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ATTRACTOR OF THE QUANTUM ZAKHAROV SYSTEM ON AN UNBOUNDED DOMAIN*

YANFENG GUO†, JINGJUN ZHANG‡, AND CHUNXIAO GUO§

Abstract. In this paper, we study the attractor of the quantum Zakharov system on unbounded domain $\mathbb{R}^d$ $(d=1,2,3)$. We first prove the existence and uniqueness of the solution by the standard energy method. Then, by making use of the particular characters of the quantum Zakharov system and the special decomposition of the solution operator, we obtain the existence of an attractor for this system.

Key words. Quantum Zakharov system, existence and uniqueness, attractor, decomposition.

AMS subject classifications. 35Q35, 60H15, 76A05.

1. Introduction

As is well known, one of the most important models in plasma physics is given by the classical Zakharov system [31] which describes the interaction between high-frequency Langmuir waves and low-frequency ion-acoustic waves. Recently, when taking quantum effects into account, a generation of the Zakharov system was derived in [19, 20], which is named the modified Zakharov system or quantum Zakharov system. Indeed, in recent years, many physicists have been increasingly interested in the application for a model taking both collective charged particle effects and quantum phenomena into account. The subject of the present work is to investigate the dynamical properties of the quantum Zakharov system. For the sake of convenience, we only study the scalar form for the system in this paper. In a dimensionless form, the quantum Zakharov system can be written as

$$iE_t+i\gamma E+\Delta E-h^2\Delta^2E=nE+f, \tag{1.1}$$

$$n_{tt}+\alpha n_t+\beta n-\Delta n+h^2\Delta^2 n=\Delta|E|^2+g, \tag{1.2}$$

where $E:\mathbb{R}\times\mathbb{R}^d\to\mathbb{C}$ denotes the envelope of the high-frequency electric field, $n:\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ denotes the plasma density measured from its equilibrium value, and $\alpha,\beta,\gamma>0$. In (1.1)–(1.2), f and g are the external forces, and the coefficient h measures the influence of quantum effects. Usually, h is an extremely small quantity, see [19].

It is well known that the classical Zakharov system has been quite extensively studied theoretically and numerically by many mathematicians and physicists in the past decades. We refer to references [1, 5, 10, 11, 13, 24, 28, 30] for the local or global well-posedness results and [4, 7, 9, 14, 23] for the existence of attractors for the classical and dissipative Zakharov system.

System (1.1)–(1.2) describes the nonlinear interaction between the quantum Langmuir and quantum ion-acoustic waves. The quantum Zakharov system plays an especially important role in intense laser plasmas and in dense astrophysical plasmas, mainly

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†Corresponding author, School of Science, Guangxi University of Science and Technology, Liuzhou, Guangxi, 545006, P.R. China (guoyan_feng@163.com).

‡College of Mathematics, Physics and Information Engineering, Jiaxing University, Jiaxing, Zhejiang, 314001, P.R. China (zjj_math@aliyun.com).

§Department of Mathematics, China University of Mining and Technology Beijing, Beijing, 100083, P.R. China (guochunxiao1983@sina.com).

due to the importance of the quantum effects in these subjects. For more comprehensive work on the dynamics of (1.1)–(1.2), see [15, 16, 17, 19, 25, 26, 27, 29] and the references cited therein. Recently, we have obtained the attractor on a bounded domain in $\mathbb{R}^d$ with $d=1,2,3$ for the quantum Zakharov system in [15]. But the results about an attractor on an unbounded domain such as $\mathbb{R}^d$ in case of $d=1,2,3$ for the quantum Zakharov system have not been obtained at present. Since the imbedding theorem is not compact on an unbounded domain, it is very difficult to obtain the compactness of an attractor by using the usual techniques. To fill this gap, the subject of this work is to investigate dynamical behaviors of the quantum Zakharov system in $\mathbb{R}^d$ $(d=1,2,3)$ complemented with initial data

$$E(0,x)=E_0(x), n(0,x)=n_0(x), n_t(0,x)=n_1(x), \quad x\in\mathbb{R}^d. \tag{1.3}$$

The method used here is different from [15].

First, we introduce some standard notation which will be used throughout the paper. We denote the spaces of complex valued functions and real valued functions by the same symbols. For $s\geq 0$, $1\leq p\leq\infty$, we denote $H^{s,p}(\mathbb{R}^d)$ (when $p=2$, we write $H^s(\mathbb{R}^d)$ for short) or $\dot{H}^s(\mathbb{R}^d)$ the usual inhomogeneous or homogeneous Sobolev spaces of order s. The notation $(\cdot,\cdot)$ is the inner product in $L^2(\mathbb{R}^d)$, and $\|\cdot\|_p$ is the norm of $L^p(\mathbb{R}^d)$, especially $\|\cdot\|=\|\cdot\|_2$. Moreover, we often write $\int_{\mathbb{R}^d}\cdot dx=\int\cdot dx$. Also, as in [16] we define the product space V_k as

$$V_k:=(H^{2k-1}(\mathbb{R}^d)\cap\dot{H}^{-1}(\mathbb{R}^d))\times H^{2k+1}(\mathbb{R}^d)\times H^{2k+2}(\mathbb{R}^d),\ k=0,1,2$$

and endow V_k with the natural norm, namely,

$$\|(u_1,u_2,u_3)\|_{V_k}:=\|u_3\|_{H^{2k-1}\cap\dot{H}^{-1}}+\|u_2\|_{H^{2k+1}}+\|u_1\|_{H^{2k+2}}.$$

We note that the imbedding of $H^s(\mathbb{R}^d)$ into $H^{s'}(\mathbb{R}^d)$ $(s>s')$ is not compact. In order to overcome this difficulty, we apply the methods in [22] and in [6] to show the asymptotic smoothness of the semigroup $S(t)$ by using the Kuratowskii α-measure of noncompactness. Then applying the theory of [21], we can prove that the quantum Zakharov system has a maximal attractor in the space V_1 which attracts bounded sets in the topology of V_2.,

In this paper, we shall repeatedly use the Gagliardo–Nirenberg inequality [8]

$$\|D^j u\|_p\leq C\|u\|_q^{1-\lambda}\|D^m u\|_r^{\lambda},\ u\in L^q\cap H^{m,r}(\mathbb{R}^d),$$

where $\frac{1}{p}=\frac{j}{d}+\lambda(\frac{1}{r}-\frac{m}{d})+\frac{1-\lambda}{q}, 1\leq q,r\leq\infty$, j is an integer, and $0\leq j\leq m$, $\frac{j}{m}\leq\lambda\leq 1$. If $m-j-\frac{d}{r}$ is a nonnegative integer, then the inequality holds for $\frac{j}{m}\leq\lambda<1$.

In the next section, by establishing some uniform estimates in the spaces V_0,V_1,V_2, we show the existence and uniqueness of solution for system (1.1)–(1.2). In order to get the compactness of the attractor, we study the decomposition of the solution operator in the third section. In Section 4, the main results of this paper are obtained by the preparation of former sections. The last section gives some further remarks on the attractor for this system. Throughout the paper, C is a generic constant, and the value of C may be different from line to line. In Section 5, some remarks are given for some work in the future.

2. Existence of bounded absorbing sets

Let $m=n_t+\varepsilon n$. Then system (1.1)–(1.2) can be written as

$$iE_t+i\gamma E+\Delta E-h^2\Delta^2 E=nE+f, \tag{2.1}$$

$$n_t+\varepsilon n=m, \tag{2.2}$$

$$m_t+(\alpha-\varepsilon)m+(\beta-\varepsilon(\alpha-\varepsilon))n-\Delta n+h^2\Delta^2 n=\Delta|E|^2+g, \tag{2.3}$$

where $(\beta-\varepsilon(\alpha-\varepsilon))I-\Delta$ is a positive self-adjoint and elliptic operator of order 2 which is a homeomorphism from $H^s(\mathbb{R}^d)$ into $H^{s-2}(\mathbb{R}^d)$, where f and g are known functions only depending on the spatial variables. The initial data of (m,n,E) is

$$(m,n,E)(0,x)=(n_1+\varepsilon n_0,n_0,E_0)(x)=(m_0,n_0,E_0)(x),\quad x\in\mathbb{R}^d. \tag{2.4}$$

LEMMA 2.1. *Let $f\in H^4(\mathbb{R}^d),g\in H^3(\mathbb{R}^d)$. Then*
(1) for any $(m_0,n_0,E_0)(x)\in V_1$, the solution of (2.1)–(2.4) belongs to $L^\infty(\mathbb{R}^+;V_1)$;
(2) for any $(m_0,n_0,E_0)(x)\in V_2$, the solution of (2.1)–(2.4) belongs to $L^\infty(\mathbb{R}^+;V_2)$.

Proof.
(1) Taking the inner product of (2.1) with $2E$ in $\mathbb{R}^d$ and choosing the imaginary part, we have

$$\frac{d}{dt}\|E\|^2+2\gamma\|E\|^2=2\mathrm{Im}\int f\bar{E}dx\le\gamma\|E\|^2+C\|f\|^2.$$

Thus we see

$$\|E\|^2=e^{-\gamma t}\|E_0\|^2+C\|f\|^2\int_0^t e^{-\gamma(t-s)}ds\le M, \tag{2.5}$$

where M is a positive constant independent of t.

Taking the inner product of (2.1) with $2(\gamma E+E_t)$ in $\mathbb{R}^d$, we can obtain

$$\begin{aligned}&\frac{d}{dt}(\|\nabla E\|^2+h^2\|\Delta E\|^2+2\mathrm{Re}\int f\bar{E}dx)+2\gamma(\|\nabla E\|^2+h^2\|\Delta E\|^2+\mathrm{Re}\int f\bar{E}dx)\\&\quad+\int n(|E|^2)_t dx+2\gamma\int n|E|^2dx=0.\end{aligned} \tag{2.6}$$

Taking inner product of (2.3) with $2(-\Delta)^{-1}m$ in $\mathbb{R}^d$ and noticing that $2\int mndx=\frac{d}{dt}\|n\|^2+2\varepsilon\|n\|^2$ and $-2\int m\Delta ndx=\frac{d}{dt}\|\nabla n\|^2+2\varepsilon\|\nabla n\|^2$, we have

$$\begin{aligned}&\frac{d}{dt}(\|m\|_{-1}^2+(\beta-\varepsilon(\alpha-\varepsilon))\|n\|_{-1}^2+\|n\|^2+h^2\|\nabla n\|^2)+2\varepsilon(\beta-\varepsilon(\alpha-\varepsilon))\|n\|_{-1}^2+2\varepsilon\|n\|^2\\&\quad+2((\alpha-\varepsilon)\|m\|_{-1}^2+\varepsilon h^2\|\nabla n\|^2+\int(n_t|E|^2+\varepsilon n|E|^2-g(-\Delta)^{-1}m)dx=0.\end{aligned} \tag{2.7}$$

Now, we set

$$\begin{aligned}H_0(t)=&2\|\nabla E\|^2+2h^2\|\Delta E\|^2+4\mathrm{Re}\int f\bar{E}dx+2\int n|E|^2dx+\|m\|_{-1}^2+\|n\|^2\\&+(\beta-\varepsilon(\alpha-\varepsilon))\|n\|_{-1}^2+h^2\|\nabla n\|^2\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}I_0(t)=&4\gamma(\|\nabla E\|^2+h^2\|\Delta E\|^2+\mathrm{Re}\int f\bar{E}dx)+4\gamma\int n|E|^2dx+2(\alpha-\varepsilon)\|m\|_{-1}^2+2\varepsilon\|n\|^2\\&+2\varepsilon(\beta-\varepsilon(\alpha-\varepsilon))\|n\|_{-1}^2+2\varepsilon h^2\|\nabla n\|^2+2\varepsilon\int n|E|^2dx-\int 2g(-\Delta)^{-1}mdx,\end{aligned} \tag{2.9}$$

then it follows from (2.6) × 2 + (2.7) that

$$\frac{dH_0(t)}{dt}+I_0(t)=0. \tag{2.10}$$

Therefore, there exist a small $\alpha_0>0$ and a positive constant K_0 satisfying

$$\frac{dH_0(t)}{dt}+\alpha_0 H_0(t)\le K_0. \tag{2.11}$$

According to Gronwall's lemma, it can be inferred that

$$H_0(t)\le e^{-\alpha_0 t}H_0(0)+\frac{K_0}{\alpha_0}(1-e^{-\alpha_0 t})\le e^{-\alpha_0 t}H_0(0)+\frac{K_0}{\alpha_0}\le M_0, \tag{2.12}$$

where M_0 is a positive constant independent of time. Then $(m,n,E)\in L^\infty(\mathbb{R}^+;V_0)$.

On the other hand, taking the inner product of (2.1) by $2\Delta^2E_t+2\gamma\Delta^2E$ in $\mathbb{R}^d$ and choosing the real part, we can obtain

$$\begin{aligned}&\frac{d}{dt}(\|\nabla\Delta E\|^2+h^2\|\Delta^2E\|^2+2\mathrm{Re}\int(f\Delta^2\bar{E}+nE\Delta^2\bar{E})dx)+2\gamma(\|\nabla\Delta E\|^2+h^2\|\Delta^2E\|^2)\\&\quad+2\gamma(\mathrm{Re}\int(f\Delta^2\bar{E}+nE\Delta^2\bar{E})dx)-2\mathrm{Re}\int(n_tE\Delta^2\bar{E}+nE_t\Delta^2\bar{E})dx=0.\end{aligned} \tag{2.13}$$

Moreover, multiplying (2.3) by $2m$ in $\mathbb{R}^d$ yields

$$\begin{aligned}&\frac{d}{dt}\|m\|^2+2(\alpha-\varepsilon)\|m\|^2+2(\beta-\varepsilon(\alpha-\varepsilon))\int mndx\\&\quad-2\int m\Delta ndx+2h^2\int m\Delta^2ndx-\int m\Delta|E|^2dx+2\int mgdx=0.\end{aligned}$$

Noticing that $2\int mndx=\frac{d}{dt}\|n\|^2+\varepsilon\|n\|^2$, $-2\int m\Delta ndx=\frac{d}{dt}\|\nabla n\|^2+2\varepsilon\|\nabla n\|^2$ and $2h^2\int m\Delta^2ndx=h^2\frac{d}{dt}\|\Delta n\|^2+2\varepsilon h^2\|\Delta n\|^2$, we have

$$\begin{aligned}&\frac{d}{dt}(\|m\|^2+(\beta-\varepsilon(\alpha-\varepsilon))\|n\|^2+\|\nabla n\|^2+h^2\|\Delta n\|^2-2\int ngdx)+2\varepsilon(\beta-\varepsilon(\alpha-\varepsilon))\|n\|^2\\&\quad+2(\alpha-\varepsilon)\|m\|^2+2\varepsilon\|\nabla n\|^2+2\varepsilon h^2\|\Delta n\|^2-2\varepsilon\int ngdx-2\int m\Delta|E|^2dx=0.\end{aligned} \tag{2.14}$$

Also, multiplying (2.3) by $2\Delta m$, we can get

$$\begin{aligned}&\frac{d}{dt}(\|\nabla m\|^2+(\beta-\varepsilon(\alpha-\varepsilon))\|\nabla n\|^2+\|\Delta n\|^2+h^2\|\nabla\Delta n\|^2+2\int g\Delta ndx)+2(\alpha-\varepsilon)\|\nabla m\|^2\\&\quad+2\varepsilon((\beta-\varepsilon(\alpha-\varepsilon))\|\nabla n\|^2+\|\Delta n\|^2+h^2\|\nabla\Delta n\|^2+\int g\Delta ndx)+2\int m\Delta^2|E|^2dx=0.\end{aligned}$$

Then we have

$$\begin{aligned}&\frac{d}{dt}(\|m\|^2+\|\nabla m\|^2+(\beta-\varepsilon(\alpha-\varepsilon))\|n\|^2+(1+(\beta-\varepsilon(\alpha-\varepsilon)))\|\nabla n\|^2+(h^2+1)\|\Delta n\|^2\\&\quad+h^2\|\nabla\Delta n\|^2+2\int g(\Delta n-n)dx)+2(\alpha-\varepsilon)(\|m\|^2+\|\nabla m\|^2)+2\varepsilon(\beta-\varepsilon(\alpha-\varepsilon))\|n\|^2\\&\quad+2\varepsilon(1+(\beta-\varepsilon(\alpha-\varepsilon)))\|\nabla n\|^2+2\varepsilon(h^2+1)\|\Delta n\|^2+2\varepsilon h^2\|\nabla\Delta n\|^2+2\varepsilon\int g(\Delta n-n)dx)\end{aligned}$$

$$-2\int m\Delta|E|^2dx+2\int m\Delta^2|E|^2dx=0. \tag{2.15}$$

Note that

$$\begin{aligned}2\int n_t\Delta^2|E|^2dx=&4\mathrm{Re}\int n_tE\Delta^2\bar{E}dx+4\int m|\Delta E|^2dx+8\mathrm{Re}\int m\nabla\Delta E\nabla\bar{E}dx\\&-4\varepsilon\int n|\Delta E|^2dx-8\mathrm{Re}\int(\varepsilon n\nabla\Delta E\nabla\bar{E}dx+\nabla m\Delta E\nabla\bar{E}-\varepsilon\nabla n\Delta E\nabla\bar{E})dx.\end{aligned} \tag{2.16}$$

Therefore, from (2.13)×2+(2.15), we obtain

$$\frac{d}{dt}H_1(t)+I_1(t)=0, \tag{2.17}$$

where

$$\begin{aligned}H_1(t)=&2\|\nabla\Delta E\|^2+2h^2\|\Delta^2E\|^2+\|m\|^2+\|\nabla m\|^2+(\beta-\varepsilon(\alpha-\varepsilon))\|n\|^2+h^2\|\nabla\Delta n\|^2\\&+(1+(\beta-\varepsilon(\alpha-\varepsilon)))\|\nabla n\|^2+(h^2+1)\|\Delta n\|^2+4\mathrm{Re}\int(nE+f)\Delta^2\bar{E}dx\\&+2\int g(\Delta n-n)dx,\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}I_1(t)=&4\gamma(\|\nabla\Delta E\|^2+h^2\|\Delta^2E\|^2+\mathrm{Re}\int f\Delta^2\bar{E}dx+\mathrm{Re}\int nE\Delta^2\bar{E}dx)+2\varepsilon h^2\|\nabla\Delta n\|^2\\&+2(\alpha-\varepsilon)(\|m\|^2+\|\nabla m\|^2)+2\varepsilon(\beta-\varepsilon(\alpha-\varepsilon))\|n\|^2+2\varepsilon(1+(\beta-\varepsilon(\alpha-\varepsilon)))\|\nabla n\|^2\\&+2\varepsilon(h^2+1)\|\Delta n\|^2-2\varepsilon\int gndx+2\varepsilon\int g\Delta ndx-2\int m\Delta|E|^2dx+4\int m|\Delta E|^2dx\\&+8\mathrm{Re}\int m\nabla\Delta E\nabla\bar{E}dx-4\varepsilon\int n|\Delta E|^2dx+2\varepsilon\int n\Delta^2|E|^2dx-4\mathrm{Re}\int nE_t\Delta^2\bar{E}dx\\&-8(\mathrm{Re}\int(\varepsilon n\nabla\Delta E\nabla\bar{E}+\nabla m\Delta E\nabla\bar{E}+\varepsilon\nabla n\Delta E\nabla\bar{E})dx.\end{aligned} \tag{2.19}$$

Hence, there exist $\alpha_1>0$ sufficiently small and $K_1>0$ satisfying

$$\frac{dH_1(t)}{dt}+\alpha_1H_1(t)\le K_1. \tag{2.20}$$

By Gronwall's lemma, we have

$$H_1(t)\le e^{-\alpha_1t}H_1(0)+\frac{K_1}{\alpha_1}(1-e^{-\alpha_1t})\le e^{-\alpha_1t}H_1(0)+\frac{K_1}{\alpha_1}\le M_1, \tag{2.21}$$

where M_1 is a positive constant independent of time. Then $(m,n,E)\in L^\infty(\mathbb{R}^+;V_1)$.
(2) On the other hand, taking inner product of (2.1) with $2\Delta^4E_t+2\gamma\Delta^4E$ in $\mathbb{R}^d$, we can obtain

$$\begin{aligned}&\frac{d}{dt}(\|\nabla\Delta^2E\|^2+h^2\|\Delta^3E\|^2+2\mathrm{Re}\int(nE+f)\Delta^4\bar{E}dx)+2\gamma\|\nabla\Delta^2E\|^2\\&+2\gamma(h^2\|\Delta^3E\|^2+\mathrm{Re}\int(nE+f)\Delta^4\bar{E}dx)-2\mathrm{Re}\int(nE_t+n_tE)\Delta^4\bar{E}dx=0.\end{aligned} \tag{2.22}$$

Meanwhile, we multiply (2.3) by $2\Delta^2 m$ in $\mathbb{R}^d$ and get

$$\frac{d}{dt}\|\Delta m\|^2+2(\alpha-\varepsilon)\|\Delta m\|^2+2(\beta-\varepsilon(\alpha-\varepsilon))\int n\Delta^2 m dx-2\int \Delta^2 m\Delta n dx$$
$$+2h^2\int \Delta^2 m\Delta^2 n dx-2\int \Delta^2 m\Delta|E|^2 dx-2\int g\Delta^2 m dx=0. \tag{2.23}$$

Note that

$$2\int \Delta^2 m n dx=\frac{d}{dt}\|\Delta n\|^2+2\varepsilon\|\Delta n\|^2,$$
$$-2\int \Delta^2 m\Delta n dx=\frac{d}{dt}\|\nabla\Delta n\|^2+2\varepsilon\|\nabla\Delta n\|^2,$$
$$2h^2\int \Delta^2 m\Delta^2 n dx=h^2\frac{d}{dt}\|\Delta^2 n\|^2+2\varepsilon h^2\|\Delta n\|^2,$$

so we have

$$\frac{d}{dt}(\|\Delta m\|^2+(\beta-\varepsilon(\alpha-\varepsilon))\|\Delta n\|^2+\|\nabla\Delta n\|^2+h^2\|\Delta^2 n\|^2-2\int g\Delta^2 n dx)$$
$$+2((\alpha-\varepsilon)\|\Delta m\|^2+\varepsilon(\beta-\varepsilon(\alpha-\varepsilon))\|\Delta n\|^2+\varepsilon\|\nabla\Delta n\|^2+\varepsilon h^2\|\Delta^2 n\|^2-\int \varepsilon g\Delta^2 n dx)$$
$$-2\int \Delta m\Delta^2|E|^2 dx=0. \tag{2.24}$$

Taking inner product of (2.3) with $2\Delta^3 m$ in $\mathbb{R}^d$ and noting

$$2\int \Delta^3 m n dx=\frac{d}{dt}\|\nabla\Delta n\|^2+2\varepsilon\|\nabla\Delta n\|^2, 2\int \Delta^3 m\Delta n dx=\frac{d}{dt}\|\Delta^2 n\|^2+2\varepsilon\|\Delta^2 n\|^2$$

and

$$-2h^2\int \Delta^3 m\Delta^2 n dx=h^2\frac{d}{dt}\|\nabla\Delta^2 n\|^2+2\varepsilon h^2\|\nabla\Delta n\|^2,$$

we can obtain

$$\frac{d}{dt}(\|\nabla\Delta m\|^2+(\beta-\varepsilon(\alpha-\varepsilon))\|\nabla\Delta n\|^2+\|\Delta^2 n\|^2+h^2\|\nabla\Delta^2 n\|^2+2\int g\Delta^3 n dx)$$
$$+2(\alpha-\varepsilon)\|\nabla\Delta m\|^2+2\varepsilon(\beta-\varepsilon(\alpha-\varepsilon))\|\nabla\Delta n\|^2+2\varepsilon\|\Delta^2 n\|^2+2\varepsilon h^2\|\nabla\Delta^2 n\|^2$$
$$+2\varepsilon\int g\Delta^3 n dx+2\int \Delta^3 m\Delta|E|^2 dx=0. \tag{2.25}$$

Since

$$2\int \Delta|E|^2\Delta^3 m dx=4\mathrm{Re}\int (n_t\bar{E}\Delta^4 E+4n_t\Delta^3 E\Delta\bar{E})dx+12\int n_t|\Delta^2 E|^2 dx, \tag{2.26}$$

we have

$$\frac{d}{dt}(\|\nabla\Delta m\|^2+(\beta-\varepsilon(\alpha-\varepsilon))\|\nabla\Delta n\|^2+\|\Delta^2 n\|^2+h^2\|\nabla\Delta^2 n\|^2+2\int g\Delta^3 n dx)$$
$$+2(\alpha-\varepsilon)\|\nabla\Delta m\|^2+2\varepsilon(\beta-\varepsilon(\alpha-\varepsilon))\|\nabla\Delta n\|^2+2\varepsilon\|\Delta^2 n\|^2+2\varepsilon h^2\|\nabla\Delta^2 n\|^2$$

$$+2\varepsilon\int g\Delta^3 n dx+4\mathrm{Re}\int(n_t\bar{E}\Delta^4E+4n_t\Delta^3E\Delta\bar{E}+3n_t|\Delta^2E|^2)dx=0. \tag{2.27}$$

Therefore, from $((2.25)+(2.27))\times 2+(2.22)$, we obtain

$$\frac{d}{dt}H_2(t)+I_2(t)=0, \tag{2.28}$$

where

$$\begin{aligned}H_2(t)=&2\|\nabla\Delta^2E\|^2+2h^2\|\Delta^3E\|^2+4\mathrm{Re}\int nE\Delta^4\bar{E}dx+4\mathrm{Re}\int f\Delta^4\bar{E}dx\\&+\|\Delta m\|^2+\|\nabla\Delta m\|^2+(\beta-\varepsilon(\alpha-\varepsilon))\|\Delta n\|^2+(1+(\beta-\varepsilon(\alpha-\varepsilon)))\|\nabla\Delta n\|^2\\&+(1+h^2)\|\Delta^2n\|^2+h^2\|\nabla\Delta^2n\|^2-2\int g\Delta^2ndx+2\int g\Delta^3ndx,\end{aligned} \tag{2.29}$$

and

$$\begin{aligned}I_2(t)=&4\gamma(\|\nabla\Delta^2E\|^2+h^2\|\Delta^3E\|^2+\mathrm{Re}\int nE\Delta^4\bar{E}dx+\mathrm{Re}\int f\Delta^4\bar{E}dx)\\&-4\mathrm{Re}\int nE_t\Delta^4\bar{E}dx+2(\alpha-\varepsilon)(\|\Delta m\|^2+\|\nabla\Delta m\|^2)+2\varepsilon(\beta-\varepsilon(\alpha-\varepsilon))\|\Delta n\|^2\\&+2\varepsilon(1+(\beta-\varepsilon(\alpha-\varepsilon)))\|\nabla\Delta n\|^2+2\varepsilon(1+h^2)\|\Delta^2n\|^2+2\varepsilon h^2\|\nabla\Delta^2n\|^2+2\varepsilon\int g\Delta^3ndx\\&-2\varepsilon\int g\Delta^2ndx-2\int\Delta m\Delta^2|E|^2dx+16\mathrm{Re}\int n_t\Delta^3E\Delta\bar{E}dx+12\int n_t|\Delta^2E|^2dx.\end{aligned}$$

With the same techniques as above, we know that there exist a small positive constant α_2 and a positive constant K_2 such that

$$\frac{dH_2(t)}{dt}+\alpha_2H_2(t)\le K_2, \tag{2.30}$$

which, by Gronwall's lemma, implies that

$$H_2(t)\le e^{-\alpha_2t}H_2(0)+\frac{K_2}{\alpha_2}(2-e^{-\alpha_2t})\le e^{-\alpha_2t}H_2(0)+\frac{K_2}{\alpha_2}\le M_2, \tag{2.31}$$

where M_2 is a positive constant independent of time. Then $(m,n,E)\in L^\infty(\mathbb{R}^+;V_2)$. □

From Lemma 2.1, it is not hard to prove the existence and uniqueness of the solutions and the bounded absorbing sets, which are stated as follows.

LEMMA 2.2. *Let $f\in H^4(\mathbb{R}^d)$, $g\in H^3(\mathbb{R}^d)$. Then, for any $(m_0,n_0,E_0)(x)\in V_1$, there exists a unique solution $(m,n,E)\in C_b(\mathbb{R}^+;V_1)$ for (2.1)–(2.4).*

Moreover, if $S(t)$ is the solution operator, that is, $(m(t),n(t),E(t))=S(t)(m_0,n_0,E_0)$ is the solution of (2.1)–(2.4) with the initial condition $(m_0,n_0,E_0)\in V_1$, then $S(t)$ is a semigroup in V_1, uniformly continuous on any compact interval $[0,T]$, and has a bounded absorbing set $B_1\subset V_1$. In addition, $S(t)$ is also a continuous semigroup in V_2 and has a bounded absorbing set $B_2\subset V_2$.

3. The decomposition of solution operators

Let $\chi_L(x)\in C_0^\infty(\mathbb{R})$ satisfying $0\le\chi_L\le 1$ and

$$\chi_L(x)=\begin{cases}1, & |x|\le L,\\ 0, & |x|\ge 1+L.\end{cases} \tag{3.1}$$

Then, for any $\sigma\in(0,1]$, there exists an $L(\sigma)>0$ (sufficiently large) such that

$$\|f-f_\sigma\|_{H^2}^4\le\sigma,\quad where\quad f_\sigma=f\chi_{L(\sigma)}, \tag{3.2}$$
$$\|g-g_\sigma\|_{H^1}^3\le\sigma,\quad where\quad g_\sigma=g\chi_{L(\sigma)}, \tag{3.3}$$
$$\|\Delta|E|^2(1-\chi_{L(\sigma)})\|_{H^3}^2\le\sigma. \tag{3.4}$$

Now, we denote by $(m_\sigma,n_\sigma,E_\sigma)=S_{1\sigma}(m_0,n_0,E_0)$ the solution of the problem:

$$\begin{aligned}&iE_{\sigma t}+\Delta E_\sigma-h^2\Delta^2E_\sigma+i\gamma E_\sigma-nE_\sigma-i\sigma\Delta E_\sigma+i\sigma h^2\Delta^2E_\sigma\\&=f-f_\sigma-i\sigma\Delta E+i\sigma h^2\Delta^2E,\end{aligned} \tag{3.5}$$
$$n_{\sigma t}+\varepsilon n_\sigma=m_\sigma, \tag{3.6}$$
$$m_{\sigma t}+(\alpha-\varepsilon)m_\sigma+(\beta-\varepsilon(\alpha-\varepsilon))n_\sigma-\Delta n_\sigma+h^2\Delta^2n_\sigma=(\Delta|E|^2+g)(1-\chi_{L(\sigma)}), \tag{3.7}$$
$$(m_\sigma,n_\sigma,E_\sigma)(0,x)=(m_0,n_0,E_0)(x),\quad x\in\mathbb{R}^d. \tag{3.8}$$

Then

$$\begin{aligned}(w_\sigma,v_\sigma,u_\sigma)&=S_{2\sigma}(t)(m_0,n_0,E_0)\\&=S(t)(m_0,n_0,E_0)-S_{1\sigma}(t)(m_0,n_0,E_0)=(m-m_\sigma,n-n_\sigma,E-E_\sigma)\end{aligned}$$

satisfies

$$iu_{\sigma t}+\Delta u_\sigma-h^2\Delta^2u_\sigma+i\gamma u_\sigma-i\sigma\Delta u_\sigma+i\sigma h^2\Delta^2u_\sigma-nu_\sigma=f_\sigma, \tag{3.9}$$
$$w_\sigma=v_{\sigma t}+\varepsilon v_\sigma, \tag{3.10}$$
$$w_{\sigma t}+(\alpha-\varepsilon)w_\sigma+(\beta-\varepsilon(\alpha-\varepsilon))v_\sigma-\Delta v_\sigma+h^2\Delta^2v_\sigma=(\Delta|E|^2+g)\chi_{L(\sigma)}, \tag{3.11}$$
$$(w_\sigma,v_\sigma,u_\sigma)(0,x)=(0,0,0),\quad x\in\mathbb{R}^d. \tag{3.12}$$

LEMMA 3.1. *There exist a constant $C>0$ and an increasing function $\omega(0)=0$ such that the solution of (3.5)–(3.8) satisfies*

$$\|E_\sigma\|_{H^4},\|n_\sigma\|_{H^3},\|m_\sigma\|_{H^1}\le C,\ for\ all\ 0<\sigma\le1\ and\ t\ge0,$$
$$\|E_\sigma\|_{H^4},\|n_\sigma\|_{H^3},\|m_\sigma\|_{H^1}\le\omega(\sigma),\ for\ all\ 0<\sigma\le1\ and\ t\ge t_*\ (\exists t_*>0).$$

Proof. Multiplying (3.5) by $2\bar{E}_\sigma$ and integrating the imaginary part, we see

$$\begin{aligned}&\frac{d}{dt}\|E_\sigma\|^2+2\gamma\|E_\sigma\|^2+2\sigma\|\nabla E_\sigma\|^2+2\sigma h^2\|\Delta E_\sigma\|^2\\&=\mathrm{Im}\int 2\bar{E}_\sigma(f-f_\sigma-i\sigma\Delta E+i\sigma h^2\Delta^2E)dx\\&\le\gamma\|E_\sigma\|^2+\sigma\|\nabla E_\sigma\|^2+\sigma h^2\|\Delta E_\sigma\|^2+\sigma\|\nabla E\|^2+\sigma h^2\|\Delta E\|^2+C\|f-f_\sigma\|^2.\end{aligned} \tag{3.13}$$

Hence, it is easy to see

$$\frac{d}{dt}\|E_\sigma\|^2+\gamma\|E_\sigma\|^2\le\sigma\|\nabla E\|^2+C\|f-f_\sigma\|^2+\sigma h^2\|\Delta E\|^2\le C\sigma. \tag{3.14}$$

By Gronwall's inequality, we obtain

$$\|E_\sigma\|^2\le\|E_0\|^2e^{-\gamma t}+\frac{C\sigma}{\gamma}(1-e^{-\gamma t}). \tag{3.15}$$

Hence, for all $t>0$ and $0<\sigma\le 1$, there holds

$$\|E_\sigma\|^2\le C. \tag{3.16}$$

Moreover, there exists a $t_1>0$ such that $e^{-\gamma t_1}\|E_0\|^2\le\sigma$ and for all $t>t_1$,

$$\|E_\sigma\|^2\le C\sigma. \tag{3.17}$$

Multiplying (3.14) by $2\bar{E}_{\sigma t}$ and integrating the real part, then we get

$$\begin{aligned}&\mathrm{Re}\int 2\bar{E}_{\sigma t}(iE_{\sigma t}+\Delta E_\sigma-h^2\Delta^2E_\sigma+i\gamma E_\sigma-i\sigma\Delta E_\sigma+i\sigma h^2\Delta^2E_\sigma)\\ =&\mathrm{Re}\int 2\bar{E}_{\sigma t}(nE_\sigma+f-f_\sigma-i\sigma\Delta E+i\sigma h^2\Delta^2E)dx.\end{aligned} \tag{3.18}$$

Substituting the identity

$$\begin{aligned}-i\bar{E}_{\sigma t}=&-\Delta\bar{E}_\sigma+h^2\Delta^2\bar{E}_\sigma+i\gamma\bar{E}_\sigma+n\bar{E}_\sigma-i\sigma\Delta\bar{E}_\sigma\\ &+i\sigma h^2\Delta^2\bar{E}_\sigma+i\sigma\Delta\bar{E}-i\sigma h^2\Delta^2\bar{E}+\bar{f}-\bar{f}_\sigma,\end{aligned}$$

into (3.18), and using Hölder's inequality, Young's inequality, and the Gagliardo–Nirenberg inequality, we can obtain

$$\frac{d}{dt}\tilde{H}_0(t)+\gamma\tilde{H}_0(t)\le C\sigma, \tag{3.19}$$

where $\tilde{H}_0(t)=\|\nabla\bar{E}_\sigma\|^2+h^2\|\Delta\bar{E}_\sigma\|^2+\int n|\bar{E}_\sigma|^2dx+2\int\bar{E}_\sigma(\bar{f}-\bar{f}_\sigma)dx$. Thus it follows from Gronwall's inequality that

$$\tilde{H}_0(t)\le e^{-\gamma t}\tilde{H}_0(0)+C\sigma(1-e^{-\gamma t}). \tag{3.20}$$

Note that

$$\int n|\bar{E}_\sigma|^2dx+2\int\bar{E}_\sigma(\bar{f}-\bar{f}_\sigma)dx\le C(\|n\|_{L^\infty}\|E_\sigma\|^2+\|E_\sigma\|^2+\|\bar{f}-\bar{f}_\sigma\|^2).$$

Therefore, there holds

$$\|\nabla\bar{E}_\sigma\|^2+h^2\|\Delta\bar{E}_\sigma\|^2\le C \tag{3.21}$$

for all $0<\sigma\le 1$ and $t\ge 0$. Also, there exists a $t_2>t_1>0$ such that

$$\|\nabla\bar{E}_\sigma\|^2+h^2\|\Delta\bar{E}_\sigma\|^2\le C\sigma \tag{3.22}$$

for all $0<\sigma\le 1$ and $t>t_2$. Similarly, with the same arguments as above, it is not difficult to obtain

$$\begin{aligned}&\|E_\sigma\|_{H^4}\le C,\ \text{for all } 0<\sigma\le 1 \text{ and } t\ge 0,\\ &\|E_\sigma\|_{H^4}\le\omega(\sigma),\ \text{for all } 0<\sigma\le 1 \text{ and } t\ge t_3\ (\exists t_3>t_2>t_1>0).\end{aligned}$$

In addition, we multiply (3.7) by $2\Delta m_\sigma$ and integrate in $\mathbb{R}^d$, to obtain the following:

$$\int 2\Delta m_\sigma(m_{\sigma t}+(\alpha-\varepsilon)m_\sigma+(\beta-\varepsilon(\alpha-\varepsilon))n_\sigma-\Delta n_\sigma+h^2\Delta^2n_\sigma)dx$$

$$=\int 2\Delta m_\sigma(\Delta|E|^2+g)(1-\chi_{L(\sigma)})dx. \tag{3.23}$$

So we can obtain

$$\begin{aligned}&\frac{d}{dt}(\|\nabla m_\sigma\|^2+(\beta-\varepsilon(\alpha-\varepsilon))\|\nabla n_\sigma\|^2+\|\Delta n_\sigma\|^2+h^2\|\nabla\Delta n_\sigma\|^2)\\&\quad+2(\alpha-\varepsilon)\|\nabla m_\sigma\|^2+2\varepsilon(\beta-\varepsilon(\alpha-\varepsilon))\|\nabla n_\sigma\|^2+2\varepsilon\|\Delta n_\sigma\|^2+2\varepsilon h^2\|\nabla\Delta n_\sigma\|^2\\&=\int 2\Delta m_\sigma(\Delta|E|^2+g)(1-\chi_{L(\sigma)})dx,\end{aligned} \tag{3.24}$$

which implies

$$\begin{aligned}&\frac{d}{dt}(\|\nabla m_\sigma\|^2+(\beta-\varepsilon(\alpha-\varepsilon))\|\nabla n_\sigma\|^2+\|\Delta n_\sigma\|^2+h^2\|\nabla\Delta n_\sigma\|^2)\\&\quad+2(\alpha-\varepsilon)\|\nabla m_\sigma\|^2+2\varepsilon(\beta-\varepsilon(\alpha-\varepsilon))\|\nabla n_\sigma\|^2+2\varepsilon\|\Delta n_\sigma\|^2+2\varepsilon h^2\|\nabla\Delta n_\sigma\|^2\\&\le\frac{\alpha-\varepsilon}{2}\|\nabla m_\sigma\|^2+C\sigma.\end{aligned}$$

Define

$$H_\sigma(t)=\|\nabla m_\sigma\|^2+(\beta-\varepsilon(\alpha-\varepsilon))\|\nabla n_\sigma\|^2+\|\Delta n_\sigma\|^2+h^2\|\nabla\Delta n_\sigma\|^2,$$

by choosing a small positive constant $\varepsilon<\min\{\frac{\alpha}{2},\frac{\beta}{\alpha}\}$ such that $\alpha-\varepsilon>\varepsilon$ and $\beta-\varepsilon(\alpha-\varepsilon)>0$. Then we have

$$\frac{d}{dt}H_\sigma(t)+\varepsilon H_\sigma(t)\le C\sigma, \tag{3.25}$$

and by Gronwall's inequality, there holds

$$H_\sigma(t)\le e^{-\varepsilon t}H_\sigma(0)+C\sigma(1-e^{-\varepsilon t})\le C,\ \forall t>0. \tag{3.26}$$

Moreover, there exists a $t_4>t_3>0$ such that

$$H_\sigma(t)\le C\sigma \tag{3.27}$$

for all $t>t_4$ and $0<\sigma\le 1$. Therefore, it is easy to obtain

$$\begin{aligned}&\|n_\sigma\|_{H^3},\|m_\sigma\|_{H^1}\le C,\ \text{for all}\ \ 0<\sigma\le 1\ \text{and}\ t\ge 0,\\&\|n_\sigma\|_{H^3},\|m_\sigma\|_{H^1}\le\omega(\sigma),\ \text{for all}\ 0<\sigma\le 1\ \ \text{and}\ t\ge t_4,\end{aligned}$$

where $\omega(0)=0$. Finally, Lemma 3.1 follows if we choose $t^*=\max\{t_3,t_4\}$. ∎

LEMMA 3.2. *There exist constants $C_1(\sigma),C_2(\sigma)$ such that*

$$\begin{aligned}&\|xu_\sigma\|,\ \|x\nabla u_\sigma\|,\ \|x\Delta u_\sigma\|,\ \|x\nabla\Delta u_\sigma\|,\ \|x\Delta^2u_\sigma\|\le C_1(\sigma),\\&\|xv_\sigma\|,\ \|x\nabla v_\sigma\|,\ \|x\Delta v_\sigma\|,\ \|x\nabla\Delta v_\sigma\|,\ \|xw_\sigma\|,\ \|x\nabla w_\sigma\|\le C_2(\sigma).\end{aligned}$$

Proof. Taking the inner product with $2|x|^2u_\sigma$ for (3.9) and taking the imaginary part, we have

$$\mathrm{Im}\int 2|x|^2\bar{u}_\sigma(iu_{\sigma t}+\Delta u_\sigma-h^2\Delta^2u_\sigma+i\gamma u_\sigma-i\sigma\Delta u_\sigma+i\sigma h^2\Delta^2u_\sigma)dx$$

$$=\mathrm{Im}\int 2|x|^2\bar{u}_\sigma(nu_\sigma+f_\sigma)dx. \tag{3.28}$$

Note that

$$\begin{aligned}
\mathrm{Im}\int 2|x|^2\bar{u}_\sigma\Delta u_\sigma dx &= -4\mathrm{Im}\int x\bar{u}_\sigma\nabla u_\sigma dx,\\
-\mathrm{Im}\int 2h^2|x|^2\bar{u}_\sigma\Delta^2 u_\sigma dx &= -4h^2\mathrm{Im}\int 2x\nabla\bar{u}_\sigma\Delta u_\sigma dx,\\
-\mathrm{Im}\int 2i\sigma|x|^2\bar{u}_\sigma\Delta u_\sigma dx &= 2\sigma\|x\nabla u_\sigma\|^2+4\sigma\mathrm{Re}\int x\bar{u}_\sigma\nabla u_\sigma dx,\\
\mathrm{Im}\int 2i\sigma h^2|x|^2\bar{u}_\sigma\Delta^2 u_\sigma dx &= 2\sigma h^2\|x\Delta u_\sigma\|^2\\
&\quad +8\sigma h^2\mathrm{Re}\int x\nabla\bar{u}_\sigma\Delta u_\sigma dx-4\sigma h^2\|\nabla u_\sigma\|^2.
\end{aligned}$$

Then, through the standard estimates for (3.28), we obtain

$$\begin{aligned}
&\frac{d}{dt}\|xu_\sigma\|^2+2\gamma\|xu_\sigma\|^2+2\sigma\|x\nabla u_\sigma\|^2+2\sigma h^2\|x\Delta u_\sigma\|^2\\
\le&\gamma\|xu_\sigma\|^2+\sigma\|x\nabla u_\sigma\|^2+\sigma h^2\|x\Delta u_\sigma\|^2+C(\sigma)(\|u_\sigma\|^2+\|\nabla u_\sigma\|^2+\|xf_\sigma\|^2),
\end{aligned}$$

which can be estimated as

$$\frac{d}{dt}\|xu_\sigma\|^2+\gamma\|xu_\sigma\|^2\le C(\sigma)(\|u_\sigma\|^2+\|\nabla u_\sigma\|^2+\|xf_\sigma\|^2)\le C(\sigma).$$

Then applying Gronwall's inequality yields

$$\|xu_\sigma\|^2\le C(\sigma)\int_0^t e^{-\gamma(t-s)}ds=\frac{C(\sigma)}{\gamma}(1-e^{-\gamma t})\le C(\sigma).$$

As before, we can further deal with the higher-order norm as follows. Taking the inner product with $2|x|^2u_{\sigma t}$ for (3.9), we have

$$\begin{aligned}
&\mathrm{Re}\int 2|x|^2\bar{u}_{\sigma t}(iu_{\sigma t}+\Delta u_\sigma-h^2\Delta^2u_\sigma+i\gamma u_\sigma-i\sigma\Delta u_\sigma+i\sigma h^2\Delta^2 u_\sigma)dx\\
=&\mathrm{Re}\int 2|x|^2\bar{u}_{\sigma t}(nu_\sigma+f_\sigma)dx.
\end{aligned} \tag{3.29}$$

Notice that

$$\begin{aligned}
\mathrm{Re}\int 2|x|^2\bar{u}_{\sigma t}\Delta u_\sigma dx&=-\frac{d}{dt}\|x\nabla u_\sigma\|^2-\mathrm{Re}\int 4x\bar{u}_{\sigma t}\nabla u_\sigma dx,\\
\mathrm{Re}\int -2|x|^2\bar{u}_{\sigma t}h^2\Delta^2u_\sigma dx&=-h^2\frac{d}{dt}\|x\Delta u_\sigma\|^2+4h^2\mathrm{Re}\int \bar{u}_{\sigma t}\Delta u_\sigma dx\\
&\quad+8h^2\mathrm{Re}\int x\bar{u}_{\sigma t}\nabla\Delta u_\sigma dx,\\
\mathrm{Re}\int 2|x|^2\bar{u}_{\sigma t}(nu_\sigma+f_\sigma)dx&=\frac{d}{dt}(\int |x|^2|u_\sigma|^2ndx+2\mathrm{Re}\int |x|^2u_\sigma f_\sigma dx)\\
&\quad-\mathrm{Re}\int |x|^2|u_\sigma|^2(m-\varepsilon n)dx,
\end{aligned}$$

and

$$\mathrm{Re}\int 2i\gamma|x|^2u_\sigma\bar{u}_{\sigma t}dx=-2\gamma\mathrm{Im}\int|x|^2u_\sigma\bar{u}_{\sigma t}dx,$$
$$\mathrm{Re}\int -2i\sigma|x|^2\bar{u}_{\sigma t}\Delta u_\sigma dx=2\sigma\mathrm{Im}\int|x|^2\bar{u}_{\sigma t}\Delta u_\sigma dx,$$
$$\mathrm{Re}\int 2i\sigma h^2|x|^2\bar{u}_{\sigma t}\Delta^2u_\sigma dx=-2\sigma h^2\mathrm{Im}\int\Delta u_\sigma(2\bar{u}_{\sigma t}+4x\nabla\bar{u}_{\sigma t}+|x|^2\Delta\bar{u}_{\sigma t})dx.$$

In addition, in view of the identity

$$\bar{u}_{\sigma t}=if_\sigma-i\Delta\bar{u}_\sigma+ih^2\Delta^2\bar{u}_\sigma-\gamma\bar{u}_\sigma+in\bar{u}_\sigma+\sigma\Delta\bar{u}_\sigma-\sigma h^2\Delta^2\bar{u}_\sigma,$$

we can obtain

$$\frac{d}{dt}G_0(t)+\gamma G_0(t)\le C(\sigma), \tag{3.30}$$

where $G_0(t)=\|x\nabla u_\sigma\|^2+h^2\|x\Delta u_\sigma\|^2+\int|x|^2|u_\sigma|^2ndx+2\mathrm{Re}\int|x|^2u_\sigma f_\sigma dx$. In the estimate (3.30), we have used the fact $u_\sigma\in H^3(\mathbb{R}^d)$, which can be easily obtained by investigating problem (3.9)–(3.12) through the standard method. Therefore, by Gronwall's inequality, it is easy to obtain $\|x\nabla u_\sigma\|^2+h^2\|x\Delta u_\sigma\|^2\le C(\sigma)$.

Moreover, using similar approach as above, one can also obtain $\|x\nabla\Delta u_\sigma\|^2+h^2\|x\Delta^2u_\sigma\|^2\le C(\sigma)$ under the condition $f\in H^4(\mathbb{R}^d)$. Therefore, we have shown that there exists a constant $C_1(\sigma)$ satisfying

$$\|xu_\sigma\|,\ \|x\nabla u_\sigma\|,\ \|x\Delta u_\sigma\|,\ \|x\nabla\Delta u_\sigma\|,\ \|x\Delta^2u_\sigma\|\le C_1(\sigma).$$

Now, we use a similar argument to deal with the terms including w_σ and v_σ. Taking the inner product with $2|x|^2w_{\sigma t}$ for (3.11), we have

$$\begin{aligned}&\int 2|x|^2w_{\sigma t}(w_{\sigma t}+(\alpha-\varepsilon)w_\sigma+(\beta-\varepsilon(\alpha-\varepsilon))v_\sigma-\Delta v_\sigma+h^2\Delta^2v_\sigma)dx\\&=\int 2|x|^2w_{\sigma t}(\Delta|E|^2+g)\chi_{L(\sigma)}dx,\end{aligned}$$

from which we can get

$$\frac{d}{dt}G_3(t)+\varepsilon G_3(t)\le C(\sigma), \tag{3.31}$$

where $G_3(t)=\|xw_\sigma\|^2+(\beta-\varepsilon(\alpha-\varepsilon))\|xv_\sigma\|^2+\|x\nabla v_\sigma\|^2+h^2\|x\Delta v_\sigma\|^2$. If we choose $\varepsilon>0$ sufficiently small to satisfy $\varepsilon<\frac{\beta}{2}$ and $\varepsilon\le(\beta-\varepsilon(\alpha-\varepsilon))$, then there exists a constant $C(\sigma)$ such that

$$\|xw_\sigma\|^2,\ \|xv_\sigma\|^2,\ \|x\nabla v_\sigma\|^2,\ \|x\Delta v_\sigma\|^2\le C(\sigma).$$

Differentiating (3.11) and taking the inner product with $2|x|^2\nabla w_{\sigma t}$ for the resulting equation, we see

$$\begin{aligned}&\int 2|x|^2\nabla w_{\sigma t}(\nabla w_{\sigma t}+(\alpha-\varepsilon)\nabla w_\sigma+(\beta-\varepsilon(\alpha-\varepsilon))\nabla v_\sigma-\nabla\Delta v_\sigma+h^2\nabla\Delta^2v_\sigma)dx\\&=\int 2|x|^2\nabla w_{\sigma t}\nabla(\Delta|E|^2+g)\chi_{L(\sigma)}dx.\end{aligned}$$

Repeating the above process, we can obtain

$$\frac{d}{dt}G_4(t)+\varepsilon G_4(t)\leq C(\sigma), \tag{3.32}$$

where $G_4(t)=\|x\nabla w_\sigma\|^2+(\beta-\varepsilon(\alpha-\varepsilon))\|x\nabla v_\sigma\|^2+\|x\Delta v_\sigma\|^2+h^2\|x\nabla\Delta v_\sigma\|^2$. Again, we select a sufficiently small $\varepsilon>0$ that $\varepsilon<\frac{\beta}{2}$ and $\varepsilon\leq(\beta-\varepsilon(\alpha-\varepsilon))$, then there exists a constant $C(\sigma)$ satisfying $\|x\nabla w_\sigma\|^2,\|x\nabla\Delta v_\sigma\|^2\leq C(\sigma)$. Hence, we can take a common $C_2(\sigma)$ such that $\|xv_\sigma\|$, $\|x\nabla v_\sigma\|$, $\|x\Delta v_\sigma\|$, $\|x\nabla\Delta v_\sigma\|$, $\|xw_\sigma\|$, $\|x\nabla w_\sigma\|\leq C_2(\sigma)$. This completes the proof of Lemma 3.2. ∎

4. The existence of the attractor

Let $S(t)$ be the semigroup generated by (2.1)–(2.4). According to the previous sections, we know that $S(t)$ has a bounded absorbing set in V_1 and V_2. In Section 3, decomposition of $S(t)$ is used in order to make use of the so-called Kuratowskii α-measure of noncompactness to prove the asymptotic smoothness of $S(t)$ and construct the maximal attractor. Recall that the α-measure of a set A in a Banach space X is defined by

$$\alpha(A)\triangleq\inf\{d|\text{ there is a finite covering of } A \text{ of diameter} < d\},$$

where d represents distance. $\alpha(A)$ has the following properties (see e.g., [2, 21]):

$$\alpha(A\cup B)\leq\alpha(A)+\alpha(B),$$
$$\alpha(A)=0,\ A \text{ is compact in } X.$$

To prove the main result, we need to give the following compact imbedding lemma.

LEMMA 4.1. *Let $s>s_1$ be an integer, then the imbedding of $H^s(\mathbb{R}^d)\cap H^{s_1}(\mathbb{R}^d,(1+|x|^2)dx)$ into $H^{s_1}(\mathbb{R}^d)$ is compact.*

The proof of Lemma 4.1 can be found in [3, 12]. Now we present the main result of the paper.

THEOREM 4.2. *Assume $f\in H^4(\mathbb{R}^d)$, $g\in H^3(\mathbb{R}^d)$ $(d=1,2,3)$, and $S(t)$ be the semigroup generated by (2.1)–(2.4). Then there exists a set $A\subset V_1$ satisfying*

(1) $S(t)A=A,\ \forall\ t\geq 0,$

(2) $\lim\limits_{t\to\infty}\operatorname{dist}_{V_1}(S(t)B,A)=\lim\limits_{t\to\infty}\sup\limits_{y\in B}\operatorname{dist}_{V_1}(S(t)y,A)=0,\quad\forall B\subset V_2$ *bounded,*

(3) A *is compact in* V_1.

That is to say, A is a maximal compact attractor in V_1 which attracts bounded sets of V_2 in the topology of V_1.

Proof. From lemmas 3.2 and 4.1, we see that $S_{2\sigma}$ defined by (3.9)–(3.12) is compact from V_2 into V_1. Then, for any bounded $B\subset V_2$, we have

$$\alpha(S_{2\sigma}(t)B)=0,\ \forall\ t\geq 0.$$

From Lemma 3.1, for any $\eta>0$, there exist σ and t_0 such that

$$\|S_{1\sigma}(t)(m_0,n_0,E_0)\|<\eta$$

for all $t \geq t_0$ and $(m_0, n_0, E_0) \in B$.

Hence, there holds $\alpha(S_{1\sigma}(t)B) \leq 2\eta$ for all $t \geq t_0$. Therefore, we can obtain

$$\alpha(S(t)B) \leq \alpha(S_{1\sigma}(t)B) + \alpha(S_{2\sigma}(t)B) = \alpha(S_{1\sigma}(t)B) \leq 2\eta, \ \forall \ t \geq t_0.$$

Namely, for any bounded set $B \subset V_2$, we have

$$\lim_{t \to \infty} \alpha(S(t)B) = 0,$$

which implies that $S(t)$ is asymptotically smooth. Therefore, Theorem 4.2 follows easily by applying Theorem 3.4 in [21]. This ends the proof of the theorem. □

5. Further remarks

In this section, we present two remarks on the attractor which will be investigated in our forthcoming work.

An interesting question is the dimensional estimate for the attractor. In the case of the periodic boundary condition, by studying the estimates of Lyapunov exponents, we obtained the finite dimension estimates of the Hausdorff dimension and the fractal dimension of the attractor for system (1.1)–(1.2) in [18]. However, the Poincaré inequality and corresponding compact operator do not hold on unbounded domain in general, which have been used for dimensional estimates in periodic boundary condition [18]. So, we must further investigate dimensional estimates in our case.

Another problem is to study the limit behavior of the attractor when $h \to 0^+$. This problem is meaningful from the view of physics, as h is an extremely small quantity in physics [19]. The classical limit of the quantum Zakharov system was shown in [16]. However, the limit of the attractor for this system has not been studied yet. In fact, concerning the limit behavior of the attractor, one has to show the existence of solutions, uniformly estimate solutions independent of h, and estimate the precompactness of union of attractors depending on h. All these properties are not obvious for this system, especially in the case of unbounded domain and higher spatial dimensions. Therefore, we need to investigate the dynamics of the attractor more deeply in the future.

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