilon\|n_R^\varepsilon\|_{H^2}+\varepsilon\|u_R^\varepsilon\|_{H^2})\|n_R^\varepsilon\|_{H^\sigma}^2. \tag{3.12}$$

For J_2 in (3.10), we have

$$\begin{aligned}J_2 = &-\int(\frac{(1-u)(1+\varepsilon a_{21}(\varepsilon n_R^\varepsilon))}{\varepsilon n})\partial_x^{\sigma+1}n_R^\varepsilon\partial_x^\sigma n_R^\varepsilon\\ &-\sum_{1\le\beta\le\sigma}C_\sigma^\beta\int\partial_x^\beta(\frac{(1-u)}{\varepsilon})\partial_x^{\sigma-\beta+1}n_R^\varepsilon\frac{1+\varepsilon a_{21}(\varepsilon n_R^\varepsilon)}{n}\partial_x^\sigma n_R^\varepsilon\\ =:& J_{21}+J_{22}.\end{aligned} \tag{3.13}$$

For J_{21}, by integrating by parts, we have

$$J_{21} = -\frac{1}{2}\int\partial_x(\frac{(1-u)(1+\varepsilon a_{21}(\varepsilon n_R^\varepsilon))}{\varepsilon n})|\partial_x^\sigma n_R^\varepsilon|^2.$$

By (3.3), there exists ε_0 such that when $\varepsilon<\varepsilon_0$

$$\|\partial_x(\frac{(1-u)(1+\varepsilon a_{21}(\varepsilon n_R^\varepsilon))}{\varepsilon n})\|_{L^\infty} \le C\|\partial_x a_{21}\|_{L^\infty}+\varepsilon(\|\partial_x n_R^\varepsilon\|_{L^\infty}+\|\partial_x u_R^\varepsilon\|_{L^\infty}).$$

This yields from Lemma A.1 that

$$\begin{aligned}J_{21} &\le C(\|\partial_x a_{21}\|_{L^\infty}+\varepsilon\|\partial_x n_R^\varepsilon\|_{L^\infty}+\varepsilon\|\partial_x u_R^\varepsilon\|_{L^\infty})\|n_R^\varepsilon\|_{H^\sigma}^2\\ &\le C(\varepsilon\|\partial_x n_R^\varepsilon\|_{L^\infty})\|n_R^\varepsilon\|_{H^\sigma}^2+C\varepsilon\|\partial_x u_R^\varepsilon\|_{L^\infty}\|n_R^\varepsilon\|_{H^\sigma}^2.\end{aligned} \tag{3.14}$$

Similarly estimates yield

$$J_{22} \le C(\varepsilon\|\partial_x n_R^\varepsilon\|_{L^\infty})(\|n_R^\varepsilon\|_{H^\sigma}^2+\|u_R^\varepsilon\|_{H^\sigma}^2)+C\varepsilon\|\partial_x u_R^\varepsilon\|_{L^\infty}(\|n_R^\varepsilon\|_{H^\sigma}^2+\|u_R^\varepsilon\|_{H^\sigma}^2). \tag{3.15}$$

Therefore

$$J_2 \le C(\varepsilon\|\partial_x n_R^\varepsilon\|_{L^\infty})(\|n_R^\varepsilon\|_{H^\sigma}^2+\|u_R^\varepsilon\|_{H^\sigma}^2)+C\varepsilon\|\partial_x u_R^\varepsilon\|_{L^\infty}(\|n_R^\varepsilon\|_{H^\sigma}^2+\|u_R^\varepsilon\|_{H^\sigma}^2).$$

For the last four terms, we have by Hölder's inequality and Lemma A.1 that

$$\begin{aligned} J_4 \le & C(\varepsilon\|n_R^\varepsilon\|_{H^2})(1+\varepsilon\|\partial_x n_R^\varepsilon\|_{L^\infty}+\varepsilon\|\partial_x u_R^\varepsilon\|_{L^\infty})(\|n_R^\varepsilon\|_{H^\sigma}^2+\|u_R^\varepsilon\|_{H^\sigma}^2) \\ \le & C_1(1+\varepsilon\|n_R^\varepsilon\|_{H^2}+\varepsilon\|u_R^\varepsilon\|_{H^2})(\|n_R^\varepsilon\|_{H^\sigma}^2+\|u_R^\varepsilon\|_{H^\sigma}^2), \end{aligned}$$

and

$$J_5+J_6+J_7 \le C_1(\varepsilon\|n_R^\varepsilon\|_{H^2})(1+\|n_R^\varepsilon\|_{H^\sigma}^2+\|u_R^\varepsilon\|_{H^\sigma}^2),$$

where $C_1 = C_1(\varepsilon\|n_R^\varepsilon\|_{H^2})$.

Adding (3.7) and (3.10) together, and using the above estimates for I_i and J_i, we have

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\int |\partial_x^\sigma u_R^\varepsilon|^2 dx + \frac{1}{2}\frac{d}{dt}\int \frac{1+\varepsilon a_{21}(\varepsilon n_R^\varepsilon)}{n}|\partial_x^\sigma n_R^\varepsilon|^2 dx \\ \le & C_1(1+\varepsilon\|n_R^\varepsilon\|_{H^2}+\varepsilon\|u_R^\varepsilon\|_{H^2})(\|n_R^\varepsilon\|_{H^\sigma}^2+\|u_R^\varepsilon\|_{H^\sigma}^2)+\mathcal{B}-J_3, \end{aligned} \tag{3.16}$$

where $\mathcal{B}$ is given in (3.8) and J_3 is given in (3.10). Therefore, we need only to estimate the following term

$$\mathcal{G} := \mathcal{B} - J_3 = -\int \frac{1+\varepsilon a_{21}(\varepsilon n_R^\varepsilon)}{\varepsilon}\partial_x(\partial_x^\sigma n_R^\varepsilon \partial_x^\sigma u_R^\varepsilon). \tag{3.17}$$

By integration by parts, we have

$$\begin{aligned} |\mathcal{G}| = & \left| \int \partial_x a_{21}(\varepsilon n_R^\varepsilon)(\partial_x^\sigma n_R^\varepsilon \partial_x^\sigma u_R^\varepsilon) \right| \\ \le & C(\varepsilon\|n_R^\varepsilon\|_{H^2})(\|n_R^\varepsilon\|_{H^\sigma}^2+\|u_R^\varepsilon\|_{H^\sigma}^2), \end{aligned}$$

thanks to Lemma A.1 in Appendix A. Plugging this into (3.16), we get the Grönwall's inequality (3.6). Completing the proof. □

Now, we prove Theorem 2.2. From Lemma A.1 and (3.2), there exists ε_0 such that $\varepsilon a_{21}(\varepsilon n_R^\varepsilon)| \le 1/2$ when $\varepsilon < \varepsilon_0$. This together with (3.3) implies that

$$\frac{1}{3} \le \frac{1+\varepsilon a_{21}(\varepsilon n_R^\varepsilon)}{n} \le 3.$$

From Proposition 3.3, by adding the inequalities (3.6) for $\sigma = 0,1,2$ together, and then integrating in time over $(0,t)$, we obtain

$$\begin{aligned} \|u_R^\varepsilon(t)\|_{H^2}^2 + \|n_R^\varepsilon(t)\|_{H^2}^2 \le & 3(\|u_R^\varepsilon(0)\|_{H^2}^2 + \|n_R^\varepsilon(0)\|_{H^2}^2) \\ & + \int_0^t C_0(\varepsilon\|n_R^\varepsilon\|_{H^2}, \varepsilon\|u_R^\varepsilon\|_{H^2})(\|n_R^\varepsilon\|_{H^2}^2+\|u_R^\varepsilon\|_{H^2}^2) d\tau, \end{aligned}$$

for some $C_0 = C_0(\varepsilon\|n_R^\varepsilon\|_{H^2}, \varepsilon\|u_R^\varepsilon\|_{H^2})$ depends on $\varepsilon\|n_R^\varepsilon\|_{H^2}$ and $\varepsilon\|u_R^\varepsilon\|_{H^2}$.

Let $\hat{C}_0 = \sup_{0\le r,s\le 1} C_0(r,s)$ and $C_2 > 3\sup_{\varepsilon<1}(\|n_R^\varepsilon(0)\|_{H^2} + \|u_R^\varepsilon(0)\|_{H^2})$. Recall that T is the maximal existence time in Proposition 2.1. Let $T' < T$ be arbitrary and $\tilde{C}$ be sufficiently large such that $\tilde{C} > C_2 e^{\hat{C}_0 T'}$. Then there exists $\varepsilon_{T'}$ such that $\varepsilon\tilde{C} \le 1$ for all $\varepsilon < \varepsilon_{T'}$, and we have

$$\sup_{0\le t\le T'} (\|n_R^\varepsilon(t)\|_{H^2} + \|u_R^\varepsilon(t)\|_{H^2}) \le C_2 e^{\hat{C}_0 T'} < \tilde{C}.$$

It is then standard to obtain uniform estimates independent of ε by the continuity method. Thus completing the proof of Theorem 2.2.

Appendix A. In this section, we derive the remainder system (2.15) for $(n_R^\varepsilon, u_R^\varepsilon)$. We first collect the equations that $(n^{(k)}, u^{(k)})(k=1,2,3)$ satisfy:

$$\begin{aligned}
&(n^{(1)}, u^{(1)}) \text{ satisfies (2.6) and (2.8)}\\
&(n^{(2)}, u^{(2)}) \text{ satisfies (2.9) and (2.11)}\\
&(n^{(3)}, u^{(3)}) \text{ satisfies (2.12) and (2.13).}
\end{aligned}$$

We only consider the derivation of the remainder equation of (2.2b). Inserting the expansion (2.14) into the equation and subtracting (2.7) multiplied by ε^2 and (2.10) multiplied by ε^3, we will have the remainder equation (2.15). For clarity, we consider Taylor expansion of the pressure term $A\gamma n^{\gamma-2} n_x$,

$$\begin{aligned}
P :=& A\gamma(1+\varepsilon\tilde{n}+\varepsilon^2 n_R^\varepsilon)^{\gamma-2}(\varepsilon\tilde{n}+\varepsilon^2 n_R^\varepsilon)_x\\
=& \Big(c_{\gamma 1} + c_{\gamma 2}(\varepsilon\tilde{n}+\varepsilon^2 n_R^\varepsilon) + c_{\gamma 3}(\varepsilon\tilde{n}+\varepsilon^2 n_R^\varepsilon)^2 + c_{\gamma 4}(\varepsilon\tilde{n}+\varepsilon^2 n_R^\varepsilon)^3\Big)(\varepsilon\tilde{n}+\varepsilon^2 n_R^\varepsilon)_x\\
&+ \left(\int_0^1 c_{\gamma 5}(1+\theta(\varepsilon\tilde{n}+\varepsilon^2 n_R^\varepsilon))^{\gamma-4}(1-\theta)^3(\varepsilon\tilde{n}+\varepsilon^2 n_R^\varepsilon)^4 d\theta\right)(\varepsilon\tilde{n}+\varepsilon^2 n_R^\varepsilon)_x,
\end{aligned} \tag{A.1}$$

where $c_{\gamma 1} = A\gamma$ and $c_{\gamma k} = A\gamma \prod_{i=2}^k (\gamma - i)/(k-1)!$ for $k=2,3,4,5$. In a power series of ε, we have

$$\begin{aligned}
P =& \varepsilon^1\left(c_{\gamma 1}\partial_x n^{(1)}\right) + \varepsilon^2(c_{\gamma 1}\partial_x n_R^\varepsilon + \cdots) + \varepsilon^3\left(c_{\gamma 2} n^{(1)}\partial_x n_R^\varepsilon + c_{\gamma 2}\partial_x n^{(1)} n_R^\varepsilon + \cdots\right)\\
&+ \varepsilon^4\left(\{c_{\gamma 2}(n^{(2)} + n_R^\varepsilon) + c_{\gamma 3}(n^{(1)})^2\}\partial_x n_R^\varepsilon + \{c_{\gamma 2} n^{(2)} + 2c_{\gamma 3}(n^{(1)})^2\} n_R^\varepsilon + \cdots\right)\\
&+ \cdots + I_R(\varepsilon^2 n_R^\varepsilon)(\varepsilon\partial_x\tilde{n}) + I_R(\varepsilon^2 n_R^\varepsilon)(\varepsilon^2\partial_x n_R^\varepsilon),
\end{aligned} \tag{A.2}$$

where $I_R(\varepsilon^2 n_R^\varepsilon)$ denotes the integral term in (A.1). Here, we only write out the terms involving n_R^ε and $\partial_x n_R^\varepsilon$, since these terms do not cancel each other out. Similarly, we can write out the other terms in a power series of ε. Adding them together, we get a power series of ε, whose coefficients depend on $\tilde{n}$, $\tilde{u}$, n_R^ε, and u_R^ε. This series is nothing but a rearrangement of (2.2b). Subtracting (2.7b) multiplied by ε^2 and (2.10b) multiplied by ε^3 from the power series of (2.2b), and then dividing the resultant by ε^3, we obtain the remainder

$$\begin{aligned}
&\frac{1}{\varepsilon}\{c_{\gamma 1}\partial_x n_R^\varepsilon\} + \{c_{\gamma 2} n^{(1)}\partial_x n_R^\varepsilon + c_{\gamma 2}\partial_x n^{(1)} n_R^\varepsilon\} + \varepsilon\{(c_{\gamma 2}(n^{(2)} + n_R^\varepsilon) + c_{\gamma 3}(n^{(1)})^2)\partial_x n_R^\varepsilon\\
&\quad + (c_{\gamma 2} n^{(2)} + c_{\gamma 3}(n^{(1)})^2) n_R^\varepsilon\} + \cdots + \frac{1}{\varepsilon^3} I_R(\varepsilon^2 n_R^\varepsilon)(\varepsilon\partial_x\tilde{n}) + \frac{1}{\varepsilon^3} I_R(\varepsilon^2 n_R^\varepsilon)(\varepsilon^2\partial_x n_R^\varepsilon),
\end{aligned} \tag{A.3}$$

where $'\cdots'$ only consists of finitely many terms. In particular, $\varepsilon^1\left(c_{\gamma 1}\partial_x n^{(1)}\right)$ in (A.2) cancels with the term $-\varepsilon^1(c_0\partial_x u^{(1)})$ from $-c_0\partial_x u$ in (2.2b). Rearranging, we obtain the remainder equation (2.15c) for (2.2b)

$$\begin{aligned}
&\partial_t u_R^\varepsilon - \frac{c_0}{\varepsilon}\partial_x u_R^\varepsilon + \frac{A\gamma}{\varepsilon}\partial_x n_R^\varepsilon + (\tilde{u} + \varepsilon u_R^\varepsilon)\partial_x u_R^\varepsilon + a_{21}(\varepsilon n_R^\varepsilon)\partial_x n_R^\varepsilon\\
&\quad + a_{22}(\varepsilon n_R^\varepsilon) n_R^\varepsilon + \partial_x\tilde{u} u_R^\varepsilon = \frac{\mu}{n}\partial_{xx} u_R^\varepsilon - \varepsilon\mu b n_R^\varepsilon - \varepsilon\mathcal{R}_2,
\end{aligned} \tag{A.4}$$

where a_{21}, a_{22} depend on $n^{(1)}$, $n^{(2)}$, $n^{(3)}$, their spatial derivatives, and $\varepsilon n_R^\varepsilon$; and

$$\begin{aligned} b =& \frac{1}{n}(\partial_{xx}u^{(1)} + \varepsilon\partial_{xx}u^{(2)} - \varepsilon\partial_{xx}u^{(1)}n^{(1)}), \\ \mathcal{R}_2 =& \partial_t u^{(3)} + \sum_{1\le i,j\le 3; i+j\ge 4} \varepsilon^{i+j-4}u^{(i)}\partial_x u^{(j)} + a_{23}(n^{(1)},n^{(2)},n^{(3)}) + a_{24}(\varepsilon n_R^\varepsilon). \end{aligned} \tag{A.5}$$

We also remark that the remainder of $\frac{1}{\varepsilon^3}I_R(\varepsilon^2 n_R^\varepsilon)(\varepsilon\partial_x\tilde{n})$ in (A.3) goes into the term $a_{22}(\varepsilon n_R^\varepsilon)n_R^\varepsilon$ and $a_{24}(\varepsilon n_R^\varepsilon)$ in (A.5) and $\frac{1}{\varepsilon^3}I_R(\varepsilon^2 n_R^\varepsilon)(\varepsilon^2\partial_x n_R^\varepsilon)$ goes into $a_{21}(\varepsilon n_R^\varepsilon)\partial_x n_R^\varepsilon$. After dividing by ε^3, the term $I_R(\varepsilon^2 n_R^\varepsilon)(\varepsilon\partial_x\tilde{n})$ depends on n_R^ε in the form of $a_{22}(\varepsilon n_R^\varepsilon)n_R^\varepsilon$ and $I_R(\varepsilon^2 n_R^\varepsilon)(\varepsilon^2\partial_x n_R^\varepsilon)$ depends on n_R^ε in the form of $a_{21}(\varepsilon n_R^\varepsilon)n_R^\varepsilon$. The dependence of a_{21} and a_{22} on $\varepsilon n_R^\varepsilon$ is important to get global in time estimates of $(u_R^\varepsilon, n_R^\varepsilon)$ uniformly in ε.

The derivation of (2.15a) is simpler and hence omitted.

Remark A.1. From the derivation of the remainder equation (A.4), we know that for any integers $\alpha, \beta \ge 0$, there exists constant C such that

$$\|\partial_{n^{(i)}}^\alpha \partial_{n_R^\varepsilon}^\beta a_{2r}\|_{L^\infty} \le \varepsilon^\beta C(\varepsilon\|n_R^\varepsilon\|_{L^\infty}).$$

Lemma A.1. *Let $n^{(1)}, n^{(2)}$ and $n^{(3)}$ be given and smooth and $n_R^\varepsilon \in H^2$. Then there exist some $\varepsilon_0 > 0$ and constant $C_1 = C_1(\varepsilon\|n_R^\varepsilon\|_{H^2})$ such that*

$$\|a_{21}\|_{H^2}, \|a_{22}\|_{H^2} \le C_1(\varepsilon\|n_R^\varepsilon\|_{H^2}),$$

for every $\varepsilon < \varepsilon_0$.

Here, the constant C_1 also depends on $n^{(i)}$ for $i = 1,2,3$, which is omitted in the parentheses to stress the dependence on n_R^ε.

Proof. By the Sobolev embedding theorem, we have $\|n_R^\varepsilon\|_{L^\infty} \le C\|n_R^\varepsilon\|_{H^1}$. Therefore, there exists constant $\varepsilon_0 > 0$ such that $\varepsilon\|\tilde{n}\|_{L^\infty} + \varepsilon^2\|n_R^\varepsilon\|_{L^\infty} < 1/2$, for any $0 < \varepsilon < \varepsilon_0$. Hence $1/2 < n < 3/2$ for every $0 < \varepsilon < \varepsilon_0$.

When $\alpha = 0$, from the definition of a_{2r} ($r = 1,2$), we have

$$|a_{2r}| \le C(|n^{(i)}| + |\varepsilon n_R^\varepsilon|),$$

where C depends on $n^{(i)}$ and $\varepsilon n_R^\varepsilon$ for $i = 1,2,3$. By Sobolev embedding, since H^1 is an algebra, we have

$$\begin{aligned} \|a_{2r}\|_{L^2} \le& C(\|n^{(i)}\|_{L^\infty}, \varepsilon\|n_R^\varepsilon\|_{L^\infty})(\|n^{(i)}\|_{L^2} + \|\varepsilon n_R^\varepsilon\|_{L^2}) \\ \le& C(\varepsilon\|n_R^\varepsilon\|_{H^1}). \end{aligned} \tag{A.6}$$

When $\alpha = 1$, we have

$$|\partial_x a_{2r}| \le |\partial_{n^{(i)}} a_{2r}||\partial_x n^{(i)}| + |\partial_{n_R^\varepsilon} a_{2r}||\varepsilon\partial_x n_R^\varepsilon|,$$

where $\partial_{n^{(i)}} a_{2r}$ and $\partial_{n_R^\varepsilon} a_{2r}$ depend on $\varepsilon n_R^\varepsilon$. By computing its L^2-norm, we have

$$\begin{aligned} \|\partial_x a_{2r}\|_{L^2} \le& \|\partial_{n^{(i)}} a_{2r}\|_{L^\infty}\|\partial_x n^{(i)}\|_{L^2} + \|\partial_{n_R^\varepsilon} a_{2r}\|_{L^\infty}\|\varepsilon\partial_x n_R^\varepsilon\|_{L^2} \\ \le& C(\varepsilon\|n_R^\varepsilon\|_{H^1}), \end{aligned} \tag{A.7}$$

thanks to Remark A.1 and the fact that H^1 is an algebra.

When $\alpha = 2$, we have similarly

$$\|\partial_{xx} a_{2r}\|_{L^2} \le C(\varepsilon \|n_R^\varepsilon\|_{H^2}). \tag{A.8}$$

Combining (A.6), (A.7), and (A.8) completes the proof of Lemma A.1. ∎

Lemma A.2. *Let $n^{(1)}i$, $n^{(2)}$ and $n^{(3)}$ be given and smooth with $n_R^\varepsilon \in H^2$. There exist $\varepsilon_0 > 0$ and constant $C_1 = C_1(\|n^{(i)}\|_{H^{\bar{s}}}, \varepsilon \|n_R^\varepsilon\|_{H^2})$ such that*

$$\|\mathcal{R}_2\|_{H^\alpha} \le C(\varepsilon \|n_R^\varepsilon\|_{H^2}),$$

for every $\varepsilon < \varepsilon_1$.

Proof. The proof is similar to that of Lemma A.1, and hence is omitted. ∎

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REFERENCES

[1] J.M. Burgers, *Application of model system to illustrate some points of the statistical theory of free turbulence*, Nederl. Akad. Wetensch., Proc. 43, 2-12, 1940.

[2] Y. Guo and X. Pu, *KdV limit of the Euler–Poisson system*, Arch. Rat. Mech. Anal., 211, 673-710, 2014.

[3] T. Kato, *Quasi-linear Equations of Evolution, with Applications to Partial Differential Equations*, Lecture Notes in Math., Springer-Verlag, Berlin, 448, 1975.

[4] A. Majda, *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables*, Appl. Math. Sci., Springer, New York-Berlin, 53, 1984.

[5] X. Pu, *Dispersive limit of the Euler–Poisson system in higher dimensions*, SIAM J. Math. Anal., 45(2), 834-878, 2013.

[6] C. Su and C. Gardner, *Korteweg–de Vries equation and generalizations. III. Derivation of the Korteweg–de Vries equation and Burgers Equation*, J. Math. Phys., 10(3), 536-539, 1969.