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LAGRANGIAN AVERAGED GYROKINETIC-WATERBAG CONTINUUM*

NICOLAS BESSE†

Abstract. In this paper, we first present the derivation of the anisotropic Lagrangian averaged gyrowaterbag continuum (LAGWBC-α) equations. The gyrowaterbag (short for *gyrokinetic-waterbag*) continuum can be viewed as a special class of exact weak solution of the gyrokinetic-Vlasov equation, allowing us to reduce the latter into an infinite-dimensional set of hydrodynamic equations while keeping its kinetic features, such as Landau damping. In order to obtain the LAGWBC-α equations from the gyrowaterbag continuum we use an Eulerian variational principle and Lagrangian averaging techniques introduced by Holm, Marsden, and Ratiu [27, 28], Marsden and Shkoller [32, 33] for the mean motion of ideal incompressible flows, extended to barotropic compressible flows by Bhat et al. [13] and some supplementary approximations for the electrical potential fluctuations. Regarding the original gyrowaterbag continuum, the LAGWBC-α equations show some additional properties and several advantages from the mathematical and physical viewpoints, which make this model a good candidate for accurately describing gyrokinetic turbulence in magnetically confined plasma. In the second part of this paper, we prove local-in-time well-posedness of an approximate version of the anisotropic LAGWBC-α equations, which we call the *isotropic* LAGWBC-α equations, by using quasilinear PDE type methods and elliptic regularity estimates for several operators.

Key words. Gyrokinetic-waterbag model, gyrowaterbag model, well-posed problem, gyrokinetic turbulence, Lagrangian averaged models, Eulerian and Lagrangian variational principles, gyrokinetic-Vlasov equations, multi-fluids systems, infinite-dimensional hyperbolic system of conservation laws in several space dimension, magnetically confined fusion plasmas.

AMS subject classifications. 35Q83, 35Q35, 35F55, 35L65, 76F02, 76N10, 76B03.

1. Introduction

The problem of turbulence in fluids and a fortiori in plasmas is a major problem of physics. Heat, particle, and momentum transport, which are crucial for plasma confinement in fusion devices, usually result from turbulent processes. One main issue of turbulent flow computations is the generation of smaller and smaller scales. In order to reduce the computationnal cost of turbulent calculations and get new physical insights in fluid dynamics, the traditional approach (such as Large Eddy Simulations or LES [34]) consists of averaging or filtering PDEs to compute accurately large-scale dynamics while small scale interactions and their effects on large scales are modeled. Unlike traditional averaging or filtering approaches, where PDEs are averaged or spatially filtered, recently Holm, Marsden, and Ratiu [27, 28] introduced a Lagrangian averaging approach based on averaging at the level of the variational principle. A nice property of this approach is that, despite the presence of model terms which arise from solving the turbulence closure problem, all geometrical properties (e.g. invariants, conservation laws) of the dynamics are retained since the Hamiltonian principle is applied after the Lagrangian averaging procedure. Since our objective is to better understand the turbulence arising from Hamiltonian chaos, as it is the case for gyrokinetic-Vlasov equations, we believe

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†Institut Jean Lamour UMR CNRS 7198, Université de Lorraine, BP 70239 54506 Vandoeuvre-lès-Nancy Cedex, France (Nicolas.Besse@univ-lorraine.fr).
Laboratoire J.-L. Lagrange, UMR CNRS/OCA/UNS 7293, Observatoire de la Côte d'Azur, Bd de l'observatoire CS 34229, 06300 Nice Cedex 4, France (Nicolas.Besse@oca.eu)

that derivation of a Hamiltonian system which preserves a modified energy and is time-reversible, by using a conservative regularization method, is more close to our turbulence framework.

Using an ensemble average over the set of solutions of the incompressible Euler and Navier–Stokes equations and a generalization of the G.I. Taylor's "frozen turbulence" assumption [35], Marsden and Shkoller [32, 33] obtained the LAE-α (Lagrangian averaged Euler equations) and LANS-α (Lagrangian averaged Navier–Stokes equations) equations, designed to accurately capture the dynamics of the incompressible Euler and Navier–Stokes equations at length scales larger than α while averaging the motion at scales smaller than α. In [13], the authors extend the derivation of LAE-α to the case of inhomogeneous barotropic compressible flows and prescribed a new flow rule, called *rotation rule*, which should be more relevant than the Lie advection rule for isotropic turbulence models.

Motivations and issues addressed in this paper. Motivations for the derivation of Lagrangian averaged gyrowaterbag continuum (LAGWBC-α) are threefold. The gyrowaterbag continuum equations on domains without boundary, such as a periodic box of $\mathbb{R}^3$, is given by

$$\left.\begin{aligned}
&\partial_t c+\nabla\cdot(cV)=0,\\
&\partial_t u+\nabla\cdot(uV)+\partial_{\parallel}\left(\frac{1}{2}\left(\frac{c^2}{4}-u^2\right)+\phi\right)=0,\\
&Q\phi=\int_0^1 da\int_{\mathbb{R}^+} d\mu\, c-n_0,\qquad V=((\nabla_\perp^\perp\phi)^T,u)^T,\\
&c(t=0,x,a,\mu)=c^0(x,a,\mu),\quad u(t=0,x,a,\mu)=u^0(x,a,\mu),
\end{aligned}\right\}\tag{1.1}$$

where the differential operator Q is defined by

$$Q\varphi=-\nabla_\perp\cdot(a_0\nabla_\perp\varphi)+b_0\varphi,\tag{1.2}$$

with a_0, b_0, and n_0 some given positive smooth functions. The velocity vector-field V is given by $V=(-\partial_{x_2}\phi,\partial_{x_1}\phi,u)^T=((\nabla_\perp^\perp\phi)^T,u)^T$, where $\phi=\phi(t,x_1,x_2,x_3)$ is the electrical potential. Here, the couple $x_\perp=(x_1,x_2)$ represents the variables associated with the transverse direction, while the variable associated with the longitudinal direction is denoted by $x_\parallel=x_3$. For each fixed value of the couple (a,μ), the quantities $c=c(t,x,a,\mu)$ and $u=u(t,x,a,\mu)$ represent, respectively, the density and the longitudinal mean velocity of the fluid labeled by the tag (a,μ) at the point $x=(x_\perp,x_\parallel)=(x_1,x_2,x_3)$ of the three-dimensional physical space. Moreover we use the notation $\nabla=(\partial_1,\partial_2,\partial_3)^T=(\partial_{x_1},\partial_{x_2},\partial_{x_3})^T=(\nabla_{x_\perp}^T,\partial_{x_\parallel})^T=(\nabla_\perp^T,\partial_\parallel)^T$. In Section 2.2 below, we deal with a more refined model which is obtained as the reduction of the gyrokinetic-Vlasov equations (see Section 2.1), by using a variational principle of least action (see Section 2.3).

Let us first notice that the gyrowaterbag continuum equation (1.1) can be seen as an infinite-dimensional hyperbolic system of first-order conservation laws in $\mathbb{R}^3$, with non-local fluxes, which is incompressible in the two-dimensional transverse direction and compressible in the one-dimensionnal longitudinal direction. The gyrowaterbag continuum equations share two common features with incompressible and compressible fluid flows that require special attention. The first one is generation of smaller and smaller scales in the transverse direction due to the intrinsic turbulent nature of Hamiltonian incompressible flow. The second one is formation of shock in the longitudinal direction coming from nonlinear convective terms (e.g. Burgers' terms) which are common in compressible models. In both cases the crucial point is to represent in a relevant way small scale effects. Instead of adding nonphysical and artificial viscous or dissipation terms to regularize shock discontinuities on one hand and removing energy contained in small scales in the other hand, the Lagrangian averaged variational principle, which adds dispersion terms (see equation (1.3) below) with a nonlinear energy redistribution mechanism, seems to be a reasonable alternative to balance the steepening effect of nonlinear convective terms in the longitudinal direction and the energy cascade to smaller scales in the transverse direction.

For instance, the isotropic LAGWBC-α equations on domains without boundary, such as a periodic box in $\mathbb{R}^3$, that we formally derive in Section 3 (cf. equations (3.29)–(3.34)) and show the well-posedness in Section 4 (cf. Theorem 4.1), are given by

$$\left.\begin{aligned}
&\partial_t c+\nabla\cdot(cV)=0,\\
&\partial_t\left([1-\alpha^2\mathcal{A}]u\right)+\nabla\cdot\left(V[1-\alpha^2\mathcal{A}]u\right)+\partial_\parallel\left(\frac{1}{2}\left(\frac{c^2}{4}-u^2\right)+\phi-\frac{\alpha^2}{2}|\nabla u|^2\right)=0,\\
&\left(Q-\frac{\alpha^2}{2}[\Delta Q+Q\Delta]\right)\phi=\int_0^1 da\int_{\mathbb{R}^+}d\mu\, c-\left(1-\frac{\alpha^2}{2}\Delta\right)n_0,\\
&c(t=0,x,a,\mu)=c^0(x,a,\mu),\quad u(t=0,x,a,\mu)=u^0(x,a,\mu),
\end{aligned}\right\}\tag{1.3}$$

where the differential operator $\mathcal{A}$ is defined by $\mathcal{A}\varphi=c^{-1}\nabla\cdot(c\nabla\varphi)$.

In the derivation of LAGWBC-α equation (1.3), we show that the parameter α can be chosen as the same order or some orders smaller or even larger than the Larmor gyroradius ρ_i. Therefore, this model offers a large perspective regarding to scale range on which we could average the motion. It should be emphasized that gyrokinetic-Vlasov equations and therefore gyrowaterbag equations are designed to accurately capture the dynamics of a magnetically confined plasma subjected to electrostatic turbulence, at length scales larger than ρ_i, while averaging the motion (namely the gyromotion) at scales smaller than ρ_i. Nevertheless, the incompressible nature of the flow in the transverse direction allows generation of scales smaller that ρ_i, with no possibility that these small scales are removed. Therefore, any model which attempts to cure this defect in a consistent and accurate way is welcome in the plasma turbulence community.

A third motivation for derivation of a Lagrangian averaged gyrowaterbag continuum is the Cauchy problem of the gyrowaterbag continuum equation (1.1). Indeed, the origin of the difficulty in the analysis of system (1.1) comes from the so-called quasineutrality limit for the Vlasov–Poisson equation, for which recent results [2–4, 24, 26] concerning stability versus instability issues and ill-posedness of the formal limit equation are discussed. In fact, combining the approachs of the papers [2, 5, 6], we could show the existence and uniqueness of the local-in-time classical solution to the gyrowaterbag continuum equation (1.1) under additional conditions which ensure the hyperbolicity of the system. As regards the existence of global weak solutions, the problem is open and actually very difficult, as it lies in the framework of infinite-dimensional hyperbolic system of first-order conservation laws in several space-dimensions. For LAGWBC-α equation (1.3), we can get rid of these additional assumptions yielding hyperbolicity, since the nature of the problem has changed by addition of dispersive terms. Moreover, in the system (1.1), the quasineutrality equation for the electrical potential fails to be elliptic in the whole space, since the operator Q (1.2) is elliptic only in the transverse direction. The loss of spatial derivatives in the longitudinal direction on the electrical potential is a truly difficult problem since, in the longitudinal direction, the coupling between fluid unknowns (density c and longitudinal mean velocity u) and electrostatic field (electrostatic potential ϕ) is algebraic (i.e. strong coupling). This is, for instance, the reason why the gyrowaterbag continuum equations (for the unknowns c, u, and ϕ)

must be considered as a hyperbolic system of first-order conservation laws with non-local fluxes and not as a weakly coupled system, which is the case for LAGWBC-α equations. Anisotropy in this kind of problem is a difficulty which has also been recently studied in [23, 25]. In addition, the Lagrangian averaged least-action principle and additional approximations for the electrical potential fluctuations allow us to recover the ellipticity of the averaged quasineutrality equation in the whole space (cf. the third equation of (1.3)). Concerning global well-posedness, at least for weak solutions, we could expect more tractable results for the LAGWBC-α model than the gyrowaterbag continuum equations. Let us note that various waterbag and gyrowaterbag models have given convincing and promising numerical results [5–11, 15, 16].

Organization of the rest of the paper. In Section 2, we first recall the gyrokinetic-Vlasov model in cylindrical geometry, from which we obtain the derivation of the gyrowaterbag continuum. The section ends by establishing the least-action principle leading to gyrowaterbag continuum equations. Section 3 is devoted to the formal derivation of LAGWBC-α equations by using the Lagrangian averaging techniques applied to the variational principle established in Section 2, with some supplementary approximations for the electrical potential fluctuations. In this section we also derive an isotropic version of the LAGWBC-α equations (cf. equations (3.29)–(3.34)) which must be seen as an approximation of the latter. Finally, in Section 4, we prove local-in-time well-posedness of the isotropic Lagrangian averaged gyrowaterbag continuum equations for periodic domains (cf. Theorem 4.1).

2. Gyrokinetic-waterbag equations

2.1. The gyrokinetic-Vlasov model. Within gyrokinetic Hamiltonian formalism and cylindrical geometry framework [18], the gyrokinetic-Vlasov equation expresses the fact that the ions gyrocenter distribution function $f = f(t, x, v_\parallel, \mu) = f(t, x_1, x_2, x_3, v_\parallel, \mu)$ is constant along gyrocenter characteristic curves in gyrocenter phase-space $(t, x, v_\parallel, \mu) \in \mathbb{R}_t^+ \times \mathbb{R}_x^3 \times \mathbb{R}_{v_\parallel} \times \Xi$:

$$\partial_t f + \mathcal{J}_\perp v_E \cdot \nabla_\perp f + v_\parallel \partial_\parallel f + \frac{q_i}{m_i} \mathcal{J}_\perp E_\parallel \partial_{v_\parallel} f = 0, \quad \forall \mu \in \Xi. \tag{2.1}$$

The ions distribution function f is coupled to the electrical potential ϕ via the quasi-neutrality equation

$$-\nabla_\perp \cdot \left(\frac{n_{i0}}{B_0 \Omega_0} \nabla_\perp \phi \right) + \frac{e \tau n_{i0}}{k_B T_{i0}} (\phi - \lambda \langle \phi \rangle_\parallel) = \int_\Xi \int_{\mathbb{R}} \mathcal{J}_\perp f(t, x, v_\parallel, \mu) dv_\parallel d\mu - n_{i0}, \tag{2.2}$$

with $\langle \phi \rangle_\parallel$ denoting the average of the electrical potential ϕ over a magnetic field line being straight lines parallel to the direction e_3. The longitudinal or parallel direction, denoted by the symbol $\|$, is the direction parallel to the magnetic field B (parallel to the axis of the cylindrical column), i.e. $b = e_3$ and the variable associated to this direction is denoted by $x_\parallel = x_3 \in \mathbb{R}$ while the transverse direction denoted by the symbol $\perp$ is perpendicular to b and the associated variable is denoted by $x_\perp = (x_1, x_2) \in \mathbb{R}^2$. In equations (2.1)–(2.2), $q_i = Z_i e$ and m_i are, respectively, the ion charge and mass, $Z_i n_{i0} = n_{e0}$ is the electronic density, $T_e = T_{e0}$ is the electronic temperature, $\tau = T_{i0}/T_{e0}$, $\lambda \in \{0, 1\}$, $E = -\nabla \phi$ is the electric field, $E_\parallel = E \cdot b$, and $v_E = E \times B / B_0^2 = \nabla_\perp^\perp \phi / B_0$ is the electrical drift velocity. Let us note that the magnetic moment μ is an invariant and thus it must be considered as a parameter or a label and not as a differential variable. The invariant μ belongs to an open subset Ξ of $\mathbb{R}^+$, while $d\mu$ stands for the Lebesgue measure. In addition of cylindrical geometry, we have supposed that the magnetic field

B is uniform and constant along the axis of the column (i.e. $B=B_0b$) which means that the perpendicular (with respect to the unit vector $b=e_3$) drift velocity does not admit any magnetic curvature or gradient effect and that the ion cyclotron frequency, $\Omega_0=q_iB_0/m_i$, is a constant.

Finally, the integral operator $\mathcal{J}_\perp$ stands for the gyroaverage operator defined by

$$\mathcal{J}_\perp f(x)=\frac{1}{2\pi}\int_0^{2\pi} d\zeta f(x+\rho(\zeta)),$$

where ζ is the gyroangle. The gyroradius vector ρ is given by $\rho(\zeta)=\sqrt{2\mu/(q_i\Omega_0)}\hat{a}(\zeta)$, where the vector $\hat{a}(\zeta)=\hat{x}\cos\zeta-\hat{y}\sin\zeta$ is defined in terms of the fixed local unit vectors basis $(\hat{x},\hat{y},b=\hat{x}\times\hat{y})$. Using the Fourier transform gyroaverage operator gives

$$\mathcal{J}_\perp f(x)=\int_{\mathbb{R}^3} dk\mathscr{F}[f](k)\exp(\mathrm{i}k\cdot x)J_0\left(k_\perp\sqrt{\frac{2\mu}{q_i\Omega_0}}\right),$$

where $k_\perp^2=k_{\hat{x}}^2+k_{\hat{y}}^2$ and J_0 is the Bessel function of the first kind and zero order.

Since the magnetic moment μ is not an independent variable but a parameter or a label related to an invariant, we can consider the plasma as a superposition of a (possibly uncountable) collection of a bunch of particles having the same initial magnetic moment μ. In other words, we can consider solution for the Vlasov equation (2.1) written in the form

$$f(t,x,v_\parallel,\mu)=\int_\Xi f_\nu(t,x,v_\parallel)\delta_\nu(\mu)\mathfrak{m}(d\nu),$$

where ν is a parameter belonging to some probability space Ξ, $\mathfrak{m}$ is a probability measure on that space, and f_ν are smooth functions which still satisfy the Vlasov equation (2.1) with $\mu=\nu$. For instance, we could take $\mathfrak{m}(d\nu)=\sum_l\varpi_\ell\delta(\nu-\mu_l)$, where ϖ_ℓ are positive constants. As a consequence, the distribution function f can be recast as

$$f(t,x,v_\parallel,\mu)=\sum_\ell\varpi_\ell f_{\mu_\ell}(t,x,v_\parallel)\delta(\mu-\mu_\ell),$$

where the function $f_{\mu_\ell}(t,x,v_\parallel)$ satisfies the Vlasov equation (2.1) with $\mu=\mu_\ell$, for all values of the index ℓ.

2.2. The gyrokinetic-waterbag model. For the description of the waterbag reduction concept applied to one-dimensional Vlasov equations, we refer the reader to [10]. We now apply it to the gyrokinetic-Vlasov equation (2.1). To this purpose, for every magnetic moment $\mu\in\Xi$, we consider two three-dimensional Lagrangian foliations of codimension one, of the four-dimensional phase-space $(x,v_\parallel)\subset\mathbb{R}^4$, to be the families of three-dimensional leaves $v_\mu^\pm(t,x,a)$, enumerated by the Lagrangian label a belonging to the one-dimensional set $[0,1]$. The collection $\{v_\mu^j\}_{\mu\in\Xi}$ with $j\in\{-,+\}$ is represented in compact form by $v^j=v^j(t,x,a,\mu)$. Therefore, the leaves $v^\pm$ can be reinterpreted as two three-dimensional Lagrangian foliations of codimension two, of the five-dimensional phase-space $(x,v_\parallel,\mu)\subset\mathbb{R}^3\times\mathbb{R}\times\mathbb{R}^+$, where the leaves are enumerated by the label $\sigma=(a,\mu)\in\Sigma=[0,1]\times\Xi$. We suppose that the leaves $v^\pm$ are smooth functions such that $v^-\leq v^+$, $\partial_a v^+\leq 0$, and $\partial_a v^-\geq 0$. If we now consider two non-closed single-valued smooth branches $v^\pm(t,x,a,\mu)$ of the $(x,v_\parallel,\mu)$-phase space, we can define $f_\mu(t,x,v_\parallel,\mu)$ as

$$f_\mu(t,x,v_\parallel,\mu)=\int_0^1\left(\mathcal{H}\left(v^+(t,x,a,\mu)-v_\parallel\right)-\mathcal{H}\left(v^-(t,x,a,\mu)-v_\parallel\right)\right)m(da), \tag{2.3}$$

where m denotes a probability measure on $[0,1]$ and $\mathcal{H}$ is the Heaviside unit step function. In the distribution function (2.3), if we choose $m(da)=\sum_i \mathcal{A}_i\delta(a-a_i)$, where $\mathcal{A}_i$ are positive constants, we recover the multiple waterbag distribution [10].

REMARK 2.1 (Generalization to a distribution function with $K-1$ changes of monotonicity in velocity). *The distribution function (2.3) is a unimodal distribution in parallel velocity, i.e., with only one change of monotonicity in parallel velocity, such as the bell-shaped or Gaussian distribution functions, for example. In order to represent an integrable unimodal velocity distribution function, we only need two branches v^j, $j\in\{1,2\}$, such as has been done for the function (2.3). We can generalize the representation (2.3) to the velocity distribution function with $K-1$ changes of monotonicity in velocity if we consider K branches (continuum) v^j, $j\in\{1,\dots,K\}$, where each branch v^j is monotonic with respect to the variable a. As an example of generalization of the representation (2.3) to a distribution function f_μ with K branches (K even), allowing us to represent a nonnegative distribution function with $K-1$ changes of monotonicity in velocity, we can define*

$$f_\mu(t,x,v_\parallel)=\sum_{j=1}^{K}(-1)^{j-1}\int_{a_j}^{b_j}\mathcal{H}\left(v^j(t,x,a,\mu)-v_\parallel\right)m(da),$$

where each branch v^j is monotonic with respect to the variable a, and the compact subsets $[a_j,b_j]\subset\mathbb{R}^+$ are such that $a_j<b_j$, $a_{2p}=a_{2p+1}$, $b_{2p+1}=b_{2p+2}$, and $a_1=a_K=0$. The representation formula (2.3) corresponds to the case $K=2$, $a_1=a_2=0$, and $b_1=b_2=1$. Another example could be the two-stream instability profile constituted by two bumps (e.g., two Maxwellians) and represented by four branches ($K=4$), such that $a_1=a_4=0$, $a_2=a_3=a$, and $b_1=b_2=b_3=b_4=b$, with $0\le a<b$. Let us notice that a branch v^j can become multivalued as time goes on because of the presence of nonlinear advection terms such as the Burgers' term in the second equation of (1.1). Multivaluedness appears when a branch ceases to be monotonic with respect to the a-variable or when contours intersect. If, after some time, a branch becomes multivalued, it means that new branches develop, and the distribution function must be represented by a larger number of branches than at the beginning because the number of oscillations of the distribution function in velocity increases. Nevertheless, we can force the number of branches in velocity to be fixed at the cost of a loss of regularity in physical space (with apparition of shocks such as in hydrodynamics or gaz dynamics) and a loss of information in phase-space (such as wave-breaking and filamentation phenomena).

As long as the contours are smooth, single-valued, and do not cross, the waterbag distribution function (2.3) is an exact weak solution of the gyrokinetic-Vlasov equation (2.1) in the sense of distribution theory if and only if the set of following equations is satisfied:

$$\partial_t v^\pm+\nabla_\perp\cdot(\mathcal{J}_\perp v_E v^\pm)+\partial_\parallel \mathrm{h}^\pm=0,\quad \forall\mu\in\Xi,\ \forall a\in[0,1], \tag{2.4}$$

with the contour Hamiltonians

$$\mathrm{h}^\pm=\frac{1}{2}{v^\pm}^2+\frac{q_i}{m_i}\mathcal{J}_\perp\phi.$$

The quasi-neutrality equation can be rewritten as

$$
\begin{aligned}
&-\nabla_\perp\cdot\left(\frac{n_{i0}}{B_0\Omega_0}\nabla_\perp\phi\right)+\frac{e\tau n_{i0}}{k_BT_{i0}}(\phi-\lambda\langle\phi\rangle_\parallel)\\
&=\int_0^1 m(da)\int_\Xi \mathfrak{m}(d\mu)\,\mathcal{J}_\perp(v^+-v^-)-n_{i0}.
\end{aligned}
\tag{2.5}
$$

Let us introduce the density $c=(v^+-v^-)$ and the average velocity $u=(v^++v^-)/2$. After a little algebra, equation (2.4) leads to continuity and Euler-type equations namely, $\forall\mu\in\Xi$, $\forall a\in[0,1]$,

$$\partial_t c+\nabla_\perp\cdot(c\mathcal{J}_\perp v_E)+\partial_\parallel(cu)=0, \tag{2.6}$$

$$\partial_t(cu)+\nabla_\perp\cdot(cu\mathcal{J}_\perp v_E)+\partial_\parallel\left(cu^2+\mathfrak{p}\right)+\frac{q_i}{m_i}c\partial_\parallel\mathcal{J}_\perp\phi=0, \tag{2.7}$$

where the partial pressure takes the form $\mathfrak{p}=c^3/12$. The connection between kinetic and fluid description clearly appears in the previous multi-fluids equations (with an exact adiabatic closure with $\gamma=3$). To complete the system (2.4)–(2.5), we need to supply an initial condition $v^\pm(t=0,x,a,\mu)=v^{0\pm}(x,a,\mu)$ or equivalently $c(t=0,x,a,\mu)=c^0(x,a,\mu)$ and $u(t=0,x,a,\mu)=u^0(x,a,\mu)$, $\forall\mu\in\Xi$, $\forall a\in[0,1]$.

2.3. The variational principle. In this section, we establish a variational principle to derive the equations of motion (2.5)–(2.7) of the gyrowaterbag continuum as the stationary point of an action functional.

For almost every $\sigma\in\Sigma$, let $\mathcal{M}_\sigma$ be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\mathcal{M}_\sigma$ containing the fluid labeled by the tag σ. Suppose we are given a Lagrangian function $L=L(\eta,\dot\eta,\mathfrak{M}^0,\phi)$ with, for almost every fixed $\sigma\in\Sigma$, $\eta_\sigma=\eta(\cdot,\sigma)\in\mathrm{Diff}(\mathcal{M}_\sigma)$, where $\mathrm{Diff}(\mathcal{M}_\sigma)$ denotes the space of diffeomorphism of $\mathcal{M}_\sigma$. For almost every fixed $\sigma\in\Sigma$, let $\mathfrak{M}^0_\sigma=\mathfrak{M}^0(\cdot,\sigma)\in\Lambda^3(\mathcal{M}_\sigma)$ be the space of 3-forms on $\mathcal{M}_\sigma$. We suppose that $\phi\in W(\mathcal{D})$, where $W(\mathcal{D})$ is the space of a real-valued function of some given Sobolev class on a bounded domain $\mathcal{D}$. Since $\mathfrak{M}^0_\sigma\in\Lambda^3(\mathcal{M}_\sigma)$, we suppose that it can be written as $\mathfrak{M}^0_\sigma=c^0_\sigma dx_1\wedge dx_2\wedge dx_3$, where $c^0_\sigma=c^0(\cdot,\sigma)$ is a smooth function on $\mathcal{M}_\sigma$. The physical interpretation of $c^0_\sigma(X)$, $X\in\mathcal{M}_\sigma$ is the initial density of the fluid σ at the material point X. Now, if we denote by $c_\sigma(t,x)=c(t,x,\sigma)$ the spatial density of the fluid σ, we define $\mathfrak{M}_\sigma=c_\sigma dx_1\wedge dx_2\wedge dx_3$, and we have the relationship $\mathfrak{M}_\sigma=(\eta_\sigma)_*\mathfrak{M}^0_\sigma$ and $\mathfrak{M}^0_\sigma=(\eta_\sigma)^*\mathfrak{M}_\sigma$, where $(\eta_\sigma)_*$ and $(\eta_\sigma)^*$ denotes, respectively, the push-forward and the pull-back operators. With compact notation, we write $\mathfrak{M}=\eta_*\mathfrak{M}^0$ and $\mathfrak{M}^0=\eta^*\mathfrak{M}$. We next suppose that for almost every $\sigma\in\Sigma$, the domains $\mathcal{M}_\sigma$ are identical to $\mathcal{D}$, and we set $\mathcal{M}=\mathcal{D}\times\Sigma$. Let us define the probability measure $\nu(d\sigma)=\mathfrak{m}(d\mu)\otimes m(da)$ on Σ, i.e.,

$$\int_\Sigma \nu(d\sigma)=1.$$

Finally, let us define the functional space

$$L^p_\nu(\Sigma)=L^p(d\nu;\Sigma)=\left\{g:\Sigma\to\mathbb{R},\nu-\text{measurable}\;\middle|\;\|g\|_{L^p_\nu(\Sigma)}=\left(\int_\Sigma\nu(d\sigma)|g|^p\right)^{1/p}<\infty\right\},$$

for $1\le p<\infty$ and set the norm $\|\cdot\|_{L^p_\nu\cap L^\infty(\Sigma)}=\|\cdot\|_{L^p_\nu(\Sigma)}+\|\cdot\|_{L^\infty(\Sigma)}$. We then suppose that $\eta\in\mathcal{C}(I;L^p_\nu(\Sigma;\mathrm{Diff}(\mathcal{D})))$ and $\mathfrak{M}^0,\mathfrak{M}\in L^p_\nu(\Sigma;\Lambda^3(\mathcal{D}))$ for $1\le p<\infty$.

Let us now establish an energy conservation law in order to determine the Lagrangian L. Before that, from equations (2.6)–(2.7) it is worthwhile to note that

$$\partial_t\int_{\mathcal{M}_\sigma}c\,dx=0,\quad\forall\sigma\in\Sigma,\quad\text{and}\quad\partial_t\int_{\mathcal{M}_\sigma}u\,dx=0,\quad\forall\sigma\in\Sigma,$$

where the first equality translates Liouville geometric invariants preservation. Now, using equations (2.6)–(2.7), we obtain the energy conservation law

$$\partial_t\left(\frac{e}{2}\right)+\nabla_\perp\cdot\left(\frac{e}{2}\mathcal{J}_\perp v_E\right)+\partial_\parallel\left(u\left[\frac{e}{2}+\mathfrak{p}\right]\right)+\frac{q_i}{m_i}cu\partial_\parallel\mathcal{J}_\perp\phi=0, \tag{2.8}$$

where $e=({v^+}^3-{v^-}^3)/3=cu^2+c^3/12$ denotes the energy density of the multi-fluids and $\mathfrak{p}=c^3/12$ the pressure. Now, from equation (2.8), using the quasineutrality equation (2.5) and continuity equation (2.6) to rewrite the last term of the left-hand side of (2.8), using integration by parts and the property $\int\psi\mathcal{J}_\perp\phi dx=\int\phi\mathcal{J}_\perp\psi dx$, after integration on $\mathcal{M}$, we obtain

$$\frac{d}{dt}\mathrm{H}(t)=0,$$

where H is the time-dependent Hamiltonian defined by

$$\begin{aligned}\mathrm{H}&=\int_{\mathcal{M}}dx\,m(da)\mathfrak{m}(d\mu)\frac{e}{2}+\frac{1}{2}\frac{q_i}{m_i}\int_{\mathcal{D}}dx\left(\frac{n_{i0}}{B_0\Omega_0}|\nabla_\perp\phi|^2+\frac{e\tau n_{i0}}{k_BT_{i0}}|\phi-\langle\phi\rangle_\parallel|^2\right)\\&=\frac{1}{2}\int_{\mathcal{M}}dx\,\nu(d\sigma)\left\{e+\frac{q_i}{m_i}\left(\frac{n_{i0}}{B_0\Omega_0}|\nabla_\perp\phi|^2+\frac{e\tau n_{i0}}{k_BT_{i0}}|\phi-\langle\phi\rangle_\parallel|^2\right)\right\}.\end{aligned}$$

In the definition of H, we recognize the kinetic energy K defined by

$$K(t)=\frac{1}{2}\int_{\mathcal{M}}dx\,\nu(d\sigma)\,cu^2,$$

and thus the Hamiltonian can be rewritten as $\mathrm{H}=K+V$, where V is the potential energy. Therefore we define the Lagrangian L as

$$\begin{aligned}L&=K-V\\&=\frac{1}{2}\int_{\mathcal{M}}dx\,\nu(d\sigma)\,cu^2-\frac{1}{2}\int_{\mathcal{M}}dx\,\nu(d\sigma)\left\{\frac{c^3}{12}+\frac{q_i}{m_i}\left(\frac{n_{i0}}{B_0\Omega_0}|\nabla_\perp\phi|^2+\frac{e\tau n_{i0}}{k_BT_{i0}}|\phi-\langle\phi\rangle_\parallel|^2\right)\right\}\\&=\frac{1}{2}\int_{\mathcal{M}}dx\,\nu(d\sigma)\,cu^2-\frac{1}{2}\int_{\mathcal{M}}dx\,\nu(d\sigma)\left\{\frac{c^3}{12}-\frac{q_i}{m_i}\left(\frac{n_{i0}}{B_0\Omega_0}|\nabla_\perp\phi|^2+\frac{e\tau n_{i0}}{k_BT_{i0}}|\phi-\langle\phi\rangle_\parallel|^2\right)\right.\\&\qquad\qquad\left.+2\frac{q_i}{m_i}\phi(\mathcal{J}_\perp c-n_{i0})\right\}.\end{aligned} \tag{2.9}$$

If X is a material point in the reference configuration of the fluid σ, we then define the Lagrangian flow (a path in $\mathrm{Diff}(\mathcal{D})$, i.e. a one-parameter family of smooth material deformation map of $\mathcal{D}$) associated to the fluid σ, $\eta_t=(\eta_\sigma)_t=\eta(t,X,\sigma)$, for $\sigma\in\Sigma$ and $X\in\mathcal{D}$, by

$$\dot\eta(t,X,\sigma)=\frac{d}{dt}\eta(t,X,\sigma)=V(t,\eta(t,X,\sigma),\sigma)=(V\circ\eta)(t,X,\sigma),$$

with the initial condition $\eta(t=0,X,\sigma)=X$, $\forall\sigma\in\Sigma$ and where the Eulerian velocity field $V=V(t,x,\sigma)=\dot\eta(t,\eta^{-1}(t,x,\sigma),\sigma)$ is defined by

$$V(t,x,\sigma)=\begin{pmatrix}\mathcal{J}_\perp v_E(t,x,\mu)\\u(t,x,\sigma)\end{pmatrix}=\begin{pmatrix}\frac{1}{B_0}b\times\nabla\mathcal{J}_\perp[\mu]\phi(t,x)\\u(t,x,\sigma)\end{pmatrix},$$

where u and ϕ are supposed to be smooth functions. Provided that velocity field V is regular in space (typically Lipschitz continuous) and the density c is integrable on $\mathcal{M}$, we can integrate the continuity equation (2.6) using the characteristic curves η_t and we find that the unique solution of (2.6) is given by

$$\begin{aligned}c(t,x,\sigma)&=c^0(X,\sigma)\det{}^{-1}(\nabla_X\eta(t,X,\sigma))\\&=c^0(\eta^{-1}(t,x,\sigma),\sigma)\det(\nabla_x\eta^{-1}(t,x,\sigma))=\eta_*c^0=\eta^{-1*}c^0,\end{aligned}$$

where

$$\begin{aligned}\det(\nabla_X\eta(t,X,\sigma))&=\exp\left(\int_0^t(\nabla\cdot V)(s,\eta(s,X,\sigma),\sigma)ds\right)\\&=\exp\left(\int_0^t(\partial_\parallel u)(s,\eta(s,X,\sigma),\sigma)ds\right).\end{aligned}$$

Therefore the variational principle in Lagrangian coordinates is determined as follows. The gyrowaterbag equations (2.5)–(2.7) are obtained as the stationary point of the action functional $S:\mathcal{C}(I;L^p_\nu(\Sigma;\mathrm{Diff}(\mathcal{D})))\times L^p_\nu(\Sigma;\Lambda^3(\mathcal{D}))\times W(\mathcal{D})\to\mathbb{R}$ defined by

$$S(\eta,\mathfrak{M}^0,\phi)=\int_0^T L(\eta(t),\dot\eta(t),\mathfrak{M}^0,\phi(t))dt, \tag{2.10}$$

where using (2.9), $L(\eta,\dot\eta,\mathfrak{M}^0,\phi)$ is given by

$$\begin{aligned}&L(\eta,\dot\eta,\mathfrak{M}^0,\phi)\\&=\int_{\mathcal{D}}dX\int_\Sigma\nu(d\sigma)\det(\nabla_X\eta)\left(\frac{1}{2}c^0(X,\sigma)\det{}^{-1}(\nabla_X\eta)\dot\eta_\parallel^2-\frac{1}{24}c^0(X,\sigma)^3\det{}^{-3}(\nabla_X\eta)\right)\\&\quad+\frac{1}{2}\frac{q_i}{m_i}\int_{\mathcal{D}}dx\int_\Sigma\nu(d\sigma)\left(\frac{n_{i0}}{B_0\Omega_0}|\nabla_\perp\phi|^2+\frac{e\tau n_{i0}}{k_BT_{i0}}|\phi-\langle\phi\rangle_\parallel|^2+2\phi n_{i0}\right)\\&\quad-\frac{q_i}{m_i}\int_{\mathcal{D}}dX\int_\Sigma\nu(d\sigma)\mathcal{J}_\perp\phi(t,\eta)c^0(X,\sigma) \qquad (2.11)\\&=\int_{\mathcal{D}}dX\int_\Sigma\nu(d\sigma)\det(\nabla_X\eta)\left(\frac{1}{2}c^0(X,\sigma)\det{}^{-1}(\nabla_X\eta)\dot\eta_\parallel^2-\frac{1}{24}c^0(X,\sigma)^3\det{}^{-3}(\nabla_X\eta)\right)\\&\quad+\frac{1}{2}\frac{q_i}{m_i}\int_{\mathcal{D}}dx\int_\Sigma\nu(d\sigma)\left(\frac{n_{i0}}{B_0\Omega_0}|\nabla_\perp\phi|^2+\frac{e\tau n_{i0}}{k_BT_{i0}}|\phi-\langle\phi\rangle_\parallel|^2\right)\\&\quad-\frac{q_i}{m_i}\int_{\mathcal{D}}dx\int_\Sigma\nu(d\sigma)\phi(\mathcal{J}_\perp c-n_{i0}). \qquad (2.12)\end{aligned}$$

Setting to zero the functional derivative $\delta S/\delta\eta_\parallel$, using (2.10)–(2.11), we obtain the equation

$$c\ddot\eta_\parallel+\partial_\parallel\mathfrak{p}+(q_i/m_i)c\partial_\parallel\mathcal{J}_\perp\phi=0,$$

which is equivalent to (2.7) (since $\ddot\eta_\parallel=D_tu$, with $D_t=\partial_t+V\cdot\nabla$), while, the functional derivative has vanished $\delta S/\delta\phi$, using (2.10) and (2.12), we get equation (2.5).

We can also derive the gyrowaterbag equations (2.5)–(2.7) from a least-action principle in Eulerian coordinates from the Lagrangian $L=L(u,\mathfrak{M},\phi)$ defined by (2.9) by using the Euler–Poincaré equations [28]. One advantage of the Euler–Poincaré theory is that computations are more straightforward than in the case of Hamilton's variational principle in Lagrangian coordinates, for which computations are often cumbersome.

3. Lagrangian averaged gyrokinetic-waterbag equations

This section is concerned with the derivation of the Lagrangian averaged gyrowaterbag equations. The idea is to construct an averaged action $S^\alpha(\eta,\mathfrak{M}^0,\phi)$ where α is the spatial length scale characterizing the coarseness of the average and then to use a least-action principle in Lagrangian (Hamilton's variational principle) or Eulerian (Euler–Poincaré theory) coordinates to obtain the equation of motion. Let us precise that the asymptotic analysis performed below remains formal and that convergence issues within a defined functional framework are not handled since the asymptotic expansions are not explicitly solvable.

3.1. Lagrangian averaging framework. Following methods exposed in [13, 32], we first introduce the deformation map $\xi^\epsilon(t,x,\sigma)$ to be the family of diffeomorphism about the identity, i.e. for each $\epsilon\geq 0$, $\xi^\epsilon(t,\cdot,\sigma)\in\mathrm{Diff}(\mathcal{D})$ for all $t\in I$ and for almost every $\sigma\in\Sigma$ and $\xi^\epsilon(t,x,\sigma)=x$ at $\epsilon=0$ for almost every $(t,x,\sigma)\in I\times\mathcal{D}\times\Sigma$. We suppose that the deformation map ξ^ϵ admits a Taylor expansion series with respect to the parameter ϵ, i.e.,

$$\xi^\epsilon=\mathrm{Id}+\epsilon\xi'+\frac{1}{2}\epsilon^2\xi''+\ldots+\frac{1}{n!}\epsilon^n\xi^{(n)}+\ldots,\quad\text{where,}\quad \xi^{(n)}=\left.\frac{\partial^n}{\partial\epsilon^n}\right|_{\epsilon=0}\xi^\epsilon. \tag{3.1}$$

Using ξ^ϵ, we construct a perturbed Lagrangian flow η^ϵ close to η by setting

$$\eta^\epsilon(t,X,\sigma)=\xi^\epsilon(t,\eta(t,X,\sigma),\sigma),\quad(\text{shortly}\quad \eta^\epsilon=\xi^\epsilon\circ\eta). \tag{3.2}$$

From the perturbed flow η^ϵ we can define a perturbed velocity field V^ϵ such that

$$V^\epsilon(t,x,\sigma)=\dot\eta^\epsilon(t,(\eta^\epsilon)^{-1}(t,x,\sigma),\sigma),\quad(\text{shortly}\quad V^\epsilon=\dot\eta^\epsilon\circ(\eta^\epsilon)^{-1},\ \text{ or }\ \dot\eta^\epsilon=V^\epsilon\circ\eta^\epsilon). \tag{3.3}$$

We now assume the existence of an ensemble averaging operation $\langle\cdot\rangle$ whose properties are described below. To this purpose, let us introduce $\mathfrak{X}(\mathcal{M})$ as a space of fluctuation fields (e.g. $\xi^{(n)}$) defined on $\mathcal{M}$ modeling the perturbations (which can been seen as multivariate random variables) and set $Y=[0,\alpha]\times\mathfrak{X}$. Let $\mathcal{F}(Y)$ be the space of smooth real-valued functions on Y. Therefore, we assume that the averaging operation $\langle\cdot\rangle:\mathcal{F}(Y)\to\mathcal{F}(\mathcal{M})$ satisfies the following properties:

1) linearity: $\langle af+bg\rangle=a\langle f\rangle+b\langle g\rangle$;
2) independence: $\langle\psi h\rangle=(1/\alpha)\langle h\rangle\int_0^\alpha\psi(\epsilon)d\epsilon$, with $\psi h\in\mathcal{F}(Y)$ being understood as the pointwise product;
3) commutativity: $\langle\int_{\mathcal{M}}f\,dxd\sigma\rangle=\int_{\mathcal{M}}\langle f\rangle dxd\sigma$, $\langle\partial f\rangle=\partial\langle f\rangle$, with $\partial\in\{\partial_t,\partial_x\}$;

where in the above properties $f,g\in\mathcal{F}(Y)$, $a,b\in\mathbb{R}$, $\psi\in\mathcal{F}([0,\alpha])$ and $h\in\mathcal{F}(\mathfrak{X})$. For examples of such averaging operation, we refer to [13, 32]. As an example, if P denotes a probability measure on the unit sphere $\mathcal{S}$ in $\mathfrak{X}(\mathcal{M})$ and if we define the ensemble average of vector-valued function $f(\epsilon,w)$ on $[0,\alpha]\times\mathcal{S}$ (i.e. $f\in\mathcal{F}(Y)$, $Y=[0,\alpha]\times\mathcal{S}$) by

$$\langle f\rangle:=\frac{1}{\alpha}\int_0^\alpha d\epsilon\int_{\mathcal{S}}P(dw)\,f(\epsilon,w),$$

then we can check that this ensemble averaging operation satisfies the above required properties.

In order to have the fluctuations ξ^ϵ centered on average about the identity (so that the average will not be skewed in an arbitrary direction), i.e. $\langle\xi^\epsilon(t,x,\sigma)\rangle=x$, for almost every $(t,x,\sigma)\in I\times\mathcal{D}\times\Sigma$, it is sufficient to assume that the nth-order fluctuation vector fields have means zero for $n\geq 1$, i.e.

$$\left\langle\left.\frac{\partial^n\xi^\epsilon}{\partial\epsilon^n}\right|_{\epsilon=0}\right\rangle=0,\quad n\geq 1. \tag{3.4}$$

Using definitions (3.2)–(3.3) and property (3.4), we obtain that $\langle \eta^\epsilon(t,X,\sigma)\rangle = \eta(t,X,\sigma)$ and $\langle V^\epsilon \circ \xi^\epsilon(t,x,\sigma)\rangle = V(t,x,\sigma)$, which means that the flow η and the velocity field V can be seen as Lagrangian means of η^ϵ and V^ϵ, respectively.

Now the perturbed fields c^ϵ, u^ϵ, ϕ^ϵ, and thus V^ϵ are Taylor expanded with respect to the small perturbation parameter ϵ,

$$c^\epsilon(t,x,\sigma) = c(t,x,\sigma) + \epsilon c'(t,x,\sigma) + \frac{1}{2}\epsilon^2 c''(t,x,\sigma) + \ldots + \frac{1}{n!}\epsilon^n c^{(n)}(t,x,\sigma) + \ldots, \tag{3.5}$$

$$u^\epsilon(t,x,\sigma) = u(t,x,\sigma) + \epsilon u'(t,x,\sigma) + \frac{1}{2}\epsilon^2 u''(t,x,\sigma) + \ldots + \frac{1}{n!}\epsilon^n u^{(n)}(t,x,\sigma) + \ldots, \tag{3.6}$$

$$\phi^\epsilon(t,x,\sigma) = \phi(t,x) + \epsilon \phi'(t,x,\sigma) + \frac{1}{2}\epsilon^2 \phi''(t,x,\sigma) + \ldots + \frac{1}{n!}\epsilon^n \phi^{(n)}(t,x,\sigma) + \ldots, \tag{3.7}$$

$$V^\epsilon(t,x,\sigma) = V(t,x,\sigma) + \epsilon V'(t,x,\sigma) + \frac{1}{2}\epsilon^2 V''(t,x,\sigma) + \ldots + \frac{1}{n!}\epsilon^n V^{(n)}(t,x,\sigma) + \ldots.,$$

where $\mathcal{X}^{(n)} = \partial_\epsilon^n \mathcal{X}^\epsilon_{|_{\epsilon=0}}$ with $\mathcal{X} \in \{c,u,\phi,V\}$ and

$$V^{(n)}(t,x,\sigma) = \begin{pmatrix} \frac{1}{B_0} b \times \nabla \mathcal{J}_\perp \phi^{(n)}(t,x,\sigma) \\ u^{(n)}(t,x,\sigma) \end{pmatrix},$$

by linearity. Since we will only retain fluctuation effects of order less than or equal to second order in ϵ, the goal is now to find expressions for u', u'', ϕ', ϕ'', c', c'', V', and V'' in terms of c, u, ϕ, V, ξ', and ξ''.

By Taylor expansion of (3.3) in x-space around the point η, using the definition (3.2) and the Taylor expansion (3.1), we obtain the following results, originally derived in [32],

$$V' = \partial_t \xi' + [V,\xi'], \tag{3.8}$$

$$V'' = \partial_t \xi'' + [V,\xi''] - 2(\xi' \cdot \nabla)V' - \xi' \otimes \xi' : \nabla\nabla V, \tag{3.9}$$

where $[A,B] = \pounds_A B = (A\cdot\nabla)B - (B\cdot\nabla)A$ is the standard Jacobi-Lie bracket of vector fields on $\mathcal{D}$ (see [1] for example) and $(\xi' \otimes \xi' : \nabla\nabla V)_k = \xi'_i \xi'_j \partial^2_{ij} V_k$ for $k \in \{1,2,3\}$. Here we use the convention that an index variable appearing twice in a single term, implies the summation of that term over all the values of the index.

By assuming that $\mathfrak{M}^0$ is independent from ϵ (i.e. $\partial_\epsilon^n \mathfrak{M}^0{}_{|_{\epsilon=0}} = 0$, $n>0$; $\mathfrak{M}^0 = \eta^* \mathfrak{M}$, $n=0$), differentiating equation $\mathfrak{M}^0 = (\eta^\epsilon)^* \mathfrak{M}^\epsilon$, with respect to ϵ and using the Lie derivative Theorem for time-dependent vector fields (see [1] for example), we obtain the following results, originally performed in the appendix of [13]:

$$c = \eta_* c^0, \tag{3.10}$$

$$c' = -\nabla \cdot (c\xi'), \tag{3.11}$$

$$c'' = \nabla\cdot(\nabla\cdot(c\xi' \otimes \xi')) - \nabla\cdot(c\xi''). \tag{3.12}$$

If we now start from the action

$$S(u^\epsilon, c^\epsilon, \phi^\epsilon) = \int_0^T dt \int_{\mathcal{M}} dx d\nu(\sigma) \left\{ \frac{1}{2}\left(c{u^\epsilon}^2 - \frac{{c^\epsilon}^3}{12} \right) - \frac{q_i}{m_i}\phi^\epsilon \mathcal{J}_\perp(c^\epsilon - c_0) \right.$$
$$\left. + \frac{1}{2}\frac{q_i}{m_i}\left(\frac{n_{i0}}{B_0\Omega_0}|\nabla_\perp \phi^\epsilon|^2 + \frac{e\tau n_{i0}}{k_B T_{i0}}|\phi^\epsilon - \langle\phi^\epsilon\rangle_\parallel|^2 \right) \right\},$$

where we have assumed that $n_{i0} = \int_\Sigma \mathcal{J}_\perp c_0 \, d\nu(\sigma)$, and plug into it the Taylor expansion series (3.5)–(3.7), we obtain at the second order in ϵ

$$
\begin{aligned}
S(u^\epsilon, c^\epsilon, \phi^\epsilon) = & \int_0^T dt \int_{\mathcal{M}} dx d\nu(\sigma) \left\{ \frac{1}{2}\left(cu^2 - \frac{c^3}{12}\right) - \frac{q_i}{m_i}\phi \mathcal{J}_\perp (c - c_0) \right. \\
& \left. + \frac{1}{2}\frac{q_i}{m_i}\left(\frac{n_{i0}}{B_0\Omega_0}|\nabla_\perp \phi|^2 + \frac{e\tau n_{i0}}{k_B T_{i0}} |\phi - \langle \phi \rangle_{\parallel}|^2 \right)\right\} \\
& + \epsilon \int_0^T dt \int_{\mathcal{M}} dx d\nu(\sigma) \left\{ cuu' + \frac{1}{2}c'u^2 - \frac{1}{8}c^2 c' - \frac{q_i}{m_i}\phi' \mathcal{J}_\perp (c - c_0) - \frac{q_i}{m_i} c' \mathcal{J}_\perp \phi \right. \\
& \left. + \frac{q_i}{m_i}\frac{n_{i0}}{B_0\Omega_0}\nabla_\perp \phi \cdot \nabla_\perp \phi' + \frac{q_i}{m_i}\frac{e\tau n_{i0}}{k_B T_{i0}} (\phi - \langle \phi \rangle_{\parallel})(\phi' - \langle \phi' \rangle_{\parallel}) \right\} \\
& + \epsilon^2 \int_0^T dt \int_{\mathcal{M}} dx d\nu(\sigma) \left\{ c'u'u + \frac{1}{4}c''u^2 + \frac{1}{2}c(u'^2 + uu'') - \frac{1}{8}\left[cc'^2 + \frac{1}{2}c^2 c'' \right] \right. \\
& + \frac{q_i}{m_i}\left(-\frac{1}{2}\phi'' \mathcal{J}_\perp (c - c_0) - c' \mathcal{J}_\perp \phi' - \frac{1}{2}\phi \mathcal{J}_\perp c'' \right) + \frac{1}{2}\frac{q_i}{m_i}\left(\frac{n_{i0}}{B_0\Omega_0} (|\nabla_\perp \phi'|^2 + \nabla_\perp \phi'' \cdot \nabla_\perp \phi) \right. \\
& \left. \left. + \frac{e\tau n_{i0}}{k_B T_{i0}} \left[(\phi' - \langle \phi' \rangle_{\parallel})^2 + (\phi - \langle \phi \rangle_{\parallel})(\phi'' - \langle \phi'' \rangle_{\parallel}) \right] \right) \right\} + \mathcal{O}(\epsilon^3). \qquad (3.13)
\end{aligned}
$$

3.2. Modeling rules. From assumption (3.4), i.e., $\langle \xi' \rangle = \langle \xi'' \rangle = 0$, and expressions (3.8)–(3.12), we observe that all linear functions of ξ' and ξ'' and their derivatives in (3.13) will vanish after averaging operation and particularly the entire $\mathcal{O}(\epsilon)$ collection. Assumption (3.4) also implies that in the $\mathcal{O}(\epsilon^2)$ collection, after averaging operation, all terms will depend on nonlinear functions of ξ' only. Therefore, we need a strategy for modeling the fluctuation ξ'. Like in the turbulence closure problem, modeling ξ' consists of specifying the Lagrangian fluctuation ξ' in terms of the mean quantities u, ϕ, and c. To do so, we shall invoke the generalization (introduced by the authors of [32]) of the classical frozen turbulence hypothesis introduced by Taylor in [35], which states that scalar fluctuation is simply advected by the mean flow. Following Taylor's ideas, in [32], Marsden and Shkoller assume that the Lagrangian fluctuation ξ' is Lie advected by the mean flow, i.e.

$$\partial_t \xi' + \pounds_V \xi' = 0. \qquad (3.14)$$

Remark 3.1. Let us note that the Lie-advection flow rule (3.14) is reminiscent to what is done in the quasilinear theory of electrostatic plasma turbulence [12, 19–21, 29, 39] where, fluctuations also satisfy quasilinear equations, i.e. advection-type equations where advection coefficients are determined through mean flow quantities and where nonlinear terms in the fluctuations are neglected since they are terms of one order smaller. Even if a rigorous mathematical proof of the validity of the quasilinear theory of electrostatic plasma turbulence is still missing, numerical [12, 21] and experimental [38] tests of quasilinear theory show that it works very well. Therefore, the classical frozen turbulence hypothesis according to which fluctuations are transported by the mean flow seems recurrent in different turbulence theories both in plasma physics and fluid mechanics, which gives a kind of rational but of course not a rigorous argument to use it, according to its numerical or experimental (a posteriori) success.

From closure Assumption (3.14), using (3.8)–(3.9), we deduce that

$$V' = 0, \quad \text{i.e.} \quad u' = 0, \text{ and } \phi' = 0,$$

and

$$\langle V''\rangle = -F : \nabla\nabla V, \quad \text{with} \quad F = \langle \xi' \otimes \xi' \rangle.$$

The previous equation is equivalent to

$$\begin{aligned} -\partial_2 \mathcal{J}_\perp \langle \phi'' \rangle &= \partial^2_{jk}(\partial_2 \mathcal{J}_\perp \phi) F^{jk} = F : \nabla\nabla \partial_2 \mathcal{J}_\perp \phi, \\ \partial_1 \mathcal{J}_\perp \langle \phi'' \rangle &= -\partial^2_{jk}(\partial_1 \mathcal{J}_\perp \phi) F^{jk} = -F : \nabla\nabla \partial_1 \mathcal{J}_\perp \phi, \\ \langle u'' \rangle &= -\partial^2_{jk} u F^{jk} = -F : \nabla\nabla u. \end{aligned}$$

From the previous equations, we infer that $-\Delta_\perp \mathcal{J}_\perp \langle \phi'' \rangle = g$ with $g := \nabla_\perp \cdot (F^{ij} \partial_{ij} \nabla_\perp \mathcal{J}_\perp \phi) = \nabla_\perp \cdot (F : \nabla\nabla\nabla_\perp \mathcal{J}_\perp \phi)$. If we now suppose the boundary condition $F_{|\partial\mathcal{D}} = 0$ for all t and $\sigma \in \Sigma$ (which is consistent with the fact that fluctuations must vanish along the boundary), then the source term g has zero mean with respect to the x-variable. Therefore, $\mathscr{F}[\mathcal{J}_\perp \langle \phi'' \rangle](k=0) = (J_0 \mathscr{F}[\langle \phi'' \rangle])(k=0)$ is unspecified, where $\mathscr{F}[\cdot]$ denotes the space Fourier transform in the x-variable. Therefore it is consistent to impose $(J_0 \mathscr{F}[\langle \phi'' \rangle])(k=0) = 0$, which is equivalent to imposing $\mathscr{F}[\langle \phi'' \rangle](k=0) = 0$ or $\int_{\mathcal{D}} dx \langle \phi'' \rangle = 0$ for all t and $\sigma \in \Sigma$. Consequently, we can invert $\Delta_\perp$ to obtain

$$\mathcal{J}_\perp \langle \phi'' \rangle = -\Delta_\perp^{-1} \nabla_\perp \cdot (F : \nabla\nabla\nabla_\perp \mathcal{J}_\perp \phi). \tag{3.15}$$

From (3.15), we would like to isolate $\langle \phi'' \rangle$, but it is not possible since the integral operator $\mathcal{J}_\perp$ is not invertible. Therefore, we have two choices to solve this problem. The first one consists of modifying the action functional (3.13) by high-order terms of order $\mathcal{O}(\epsilon^\beta)$, with $\beta > 2$, in such a way that we can replace $\langle \phi'' \rangle$ by $\mathcal{J}_\perp \langle \phi'' \rangle$ in (3.13). This procedure is easy to perform but weakens the regularity on ϕ, and even worse it does not make the quasineutrality operator (right-hand side of (2.5)) elliptic in $\mathbb{R}^3$ since the zeros of the Bessel function J_0 will cancel elliptic estimates. The second method consists of approximating the integral operator $\mathcal{J}_\perp$ by an invertible operator with an approximation error of order at least $\mathcal{O}(\alpha^\beta)$, with $\beta > 0$. Provided that $k_\perp \rho_i < 1$, by using first-order Taylor expansion of the Bessel function J_0 and first-order Padé rational approximation, we have

$$\mathscr{F}[\mathcal{J}_\perp] = J_0(k_\perp \rho_i) = 1 - \frac{1}{4}(k_\perp \rho_i)^2 + \mathcal{O}\left((k_\perp \rho_i)^4\right) = \left(1 + \frac{1}{4}(k_\perp \rho_i)^2\right)^{-1} + \mathcal{O}\left((k_\perp \rho_i)^4\right),$$

and thus we obtain

$$\mathcal{J}_\perp = \widetilde{\Delta}_\perp^{-1} + \mathcal{O}\left(\rho_i^4\right) := \left(1 - \frac{\rho_i^2}{4}\Delta_\perp\right)^{-1} + \mathcal{O}\left(\rho_i^4\right).$$

Substituting operator $\mathcal{J}_\perp$ by $\widetilde{\Delta}_\perp^{-1} := (1 - \rho_i^2 \Delta_\perp / 4)^{-1}$ into (3.15) leads to an error term of order $\mathcal{O}(\alpha^2 \rho_i^4)$ in (3.13). Therefore, we can choose α such that $\alpha^2 \rho_i^4 \lesssim \alpha^{2+\beta}$, with $\beta > 0$, i.e. we can take $\alpha \in]\rho_i^{4/\beta}, \rho_0[$, with $\rho_0 < 1$. Typically, we are allowed to take $\alpha > \rho_i$. It should also be interesting to consider high-order Taylor and Padé expansion series of the Bessel function J_0 to obtain higher-order approximation of the gyroaverage operator $\mathcal{J}_\perp$. Now, in order to regularize the system in the parallel direction, we will modify the coupling terms (the fifth and seventh terms in the $\mathcal{O}(\epsilon^2)$ collection in the right-hand side of (3.13)) in the action functional (3.13). To this purpose, we introduce regularizing linear operators $\mathcal{R}_i^\alpha$ in $\mathbb{R}^3$ with $i = 1, 2$, such that

1) Regularity: $\mathcal{R}_i^\alpha : H^s(\mathcal{D}) \to H^{s+s_i}(\mathcal{D})$ with $s_i \geq 0$,

2) Self-adjointness: $(\mathcal{R}_i^\alpha)^* = \mathcal{R}_i^\alpha$,

3) Commutativity: $[\mathcal{J}_\perp, \mathcal{R}_i^\alpha] = 0$, $[\partial^\ell, \mathcal{R}_i^\alpha] = 0$,

4) Approximation error: $\mathrm{Id} - \mathcal{R}_i^\alpha = \mathcal{O}(\alpha^\beta)$ with $\beta > 0$.

An example of such operator could be $\mathcal{J}_{m,\alpha^{\beta/2}}$, where $\mathscr{F}[\mathcal{J}_{m,\alpha^{\beta/2}}](k) = J_0^m(k\alpha^{\beta/2})$, with $\beta, m > 0$, and J_0 the Bessel function of the first kind and zero-order.

3.3. Lagrangian averaged gyrokinetic-waterbag model. From modeling rules of the previous section (3.2), after averaging the action functional (3.13) and using the rescaling $\alpha \mapsto \alpha/\sqrt{3}$ to get rid of the factor of $1/3$ coming from ϵ-integration we obtain

$$
\begin{aligned}
S^\alpha(u,c,\phi) &= \langle S(u^\epsilon, c^\epsilon, \phi^\epsilon)\rangle \\
&= \int_0^T dt \int_{\mathcal{M}} dx d\nu(\sigma) \left\{ \frac{1}{2}\left(cu^2 - \frac{c^3}{12}\right) - \frac{q_i}{m_i}\phi \mathcal{J}_\perp (c - c_0) \right. \\
&\qquad \left. + \frac{1}{2}\frac{q_i}{m_i}\left(\frac{n_{i0}}{B_0\Omega_0}|\nabla_\perp \phi|^2 + \frac{e\tau n_{i0}}{k_B T_{i0}}|\phi - \langle\phi\rangle_\parallel|^2\right)\right\} \\
&\quad + \alpha^2 \int_0^T dt \int_{\mathcal{M}} dx d\nu(\sigma) \left\{ \frac{1}{4}\left(u^2 - \frac{c^2}{4}\right)\nabla\cdot(\nabla\cdot(cF)) \right. \\
&\qquad - \frac{1}{2} cuF : \nabla\nabla u \; - \frac{1}{8}c\left(c^2 H + \nabla^T cF\nabla c + 2cG\cdot\nabla c\right) \\
&\quad + \frac{1}{2}\frac{q_i}{m_i}\left(\mathcal{J}_\perp \widetilde{\Delta}_\perp \Delta_\perp^{-1}\nabla_\perp\cdot(F:\nabla\nabla\nabla_\perp \widetilde{\Delta}_\perp^{-1}\phi)\right)\mathcal{R}_1^\alpha(c - c_0) \\
&\qquad - \frac{1}{2}\frac{q_i}{m_i}(\mathcal{J}_\perp\phi)\mathcal{R}_2^\alpha \nabla\cdot(\nabla\cdot(cF)) \\
&\quad + \frac{1}{2}\frac{q_i}{m_i}\left[-\frac{n_{i0}}{B_0\Omega_0}\nabla_\perp\phi\cdot\nabla_\perp\widetilde{\Delta}_\perp\Delta_\perp^{-1}\nabla_\perp\cdot(F:\nabla\nabla\nabla_\perp\widetilde{\Delta}_\perp^{-1}\phi)\right. \\
&\qquad - \frac{e\tau n_{i0}}{k_B T_{i0}}(\phi - \langle\phi\rangle_\parallel)\left(\widetilde{\Delta}_\perp\Delta_\perp^{-1}\nabla_\perp\cdot(F:\nabla\nabla\nabla_\perp\widetilde{\Delta}_\perp^{-1}\phi)\right. \\
&\qquad \left.\left.\left. - \langle \widetilde{\Delta}_\perp\Delta_\perp^{-1}\nabla_\perp\cdot(F:\nabla\nabla\nabla_\perp\widetilde{\Delta}_\perp^{-1}\phi)\rangle_\parallel\right)\right]\right\} + \varepsilon(\alpha) \\
&= \widehat{S}^\alpha(u,c,\phi) + \varepsilon(\alpha) = \int_0^T \widehat{L}^\alpha(u(t),c(t),\phi(t))dt + \varepsilon(\alpha),
\end{aligned}
\tag{3.16}
$$

where

$$
F^{ij} = \langle \xi'^i \xi'^j \rangle, \quad G^i = \langle \xi'^i \partial_j \xi'^j\rangle, \quad \text{and} \quad H = \langle \partial_i \xi'^i \partial_j \xi'^j \rangle,
\tag{3.17}
$$

and from the modeling rules Section 3.2, with $\beta > 0$,

$$
\varepsilon(\alpha) = \mathcal{O}\left(\alpha^3\right) + \mathcal{O}\left(\alpha^2 \rho_i^4\right) + \mathcal{O}\left((I - \mathcal{R}_i^\alpha)\alpha^2\right) = \mathcal{O}\left(\alpha^{\min(2+\beta,3)}\right).
$$

Applying Euler–Poincaré equations [28] to the Lagrangian $\widehat{L}^\alpha$ defined by (3.16) or equivalently applying the Hamilton's variational principle in Lagrangian coordinates directly to the action functional $\widehat{S}^\alpha$ defined by (3.16), we obtain the following Lagrangian averaged gyrowaterbag equations:

$$\partial_t\left(c\left[1-\alpha^2\mathcal{A}\right]u\right)+\nabla\cdot\left(Vc\left[1-\alpha^2\mathcal{A}\right]u\right)+\partial_\parallel\left(\frac{c^3}{12}\right)+\frac{q_i}{m_i}c\partial_\parallel\mathcal{J}_\perp\phi-\alpha^2c\partial_\parallel u\mathcal{A}u$$
$$-\frac{\alpha^2}{2}c\partial_\parallel\left(\nabla^T uF\nabla u-\frac{c^2}{4}\left[\nabla\cdot(\nabla\cdot F)+3H-2\nabla\cdot G\right]\right.$$
$$\left.+\frac{q_i}{m_i}\mathcal{R}_1^\alpha\mathcal{J}_\perp\mathfrak{D}_F^*\phi-\frac{q_i}{m_i}F:\nabla\nabla\mathcal{R}_2^\alpha\mathcal{J}_\perp\phi\right)=0, \quad (3.18)$$

$$\left(Q-\frac{\alpha^2}{2}\left[\mathfrak{D}_{\overline{F}}Q+(\mathfrak{D}_{\overline{F}}Q)^*\right]\right)\phi$$
$$=\int_\Sigma\nu(d\sigma)\left\{\left(1-\frac{\alpha^2}{2}\mathfrak{D}_F\mathcal{R}_1^\alpha\right)\mathcal{J}_\perp(c-c_0)+\frac{\alpha^2}{2}\mathcal{R}_2^\alpha\mathcal{J}_\perp\nabla\cdot(\nabla\cdot(cF))\right\}, \quad (3.19)$$

$$\partial_t c=-\nabla\cdot(cV), \quad (3.20)$$
$$\partial_t F=-\pounds_V F=-(V\cdot\nabla)F-\nabla V\cdot F-[\nabla V\cdot F]^T, \quad (3.21)$$
$$\partial_t G=-\pounds_V G+F\nabla(\nabla\cdot V)=-(V\cdot\nabla)G-(G\cdot\nabla)V+F\nabla\partial_\parallel u, \quad (3.22)$$
$$\partial_t H=-(V\cdot\nabla)H+2G\cdot\nabla(\nabla\cdot V)=-(V\cdot\nabla)H+2G\cdot\nabla\partial_\parallel u. \quad (3.23)$$

Let us recall that $V=(\mathcal{J}_\perp {v_E}^T,u)^T$. Let $\psi:\mathcal{M}\to\mathbb{R}$ and $\varphi:\mathcal{D}\to\mathbb{R}$, regular enough functions which are integrable on $\mathcal{M}$ and $\mathcal{D}$, respectively. In equation (3.18) the differential operator $\mathcal{A}$ is defined by

$$\mathcal{A}\psi=\frac{1}{c}\nabla\cdot(cF\nabla\psi). \quad (3.24)$$

In equations (3.18)–(3.19), the operator $\mathfrak{D}_F^*$ is the dual of the differential operator $\mathfrak{D}_F$ defined by

$$\mathfrak{D}_F\psi=\widetilde{\Delta}_\perp^{-1}\nabla\cdot\nabla\cdot\nabla_\perp\cdot(F\nabla_\perp\Delta_\perp^{-1}\widetilde{\Delta}_\perp\psi), \quad (3.25)$$

i.e.

$$\mathfrak{D}_F^*\psi=\widetilde{\Delta}_\perp\Delta_\perp^{-1}\nabla_\perp\cdot(F:\nabla\nabla\nabla_\perp\widetilde{\Delta}_\perp^{-1}\psi). \quad (3.26)$$

In equation (3.19), the symmetric differential operator Q is defined by

$$Q\varphi=-\nabla_\perp\cdot\left(\frac{n_{i0}}{B_0\Omega_0}\nabla_\perp\varphi\right)+\frac{e\tau n_{i0}}{k_BT_{i0}}(\varphi-\langle\varphi\rangle_\parallel), \quad (3.27)$$

while $\overline{\psi}$ denotes the ν-average of ψ over Σ, i.e.

$$\overline{\psi}=\int_\Sigma\nu(d\sigma)\psi.$$

The derivation of equations (3.21)–(3.23) comes simply from the linear flow rule (3.14) and the definitions (3.17), while continuity equation (3.20) is equivalent to (3.10) and translates preservation of Liouville and waterbag geometric invariants. Using continuity equation (3.20), equation (3.18) can be recast as

$$\partial_t\left(\left[1-\alpha^2\mathcal{A}\right]u\right)+\nabla\cdot\left(V\left[1-\alpha^2\mathcal{A}\right]u\right)+\partial_\parallel\left(-\frac{1}{2}u^2+\frac{c^2}{8}+\frac{q_i}{m_i}\mathcal{J}_\perp\phi\right)$$

$$-\frac{\alpha^2}{2}\partial_\| \left(\nabla^T uF\nabla u - \frac{c^2}{4}[\nabla\cdot(\nabla\cdot F)+3H-2\nabla\cdot G] \right.$$
$$\left. +\frac{q_i}{m_i}\mathcal{R}_1^\alpha \mathcal{J}_\perp \mathfrak{D}_F^*\phi - \frac{q_i}{m_i}F:\nabla\nabla\mathcal{R}_2^\alpha\mathcal{J}_\perp\phi\right)=0.$$

In addition, by using the property

$$c\left[\frac{d}{dt},(1-\alpha^2\mathcal{A})\right]u=\alpha^2 c(\partial_\| u)\mathcal{A}u+\alpha^2\left(\frac{dc}{dt}\right)\mathcal{A}u=0,$$

where $[\cdot,\cdot]$ denotes the commutator and $d_t=\partial_t+V\cdot\nabla=\partial_t+v_E\cdot\nabla_\perp+u\partial_\|$ is the material derivative, equation (3.18) can be rewritten as

$$(1-\alpha^2\mathcal{A})\frac{du}{dt}=\alpha^2(\partial_\| u)\mathcal{A}u-\partial_\|\left(\frac{c^2}{8}+\frac{q_i}{m_i}\mathcal{J}_\perp\phi-\frac{\alpha^2}{2}\left\{\nabla^T uF\nabla u\right.\right.$$
$$\left.\left.-\frac{c^2}{4}[\nabla\cdot(\nabla\cdot F)+3H-2\nabla\cdot G]+\frac{q_i}{m_i}\mathcal{R}_1^\alpha\mathcal{J}_\perp\mathfrak{D}_F^*\phi-\frac{q_i}{m_i}F:\nabla\nabla\mathcal{R}_2^\alpha\mathcal{J}_\perp\phi\right\}\right).$$

Equations (3.18)–(3.27) constitute the anisotropic Lagrangian averaged gyrowaterbag equations, where anisotropy appears through the covariance tensor F. Now for an isotropic version of equations (3.18)–(3.23), we can assume that the covariance tensor F is a multiple of the identity, G is zero, and H is a constant which, without loss of generality, can be assumed to be zero. More precisely, we make the assumptions

$$F=\mathrm{Id},\quad G=0,\quad \text{and}\quad H=0. \tag{3.28}$$

Let us note that the assumptions (3.28) are inconsistent with the Lie advection flow rule (3.14) since (3.28) is not a solution of (3.21)–(3.22). Therefore, the modeling assumptions (3.28) can only be seen as an approximation of the isotropic model and remain valid only for flows which almost preserve these properties. If we take $\mathcal{R}_i^\alpha=\mathrm{Id}$, for $i=1,2$, then from the modeling assumptions (3.28) and equations (3.21)–(3.22), we obtain the following isotropic model:

$$\partial_t c+\nabla\cdot(cV)=0, \tag{3.29}$$
$$(1-\alpha^2\mathcal{A})(\partial_t+V\cdot\nabla)u=\alpha^2\partial_\| u\mathcal{A}u-\partial_\|\left(\frac{c^2}{8}+\frac{q_i}{m_i}\mathcal{J}_\perp\phi-\frac{\alpha^2}{2}|\nabla u|^2\right), \tag{3.30}$$
$$\left(Q-\frac{\alpha^2}{2}[\Delta Q+Q\Delta]\right)\phi=\int_\Sigma \nu(d\sigma)\mathcal{J}_\perp c-\left(1-\frac{\alpha^2}{2}\Delta\right)n_{i0}, \tag{3.31}$$

where for all $\varphi:\mathcal{D}\to\mathbb{R}$, regular enough, the differential operator Q is defined by

$$Q\varphi=-\nabla_\perp\cdot\left(\frac{n_{i0}}{B_0\Omega_0}\nabla_\perp\varphi\right)+\frac{e\tau n_{i0}}{k_B T_{i0}}(\varphi-\langle\varphi\rangle_\|), \tag{3.32}$$

and for all $\psi:\mathcal{M}\to\mathbb{R}$, regular enough, the differential operator $\mathcal{A}$ is defined by

$$\mathcal{A}\psi=\frac{1}{c}\nabla\cdot(c\nabla\psi), \tag{3.33}$$

and with

$$V(t,x,\sigma)=\begin{pmatrix}\frac{1}{B_0}b\times\nabla\phi(t,x)\\ u(t,x,\sigma)\end{pmatrix}=\begin{pmatrix}-\frac{1}{B_0}\partial_2\mathcal{J}_\perp\phi(t,x)\\ \frac{1}{B_0}\partial_1\mathcal{J}_\perp\phi(t,x)\\ u(t,x,\sigma)\end{pmatrix}. \tag{3.34}$$

We believe that such an isotropic model could accurately describe turbulent flow only on domains without boundary, such as a periodic box.

4. Well-posedness of the isotropic Lagrangian averaged gyrokinetic-waterbag equations

In this section, we prove the existence and uniqueness of classical solutions to the dimensionless isotropic Lagrangian averaged gyrowaterbag continuum (ILAGWBC-α) formed by equations (3.29)–(3.34). Here, we consider the three-dimensional periodic-box $\Omega=\mathbb{T}_L^3=\Pi_{i=1}^3(\mathbb{R}/L_i\mathbb{Z})$ with $L_i>0$, $1\le i\le 3$. We set $\Omega_\perp=\mathbb{T}_L^2=\Pi_{i=1}^2(\mathbb{R}/L_i\mathbb{Z})$. In order to prove well-posedness of (3.29)–(3.34), we need to get a priori estimates on the density c and its inverse $\varrho:=1/c$ through (3.29), the velocity u through (3.30), and finally the electrical potential ϕ through (3.31). In order to show existence, we define an iteration scheme, which consists of a regularized approximation (with some linearizations) of the system (3.29)–(3.34), and we use a priori estimates to deduce the existence of weakly convergent solution sub-sequences. To pass to the limit in nonlinear terms (to show that the limit points of these sub-sequences satisfy the original model) we need strong convergence of these sub-sequences, which is obtained by proving directly that these sequences are Cauchy sequences in Banach spaces, since we cannot use Sobolev embeddings and the Arzelà–Ascoli theorem to get compactness (due to the lack of regularity estimates with respect to the variable σ). Finally, we show uniqueness of the solutions.

4.1. Main theorem. Here, we state the theorem for existence and uniqueness of classical solutions (local-in-time) to the isotropic Lagrangian averaged gyrowaterbag continuum formed by equations (3.29)–(3.34).

THEOREM 4.1. *Let us assume that $n_{i0}=n_{i0}(x_\perp)$, $a_0=a_0(x_\perp)=n_{i0}/B_0\Omega_0$, and $b_0=b_0(x_\perp)=e\tau/(k_B\Omega_0)$ are positive functions in $H^{s+2}(\Omega_\perp)$ with $s>5/2$ which depend only on the transverse variables and are such that $\alpha\|\nabla_\perp \log p_0\|_{L^\infty(\Omega_\perp)}\ll 1$ for $p_0\in\{n_{i0},a_0,b_0\}$. Let us assume that $c_0,u_0\in L^p_\nu\cap L^\infty(\Sigma;H^s(\Omega))$ for $s>5/2$ and $1\le p<\infty$, and $c_0\ge\mathfrak{c}_\circ$ with the constant $\mathfrak{c}_\circ>0$, then there exists a time $T>0$, which depends on initial data, such that the system (3.29)–(3.31) admits a unique solution (c,u,ϕ), and*

$$
\begin{aligned}
&c\in L^\infty([0,T];L^p_\nu\cap L^\infty(\Sigma;H^s(\Omega)))\cap \mathrm{Lip}([0,T];L^p_\nu\cap L^\infty(\Sigma;H^{s-1}(\Omega))),\ c>0,\\
&u\in L^\infty([0,T];L^p_\nu\cap L^\infty(\Sigma;H^{s+1}(\Omega))\cap \mathrm{Lip}([0,T];L^p_\nu\cap L^\infty(\Sigma;H^{s}(\Omega))),\\
&\phi\in L^\infty([0,T];H^{s+2}(\Omega))\cap \mathrm{Lip}([0,T];H^{s+1}(\Omega)).
\end{aligned}
$$

4.2. A priori estimates. Energy estimates on the density c and its inverse ϱ obtained from continuity equation (3.29), are standard. On the contrary elliptic estimates in Sobolev spaces H^s of high index for operators $(1-\alpha^2\mathcal{A})$, and $\big(Q-\alpha^2[\Delta Q+Q\Delta]/2\big)$ (see quasineutrality equation (3.31)) are not standard and are more technical to establish. In fact, before being allowed to use Kato–Ponce type commutator estimates and Sobolev embeddings to deal with H^s-estimates of high index s, we have to use particular methods to treat intermediate indices. For the operator $(1-\alpha^2\mathcal{A})$ and $1\le s\le 3$, we use mainly

Gagliardo–Nirenberg and Young inequalities on one hand and Sobolev embeddings on the other hand, while for the operator $\big(Q-\alpha^2[\Delta Q+Q\Delta]/2\big)$ and $1\le s\le 4$, we use multiple integration by parts and equivalence between some semi-norms. Of course, the condition $\alpha\|\nabla_\perp \log p_0\|_{L^\infty(\Omega_\perp)}\ll 1$ of Theorem 4.1 is crucial to prove the coercivity of quasineutrality equation (3.31). Energy estimates for the longitudinal mean velocity u obtained from equation (3.30) have to be done carefully since they involve the operator $(1-\alpha^2\mathcal{A})$ and additional nonlinear terms.

We start with a basic estimate that we use many times in the following section.

LEMMA 4.2. *Let β and γ, some multi-indices such that $|\beta|+|\gamma|=m$ with $m\geq 0$ an integer. Then for all $f,\, g\in\mathscr{C}_0(\mathbb{R}^n)\cap H^m(\mathbb{R}^n)$*

$$\|(\partial^\beta f)(\partial^\gamma g)\|_{L^2(\mathbb{R}^n)}\leq C\left(\|f\|_{L^\infty(\mathbb{R}^n)}\|g\|_{H^m(\mathbb{R}^n)}+\|f\|_{H^m(\mathbb{R}^n)}\|g\|_{L^\infty(\mathbb{R}^n)}\right). \tag{4.1}$$

Proof. See the proof of Proposition 3.6 in Chapter 13 of [36]. □

We next establish some a priori elliptic estimates for the operators $\mathcal{A}$ and $(1-\alpha^2\mathcal{A})^{-1}$.

4.2.1. A priori estimates for the operators $\mathcal{A}$ and $(1-\alpha^2\mathcal{A})^{-1}$.

LEMMA 4.3. *If we assume $c(t,\cdot,\sigma)\in H^{s+1}(\Omega)$ and $\varrho(t,\cdot,\sigma)\in H^s(\Omega)$ for almost every $\sigma\in\Sigma$ and time $t\geq 0$, with $s>3/2$, then there exists a constant C, such that*

$$\|\mathcal{A}\psi\|_{H^s(\Omega)}\leq C\|\psi\|_{H^{s+2}(\Omega)}(1+\|\varrho\|_{H^s(\Omega)}\|c\|_{H^{s+1}(\Omega)}). \tag{4.2}$$

Proof. Let β be a multi-index such that $|\beta|\leq s$. Then, using Lemma 4.2 and Sobolev imbedding $H^s(\Omega)\hookrightarrow L^\infty(\Omega)$ with $s>3/2$, we get

$$\begin{aligned}\|\partial_x^\beta\mathcal{A}\psi\|_{L^2(\Omega)}&\leq\|\partial_x^\beta(\Delta\psi+\varrho\nabla c\cdot\nabla\psi)\|_{L^2(\Omega)}\\&\leq\|\psi\|_{H^{s+2}(\Omega)}+\sum_{\gamma\geq 0}\binom{\beta}{\gamma}\|\partial_x^\gamma(\varrho\nabla c)\cdot\partial_x^{\beta-\gamma}(\nabla\psi)\|_{L^2(\Omega)}\\&\leq\|\psi\|_{H^{s+2}(\Omega)}+C\|\psi\|_{H^{s+1}(\Omega)}\|\varrho\nabla c\|_{H^s(\Omega)}\\&\leq\|\psi\|_{H^{s+2}(\Omega)}+C\|\psi\|_{H^{s+1}(\Omega)}\|c\|_{H^{s+1}(\Omega)}\|\varrho\|_{H^s(\Omega)},\end{aligned}$$

which ends the proof of (4.2). □

PROPOSITION 4.4. *If we assume that $c(t,\cdot,\sigma),\ \varrho(t,\cdot,\sigma)\in H^{\max(s-1,5/2)}(\Omega)$ for all almost every $\sigma\in\Sigma$ and time $t\geq 0$, with $s\geq 1$, then, for $f\in H^{-1}(\Omega)$, the equation*

$$(1-\alpha^2\mathcal{A})\psi=f,\quad in\ \Omega, \tag{4.3}$$

with periodic boundary conditions, has a unique solution $\psi\in H^1(\Omega)$. Moreover, if $f\in H^{s-2}(\Omega)$, then there exists a nondecreasing polynomial function $G:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ such that

$$\|(1-\alpha^2\mathcal{A})^{-1}f\|_{H^s(\Omega)}\leq G\left(\|c\|_{H^{\max(s-1,5/2)}(\Omega)},\|\varrho\|_{H^{\max(s-1,5/2)}(\Omega)}\right)\|f\|_{H^{s-2}(\Omega)}.$$

Proof. Let us define the scalar product of $L^2(\Omega)$ by $(\varphi,\psi)=\int_\Omega\varphi\psi dx$. Here, integration and derivation are always performed with respect to the space variables $x\in\Omega$. Moreover we denote by D^k any partial derivative of order k with respect to $x\in\Omega$. If we multiply the equation (4.3) by $c\varphi$, with $\varphi\in H^1(\Omega)$ and periodic, integrate it with respect to x over Ω, and use integration by parts with periodic boundary conditions, we obtain

$$a(\psi,\varphi):=(c\psi,\varphi)+\alpha^2(c\nabla\psi,\nabla\varphi)=(cf,\varphi). \tag{4.4}$$

<u>H^1 estimate</u>: Taking $\varphi=\psi$ in (4.4), we obtain

$$\begin{aligned}\|\psi\|_{H^1(\Omega)}&\leq C(\alpha)\|\varrho\|_{L^\infty(\Omega)}\|c\|_{W^{1,\infty}(\Omega)}\|f\|_{H^{-1}(\Omega)}\\&\leq C(\alpha)\|\varrho\|_{L^\infty(\Omega)}\|c\|_{H^{5/2}(\Omega)}\|f\|_{H^{-1}(\Omega)}.\end{aligned} \tag{4.5}$$

Since a is a continuous bilinear form on $H^1(\Omega)\times H^1(\Omega)$ elliptic or coercive on $H^1(\Omega)$, then from the Lax–Milgram theorem, for any $f\in H^{-1}(\Omega)$, there exists a unique solution $\psi\in H^1(\Omega)$ to (4.3) and we have the isomorphism $(1-\alpha^2\mathcal{A}):H^1(\Omega)\to H^{-1}(\Omega)$.
H^2 estimate: Taking $\varphi=-D^2\psi$ in (4.4), after integration by parts, we obtain

$$(cD\psi,D\psi)+\alpha^2(cD\nabla\psi,D\nabla\psi)=-(Dc\psi,D\psi)-\alpha^2(Dc\nabla\psi,D\nabla\psi)-(cf,D^2\psi). \quad (4.6)$$

Using the Poincaré inequality, there exists a constant $\lambda=\lambda(\Omega)$ depending on Ω, such that the left-hand side of (4.6) is bounded below by

$$(cD\psi,D\psi)+\alpha^2(cD\nabla\psi,D\nabla\psi)\geq C(\lambda,\alpha)\|\varrho\|^{-1}_{L^\infty(\Omega)}\|\psi\|^2_{H^2(\Omega)}. \quad (4.7)$$

Using Young's inequality, the third term of the right-hand side of (4.6) is bounded by

$$(cf,D^2\psi)\leq C(\varepsilon)(\|c\|_{L^\infty(\Omega)}\|f\|_{L^2(\Omega)})^2+\varepsilon\|\psi\|^2_{H^2(\Omega)}.$$

Using the Cauchy–Schwarz inequality, Sobolev embedding $H^{3/4}(\Omega)\hookrightarrow L^4(\Omega)$, and (4.5), the first term of the right-hand side of (4.6) is bounded by

$$\begin{aligned}(Dc\psi,D\psi)&\leq\|Dc\|_{L^4(\Omega)}\|\psi\|_{L^4(\Omega)}\|\psi\|_{H^1(\Omega)}\\&\leq C\|c\|_{H^2(\Omega)}(\|c\|_{W^{1,\infty}(\Omega)}\|\varrho\|_{L^\infty(\Omega)}\|f\|_{H^{-1}(\Omega)})^2.\end{aligned}$$

Using the Cauchy–Schwarz inequality, Sobolev embedding $H^{3/4}(\Omega)\hookrightarrow L^4(\Omega)$, the Gagliardo–Nirenberg inequality $\|D\varphi\|_{L^4(\Omega)}\leq C\|D^2\varphi\|^{3/4}_{L^2(\Omega)}\|D\varphi\|^{1/4}_{L^2(\Omega)}$ (see Proposition 3.4, Chapter 13 of [36] for example), and Young's inequality, we get for the second term of the right-hand side of (4.6)

$$\begin{aligned}(Dc\nabla\psi,D\nabla\psi)&\leq\|Dc\|_{L^4(\Omega)}\|\nabla\psi\|_{L^4(\Omega)}\|\psi\|_{H^2(\Omega)}\\&\leq C\|c\|_{H^{7/4}(\Omega)}\|\psi\|^{1/4}_{H^1(\Omega)}\|\psi\|^{7/4}_{H^2(\Omega)}\\&\leq C(\varepsilon)\|c\|^8_{H^2(\Omega)}\|\psi\|^2_{H^1(\Omega)}+\varepsilon\|\psi\|^2_{H^2(\Omega)}.\end{aligned} \quad (4.8)$$

Gathering estimates (4.7)–(4.8), using Sobolev embedding $H^{s-1}(\Omega)\hookrightarrow L^\infty(\Omega)$, $s>5/2$, and (4.5), equation (4.6) gives

$$\begin{aligned}\|\psi\|^2_{H^2(\Omega)}\leq{}& C(\varepsilon,\lambda,\alpha)\|\varrho\|_{H^2(\Omega)}\|c\|^2_{H^{5/2}(\Omega)}\\&\left(1+\|\varrho\|^2_{H^2(\Omega)}\left[\|c\|_{H^2(\Omega)}+\|c\|^8_{H^2(\Omega)}\right]\right)\|f\|^2_{L^2(\Omega)}.\end{aligned} \quad (4.9)$$

H^3 estimate: Taking $\varphi=D^4\psi$ in (4.4), after integration by parts, we obtain

$$\begin{aligned}(cD^2\psi,D^2\psi)+\alpha^2(cD^2\nabla\psi,D^2\nabla\psi)={}&(cf,D^4\psi)-(D^2c\psi,D^2\psi)\\&-2(DcD\psi,D^2\psi)-\alpha^2(D^2c\nabla\psi,D^2\nabla\psi)-2\alpha^2(DcD\nabla\psi,D^2\nabla\psi).\end{aligned} \quad (4.10)$$

Using the Poincaré inequality, there exists a constant $\lambda=\lambda(\Omega)$ such that the left-hand side of (4.10) is bounded below by

$$(cD^2\psi,D^2\psi)+\alpha^2(cD^2\nabla\psi,D^2\nabla\psi)\geq C(\lambda,\alpha)\|\varrho\|^{-1}_{L^\infty(\Omega)}\|\psi\|^2_{H^3(\Omega)}. \quad (4.11)$$

Using Young's inequality, the first term of the right-hand side of (4.10) is bounded by

$$
\begin{aligned}
(cf,D^4\psi) &\leq \|c\|_{W^{1,\infty}(\Omega)}\|f\|_{H^1(\Omega)}\|D^4\psi\|_{H^{-1}(\Omega)} \\
&\leq C(\varepsilon)\|c\|^2_{W^{1,\infty}(\Omega)}\|f\|^2_{H^1(\Omega)}+\varepsilon\|\psi\|^2_{H^3(\Omega)}. \qquad (4.12)
\end{aligned}
$$

From Sobolev embedding $H^s(\Omega)\hookrightarrow L^\infty(\Omega)$, $s>3/2$ and using the Cauchy–Schwarz inequality, the second term of the right-hand side of (4.10) is bounded by

$$
(D^2c\psi,D^2\psi)\leq \|D^2c\|_{L^2(\Omega)}\|\psi\|_{L^\infty(\Omega)}\|D^2\psi\|_{L^2(\Omega)}\leq C\|c\|_{H^2(\Omega)}\|\psi\|^2_{H^2(\Omega)}. \qquad (4.13)
$$

From Sobolev embedding $H^{3/4}(\Omega)\hookrightarrow L^4(\Omega)$, and using the Cauchy–Schwarz inequality the third term of the right-hand side of (4.10) is bounded by

$$
(DcD\psi,D^2\psi)\leq \|Dc\|_{L^4(\Omega)}\|D\psi\|_{L^4(\Omega)}\|D^2\psi\|_{L^2(\Omega)}\leq C\|c\|_{H^2(\Omega)}\|\psi\|^2_{H^2(\Omega)}. \qquad (4.14)
$$

Using the Cauchy–Schwarz inequality, Sobolev embedding $W^{1,4}(\Omega)\hookrightarrow L^\infty(\Omega)$, the Gagliardo–Nirenberg inequality $\|D^2\varphi\|_{L^4(\Omega)}\leq C\|\varphi\|^{7/8}_{H^3(\Omega)}\|\varphi\|^{1/8}_{H^1(\Omega)}$ (see Proposition 3.4, Chapter 13 of [36] for example), and Young's inequality we get for the fourth term of the right-hand side of (4.10)

$$
\begin{aligned}
(D^2c\nabla\psi,D^2\nabla\psi) &\leq \|D^2c\|_{L^2(\Omega)}\|\nabla\psi\|_{L^\infty(\Omega)}\|\psi\|_{H^3(\Omega)} \\
&\leq \|c\|_{H^2(\Omega)}\|D^2\psi\|_{L^4(\Omega)}\|\psi\|_{H^3(\Omega)} \\
&\leq C\|c\|_{H^2(\Omega)}\|\psi\|^{1/8}_{H^1(\Omega)}\|\psi\|^{15/8}_{H^3(\Omega)} \\
&\leq C(\varepsilon)\|c\|^{16}_{H^2(\Omega)}\|\psi\|^2_{H^1(\Omega)}+\varepsilon\|\psi\|^2_{H^3(\Omega)} \qquad (4.15)
\end{aligned}
$$

and for fifth term of the right-hand side of (4.10)

$$
\begin{aligned}
(DcD\nabla\psi,D^2\nabla\psi) &\leq \|Dc\|_{L^4(\Omega)}\|D\nabla\psi\|_{L^4(\Omega)}\|\psi\|_{H^3(\Omega)} \\
&\leq C\|Dc\|_{H^{3/4}(\Omega)}\|\psi\|^{1/8}_{H^1(\Omega)}\|\psi\|^{15/8}_{H^3(\Omega)} \\
&\leq C(\varepsilon)\|c\|^{16}_{H^2(\Omega)}\|\psi\|^2_{H^1(\Omega)}+\varepsilon\|\psi\|^2_{H^3(\Omega)}. \qquad (4.16)
\end{aligned}
$$

Gathering estimates (4.11)–(4.16), using Sobolev embedding $H^{s-1}(\Omega)\hookrightarrow L^\infty(\Omega)$, $s>5/2$, and estimates (4.5) and (4.9), equation (4.10) gives

$$
\begin{aligned}
\|\psi\|^2_{H^3(\Omega)} &\leq C(\varepsilon,\lambda,\alpha)\|\varrho\|_{H^2(\Omega)}\|c\|^2_{H^{5/2}(\Omega)}\Big\{1+\|\varrho\|^2_{H^2(\Omega)}\|c\|^{16}_{H^2(\Omega)} \\
&\quad +\|\varrho\|_{H^2(\Omega)}\|c\|_{H^2(\Omega)}\Big(1+\|\varrho\|^2_{H^2(\Omega)}\Big[\|c\|_{H^2(\Omega)}+\|c\|^8_{H^2(\Omega)}\Big]\Big)\Big\}\|f\|^2_{H^1(\Omega)}. \qquad (4.17)
\end{aligned}
$$

H^k estimate, $k\geq 4$: Taking $\varphi=(-1)^{k-1}D^{2k-2}\psi$ in (4.4), after integration by parts, we obtain

$$
\begin{aligned}
(cD^{k-1}\psi,D^{k-1}\psi)+\alpha^2(cD^{k-1}\nabla\psi,D^{k-1}\nabla\psi) &= (D^{k-1}(cf),D^{k-1}\psi) \\
&\quad -\alpha^2([D^{k-1},c]\nabla\psi,D^{k-1}\nabla\psi)-([D^{k-1},c]\psi,D^{k-1}\psi), \qquad (4.18)
\end{aligned}
$$

where $[\cdot,\cdot]$ denotes the commutator. Using the Poincaré inequality, there exists a constant $\lambda=\lambda(\Omega)$ such that the left-hand side of (4.18) is bounded below by

$$
(cD^{k-1}\psi,D^{k-1}\psi)+\alpha^2(cD^{k-1}\nabla\psi,D^{k-1}\nabla\psi)\geq C(\lambda,\alpha)\|\varrho\|^{-1}_{L^\infty(\Omega)}\|\psi\|^2_{H^k(\Omega)}. \qquad (4.19)
$$

Using duality between $H^1(\Omega)$ and $H^{-1}(\Omega)$, Lemma 4.2, and Sobolev embedding $H^s(\Omega)\hookrightarrow L^\infty(\Omega)$, $s>3/2$, the first term of the right-hand side of (4.18) is bounded by

$$\begin{aligned}(D^{k-1}(cf),D^{k-1}\psi)&\leq \|D^{k-2}(cf)\|_{L^2(\Omega)}\|\psi\|_{H^k(\Omega)}\\&\leq C\|\psi\|_{H^k(\Omega)}\sum_{\gamma\geq 0}\binom{k-2}{\gamma}\|\partial_x^\gamma c\partial_x^{k-2-\gamma}f\|_{L^2(\Omega)}\\&\leq C\|c\|_{H^{k-2}(\Omega)}\|f\|_{H^{k-2}(\Omega)}\|\psi\|_{H^k(\Omega)}.\end{aligned}\tag{4.20}$$

Using the Cauchy–Schwarz inequality, commutator estimates (see Proposition 3.7 of Chapter 13 of [36], for example), and Sobolev embedding $H^{k-2}(\Omega)\hookrightarrow L^\infty(\Omega)$, $k>7/2$, we get for the second term of the right-hand side of (4.18)

$$\begin{aligned}([D^{k-1},c]\nabla\psi,D^{k-1}\nabla\psi)&\leq \|[D^{k-1},c]\nabla\psi\|_{L^2(\Omega)}\|\psi\|_{H^k(\Omega)}\\&\leq C\|\psi\|_{H^k(\Omega)}\left(\|c\|_{H^{k-1}(\Omega)}\|\nabla\psi\|_{L^\infty(\Omega)}+\|c\|_{W^{1,\infty}(\Omega)}\|\nabla\psi\|_{H^{k-2}(\Omega)}\right)\\&\leq C\|c\|_{H^{k-1}(\Omega)}\|\psi\|_{H^{k-1}(\Omega)}\|\psi\|_{H^k(\Omega)}\end{aligned}\tag{4.21}$$

and for the third term of the right-hand side of (4.18)

$$\begin{aligned}([D^{k-1},c]\psi,D^{k-1}\psi)&\leq \|[D^{k-1},c]\psi\|_{L^2(\Omega)}\|\psi\|_{H^k(\Omega)}\\&\leq C\|c\|_{H^{k-1}(\Omega)}\|\psi\|_{H^{k-1}(\Omega)}\|\psi\|_{H^k(\Omega)}.\end{aligned}\tag{4.22}$$

Gathering estimates (4.19)–(4.22) and using Sobolev embedding $H^{k-1}(\Omega)\hookrightarrow L^\infty(\Omega)$, $k>5/2$, equation (4.18) gives

$$\|\psi\|_{H^k(\Omega)}\leq C(\varepsilon,\lambda,\alpha)\|\varrho\|_{H^{k-1}(\Omega)}\|c\|_{H^{k-1}(\Omega)}\left(\|f\|_{H^{k-2}(\Omega)}+\|\psi\|_{H^{k-1}(\Omega)}\right).$$

Using the previous recurrence formula and (4.17), we obtain that

$$\begin{aligned}\|\psi\|_{H^k(\Omega)}\leq&\sum_{i=0}^{k-4}\|f\|_{H^{k-2-i}(\Omega)}\prod_{j=0}^{i}\|\varrho\|_{H^{k-1-j}(\Omega)}\|c\|_{H^{k-1-j}(\Omega)}\\&+\|\psi\|_{H^3(\Omega)}\prod_{i=4}^{k}\|\varrho\|_{H^{k-i}(\Omega)}\|c\|_{H^{k-i}(\Omega)}\\\leq&\,\mathcal{P}\left(\|c\|_{H^{k-1}(\Omega)},\|\varrho\|_{H^{k-1}(\Omega)}\right)\|f\|_{H^{k-2}(\Omega)},\end{aligned}$$

where $\mathcal{P}:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ is a polynomial with nonnegative coefficients and thus is a nondecreasing function in its arguments, which completes the proof. ∎

4.2.2. A priori estimates for the electrical potential ϕ. In this section, we obtain a priori estimates on the electrical potential ϕ through the Lagrangian averaged quasineutrality equation (3.31), where $n_{i0}=n_{i0}(x_\perp)>0$, $a_0=a_0(x_\perp)=n_{i0}/B_0\Omega_0>0$, and $b_0=b_0(x_\perp)=e\tau/(k_B\Omega_0)>0$ are smooth functions which depends only on the transverse (with respect to the magnetic field direction b) variables $x_\perp$. We now establish the following proposition.

PROPOSITION 4.5. *Let us assume that n_{i0}, a_0, and b_0 are positive functions in $H^{s+2}(\Omega_\perp)$ which depend only on the transverse variables and such that $\alpha\|\nabla_\perp\log p_0\|_{L^\infty(\Omega_\perp)}\ll 1$ for $p_0\in\{n_{i0},a_0,b_0\}$. Let us assume that, for $s\geq 1$, $c(t,\cdot,\cdot)\in L^2_\nu(\Sigma;H^{s-2}(\Omega))$ for almost every time $t\geq 0$. Then the quasineutrality equation (3.31)* *has a unique solution in $\phi \in H^1(\Omega)$ and there exists a constant C depending on s, α, a_0, b_0, n_{i0}, Σ, and Ω such that*

$$\|\phi\|_{H^s(\Omega)} \leq C(s,\alpha,a_0,b_0,n_{i0},\Omega)(1+\|c\|_{L^2_\nu(\Sigma;H^{s-2}(\Omega))}), \quad \forall s\geq 1. \tag{4.23}$$

REMARK 4.6. It is well known in the magnetic fusion community [18] that transverse equilibrium profiles $p_0 \in \{n_{i0},B_0,a_0,b_0\}$ have length scales much larger than the Larmor gyroradius ρ_i, typically with the ratio $\rho_i\|\nabla_\perp \log p_0\|_{L^\infty(\Omega_\perp)} \simeq 10^{-3} \ll 1$. Since the parameter α could be of the same order as or even some orders larger than ρ_i, we then get $\alpha\|\nabla_\perp \log p_0\|_{L^\infty(\Omega_\perp)} \ll 1$, too. The condition $\alpha\|\nabla_\perp \log p_0\|_{L^\infty(\Omega_\perp)} \ll 1$ is linked to the existence and stability of an equilibrium configuration and thus to the existence of a stable confinement in the sense that, if the length scale of perturbation of order α, represented here by the electrical potential ϕ, were of the same order of length scale of equilibrium quantities, represented by p_0, then the plasma could not be confined.

Proof. By taking the parallel average of (3.31) along the $\parallel$-direction, we obtain for $\langle\phi\rangle_\parallel$ the equation

$$\begin{aligned}\mathcal{B}\langle\phi\rangle_\parallel &= -\nabla_\perp\cdot(a_0\nabla_\perp\langle\phi\rangle_\parallel) - \frac{\alpha^2}{2}[-\Delta_\perp\nabla_\perp\cdot(a_0\nabla_\perp\langle\phi\rangle_\parallel) - \nabla_\perp\cdot(a_0\nabla_\perp\Delta_\perp\langle\phi\rangle_\parallel)]\\ &= \langle\rho\rangle_\parallel,\end{aligned} \tag{4.24}$$

where we have set

$$\rho = \int_\Sigma \nu(d\sigma)\mathcal{J}_\perp c - \left(1-\frac{\alpha^2}{2}\Delta\right)n_{i0}. \tag{4.25}$$

Therefore the potential ϕ satisfies the equation

$$\begin{aligned}\mathfrak{B}\phi &= -\nabla_\perp\cdot(a_0\nabla_\perp\phi) + b_0\phi - \frac{\alpha^2}{2}[-\Delta\nabla_\perp\cdot(a_0\nabla_\perp\phi) + \Delta(b_0\phi) - \nabla_\perp\cdot(a_0\nabla_\perp\Delta\phi) + b_0\Delta\phi]\\ &= \tilde{\rho},\end{aligned} \tag{4.26}$$

where

$$\tilde{\rho} = \rho + b_0\langle\phi\rangle_\parallel - \frac{\alpha^2}{2}[-\Delta_\perp(b_0\langle\phi\rangle_\parallel) + b_0\Delta_\perp\langle\phi\rangle_\parallel]. \tag{4.27}$$

From (4.25) and using the Cauchy–Schwarz inequality, a simple computation shows that, for $s\geq -2$, we have

$$\|\langle\rho\rangle_\parallel\|_{H^s(\Omega_\perp)} \leq C(s,\alpha,n_0,L_3)(1+\|c\|_{L^2_\nu(\Sigma;H^s(\Omega))}). \tag{4.28}$$

<u>H^1 estimate</u>: By taking $\psi=\varphi$, using integration by parts, we obtain

$$\begin{aligned}&\int_{\Omega_\perp} dx_\perp \left\{\alpha^2 a_0|\nabla^2_\perp\varphi|^2 + a_0\left(1-\frac{\alpha^2}{2}\frac{\Delta_\perp a_0}{a_0}\right)|\nabla_\perp\varphi|^2\right\}\\ &= (\mathcal{B}\varphi,\psi) \leq \|\mathcal{B}\varphi\|_{H^{-2}(\Omega)}\|\varphi\|_{H^2(\Omega)}.\end{aligned}$$

Since $a_0>0$ and $\alpha\|\nabla_\perp \log a_0\|_{L^\infty(\Omega_\perp)} \ll 1$, using the Poincaré inequality, we get from the previous equation that

$$\|\langle\phi\rangle_\parallel\|_{H^2(\Omega_\perp)} \leq C(\alpha,a_0,\Omega_\perp)\|\langle\rho\rangle_\parallel\|_{H^{-2}(\Omega_\perp)}. \tag{4.29}$$

Using Theorem 3.1 of Chapter 2, Section 3.2 of [30], we get for $s\geq 0$,

$$\|\langle\phi\rangle_{\shortparallel}\|_{H^s(\Omega_\perp)}\leq C(\|\langle\rho\rangle_{\shortparallel}\|_{H^{s-4}(\Omega_\perp)}+\|\langle\phi\rangle_{\shortparallel}\|_{H^{s-1}(\Omega_\perp)}).$$

Therefore, from the previous estimates and (4.29), we obtain for $s\geq 3$

$$\|\langle\phi\rangle_{\shortparallel}\|_{H^s(\Omega_\perp)}\leq C(s,\alpha,a_0,\Omega_\perp)\|\langle\rho\rangle_{\shortparallel}\|_{H^{s-4}(\Omega_\perp)}. \tag{4.30}$$

Using (4.28) and (4.30), we obtain that, for $s\geq -1$,

$$\begin{aligned}\|\tilde\rho\|_{H^s(\Omega)}&\leq\|\rho\|_{H^s(\Omega)}+\|b_0\langle\phi\rangle_{\shortparallel}\|_{H^s(\Omega)}+\frac{\alpha^2}{2}\big(\|\Delta_\perp(b_0\langle\phi\rangle_{\shortparallel})\|_{H^s(\Omega)}+\|b_0\Delta_\perp\langle\phi\rangle_{\shortparallel}\|_{H^s(\Omega)}\big)\\&\leq C(s,\alpha,n_0,a_0,b_0,\Omega)(1+\|c\|_{L^2_\nu(\Sigma;H^s(\Omega))}).\end{aligned} \tag{4.31}$$

Now, if we take $\psi=\varphi$, using some integration by parts, we obtain

$$\begin{aligned}\int_\Omega dx\bigg\{&b_0\left(1-\frac{\alpha^2}{2}\frac{\Delta_\perp b_0}{b_0}\right)|\varphi|^2+a_0\left(1-\frac{\alpha^2}{2}\frac{\Delta_\perp a_0}{a_0}\right)|\nabla_\perp\varphi|^2\\&+\alpha^2b_0|\nabla\varphi|^2+a_0|\nabla\nabla_\perp\varphi|^2\bigg\}=(\mathfrak{B}\varphi,\psi)\leq\|\mathfrak{B}\varphi\|_{H^{-1}(\Omega)}\|\varphi\|_{H^1(\Omega)}.\end{aligned} \tag{4.32}$$

Since $a_0,b_0>0$, $\alpha\|\nabla_\perp\log a_0\|_{L^\infty(\Omega_\perp)}\ll 1$, and $\alpha\|\nabla_\perp\log b_0\|_{L^\infty(\Omega_\perp)}\ll 1$, then $(\mathfrak{B}\cdot,\cdot)$ is a continuous bilinear form on $H^1(\Omega)\times H^1(\Omega)$ elliptic or coercive on $H^1(\Omega)$. Therefore, from Lax–Milgram theorem, for any $f\in H^{-1}(\Omega)$, there exists a unique solution $\varphi\in H^1(\Omega)$ to $\mathfrak{B}\varphi=f$, and we have the isomorphism $\mathfrak{B}:H^1(\Omega)\to H^{-1}(\Omega)$. Moreover, from (4.32), we get

$$\|\varphi\|_{H^1(\Omega)}\leq C(\alpha,a_0,b_0)\|\mathfrak{B}\varphi\|_{H^{-1}(\Omega)}. \tag{4.33}$$

Equations (4.33) and (4.31) yield estimates (4.23) for $s=1$.
$\underline{H^2\text{ estimate}}$: As regards the H^2-norm, if we take $\psi=-\Delta\varphi$, after some integration by parts, we obtain

$$\begin{aligned}&\int_\Omega dx\bigg\{-\frac{1}{2}|\nabla_\perp\varphi|^2\Delta_\perp\left(1-\frac{\alpha^2}{2}\Delta_\perp\right)a_0-\alpha^2\nabla\varphi\cdot\nabla^2b_0\nabla\varphi+a_0|\nabla_\perp\nabla\varphi|^2+b_0|\nabla\varphi|^2\\&\quad-\alpha^2\nabla\nabla_\perp\varphi\cdot\nabla^2a_0\nabla\nabla_\perp\varphi-\frac{1}{2}|\varphi|^2\Delta_\perp\left(1-\frac{\alpha^2}{2}\Delta_\perp\right)b_0+\alpha^2b_0|\Delta\varphi|^2+\alpha^2a_0|\nabla_\perp\Delta\varphi|^2\bigg\}\\&=(\mathfrak{B}\varphi,\psi)\leq\|\mathfrak{B}\varphi\|_{L^2(\Omega)}\|\Delta\varphi\|_{L^2(\Omega)},\end{aligned}$$

which, using $a_0>0$, leads to

$$\begin{aligned}&\int_\Omega dx\left\{\alpha^2b_0|\Delta\varphi|^2+a_0|\nabla\nabla_\perp\varphi|^2\left(1-C\alpha^2\frac{\|\nabla^2_\perp a_0\|_{L^\infty(\Omega_\perp)}}{a_0}\right)\right\}\\&\leq\|\mathfrak{B}\varphi\|_{L^2(\Omega)}\|\Delta\varphi\|_{L^2(\Omega)}+C(\alpha,a_0,b_0)\|\varphi\|^2_{H^1(\Omega)}.\end{aligned} \tag{4.34}$$

Above and in the sequel, we use the notation $\nabla\nabla_\perp\varphi\cdot\nabla^2a_0\nabla\nabla_\perp\varphi=\sum_{i,j,k}\partial_i\partial_{\perp j}\varphi\partial^2_{ik}a_0\partial_k\partial_{\perp j}\varphi$. Using $a_0,b_0>0$, $\alpha\|\nabla_\perp\log a_0\|_{L^\infty(\Omega_\perp)}\ll 1$, (4.33) and the Poincaré inequality, since the semi-norm $\|\Delta\cdot\|_{L^2(\Omega)}$ and semi-norm $|\cdot|_{H^2(\Omega)}$ are equivalent, expression (4.34) leads to

$$\|\varphi\|_{H^2(\Omega)}\leq C(\alpha,a_0,b_0,\Omega)\|\mathfrak{B}\varphi\|_{L^2(\Omega)}. \tag{4.35}$$

<u>H^3 estimate</u>: Concerning the H^3-norm estimate, if we take $\psi=\Delta^2\varphi$, using some integration by parts, we obtain

$$\begin{aligned}&\int_\Omega dx\Big\{\frac{1}{2}|\varphi|^2\Delta_\perp^2 b_0+\frac{1}{2}|\nabla_\perp\varphi|^2\Delta_\perp^2 a_0+b_0|\Delta\varphi|^2-2\nabla\varphi\cdot\nabla^2 b_0\nabla\varphi+a_0|\nabla_\perp\Delta\varphi|^2\\
&\quad-2\nabla\nabla_\perp\varphi\cdot\nabla^2 a_0\nabla\nabla_\perp\varphi-\frac{\alpha^2}{2}\Big(-\frac{1}{2}\Delta_\perp^2 a_0|\nabla\nabla_\perp\varphi|^2+\Delta_\perp a_0|\nabla\nabla_\perp\varphi|^2\\
&\quad+\frac{1}{2}\Delta_\perp^3 a_0|\nabla_\perp\phi|^2-2\nabla\nabla_\perp\varphi\cdot\nabla^2\Delta a_0\nabla\nabla_\perp\varphi-\frac{1}{2}\Delta_\perp^2 a_0|\nabla\nabla_\perp\varphi|^2\\
&\quad+\Delta_\perp a_0|\nabla_\perp\Delta\varphi|^2+4\nabla^2\nabla_\perp\varphi\cdot\nabla^2 a_0\nabla^2\nabla_\perp\varphi-\Delta_\perp a_0|\nabla^2\nabla_\perp\varphi|^2\\
&\quad+\frac{1}{2}\Delta_\perp^3 b_0|\varphi|^2-\Delta_\perp^2 b_0|\nabla\varphi|^2-\nabla\varphi\cdot\nabla^2\Delta b_0\nabla\varphi+\Delta_\perp b_0|\Delta\varphi|^2-2b_0|\nabla\Delta\varphi|^2\\
&\quad+4\nabla^2\varphi\cdot\nabla^2 b_0\nabla^2\varphi+2\nabla^2\varphi\cdot\nabla^3 b_0\nabla\varphi-2a_0|\nabla_\perp\nabla\Delta\varphi|^2\Big)\Big\}\\
&=(\mathfrak{B}\varphi,\psi)\le\|\mathfrak{B}\varphi\|_{H^1(\Omega)}\|\Delta^2\varphi\|_{H^{-1}(\Omega)},\end{aligned}$$

which, using $a_0>0$, semi-norm equivalence between $\|\Delta^s\cdot\|_{L^2(\Omega)}$ (resp. $\|\nabla\Delta^s\cdot\|_{L^2(\Omega)}$) and $|\cdot|_{H^{2s}(\Omega)}$ (resp. $|\cdot|_{H^{2s+1}(\Omega)}$), and the Cauchy–Schwarz inequality, leads to

$$\begin{aligned}&\int_\Omega dx\Big\{\alpha^2 b_0|\nabla\Delta\varphi|^2+a_0|\nabla_\perp\Delta\varphi|^2\Big(1-C\alpha^2\frac{\|\nabla_\perp^2 a_0\|_{L^\infty(\Omega_\perp)}}{a_0}\Big)\Big\}\\
&\le\|\mathfrak{B}\varphi\|_{H^1(\Omega)}\|\Delta^2\varphi\|_{H^{-1}(\Omega)}+C(\alpha,a_0,b_0)\|\varphi\|^2_{H^2(\Omega)}.\end{aligned}\tag{4.36}$$

Above and in the sequel we use the notations
$\nabla^2\nabla_\perp\varphi\cdot\nabla^2 a_0\nabla^2\nabla_\perp\varphi=\sum_{i,j,k,l}\partial^2_{il}\partial_{\perp j}\varphi\partial^2_{ik}a_0\partial^2_{kl}\partial_{\perp j}\varphi$,
$\nabla^2\varphi\cdot\nabla^2 b_0\nabla^2\varphi=\sum_{i,j,k}\partial^2_{ik}\varphi\partial^2_{ij}b_0\partial^2_{jk}\varphi$, and $\nabla^2\varphi\cdot\nabla^3 b_0\nabla\varphi=\sum_{i,j,k}\partial^2_{ij}\varphi\partial^3_{ijk}b_0\partial_k\varphi$.
Using $a_0,b_0>0$, $\alpha\|\nabla_\perp\log a_0\|_{L^\infty(\Omega_\perp)}\ll 1$, (4.35), and the Poincaré inequality, since the semi-norm $\|\Delta\cdot\|_{L^2(\Omega)}$, and semi-norm $|\cdot|_{H^2(\Omega)}$ are equivalent, expression (4.36) leads to

$$\|\varphi\|_{H^3(\Omega)}\le C(\alpha,a_0,b_0,\Omega)\|\mathfrak{B}\varphi\|_{H^1(\Omega)}.\tag{4.37}$$

<u>H^4 estimate</u>: Finally, we deal with the H^4-norm estimate. If we take $\psi=-\Delta^3\varphi$, using some integration by parts, we obtain

$$\begin{aligned}&\int_\Omega dx\Big\{\big(\nabla\Delta a_0\nabla_\perp\varphi+\Delta a_0\nabla\nabla_\perp\varphi+2\nabla^2 a_0\nabla\nabla_\perp\varphi\big)\cdot\nabla\nabla_\perp\Delta\varphi\\
&\quad+(2\nabla a_0\cdot\nabla(\nabla\nabla_\perp\varphi)+\nabla a_0\nabla_\perp\Delta\varphi)\cdot\nabla\nabla_\perp\Delta\varphi+a_0|\nabla\nabla_\perp\Delta\varphi|^2\\
&\quad+\big(\varphi\nabla\Delta b_0+\Delta b_0\nabla\varphi+2\nabla^2 b_0\nabla\varphi+2\nabla^2\varphi\nabla b_0+\Delta\varphi\nabla b_0+b_0\nabla\Delta\varphi\big)\cdot\nabla\Delta\varphi\\
&\quad-\frac{\alpha^2}{2}\Big[-(\Delta^2 a_0\nabla_\perp\varphi+4\nabla\Delta a_0\cdot\nabla\nabla_\perp\varphi)\cdot\nabla_\perp\Delta^2\varphi+3\nabla\Delta a_0\cdot\nabla\nabla_\perp\Delta\varphi\cdot\nabla_\perp\Delta\varphi\\
&\quad+3\Delta_\perp a_0|\nabla\nabla_\perp\Delta\varphi|^2+3\nabla\nabla_\perp\Delta\varphi\cdot\nabla^2 a_0\nabla\nabla_\perp\Delta\varphi-2a_0|\Delta^2\nabla_\perp\varphi|^2\\
&\quad-(\varphi\Delta^2 b_0+4\nabla\Delta b_0\cdot\nabla\varphi+3\Delta b_0\Delta\varphi+6\nabla b_0\cdot\nabla\Delta\varphi+4\nabla^2 b_0\cdot\nabla^2\varphi)\Delta^2\varphi\\
&\quad-4\nabla^2\nabla_\perp a_0\cdot\nabla^2\nabla_\perp\varphi\Delta^2\varphi-4\nabla^2 a_0\cdot\nabla^2\Delta_\perp\varphi\Delta^2\varphi-2b_0|\Delta^2\varphi|^2\Big]\Big\}\\
&=(\mathfrak{B}\varphi,\psi)\le\|\mathfrak{B}\varphi\|_{H^2(\Omega)}\|\Delta^3\varphi\|_{H^{-2}(\Omega)},\end{aligned}\tag{4.38}$$

which, using $a_0>0$, some integrations by parts for the penultimate term of left-hand side of equality (4.38), semi-norm equivalence between $\|\Delta^s\cdot\|_{L^2(\Omega)}$ (resp. $\|\nabla\Delta^s\cdot\|_{L^2(\Omega)}$) and $|\cdot|_{H^{2s}(\Omega)}$ (resp. $|\cdot|_{H^{2s+1}(\Omega)}$), and the Cauchy–Schwarz inequality, leads to

$$\int_\Omega dx\left\{\alpha^2 b_0|\Delta^2\varphi|^2+a_0|\nabla\nabla_\perp\Delta\varphi|^2\left(1-C\alpha^2\frac{\|\nabla_\perp^2 a_0\|_{L^\infty(\Omega_\perp)}}{a_0}\right)\right\}$$
$$\leq\|\mathfrak{B}\varphi\|_{H^2(\Omega)}\|\Delta^3\varphi\|_{H^{-2}(\Omega)}+C(\alpha,a_0,b_0)\|\varphi\|_{H^3(\Omega)}\|\varphi\|_{H^4(\Omega)}. \tag{4.39}$$

Using $a_0,b_0>0$, $\alpha\|\nabla_\perp\log a_0\|_{L^\infty(\Omega_\perp)}\ll 1$, (4.37), and the Poincaré inequality, expression (4.39) leads to

$$\|\varphi\|_{H^4(\Omega)}\leq C(\alpha,a_0,b_0,\Omega)\|\mathfrak{B}\varphi\|_{H^2(\Omega)}.$$

H^s estimate, $s>4$: From estimates (4.34), (4.36), and (4.39) we observe, for $s>1$, a recurrence formula given by

$$\int_\Omega dx\left\{\alpha^2 b_0|\mathfrak{D}_\mathfrak{b}\varphi|^2+a_0|\mathfrak{D}_\mathfrak{a}\nabla_\perp\varphi|^2\left(1-C\alpha^2\frac{\|\nabla_\perp^2 a_0\|_{L^\infty(\Omega_\perp)}}{a_0}\right)\right\}$$
$$\leq\|\mathfrak{B}\varphi\|_{H^{s-2}(\Omega)}\|(-1)^{s-1}\Delta^{s-1}\varphi\|_{H^{-s+2}(\Omega)}+C(\alpha,a_0,b_0)\|\varphi\|_{H^{s-1}(\Omega)}\|\varphi\|_{H^s(\Omega)}, \tag{4.40}$$

with $\mathfrak{D}_\mathfrak{b}=\Delta^{s/2}$ (resp. $\mathfrak{D}_\mathfrak{b}=\nabla\Delta^{(s-1)/2}$) and $\mathfrak{D}_\mathfrak{a}=\nabla\Delta^{(s-2)/2}$ (resp. $\mathfrak{D}_\mathfrak{a}=\Delta^{(s-1)/2}$) for s even (resp. odd). Before proving (4.40), let us see the consequence of (4.40). Using $a_0,b_0>0$, $\alpha\|\nabla_\perp\log a_0\|_{L^\infty(\Omega_\perp)}\ll 1$, (4.33), and the Poincaré inequality, we obtain

$$\|\varphi\|_{H^s(\Omega)}\leq C(\alpha,a_0,b_0,\Omega)\|\mathfrak{B}\varphi\|_{H^{s-2}(\Omega)},\quad\forall s\geq 1,$$

which combined with estimate (4.31), ends the proof. We now prove the estimate (4.40) with $s\geq 5$, for s even, the proof being the same for s odd. If we take $\psi=(-1)^{s-1}\Delta^{s-1}\varphi$, after some integrations by parts, we obtain

$$\begin{aligned}\int_\Omega dx\Big\{(-1)^s\Big[&\Delta^{\frac{s-2}{2}}(\nabla a_0\nabla_\perp\varphi)\cdot\Delta^{\frac{s-2}{2}}\nabla\nabla_\perp\varphi\\
&+\Delta^{\frac{s-2}{2}}(a_0\nabla\nabla_\perp\varphi)\cdot\Delta^{\frac{s-2}{2}}\nabla\nabla_\perp\varphi+\Delta^{\frac{s-2}{2}}(\nabla b_0\varphi)\cdot\Delta^{\frac{s-2}{2}}\nabla\varphi\\
&+\Delta^{\frac{s-2}{2}}(b_0\nabla\varphi)\cdot\Delta^{\frac{s-2}{2}}\nabla\varphi\Big]-\frac{\alpha^2}{2}(-1)^{s-1}\Big[2\Delta^{\frac{s}{2}}(a_0\nabla_\perp\varphi)\cdot\Delta^{\frac{s}{2}}\nabla_\perp\varphi\\
&+\Delta^{\frac{s}{2}}(b_0\varphi)\Delta^{\frac{s}{2}}\varphi-2\Delta^{\frac{s-1}{2}}(\nabla a_0\cdot\nabla\nabla_\perp\varphi)\cdot\Delta^{\frac{s-1}{2}}\nabla_\perp\varphi+\Delta^{\frac{s-2}{2}}(b_0\Delta\varphi)\Delta^{\frac{s-2}{2}}\Delta\varphi\Big]\Big\}\\
=(\mathfrak{B}\varphi,\psi)\leq\|\mathfrak{B}\varphi\|_{H^{s-2}(\Omega)}&\|(-1)^{s-1}\Delta^{s-1}\varphi\|_{H^{-s+2}(\Omega)}.\end{aligned} \tag{4.41}$$

In the sequel, we use the following Kato–Ponce type commutator estimates, whose proof can be found in Section 3.6 of Chapter 3 of [37]. Let $P\in OPS^s_{1,0}$, then

$$\|[P,f]g\|_{L^2(\Omega)}\leq C(\|f\|_{\mathrm{Lip}(\Omega)}\|g\|_{H^{s-1}(\Omega)}+\|f\|_{H^s(\Omega)}\|g\|_{L^\infty(\Omega)}), \tag{4.42}$$

where $[P,f]g=P(fg)-fPg$. Using (4.42) with $s>7/2$, the first term of the left-hand side of (4.41) can be bounded by

$$\|\nabla_\perp a_0\|_{L^\infty(\Omega_\perp)}\|\varphi\|_{H^{s-1}(\Omega)}\|\varphi\|_{H^{s-1}(\Omega)}\|\varphi\|_{H^s(\Omega)}+\|[\Delta^{\frac{s-2}{2}},\nabla_\perp a_0]\nabla_\perp\varphi\|_{L^2(\Omega)}\|\varphi\|_{H^s(\Omega)}$$
$$\leq C(s,a_0)\|\varphi\|_{H^{s-1}(\Omega)}\|\varphi\|_{H^s(\Omega)}. \tag{4.43}$$

For the second term of the left-hand side of (4.41), we get

$$
\begin{aligned}
&(-1)^s \int_\Omega dx\, \Delta^{\frac{s-2}{2}}(a_0 \nabla\nabla_\perp \varphi)\cdot \Delta^{\frac{s-2}{2}} \nabla\nabla_\perp \varphi \\
&= \int_\Omega dx \Big\{ a_0 |\Delta^{\frac{s-2}{2}} \nabla\nabla_\perp \varphi|^2 + \Delta^{\frac{s-2}{2}} \nabla\nabla_\perp \varphi \cdot [\Delta^{\frac{s-2}{2}}, a_0] \nabla\nabla_\perp \varphi \Big\},
\end{aligned}
\tag{4.44}
$$

where the second term of the right-hand side of (4.44) is bounded by

$$
\|[\Delta^{\frac{s-2}{2}}, a_0]\nabla\nabla_\perp \varphi\|_{L^2(\Omega)} \|\varphi\|_{H^s(\Omega)} \le C(a_0) \|\varphi\|_{H^{s-1}(\Omega)} \|\varphi\|_{H^s(\Omega)}, \tag{4.45}
$$

by using (4.42) with $s > 9/2$. The third term of the left-hand side of (4.41) is bounded by

$$
\begin{aligned}
&\|\nabla_\perp b_0\|_{L^\infty(\Omega_\perp)} \|\varphi\|_{H^{s-2}(\Omega)} \|\varphi\|_{H^{s-1}(\Omega)} + \|[\Delta^{\frac{s-2}{2}}, \nabla b_0]\varphi\|_{L^2(\Omega)} \|\varphi\|_{H^{s-1}(\Omega)} \\
&\le C(s, b_0) \|\varphi\|_{H^{s-1}(\Omega)} \|\varphi\|_{H^s(\Omega)},
\end{aligned}
\tag{4.46}
$$

using (4.42) with $s > 3/2$, while the fourth term is bounded by

$$
\begin{aligned}
&\|b_0\|_{L^\infty(\Omega_\perp)} \|\varphi\|^2_{H^{s-1}(\Omega)} + \|[\Delta^{\frac{s-2}{2}}, b_0]\nabla\varphi\|_{L^2(\Omega)} \|\varphi\|_{H^{s-1}(\Omega)} \\
&\le C(s, b_0) \|\varphi\|_{H^{s-1}(\Omega)} \|\varphi\|_{H^s(\Omega)},
\end{aligned}
\tag{4.47}
$$

using (4.42) with $s > 5/2$. By using the fact that $\Delta^{1/2}[\Delta^{s/2}, a_0] = [\Delta^{s/2}, \Delta^{1/2} a_0]$, and integration by parts, we obtain, for the fifth term of the left-hand side of (4.41), the expression

$$
(-1)^s \alpha^2 \Big\{ \int_\Omega dx\, a_0 |\Delta^{s/2} \nabla_\perp \varphi|^2 + \int_\Omega dx\, \Delta^{\frac{s-1}{2}} \nabla_\perp \varphi \cdot [\Delta^{s/2}, \Delta^{1/2} a_0] \nabla_\perp \varphi \Big\}. \tag{4.48}
$$

Using equivalence between semi-norms $|\nabla_\perp \cdot|_{H^{s-1}(\Omega)}$, $|\nabla\nabla_\perp \cdot|_{H^{s-2}(\Omega)}$, and $\|\Delta^{\frac{s-2}{2}} \nabla\nabla_\perp \cdot \|_{L^2(\Omega)}$, the Poincaré inequality $\|\nabla_\perp \varphi\|_{H^{s-1}(\Omega)} \le C(\Omega) |\nabla_\perp \varphi|_{H^{s-1}(\Omega)}$, and commutator estimate (4.42) with $s > 5/2$, the second term of (4.48) is bounded by

$$
C\alpha^2 \|\nabla^2_\perp a_0\|_{L^\infty(\Omega_\perp)} \int_\Omega dx\, |\Delta^{\frac{s-2}{2}} \nabla\nabla_\perp \varphi|^2 + C(\alpha, a_0) \|\varphi\|_{H^{s-1}(\Omega)} \|\varphi\|_{H^s(\Omega)}. \tag{4.49}
$$

The sixth term of the left-hand side of (4.41) can be rewritten as

$$
(-1)^s \frac{\alpha^2}{2} \Big\{ \int_\Omega b_0 |\Delta^{s/2} \varphi|^2 + \int_\Omega \Delta^{s/2} \varphi [\Delta^{s/2}, b_0] \varphi \Big\}, \tag{4.50}
$$

where the second term of (4.50) can be bounded by

$$
C(\alpha) \|[\Delta^{\frac{s}{2}}, b_0]\varphi\|_{L^2(\Omega)} \|\varphi\|_{H^s(\Omega)} \le C(\alpha, b_0) \|\varphi\|_{H^{s-1}(\Omega)} \|\varphi\|_{H^s(\Omega)} \tag{4.51}
$$

by using (4.42) with $s > 5/2$. Using integration by parts, commutator estimate (4.42) with $s > 9/2$, the Poincaré inequality, and equivalence between some semi-norms, the seventh term of the left-hand side of (4.41) can be estimated as

$$
(-1)^s \alpha^2 \int_\Omega dx\, \Delta^{\frac{s-1}{2}} (\nabla a_0 \cdot \nabla\nabla_\perp \varphi) \cdot \Delta^{\frac{s-1}{2}} \nabla_\perp \varphi
$$

$$\leq \alpha^2 \left\{ \int_\Omega dx \frac{1}{2} |\Delta a_0| |\Delta^{\frac{s-1}{2}} \nabla_\perp \varphi|^2 + \|[\Delta^{\frac{s-1}{2}}, \nabla a_0] \nabla \nabla_\perp \varphi\|_{L^2(\Omega)} \|\Delta^{\frac{s-1}{2}} \nabla_\perp \varphi\|_{L^2(\Omega)} \right\}$$
$$\leq C\alpha^2 \|\nabla_\perp^2 a_0\|_{L^\infty(\Omega_\perp)} \int_\Omega dx |\nabla \Delta^{\frac{s-2}{2}} \nabla_\perp \varphi|^2$$
$$+ C\alpha^2 \Big(\|\nabla_\perp^2 a_0\|_{L^\infty(\Omega_\perp)} \|\nabla \nabla_\perp \varphi\|_{H^{s-2}(\Omega)} \|\Delta^{\frac{s-2}{2}} \nabla \nabla_\perp \varphi\|_{L^2(\Omega)}$$
$$+ \|a_0\|_{H^{s+1}(\Omega_\perp)} \|\varphi\|_{H^{s-1}(\Omega)} \|\varphi\|_{H^s(\Omega)} \Big)$$
$$\leq C\alpha^2 \|\nabla_\perp^2 a_0\|_{L^\infty(\Omega_\perp)} \int_\Omega dx |\nabla \Delta^{\frac{s-2}{2}} \nabla_\perp \varphi|^2 + C(\alpha, a_0) \|\varphi\|_{H^{s-1}(\Omega)} \|\varphi\|_{H^s(\Omega)}. \tag{4.52}$$

Finally, the eighth term of the left-hand side of (4.41) can be rewritten as

$$(-1)^s \frac{\alpha^2}{2} \left\{ \int_\Omega b_0 |\Delta^{s/2} \varphi|^2 + \int_\Omega \Delta^{s/2} \varphi [\Delta^{(s-2)/2}, b_0] \Delta \varphi \right\}, \tag{4.53}$$

where the second term of (4.53) can be bounded by

$$C(\alpha) \|[\Delta^{\frac{s-2}{2}}, b_0] \Delta \varphi\|_{L^2(\Omega)} \|\varphi\|_{H^s(\Omega)} \leq C(\alpha, b_0) \|\varphi\|_{H^{s-1}(\Omega)} \|\varphi\|_{H^s(\Omega)}, \tag{4.54}$$

by using (4.42) with $s > 9/2$. Gathering all estimates (4.43)–(4.54), (4.41) leads to (4.40), which ends the proof. ∎

4.2.3. A priori estimates for the density c and its inverse ϱ. In this section, we obtain a priori estimates on the density c and its inverse ϱ. The density c satisfies equation (3.29), which can be recast as

$$\partial_t c + V \cdot \nabla c + c \partial_\parallel u = 0. \tag{4.55}$$

In order to obtain a priori estimates on $\varrho := 1/c$, we easily deduce from (3.29) that ϱ satisfies

$$\partial_t \varrho + V \cdot \nabla \varrho - \varrho \partial_\parallel u = 0. \tag{4.56}$$

If we now consider an application $\psi : \mathbb{R}^2 \to \mathbb{R}$, then thanks to the properties of the Bessel function J_0, we have $\|\mathcal{J}_\perp \psi\|_{H^s(\mathbb{R}^2)} \leq \|\psi\|_{H^s(\mathbb{R}^2)}$ which leads to

$$\|v_E\|_{H^s(\Omega)} \leq \|\mathcal{J}_\perp \nabla_\perp^\perp \phi\|_{H^s(\Omega)} \leq \|\phi\|_{H^{s+1}(\Omega)} \leq C(1 + \|c\|_{L^2_\nu(\Sigma; H^{s-1}(\Omega))}). \tag{4.57}$$

Using (4.57) and standard energy estimates (see [17, 31] for examples) for continuity equation (4.55), we get, for $s > 5/2$,

$$\frac{d}{dt} \|c\|_{H^s(\Omega)} \leq C(s, \alpha, a_0, b_0, n_{i0}, \Omega, \Sigma) \|c\|_{H^s(\Omega)}$$
$$(1 + \|u\|_{H^{s+1}(\Omega)} + \|c\|_{L^2_\nu(\Sigma; H^{s-1}(\Omega))}). \tag{4.58}$$

Since equations (4.56) and (4.55) have the same mathematical structure, we also obtain, for $s > 5/2$,

$$\frac{d}{dt} \|\varrho\|_{H^s(\Omega)} \leq C(s, \alpha, a_0, b_0, n_{i0}, \Omega, \Sigma) \|\varrho\|_{H^s(\Omega)}$$
$$(1 + \|u\|_{H^{s+1}(\Omega)} + \|c\|_{L^2_\nu(\Sigma; H^{s-1}(\Omega))}). \tag{4.59}$$

Thus we arrive at estimates for c and ϱ.

4.2.4. A priori estimates for the velocity u. In this section, we obtain a priori estimate on the velocity u which satisfies equation (3.30). Let β be a multi-index such that $|\beta|\le s+1$. From Proposition 4.4, we know that the operator $(1-\alpha^2\mathcal{A})$ is invertible as long as c and ϱ are regular enough. Therefore, if we apply the operator $\partial_x^\beta(1-\alpha^2\mathcal{A})^{-1}$ to (3.30), multiply the result by $\partial_x^\beta u$, and integrate over Ω, we obtain, for almost $\sigma\in\Sigma$ and $t>0$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_x^\beta u\|^2_{L^2(\Omega)} &+ \int_\Omega dx\,\partial_x^\beta u\partial_x^\beta(u\partial_\parallel u) + \int_\Omega dx\,\partial_x^\beta u\partial_x^\beta(v_E\cdot\nabla_\perp u)\\ &-\alpha^2\int_\Omega dx\,\partial_x^\beta u\partial_x^\beta\left((1-\alpha^2\mathcal{A})^{-1}(\partial_\parallel u\mathcal{A}u)\right)+\int_\Omega dx\,\partial_x^\beta u\partial_x^\beta\left((1-\alpha^2\mathcal{A})^{-1}\partial_\parallel(c^2/8)\right)\\ \int_\Omega dx\,\partial_x^\beta u\partial_x^\beta&\left((1-\alpha^2\mathcal{A})^{-1}\partial_\parallel\mathcal{J}_\perp\phi\right)-\frac{\alpha^2}{2}\int_\Omega dx\,\partial_x^\beta u\partial_x^\beta\left((1-\alpha^2\mathcal{A})^{-1}\partial_\parallel|\nabla u|^2\right)=0. \quad (4.60)\end{aligned}$$

Using integration by parts, the Cauchy–Schwarz inequality, estimate (4.1), and Sobolev embedding $H^s(\Omega)\hookrightarrow L^\infty(\Omega)$, with $s>3/2$, the second term of the left-hand side of (4.60) with $|\ell'|=1$ can be bounded by

$$\begin{aligned}\int_\Omega dx\,\partial_x^\beta u\partial_x^\beta(u\partial_\parallel u) &= -\frac{1}{2}\int_\Omega dx\,|\partial_x^\beta u|^2\partial_\parallel u\\ &+\int_\Omega dx\,\partial_x^\beta u\sum_{\ell>0}^{\beta}\binom{\beta}{\ell}\partial_x^{\ell-\ell'}(\partial_x^{\ell'}u)\cdot\partial_x^{\beta-\ell}\partial_\parallel u\le C\|u\|^3_{H^{s+1}(\Omega)}. \quad (4.61)\end{aligned}$$

Using integration by parts associated with the fact that $\nabla_\perp\cdot v_E=0$ and the Cauchy–Schwarz inequality, the third term of the left-hand side of (4.60), with $|\ell'|=1$, can be bounded by

$$\begin{aligned}\int_\Omega dx\,\partial_x^\beta u\partial_x^\beta(v_E\cdot\nabla_\perp u) &= \int_\Omega dx\,\partial_x^\beta u(\nabla_\perp\partial_x^\beta u)\cdot v_E+\int_\Omega dx\,\partial_x^\beta u\sum_{\ell>0}^{\beta}\binom{\beta}{\ell}\partial_x^\ell v_E\cdot\partial_x^{\beta-\ell}\nabla_\perp u\\ &\le\sum_{\ell>0}^{\beta}\binom{\beta}{\ell}\|\partial_x^{\ell-\ell'}(\partial_x^{\ell'}v_E)\cdot\partial_x^{\beta-\ell}\nabla_\perp u\|_{L^2(\Omega)}\|u\|_{H^{s+1}(\Omega)}\\ &\le C\|u\|^2_{H^{s+1}(\Omega)}(1+\|c\|_{L^2_\nu(\Sigma;H^s(\Omega))}), \quad (4.62)\end{aligned}$$

where we have used estimate (4.1), Sobolev embedding $H^s(\Omega)\hookrightarrow L^\infty(\Omega)$, with $s>3/2$ and estimate (4.57). Using Cauchy–Schwarz inequality, Lemma 4.3, Proposition 4.4, estimate (4.1) and Sobolev embedding $H^{s-1}(\Omega)\hookrightarrow L^\infty(\Omega)$, with $s>5/2$, the fourth term of the left-hand side of (4.60), can be bounded by

$$\begin{aligned}&\alpha^2\int_\Omega dx\,\partial_x^\beta u\partial_x^\beta\left((1-\alpha^2\mathcal{A})^{-1}(\partial_\parallel u\mathcal{A}u)\right)\le\|u\|_{H^{s+1}(\Omega)}\|\partial_x^\beta\left((1-\alpha^2\mathcal{A})^{-1}(\partial_\parallel u\mathcal{A}u)\right)\|_{L^2(\Omega)}\\ &\le\|u\|_{H^{s+1}(\Omega)}G\left(\|c\|_{H^{\max(s,5/2)}(\Omega)},\|\varrho\|_{H^{\max(s,5/2)}(\Omega)}\right)\|\partial_\parallel u\mathcal{A}u\|_{H^{s-1}(\Omega)}\\ &\le\|u\|_{H^{s+1}(\Omega)}G\left(\|c\|_{H^{\max(s,5/2)}(\Omega)},\|\varrho\|_{H^{\max(s,5/2)}(\Omega)}\right)\\ &\qquad\qquad\sum_{|\gamma|\le s-1}\sum_{\ell\ge0}^{\gamma}\binom{\gamma}{\ell}\|\partial_x^\ell(\partial_\parallel u)\partial_x^{\gamma-\ell}(\mathcal{A}u)\|_{L^2(\Omega)}\\ &\le C\|u\|^2_{H^{s+1}(\Omega)}\|u\|_{H^s(\Omega)}\end{aligned}$$

$$G\left(\|c\|_{H^{\max(s,5/2)}(\Omega)},\|\varrho\|_{H^{\max(s,5/2)}(\Omega)}\right)(1+\|\varrho\|_{H^{s-1}(\Omega)}\|c\|_{H^s(\Omega)}). \quad (4.63)$$

Using Cauchy–Schwarz inequality, Proposition 4.4, estimate (4.1) and Sobolev embedding $H^{s-1}(\Omega)\hookrightarrow L^\infty(\Omega)$, with $s>5/2$, the fifth term of the left-hand side of (4.60), can be bounded by

$$\begin{aligned}
&\int_\Omega dx\,\partial_x^\beta u \partial_x^\beta \left((1-\alpha^2\mathcal{A})^{-1}\partial_\parallel(c^2/8)\right) \le \|u\|_{H^{s+1}(\Omega)}\|\partial_x^\beta\left((1-\alpha^2\mathcal{A})^{-1}\partial_\parallel(c^2/8)\right)\|_{L^2(\Omega)}\\
&\le C\|u\|_{H^{s+1}(\Omega)}G\left(\|c\|_{H^{\max(s,5/2)}(\Omega)},\|\varrho\|_{H^{\max(s,5/2)}(\Omega)}\right)\|c\partial_\parallel c\|_{H^{s-1}(\Omega)}\\
&\le C\|u\|_{H^{s+1}(\Omega)}G\left(\|c\|_{H^{\max(s,5/2)}(\Omega)},\|\varrho\|_{H^{\max(s,5/2)}(\Omega)}\right)\\
&\qquad\qquad\sum_{|\gamma|\le s-1}\sum_{\ell\ge 0}^{\gamma}\binom{\gamma}{\ell}\|\partial_x^\ell c\partial_x^{\gamma-\ell}(\partial_\parallel c)\|_{L^2(\Omega)}\\
&\le C\|c\|^2_{H^s(\Omega)}\|u\|_{H^{s+1}(\Omega)}G\left(\|c\|_{H^{\max(s,5/2)}(\Omega)},\|\varrho\|_{H^{\max(s,5/2)}(\Omega)}\right). \quad (4.64)
\end{aligned}$$

Using Cauchy–Schwarz inequality, Proposition 4.4, and estimate (4.57) the sixth term of the left-hand side of (4.60), for $s\ge 1$, can be bounded by

$$\begin{aligned}
&\int_\Omega dx\,\partial_x^\beta u \partial_x^\beta \left((1-\alpha^2\mathcal{A})^{-1}\partial_\parallel \mathcal{J}_\perp\phi\right)\\
&\le \|u\|_{H^{s+1}(\Omega)}\|\partial_x^\beta\left((1-\alpha^2\mathcal{A})^{-1}(\partial_\parallel\mathcal{J}_\perp\phi)\right)\|_{L^2(\Omega)}\\
&\le C\|u\|_{H^{s+1}(\Omega)}G\left(\|c\|_{H^{\max(s,5/2)}(\Omega)},\|\varrho\|_{H^{\max(s,5/2)}(\Omega)}\right)\|\phi\|_{H^s(\Omega)}\\
&\le C\|u\|_{H^{s+1}(\Omega)}(1+\|c\|_{L^2_\nu(\Sigma;H^{s-2}(\Omega))})G\left(\|c\|_{H^{\max(s,5/2)}(\Omega)},\|\varrho\|_{H^{\max(s,5/2)}(\Omega)}\right). \quad (4.65)
\end{aligned}$$

Using the Cauchy–Schwarz inequality, Proposition 4.4, estimate (4.1), and Sobolev embedding $H^{s-1}(\Omega)\hookrightarrow L^\infty(\Omega)$, with $s>5/2$, the seventh term of the left-hand side of (4.60), can be bounded by

$$\begin{aligned}
&\frac{\alpha^2}{2}\int_\Omega dx\,\partial_x^\beta u \partial_x^\beta \left((1-\alpha^2\mathcal{A})^{-1}\partial_\parallel|\nabla u|^2\right)\\
&\le \|u\|_{H^{s+1}(\Omega)}\|\partial_x^\beta\left((1-\alpha^2\mathcal{A})^{-1}(\partial_\parallel|\nabla u|^2)\right)\|_{L^2(\Omega)}\\
&\le \|u\|_{H^{s+1}(\Omega)}G\left(\|c\|_{H^{\max(s,5/2)}(\Omega)},\|\varrho\|_{H^{\max(s,5/2)}(\Omega)}\right)\|\partial_\parallel|\nabla u|^2\|_{H^{s-1}(\Omega)}\\
&\le \|u\|_{H^{s+1}(\Omega)}G\left(\|c\|_{H^{\max(s,5/2)}(\Omega)},\|\varrho\|_{H^{\max(s,5/2)}(\Omega)}\right)\\
&\qquad\qquad\sum_{|\gamma|\le s-1}\sum_{\ell\ge 0}^{\gamma}\binom{\gamma}{\ell}\|\partial_x^\ell(\nabla u)\cdot\partial_x^{\gamma-\ell}(\nabla\partial_\parallel u)\|_{L^2(\Omega)}\\
&\le C\|u\|^2_{H^{s+1}(\Omega)}\|u\|_{H^s(\Omega)}G\left(\|c\|_{H^{\max(s,5/2)}(\Omega)},\|\varrho\|_{H^{\max(s,5/2)}(\Omega)}\right). \quad (4.66)
\end{aligned}$$

From expression (4.60) and estimates (4.61)–(4.66), we obtain, for $s>5/2$,

$$\begin{aligned}
\frac{d}{dt}\|u\|_{H^{s+1}(\Omega)} &\le C\Big\{\|u\|^2_{H^{s+1}(\Omega)}+\|u\|_{H^{s+1}(\Omega)}(1+\|c\|_{L^2_\nu(\Sigma;H^s(\Omega))})\\
&\quad + G\left(\|c\|_{H^{\max(s,5/2)}(\Omega)},\|\varrho\|_{H^{\max(s,5/2)}(\Omega)}\right)\Big[\|c\|^2_{H^s(\Omega)}+\big(1+\\
&\quad \|c\|_{L^2_\nu(\Sigma;H^{s-2}(\Omega))}\big)+\|u\|_{H^s(\Omega)}\|u\|_{H^{s+1}(\Omega)}(1+\|\varrho\|_{H^s(\Omega)}\|c\|_{H^s(\Omega)})\Big]\Big\}\\
&\le \mathcal{G}\left(\|u\|_{H^{s+1}(\Omega)},\|c\|_{H^{\max(s,5/2)}(\Omega)},\|\varrho\|_{H^{\max(s,5/2)}(\Omega)},\|c\|_{L^2(\Sigma;H^s(\Omega))}\right), \quad (4.67)
\end{aligned}$$

where the function $\mathcal{G}$ is positive and nondecreasing with respect to its arguments.

If we set $X(t)=\|c(t)\|_{L^\infty(\Sigma;H^s(\Omega))}+\|\varrho(t)\|_{L^\infty(\Sigma;H^s(\Omega))}+\|u(t)\|_{L^\infty(\Sigma;\ H^{s+1}(\Omega))}$, since $\|\cdot\|_{L^p_\nu(\Sigma;H^s(\Omega))}\le\|\cdot\|_{L^\infty(\Sigma;H^s(\Omega))}$ for $1\le p<\infty$ (because ν is a finite measure), then, using (4.58), (4.59), and (4.67), after time integration, we obtain

$$X(t)\le X(0)+\int_0^t \mathcal{F}(X(\tau))d\tau,$$

where the function $\mathcal{F}:\mathbb{R}^+\to\mathbb{R}^+$ is a positive and nondecreasing function. Therefore from Gronwall's lemma (see Appendix A of [10]), there exist a time $T>0$ and a positive function $\mathcal{K}(t)$, finite on $[0,T]$, such that $X(t)<\mathcal{K}(t)$, $\forall t\in[0,T]$. Consequently, using equations (3.30)–(3.31), (4.56), and (4.55) we obtain that, for $s>5/2$ and $1\le p<\infty$,

$$\left.\begin{aligned}
&c,\varrho\in L^\infty([0,T];L^p_\nu\cap L^\infty(\Sigma;H^s(\Omega)))\cap\mathrm{Lip}([0,T];L^p_\nu\cap L^\infty(\Sigma;H^{s-1}(\Omega))),\\
&u\in L^\infty([0,T];L^p_\nu\cap L^\infty(\Sigma;H^{s+1}(\Omega)))\cap\mathrm{Lip}([0,T];L^p_\nu\cap L^\infty(\Sigma;H^{s}(\Omega))),\\
&\phi\in L^\infty([0,T];H^{s+2}(\Omega))\cap\mathrm{Lip}([0,T];H^{s+1}(\Omega)).
\end{aligned}\right\}\tag{4.68}$$

4.3. Existence. In this section, using a priori estimates (4.68) and an iterative scheme, we prove the existence of sequences of solutions for a modified problem and their convergence to limit points which are solutions of the unmodified problem formed by equations (3.29)–(3.31) and (4.56). Let ζ_{δ_k} be a sequence of mollifier such that $\zeta_\delta=\zeta(x/\delta)/\delta^3$ $(0<\delta<1,\ \int_{\mathbb{R}^3}\zeta dx=1)$ and $\delta_k\to0$ as $k\to\infty$. We define the iteration scheme through the solution $(c^{k+1},\varrho^{k+1},u^{k+1},\phi^{k+1})$ of the following modified problem

$$\left.\begin{aligned}
&\partial_t c^{k+1}+{v_E}^k\cdot\nabla_\perp c^{k+1}+u^k\partial_\parallel c^{k+1}+c^k\partial_\parallel u^k=0,\qquad c^{k+1}(t=0)=\zeta_{\delta_{k+1}}*c_0,\\
&\partial_t\varrho^{k+1}+{v_E}^k\cdot\nabla_\perp\varrho^{k+1}+u^k\partial_\parallel\varrho^{k+1}-\varrho^k\partial_\parallel u^k=0,\qquad \varrho^{k+1}(t=0)=\zeta_{\delta_{k+1}}*\varrho_0,\\
&\left(Q-\tfrac{\alpha^2}{2}[\Delta Q+Q\Delta]\right)\phi^k=\int_\Sigma d\nu(\sigma)\mathcal{J}_\perp c^k-\left(1-\frac{\alpha^2}{2}\Delta\right)n_{i0},\\
&\partial_t u^{k+1}+{v_E}^k\cdot\nabla_\perp u^{k+1}+u^k\partial_\parallel u^{k+1}-\alpha^2(1-\alpha^2\mathcal{A}^k)^{-1}(\partial_\parallel u^k\mathcal{A}^k u^k)\\
&\quad+(1-\alpha^2\mathcal{A}^k)^{-1}\partial_\parallel\left(\tfrac{{c^k}^2}{8}+\mathcal{J}_\perp\phi^k-\tfrac{\alpha^2}{2}|\nabla u^k|^2\right)=0,\ u^{k+1}(t=0)=\zeta_{\delta_{k+1}}*u_0.
\end{aligned}\right\}\tag{4.69}$$

Let us show the existence and convergence of the sequences $(\{c^k\}_{k\ge0},\{\varrho^k\}_{k\ge0},\{u^k\}_{k\ge0},\{\phi^k\}_{k\ge0})$. Following the proof of a priori estimates (4.68) obtained in the previous section, we can show in the same way that the sequences of solutions $(\{c^k\}_{k\ge0},\{\varrho^k\}_{k\ge0},\{u^k\}_{k\ge0},\{\phi^k\}_{k\ge0})$ of (4.69) satisfied the following bounds, for all $k\ge0$, $s>5/2$, and $1\le p<\infty$:

$$\left.\begin{aligned}
&c^k,\varrho^k\in L^\infty([0,T];L^p_\nu\cap L^\infty(\Sigma;H^s(\Omega))\cap\mathrm{Lip}([0,T];L^p_\nu\cap L^\infty(\Sigma;H^{s-1}(\Omega))),\\
&u^k\in L^\infty([0,T];L^p_\nu\cap L^\infty(\Sigma;H^{s+1}(\Omega))\cap\mathrm{Lip}([0,T];L^p_\nu\cap L^\infty(\Sigma;H^{s}(\Omega))),\\
&\phi^k\in L^\infty([0,T];H^{s+2}(\Omega))\cap\mathrm{Lip}([0,T];H^{s+1}(\Omega)).
\end{aligned}\right\}\tag{4.70}$$

The lack of regularity of a priori estimates with respect to the variable $\sigma\in\Sigma$ prevents us from using Ascoli–Arzelà compactness theorem and compact Sobolev embeddings to get strong convergence of the sub-sequences of $(\{c^k\}_{k\ge0},\{\varrho^k\}_{k\ge0},\ \{u^k\}_{k\ge0},\{\phi^k\}_{k\ge0})$ (which converge weakly thanks to weak compactness and bounds (4.70)). In order to recover strong convergence, we have to show directly that the sequences are Cauchy sequences in a Banach space. Let us define $\delta c^k=c^k-c^{k-1}$, $\delta u^k=u^k-u^{k-1}$, $\delta\phi^k=\phi^k-\phi^{k-1}$ and the operator $\mathcal{A}^{\delta k}=(\delta c^k)^{-1}\nabla\cdot(\delta c^k\nabla)$. By subtraction of two consecutive stages k and $k-1$ of the system (4.69) we obtain the system for the difference sequences $(\{\delta c^k\}_{k>0}$, $\{\delta u^k\}_{k>0}, \{\delta\phi^k\}_{k>0})$

$$\left.\begin{aligned}&\partial_t\delta c^{k+1}+{v_E}^k\cdot\nabla_\perp\delta c^{k+1}+u^k\partial_\parallel\delta c^{k+1}+\delta{v_E}^k\cdot\nabla_\perp c^k\\&\qquad\qquad\qquad\qquad+\delta u^k\partial_\parallel c^k+c^k\partial_\parallel\delta u^k+\delta c^k\partial_\parallel u^{k-1}=0,\\&\delta c^{k+1}(t=0)=(\zeta_{\delta_{k+1}}-\zeta_{\delta_k})*c_0,\end{aligned}\right\}\tag{4.71}$$

$$\left.\begin{aligned}&\partial_t\delta u^{k+1}+{v_E}^k\cdot\nabla_\perp\delta u^{k+1}+u^k\partial_\parallel\delta u^{k+1}+\delta{v_E}^k\cdot\nabla_\perp u^k+\delta u^k\partial_\parallel u^k\\&\quad-\alpha^2(1-\alpha^2\mathcal{A}^{k-1})^{-1}\Big\{\partial_\parallel\delta u^k\mathcal{A}^k u^k+\partial_\parallel u^{k-1}\mathcal{A}^{k-1}\delta u^k+\partial_\parallel u^{k-1}\varrho^k\delta c^k\mathcal{A}^{\delta k}u^k\\&\qquad\qquad-\ \partial_\parallel u^{k-1}\varrho^k\delta c^k\mathcal{A}^{k-1}u^k-\delta c^k\partial_\parallel(c^k+c^{k-1})/8\\&\qquad\qquad-\ \partial_\parallel\mathcal{J}_\perp\delta\phi^k+\nabla\delta u^k\cdot\partial_\parallel\nabla u^k+\nabla u^{k-1}\cdot\partial_\parallel\nabla\delta u^k\\&\qquad\qquad+\ \delta c^k\varrho^k\mathcal{A}^{\delta k}(\partial_t u^{k+1}+{v_E}^k\cdot\nabla_\perp u^{k+1}+u^k\partial_\parallel u^{k+1})\\&\qquad\qquad-\ \delta c^k\varrho^k\mathcal{A}^{k-1}(\partial_t u^{k+1}+{v_E}^k\cdot\nabla_\perp u^{k+1}+u^k\partial_\parallel u^{k+1})\Big\}=0,\\&\delta u^{k+1}(t=0)=(\zeta_{\delta_{k+1}}-\zeta_{\delta_k})*u_0,\end{aligned}\right\}\tag{4.72}$$

and

$$\left(Q-\frac{\alpha^2}{2}[\Delta Q+Q\Delta]\right)\delta\phi^k=\int_\Sigma d\nu(\sigma)\mathcal{J}_\perp\delta c^k,\tag{4.73}$$

Starting from equations (4.71) and (4.73) and thus performing the same kind of energy estimates done in Section 4.2.3 except for the second (resp. third) term of (4.71) for which we rewrite the terms $\nabla_\perp\delta c^{k+1}$ (resp. $\partial_\parallel\delta c^{k+1}$) as $\Lambda(\Lambda^{-1}\nabla_\perp\delta c^{k+1})$ (resp. $\Lambda(\Lambda^{-1}\partial_\parallel\delta c^{k+1})$) with $\Lambda=(1-\Delta)^{1/2}$ before applying estimate (4.1), we obtain for $s>5/2$,

$$\begin{aligned}&\frac{d}{dt}\|\delta c^{k+1}\|_{H^{s-1}(\Omega)}\\&\le C\left(s,\alpha,a_0,b_0,n_0,\Omega,\Sigma,\|c^k\|_{L^\infty([0,T]\times\Sigma;H^s(\Omega))},\{\|u^\ell\|_{L^\infty([0,T]\times\Sigma;H^s(\Omega))}\}_{\ell=k-1,k}\right)\\&\qquad\qquad\left(\|\delta c^{k+1}\|_{H^{s-1}(\Omega)}+\|\delta c^k\|_{H^{s-1}(\Omega)}+\|\delta u^k\|_{H^s(\Omega)}+\|\delta c^k\|_{L^2_\nu(\Sigma;H^{s-2}(\Omega))}\right).\end{aligned}\tag{4.74}$$

Considering equations (4.72) and (4.73) and then performing the same kind of energy estimates done in Section 4.2.4, except for the seventh, thirteenth, fourteenth, and fifteenth terms of (4.72), for which we rewrite the terms $\mathcal{A}^{k-1}\delta u^k$, $\partial_\parallel\nabla\delta u^k$, $\mathcal{A}^{\delta k}(\partial_t u^{k+1}+{v_E}^k\cdot\nabla_\perp u^{k+1}+u^k\partial_\parallel u^{k+1})$ and $\mathcal{A}^{k-1}(\partial_t u^{k+1}+{v_E}^k\cdot\nabla_\perp u^{k+1}+u^k\partial_\parallel u^{k+1})$ respectively as $\Lambda(\Lambda^{-1}\mathcal{A}^{k-1}\delta u^k)$, $\Lambda(\Lambda^{-1}\partial_\parallel\nabla\delta u^k)$, $\Lambda(\Lambda^{-1}\mathcal{A}^{\delta k}(\partial_t u^{k+1}+{v_E}^k\cdot\nabla_\perp u^{k+1}+u^k\partial_\parallel u^{k+1}))$ and $\Lambda(\Lambda^{-1}\mathcal{A}^{k-1}(\partial_t u^{k+1}+{v_E}^k\cdot\nabla_\perp u^{k+1}+u^k\partial_\parallel u^{k+1}))$ with $\Lambda=(1-\Delta)^{1/2}$ before applying estimate (4.1), we obtain, for $s>5/2$,

$$\begin{aligned}\frac{d}{dt}\|\delta u^{k+1}\|_{H^s(\Omega)}&\le C\big(s,\alpha,a_0,b_0,n_0,\Omega,\Sigma,\\&\{\|c^\ell\|_{L^\infty([0,T]\times\Sigma;H^{s+1}(\Omega))}\}_{\ell=k-1,k},\{\|u^\ell\|_{L^\infty([0,T]\times\Sigma;H^{s+1}(\Omega))}\}_{\ell=k-1,k,k+1},\\&\quad\{\|\varrho^\ell\|_{L^\infty([0,T]\times\Sigma;H^s(\Omega))}\}_{\ell=k-1,k},\|\partial_t u^{k+1}\|_{L^\infty([0,T]\times\Sigma;H^s(\Omega))}\big)\\&\quad\left(\|\delta u^{k+1}\|_{H^s(\Omega)}+\|\delta c^k\|_{H^{s-1}(\Omega)}+\|\delta u^k\|_{H^s(\Omega)}+\|\delta c^k\|_{L^2_\nu(\Sigma;H^{s-1}(\Omega))}\right).\end{aligned}\tag{4.75}$$

If we set $Y^{k+1}(t)=\|\delta c^{k+1}(t)\|_{L^p_\nu\cap L^\infty(\Sigma;H^{s-1}(\Omega))}+\|\delta u^{k+1}(t)\|_{L^p_\nu\cap L^\infty(\Sigma;H^{s}(\Omega))}$, with $2\leq p<\infty$, then, using (4.74)–(4.75), we get the differential inequality

$$\frac{d}{dt}Y^{k+1}(t)\leq C(Y^k(t)+Y^{k+1}(t)),$$

which leads, after using a Gronwall lemma to the estimate

$$\sup_{t\in[0,T]} Y^{k+1}(t)\leq Y^{k+1}(0)\exp(CT)+CT\exp(CT)\sup_{t\in[0,T]} Y^k(t), \tag{4.76}$$

where

$$\begin{aligned}&Y^{k+1}(0)\exp(CT)\\&=\exp(CT)\left(\|(\zeta_{\delta_{k+1}}-\zeta_{\delta_k})*c_0\|_{L^p_\nu\cap L^\infty(\Sigma;H^{s-1}(\Omega))}+\|(\zeta_{\delta_{k+1}}-\zeta_{\delta_k})*u_0\|_{L^p_\nu\cap L^\infty(\Sigma;H^{s}(\Omega))}\right)\\&\leq C\left(T,\|c_0\|_{L^p_\nu\cap L^\infty(\Sigma;H^{s}(\Omega))},\|u_0\|_{L^p_\nu\cap L^\infty(\Sigma;H^{s+1}(\Omega))}\right)|\delta_{k+1}-\delta_k|\delta_{k+1}/\delta_k=\epsilon_k.\end{aligned}$$

As a result, any good choice of δ_k (e.g. $\delta_k=1/k$) makes the series $\sum_k\epsilon_k$ convergent ($\epsilon_k=\mathcal{O}(k^{-2})$ if $\delta_k=1/k$). Therefore, using (4.76), if T is small enough, there exists a constant $\kappa<1$ such that, for $k>0$,

$$\sup_{t\in[0,T]} Y^{k+1}(t)\leq\kappa\sup_{t\in[0,T]} Y^k(t)+\epsilon_k, \tag{4.77}$$

which proves that $\|c^{k+1}-c^k\|_{L^\infty([0,T];L^p_\nu\cap L^\infty(\Sigma;H^{s-1}(\Omega)))}$, $\|\varrho^{k+1}-\varrho^k\|_{L^\infty([0,T];L^p_\nu\cap L^\infty(\Sigma;H^{s-1}(\Omega)))}$ (since $\varrho^{k+1}-\varrho^k=-(c^{k+1}-c^k)\varrho^{k+1}\varrho^k$), $\|u^{k+1}-u^k\|_{L^\infty([0,T];L^p_\nu\cap L^\infty(\Sigma;H^{s}(\Omega)))}$ and $\|\phi^{k+1}-\phi^k\|_{L^\infty([0,T];H^{s+1}(\Omega))}$ (using equation (4.73)) are bounded for any $k\geq 0$, and $1\leq p<\infty$. Using (4.77) with $\delta_k=1/k$, we obtain, for $s>5/2$, and $1\leq p<\infty$, that

$$\begin{aligned}&\|c^q-c^{q'}\|_{L^\infty([0,T];L^p_\nu\cap L^\infty(\Sigma;H^{s-1}(\Omega)))}\\&\leq\textstyle\sum_{k=q'+1}^q\|c^k-c^{k-1}\|_{L^\infty([0,T];L^p_\nu\cap L^\infty(\Sigma;H^{s-1}(\Omega)))}\leq C\frac{|q-q'|}{qq'},\\&\|\varrho^q-\varrho^{q'}\|_{L^\infty([0,T];L^p_\nu\cap L^\infty(\Sigma;H^{s-1}(\Omega)))}\\&\leq\textstyle\sum_{k=q'+1}^q\|\varrho^k-\varrho^{k-1}\|_{L^\infty([0,T];L^p_\nu\cap L^\infty(\Sigma;H^{s-1}(\Omega)))}\leq C\frac{|q-q'|}{qq'},\\&\|u^q-u^{q'}\|_{L^\infty([0,T];L^p_\nu\cap L^\infty(\Sigma;H^{s}(\Omega)))}\\&\leq\textstyle\sum_{k=q'+1}^q\|u^k-u^{k-1}\|_{L^\infty([0,T];L^p_\nu\cap L^\infty(\Sigma;H^{s}(\Omega)))}\leq C\frac{|q-q'|}{qq'},\\&\|\phi^q-\phi^{q'}\|_{L^\infty([0,T];H^{s+1}(\Omega))}\\&\leq\textstyle\sum_{k=q'+1}^q\|\phi^k-\phi^{k-1}\|_{L^\infty([0,T];H^{s+1}(\Omega))}\leq C\frac{|q-q'|}{qq'},\end{aligned}$$

which proves that the sequences $\{c^k\}_{k\geq 0}$, $\{\varrho^k\}_{k\geq 0}$, $\{u^k\}_{k\geq 0}$, and $\{\phi^k\}_{k\geq 0}$ are a Cauchy sequences in the Banach space $L^\infty([0,T];L^p_\nu\cap L^\infty(\Sigma;H^{s-1}(\Omega)))$, $L^\infty([0,T];L^p_\nu\cap L^\infty(\Sigma;H^{s-1}(\Omega)))$, $L^\infty([0,T];L^p_\nu\cap L^\infty(\Sigma;H^{s}(\Omega)))$, and $L^\infty([0,T];H^{s+1}(\Omega))$, respectively, with $s>5/2$ and $1\leq p<\infty$, and have strong limit points $(c,\varrho,u,\phi)\in(L^\infty([0,T];L^p_\nu\cap L^\infty(\Sigma;H^{s}(\Omega))))^2\times L^\infty([0,T];L^p_\nu\cap L^\infty(\Sigma;H^{s+1}(\Omega)))\times L^\infty([0,T];H^{s+2}(\Omega))$, with $s>5/2$, $1\leq p<\infty$ and of course $\varrho=1/c$. Using strong convergence of the sequences $\{c^k\}_{k\geq 0}$, $\{\varrho^k\}_{k\geq 0}$, $\{u^k\}_{k\geq 0}$ and $\{\phi^k\}_{k\geq 0}$, we can pass to the limit in the iterative scheme (4.69) and consequently the limit point (c,ϱ,u,ϕ) satisfies the original unmodified system (3.29)–(3.31) and (4.56). In addition, from equations (3.29)–(3.31), and (4.56), we deduce that $(c,\varrho,u,\phi)\in(\mathrm{Lip}([0,T];L^p_\nu\cap L^\infty(\Sigma;H^{s-1}(\Omega))))^2\times\mathrm{Lip}([0,T];L^p_\nu\cap L^\infty(\Sigma;H^{s}(\Omega)))\times\mathrm{Lip}([0,T];H^{s+1}(\Omega))$ with $s>5/2$ and $1\leq p<\infty$.

4.4. Uniqueness. Here, we prove the uniqueness of solutions built in the previous section. For this purpose, we consider two solutions (c_i,u_i,ϕ_i), $i=1,2$, of the system formed by equations (3.29)–(3.31). If we set $c=c_1-c_2$, $u=u_1-u_2$, and $\phi=\phi_1-\phi_2$, then after subtracting equations (3.29)–(3.31) for each solution, we get a system similar to (4.71)–(4.73). Therefore, following the same kind of analysis done for the existence proof where we have obtained a priori estimates (4.74)–(4.75), we show that, for $s>5/2$, there exists a constant $C_\star$, depending on s, α, a_0, b_0, n_0, Ω, Σ, $\{\|c_i\|_{L^\infty([0,T]\times\Sigma;H^s(\Omega))}\}_{i=1,2}$, $\{\|u_i\|_{L^\infty([0,T]\times\Sigma;H^{s+1}(\Omega))}\}_{i=1,2}$, $\{\|\varrho_i\|_{L^\infty([0,T]\times\Sigma;H^s(\Omega))}\}_{i=1,2}$, and $\{\|\partial_t u_i\|_{L^\infty([0,T]\times\Sigma;H^s(\Omega))}\}_{i=1,2}$ such that

$$\frac{d}{dt}\|c\|_{H^{s-1}(\Omega)}\leq C_\star\left(\|c\|_{H^{s-1}(\Omega)}+\|u\|_{H^s(\Omega)}+\|c\|_{L^2(\Sigma;H^{s-2}(\Omega))}\right) \tag{4.78}$$

and

$$\frac{d}{dt}\|u\|_{H^s(\Omega)}\leq C_\star\left(\|c\|_{H^{s-1}(\Omega)}+\|u\|_{H^s(\Omega)}+\|c\|_{L^2(\Sigma;H^{s-1}(\Omega))}\right). \tag{4.79}$$

If we set $Z(t)=\|c(t)\|_{L^p_\nu\cap L^\infty(\Sigma;H^{s-1}(\Omega))}+\|u(t)\|_{L^p_\nu\cap L^\infty(\Sigma;H^s(\Omega))}$, with $2\leq p<\infty$, then, using (4.78)–(4.79), we get the differential inequality $d_tZ(t)\leq C(t)Z(t)$, which leads, after using a Gronwall lemma, to the estimate

$$\sup_{t\in[0,T]} Z(t)\leq Z(0)\exp\left(\int_0^T C(\tau)d\tau\right)\leq CZ(0).$$

This last stability inequality proves uniqueness.

5. Conclusion

In this paper, we have derived a new model to deal with small scales in gyrokinetic turbulence problems, using Lagrangian averaging techniques. It turns out that this modeling also improves the well-posedness of the model, which should be numerically and physically more robust than the original one. There are two prospects following this work. The first one is numerical simulations of this new model and its comparison with orginal one [8,15,16] and the gyrokinetic-Vlasov model [22]. The second one is to understand the long-time behaviour of solutions and thus global-in-time well-posedness of at least weak solutions.

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