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ON THE CAMASSA–HOLM SYSTEM WITH ONE MEAN ZERO COMPONENT*

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Abstract. In this paper, a generalized two-component Camassa–Holm model, closely connected to the shallow water theory, is discussed. This two-component Camassa–Holm system is investigated on the local well-posedness and blow-up phenomena. The present work is mainly concerned with the detailed blow-up criteria where some special classes of initial data are involved. Moreover, as a by-product, the blow-up rate is established.

Key words. Two-component Camassa–Holm system, well-posedness, blow-up criteria.

AMS subject classifications. 35Q35, 37L05, 37J35, 58E35.

1. Introduction

We consider the following two-component Camassa–Holm system:

$$\begin{cases} y_t + y_x u + 2y u_x + \pi(\rho)\rho_x = 0, \\ \pi(\rho)_t + (\pi(\rho)u)_x = 0, \end{cases} \tag{1.1}$$

where

$$y = u - u_{xx}, \quad \pi(\rho) = \rho - \mu(\rho) = \rho - \int_0^1 \rho dx,$$

with $u(x,t)$ and $\rho(x,t)$ depending on a space variable $x \in \mathbb{S} = \mathbb{R}/\mathbb{Z}$ and a time variable $t \geq 0$. For convenience, we call system (1.1) the π-CH2 equation. It is obvious that equation (1.1) for $\mu(\rho) = 0$ reduces to the two-component Camassa–Holm (CH2) equation studied in [8, 9, 12, 13, 33, 37], and reduces to the Camassa–Holm (CH) equation for $\pi(\rho) = 0$ investigated in [1–4, 27, 32, etc.]

The CH equation was derived physically by Camassa and Holm in [2] (found earlier by Fokas and Fuchssteniner [14] as a bi-Hamiltonian generalization of the KdV equation) by directly approximating the Hamiltonian for Euler's equation in the shallow water region with $u(x,t)$ represents the free surface above a flat bottom. An important feature of CH equation is its integrability [2]. Recently, an alternative derivation of the CH equation as a model for water waves was presented by Johnson [28]. Some satisfactory results were obtained by several authors concerning the local well-posedness [3,26,31,34], wave breaking phenomena (the solution itself remains bounded while its slope becomes infinity in finite time) [3, 4, 7, 29, 31, 32, 42, 43] and global in time solutions [3, 4], the aspects of weak solutions [6,35,40], the global conservative solutions [1] and the solitary waves [10]. It is worthy of being mentioned here is the property of propagation speed

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of solutions to the CH equation, which was presented by Zhou and his collaborators in their work [27], and the asymptotic profile in [27] was improved by [36].

The CH2 equation is also an integrable system [8]. It has been discussed under geometric aspects [11, 24]. Some qualitative research of solutions to the CH2 equation and its generalizations are the subjects of [12,15–22,38,39,41,46] . The inverse scattering and the derivation of the solitons are in [25] and references therein. Let's now draw our attention to (1.1). It is obvious that (1.1) can be viewed as the CH2 equation by substituting ρ with $\pi(\rho)$, or equivalently, viewed as the CH2 equation with one mean zero component. Note that the geometric structure of (1.1) is different from CH2 equation with the incorporation of the term $\mu(\rho)$, this restricts our discussion only on the unit circle. Furthermore, the discussion of this work shows that π-CH2 system possesses wave breaking phenomenon which is described with different blow-up criteria while the CH2 system admits not only breaking wave solutions but also solutions defined for all times. At least, we are not sure the existence of global solutions of π-CH2 equation for the time being. It is the structure of (1.1) that breaks some properties which previously holds for the CH2 system. We are interested in (1.1) based on the following considerations. It is known that for the CH2 system (1.1)

$$\frac{d}{dt}\int_{\mathbb{S}}\rho dx = -\int_{\mathbb{S}}(\rho u)_x dx = 0,$$

it follows that $\int_{\mathbb{S}}\rho dx$ is an invariant with respect to time. Particularly, if ρ has mean zero at initial time $t=0$, then the solution ρ to CH2 system will preserve zero mean for all time. This is one motivation of our present work. Secondly, we consider the system (1.1) in the spaces $H^s \times H^{s-1}/\mathbb{R}$ for $s>5/2$ on the circle, where $H^s = H^s(\mathbb{S})$ denotes the L^2-Sobolev space of regularity. The basic idea of this variation is the decomposition of $\rho \in H^{s-1} = \hat{H}^{s-1} \otimes \mathbb{R}$ into $\pi(\rho)$ and $\mu(\rho)$, where $\mu(\rho) \in \mathbb{R}$ is independent on variable x, $\pi(\rho)$ belongs to $\hat{H}^{s-1}$, a subspace of H^{s-1} containing all zero mean functions. The interesting aspect is that the CH2 equation has a meaningful geometric interpretation on the entire space $H^s \times H^{s-1}$ as well as on the component $H^s \times \hat{H}^{s-1}$. In [30], the author investigated the π-CH2 equation from geometric point of view and the local well-posedness of the system (1.1) was established. No further results were obtained for (1.1) for the moment. The main purpose of our work is to investigate formation of singularities of solutions to (1.1) where the conservation laws play crucial roles.

The rest of this paper is organized as follows. In Section 2, we recall the local well-posedness theorem and show some auxiliary results which will be used in the sequel. In Section 3, the blow-up criteria are established via various initial conditions and the blow-up rate is also shown. The final section, Section 4, is a brief conclusion and remarks to our results.

2. Preliminaries

We now provide the framework in which we shall reformulate (1.1). Let

$$G(x) = \frac{\cosh(x-[x]-1/2)}{2\sinh(1/2)}, \quad x \in \mathbb{R}.$$

where $[x]$ means the integer part of x. Then $(1-\partial_x^2)^{-1}f = G*f$ for all $f \in L^2(\mathbb{S})$ and $G*y = u$ where $*$ is the convolution. With these in hand, one can rewrite (1.1) as follows

$$\begin{cases} u_t + uu_x = -\partial_x G * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\pi(\rho)^2\right) \\ \pi(\rho)_t = -(\pi(\rho)u)_x \end{cases} \tag{2.1}$$

We first recall the elementary result due to Kohlmann on the local well-posedness theorem for system (1.1). For sake of convenience of readers, we denote by $H^{s-1}/\mathbb{R}$ the space H^{s-1} with two functions being identified if they differ by a constant, and write $[\rho]$ for the element of $H^{s-1}/\mathbb{R}$ which can be represented by $\pi(\rho)$.

THEOREM 2.1 ([30]). *Let $s>5/2$. There is an open neighborhood U containing $(0,[0])$ in $H^s\times H^{s-1}/\mathbb{R}$ such that for any $(u_0,[\rho_0])\in U$ there is $T>0$ and a unique solution $(u,[\rho])$ to the initial value problem for the system (1.1) with*

$$(u,[\rho])\in C\left([0,T);H^s\times H^{s-1}/\mathbb{R}\right)\cap C^1\left([0,T);H^{s-1}\times H^{s-2}/\mathbb{R}\right),$$

$(u,[\rho])(0)=(u_0,[\rho_0])$ and with continuous dependence on $(u_0,[\rho_0])$, i.e., the mapping

$$(u_0,[\rho_0])\longmapsto(u,[\rho]),\ U\to C\left([0,T);H^s\times H^{s-1}/\mathbb{R}\right)\cap C^1\left([0,T);H^{s-1}\times H^{s-2}/\mathbb{R}\right)$$

is continuous.

The associated Lagrangian scale of (1.1) is established by the initial-value problem

$$\begin{cases} q_t=u(q,t), & 0<t<T,\ x\in\mathbb{R},\\ q(x,0)=x, & x\in\mathbb{R}. \end{cases} \tag{2.2}$$

where u denotes the first component of the solution to (1.1) with certain initial data and T is the lifespan of the solution, then q is a diffeomorphism of the line. This implies that the L^∞-norm of any function $v(\cdot,t)\in L^\infty(\mathbb{R})$, $t\in[0,T)$ is preserved under the family of diffeomorphism $q(\cdot,t)$, i.e.,

$$||v(\cdot,t)||_{L^\infty}=||v(q(\cdot,t),t)||_{L^\infty},\quad t\in[0,T),$$

which will be used in the sequel without mention. Moreover, we know the map $q(\cdot,t)$ is an increasing diffeomorphism of $\mathbb{R}$ with

$$q_x(x,t)=\exp\Big(\int_0^t u_x(q,s)ds\Big)>0,\quad (x,t)\in\mathbb{R}\times[0,T). \tag{2.3}$$

This is usually called the particle trajectory method, and it is important in the discussion of blow-up phenomena. Now, in the framework of the well-posedness result, we are in a position to state.

LEMMA 2.2. *Let U be an open neighborhood containing $(0,[0])$ in $H^s\times H^{s-1}/\mathbb{R}$, $s>5/2$, and $X_0=(u_0,\pi(\rho_0))\in U$, $T>0$ is assumed to be the maximal existence time of the corresponding solution $X=(u,\pi(\rho))$ to (1.1). Then we have*

$$\pi(\rho(q,t))q_x(x,t)=\pi(\rho_0(x)),\quad (x,t)\in\mathbb{S}\times[0,T).$$

The proof follows from the direct application of (2.2). Generally, we say a solution $(u,[\rho])$ for (1.1) which corresponds to certain initial data blows up in finite time, that is the $H^s\times H^{s-1}/\mathbb{R}$ Sobolev norm of $(u,\pi(\rho))$ goes to infinity in finite time. To be precise, it is shown that the behavior of the first order spatial derivative of $u(x,t)$ is sufficient to determine wave breaking of the considered solutions in finite time while ρ is not involved. This is the content of the following result.

THEOREM 2.3. *Let U be an open neighborhood containing $(0,[0])$ in $H^s\times H^{s-1}/\mathbb{R}$, $s>5/2$, and $X_0=(u_0,\pi(\rho_0))\in U$. Let T be the maximal existence time of the solution*

$X=(u,\pi(\rho))$ to (1.1) with the initial data X_0. Then the solution X blows up in finite time if and only if

$$\lim_{t\to T}\inf_{x\in\mathbb{S}}\{u_x(x,t)\}=-\infty. \tag{2.4}$$

Proof. It is sufficient to consider the case of $s=3$ for the solution $(u,\pi(\rho))$ due to the density argument. Multiplying the first equation in (1.1) by y and integrating by parts, we get

$$\int_{\mathbb{S}} yy_t dx+\int_{\mathbb{S}} yy_x u dx+2\int_{\mathbb{S}} y^2u_x dx+\int_{\mathbb{S}} y\pi(\rho)\pi(\rho)_x dx=0.$$

It follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}} y^2dx&=-\frac{3}{2}\int_{\mathbb{S}} y^2u_xdx+\frac{1}{2}\int_{\mathbb{S}}\pi(\rho)^2y_xdx\\&=-\frac{3}{2}\int_{\mathbb{S}} y^2u_xdx+\frac{1}{2}\int_{\mathbb{S}}\pi(\rho)^2u_xdx-\frac{1}{2}\int_{\mathbb{S}}\pi(\rho)^2u_{xxx}dx.\end{aligned} \tag{2.5}$$

Differentiating the first equation in (1.1) with respect to x, multiplying by y_x, and integrating by parts yield

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}} y_x^2dx&=-\frac{5}{2}\int_{\mathbb{S}} u_xy_x^2dx+\int_{\mathbb{S}} y^2u_xdx\\&\quad+\int_{\mathbb{S}} u_{xxx}(\pi(\rho)_x^2+\pi(\rho)\pi(\rho)_{xx}-\frac{1}{2}\pi(\rho)^2)dx.\end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6) together, we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{S}}\left(y^2+y_x^2\right)dx&=\int_{\mathbb{S}}\left(\pi(\rho)^2-5y_x^2-y^2\right)u_xdx\\&\quad+2\int_{\mathbb{S}} u_{xxx}(\pi(\rho)_x^2+\pi(\rho)\pi(\rho)_{xx}-\pi(\rho)^2)dx.\end{aligned} \tag{2.7}$$

Similar arguments made on the second equation in (1.1) yield

$$\frac{d}{dt}\int_{\mathbb{S}}\pi(\rho)^2dx=-\int_{\mathbb{S}}\pi(\rho)^2u_xdx, \tag{2.8}$$

and

$$\frac{d}{dt}\int_{\mathbb{S}}\pi(\rho)_x^2dx=-3\int_{\mathbb{S}}\pi(\rho)_x^2u_xdx+\int_{\mathbb{S}}\pi(\rho)^2u_{xxx}dx, \tag{2.9}$$

$$\frac{d}{dt}\int_{\mathbb{S}}\pi(\rho)_{xx}^2dx=-5\int_{\mathbb{S}}\pi(\rho)_{xx}^2u_xdx-\int_{\mathbb{S}}(2\pi(\rho)\pi(\rho)_{xx}+6\pi(\rho)_x\pi(\rho)_{xx})u_{xxx}dx. \tag{2.10}$$

It follows by combining (2.7)–(2.10) that

$$\frac{d}{dt}\int_{\mathbb{S}}\left(y^2+y_x^2+\pi(\rho)^2+\pi(\rho)_x^2+\pi(\rho)_{xx}^2\right)dx$$

$$= -\int_{\mathbb{S}} u_x \left(y^2 + 5y_x^2 + 3\pi(\rho)_x^2 + 5\pi(\rho)_{xx}^2\right) dx$$
$$+ \int_{\mathbb{S}} u_{xxx} \left(2\pi(\rho)_x^2 - 6\pi(\rho)_x \pi(\rho)_{xx} - \pi(\rho)^2\right) dx.$$

Assume the solution X blows up in finite time, but (2.4) does not hold. Then there exists $\mathcal{M} > 0$ such that

$$\inf u_x(x,t) \geq -\mathcal{M}, \quad (x,t) \in \mathbb{S} \times [0,T). \tag{2.11}$$

We claim that

$$||\pi(\rho)||_{L^\infty} \leq e^{\mathcal{M}t} ||\pi(\rho_0)||_{L^\infty} \leq c e^{\mathcal{M}t} ||\rho_0||_{L^\infty}, \quad \forall t \in [0,T), \tag{2.12}$$

and

$$||\pi(\rho)_x||_{L^\infty} = ||\rho_x||_{L^\infty} \leq c e^{\left(2\mathcal{M} + \frac{1}{2}\right)t}, \quad \forall t \in [0,T), \tag{2.13}$$

where c is a suitable constant. In fact, (2.12) is a direct consequence of Lemma 2.2 and the properties of $q(x,t)$. In order to show (2.13), we proceed as follows. First, note that

$$\begin{aligned} \frac{d}{dt} u_x(q(x,t),t) &= (u_{xt} + u u_{xx})(q,t) \\ &= -\frac{1}{2} u_x^2(q,t) + \frac{1}{2}\pi(\rho)^2 + g(q,t), \end{aligned}$$

and

$$\frac{d}{dt} \pi(\rho(q,t))_t = -\pi(\rho) u_x(q,t)$$

where $g(q,t) = u^2(q,t) - G * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\pi(\rho)^2\right)(q,t)$. It is not difficult to show that $g(q,t)$ is bounded due to Sobolev embedding theorem and the conservation of E_1 (see below). It follows that

$$\begin{aligned} & u_x(q(x,t),t) \frac{d}{dt} u_x(q(x,t),t) \\ = & -\frac{1}{2} u_x(q(x,t),t) u_x^2(q,t) + \frac{1}{2} u_x(q(x,t),t)\pi(\rho)^2 + u_x(q(x,t),t) g(q,t) \\ \leq & \frac{1}{2}\mathcal{M} u_x^2(q,t) + \frac{1}{2} u_x(q,t)\pi(\rho(q,t))^2 + \frac{1}{2}\left(u_x^2(q,t) + g^2(q,t)\right). \end{aligned}$$

Similarly,

$$\pi(\rho(q,t)) \frac{d}{dt} \pi(\rho(q,t))_t = -\pi(\rho(q,t))^2 u_x(q,t).$$

Then

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\left(u_x^2(q(x,t),t) + \pi(\rho(q,t))^2\right) \\ \leq & \frac{1}{2}\mathcal{M} u_x^2(q,t) + \frac{1}{2}\mathcal{M}\pi(\rho(q,t))^2 + \frac{1}{2}\left(u_x^2(q,t) + g^2(q,t)\right) \\ \leq & \frac{1}{2}(\mathcal{M}+1)\left(u_x^2(q,t) + \pi(\rho(q,t))^2\right) + \frac{1}{2} g^2(q,t). \end{aligned}$$

It follows by Gronwall's inequality

$$\begin{aligned} u_x^2(q(x,t),t)+\pi(\rho(q,t))^2 &\le e^{\int_0^t(\mathcal{M}+1)d\tau}\left(\int_0^t g^2(q,t)e^{-\int_0^\tau(\mathcal{M}+1)ds}d\tau+u_{0x}^2(x)+\pi(\rho_0(x))^2\right)\\ &\le e^{(\mathcal{M}+1)t}\left(\int_0^t g^2(q,t)e^{-(\mathcal{M}+1)\tau}d\tau+u_{0x}^2(x)+\pi(\rho_0(x))^2\right)\\ &\le ce^{(\mathcal{M}+1)t}, \end{aligned}$$

for $t\in[0,T)$ with positive constant c depending on $\mathcal{M}$ and the norm of initial value, this constant may be different from instance to instance, changing even within the same line, we still use this notation without mention in the following. Similar arguments made on $\pi(\rho)_x$ yield

$$\pi(\rho(q))_x^2+u_{xx}^2(q,t)\le ce^{(4\mathcal{M}+1)t}.$$

This gives the bound (2.13).

Now we have

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{S}}\left(y^2+y_x^2+\pi(\rho)^2+\pi(\rho)_x^2+\pi(\rho)_{xx}^2\right)dx\\ \le\, &c\mathcal{M}\int_{\mathbb{S}}\left(y^2+y_x^2+\pi(\rho)_x^2+\pi(\rho)_{xx}^2\right)dx\\ &+c\int_{\mathbb{S}}\left(u_{xxx}^2+\pi(\rho)^2+\pi(\rho)_x^2+\pi(\rho)_{xx}^2\right)dx\\ \le\, &c\int_{\mathbb{S}}\left(y^2+y_x^2+\pi(\rho)^2+\pi(\rho)_x^2+\pi(\rho)_{xx}^2\right)dx. \end{aligned}$$

By Gronwall's inequality, we get

$$\begin{aligned} ||u||_{H^3}^2+||\pi(\rho)||_{H^2}^2 &\le ||y||_{H^1}^2+||\pi(\rho)||_{H^2}^2\\ &\le e^{ct}\left(||y_0||_{H^1}^2+||\pi(\rho_0)||_{H^2}^2\right),\quad \forall t\in[0,T), \end{aligned} \tag{2.14}$$

which is a contradiction of our assumption that $T<\infty$ with the maximum time T of existence.

On the other hand, due to the Sobolev embedding of H^1 into L^∞, we observe that $u_x(x,t)\to-\infty$ will lead to blow-up of solutions. □

After the local well-posedness of strong solutions (see Theorem 2.1) is established, a natural question is whether this local solution can exist globally. If the solution only exists in finite time, what induces the blow-up? On the other hand, to find sufficient conditions to guarantee the finite time singularities or global existence is of great interest, especially for sufficient conditions added on certain initial value. The following results will give positive answers.

3. Blow-up phenomena

In this section we pay more attention to the formation of singularities for strong solutions to our system. The following theorems will show that wave breaking is the one way that singularities arise in smooth solutions. Let us start this section with the following useful lemmas.

LEMMA 3.1. *Let U be an open neighborhood containing $(0,[0])$ in $H^s\times H^{s-1}/\mathbb{R}$, $s>5/2$, and $X_0=(u_0,\pi(\rho_0))\in U$. We assume T is the maximal existence time of the* *corresponding solution $X=(u,\pi(\rho))$ to (1.1) with initial data X_0. Then we have the following conservation laws*

$$E_1=\int_{\mathbb{S}}(u^2+u_x^2+\pi(\rho)^2)dx \quad \textit{and} \quad E_2=\int_{\mathbb{S}}(u^3+uu_x^2+u\pi(\rho)^2)dx.$$

The proof is very similar to the one in [18] by energy method, we are not going to repeat it here. The conservation of E_1 guarantees the uniform bound of $u(x,t)$, then Theorem 2.3 is also interpreted as wave breaking.

LEMMA 3.2 ([44]). *For all $f\in H^1(\mathbb{S})$, the following inequality holds*

$$G*\left(f^2+\frac{f_x^2}{2}\right)\geq C_0f^2(x),$$

with

$$C_0=\frac{1}{2}+\frac{\arctan(\sinh(1/2))}{2\sinh(1/2)+2\arctan(\sinh(1/2))\sinh^2(1/2)}\approx 0.869.$$

Moreover, C_0 is the optimal constant obtained by the function

$$f_0=\frac{1+\arctan(\sinh(x-[x]-1/2))\sinh(x-[x]-1/2)}{1+\arctan(\sinh(1/2))\sinh(1/2)}.$$

LEMMA 3.3 ([43]). *For all $f\in H^1(\mathbb{S})$, the following inequality holds*

$$\max_{x\in[0,1]}f^2(x)\leq C_1||f||^2_{H^1(\mathbb{S})},$$

where

$$C_1=\frac{e^{1/2}+e^{-1/2}}{2(e^{1/2}-e^{-1/2})}\approx 1.082.$$

Moreover, C_1 is the minimum value, so in this sense, C_1 is the optimal constant which is obtained by the associated Green's function

$$G(x)=\frac{\cosh(x-[x]-1/2)}{2\sinh(1/2)}$$

LEMMA 3.4 ([43]). *For any function $f\in H^2(\mathbb{S})$, the following inequality holds*

$$\|f(x)\|^2_{L^\infty(\mathbb{S})}\leq\left(\int_{\mathbb{S}}f(x)dx\right)^2+\|f(x)\|^2_{H^1(\mathbb{S})}.$$

LEMMA 3.5 ([45]). *Assume $f(x)\in H^s(\mathbb{S})$, $s>2$. If $\int_{\mathbb{S}}f(x)dx=0$, then*

$$||f(x)||^2_{L^\infty(\mathbb{S})}\leq\frac{1}{12}\int_{\mathbb{S}}f_x^2(x)dx,\qquad \int_{\mathbb{S}}f^2(x)dx\leq\frac{1}{12}\int_{\mathbb{S}}f_x^2(x)dx,$$
$$\textit{and}\int_{\mathbb{S}}f^2(x)f_x^2(x)dx\leq\frac{1}{12}\left(\int_{\mathbb{S}}f_x^2(x)dx\right)^2.$$

We now state our result.

THEOREM 3.6. *Let U be an open neighborhood containing $(0,[0])$ in $H^s\times H^{s-1}/\mathbb{R}$, $s>5/2$, and $X_0=(u_0,\pi(\rho_0))\in U$. Let T be the maximal existence time of solution $X=(u,\pi(\rho))$ to system (1.1) with the initial data X_0. If the following inequality holds*

$$\int_{\mathbb{S}} u_x^3(x,0)dx < -M_0,$$

where the constant $M_0=\sqrt{2E_1(0)\varkappa}$, and $\varkappa$ is a constant which is determined later, then the corresponding solution X blows up in finite time.

Proof. Differentiating the first equation in system (1.1) with respect to x, we obtain

$$u_{xt}+u_x^2+uu_{xx}+\partial_x^2\left(G*(u^2+\frac{1}{2}u_x^2+\frac{1}{2}\pi(\rho)^2)\right)=0. \tag{3.1}$$

Applying the relation $\partial_x^2 G*f=G*f-f$ to (3.1) gives

$$u_{xt}+\frac{1}{2}u_x^2+uu_{xx}+G*\left(u^2+\frac{1}{2}u_x^2+\frac{1}{2}\pi(\rho)^2\right)-u^2-\frac{1}{2}\pi(\rho)^2=0. \tag{3.2}$$

Multiplying (3.2) by u_x^2 and integrating by parts subsequently, we obtain

$$\begin{aligned}\frac{1}{3}\frac{d}{dt}\int_{\mathbb{S}}u_x^3dx&=-\int_{\mathbb{S}}u_x^2\left(G*(u^2+\frac{1}{2}u_x^2+\frac{1}{2}\pi(\rho)^2)\right)dx\\&\quad-\frac{1}{6}\int_{\mathbb{S}}u_x^4dx+\int_{\mathbb{S}}u_x^2\left(u^2+\frac{1}{2}\pi(\rho)^2\right)dx\\&\leq-\frac{1}{6}\int_{\mathbb{S}}u_x^4dx+\frac{1}{2}\int_{\mathbb{S}}u_x^2u^2dx+\frac{1}{2}\int_{\mathbb{S}}u_x^2\pi(\rho)^2dx,\end{aligned} \tag{3.3}$$

where we have used the facts

$$\int_{\mathbb{S}}u_x^2\left(G*\pi(\rho)^2\right)dx\geq 0,$$

$$G*\left(u^2+\frac{1}{2}u_x^2\right)\geq\frac{1}{2}u^2(x).$$

In the following, we estimate the three terms on the right-hand side of (3.3) one by one. The Cauchy–Schwartz inequality implies that

$$\left|\int_{\mathbb{S}}u_x^3dx\right|\leq\left(\int_{\mathbb{S}}u_x^4dx\right)^{1/2}\left(\int_{\mathbb{S}}u_x^2dx\right)^{1/2},$$

hence

$$\left|\int_{\mathbb{S}}u_x^4dx\right|\geq\frac{1}{E_1(0)}\left(\int_{\mathbb{S}}u_x^3dx\right)^2. \tag{3.4}$$

Using Lemma 3.3 and the invariant property of E_1, we have

$$\int_{\mathbb{S}}u_x^2u^2dx\leq\|u\|_{L^\infty}^2\left|\int_{\mathbb{S}}u_x^2dx\right|\leq C_1E_1^2(0). \tag{3.5}$$

Suppose the solution does not blow-up in finite time, it follows that there exists a constant $\mathcal{M}^*>0$ such that $u_x(x,t)>-\mathcal{M}^*$, and $\pi(\rho)$ is bounded thanks to (2.12) by some constant $\mathcal{M}_1$. Thus

$$\left|\int_{\mathbb{S}} u_x^2\pi(\rho)^2dx\right|\le\mathcal{M}_1^2\left|\int_{\mathbb{S}} u_x^2dx\right|\le\mathcal{M}_1^2E_1(0). \tag{3.6}$$

By (3.4)–(3.6), we get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{S}} u_x^3dx &\le -\frac{1}{2}\int_{\mathbb{S}} u_x^4dx+\frac{3}{2}\int_{\mathbb{S}} u_x^2u^2dx+\frac{3}{2}\int_{\mathbb{S}} u_x^2\pi(\rho)^2dx\\ &\le -\frac{1}{2E_1(0)}\left(\int_{\mathbb{S}} u_x^3dx\right)^2+\frac{3C_1}{2}E_1^2(0)+\frac{3}{2}\mathcal{M}_1^2E_1(0).\end{aligned}$$

For convenience of notations, we denote by $\varkappa$ the quantity

$$\frac{3C_1}{2}E_1^2(0)+\frac{3}{2}\mathcal{M}_1^2E_1(0).$$

That is

$$\frac{d}{dt}\int_{\mathbb{S}} u_x^3dx\le-\frac{1}{2E_1(0)}\left(\int_{\mathbb{S}} u_x^3dx\right)^2+\varkappa. \tag{3.7}$$

Note that if the initial quantity satisfies

$$\int_{\mathbb{S}} u_x^3(x,0)dx<-\sqrt{2E_1(0)\varkappa},$$

then by (3.7)

$$\int_{\mathbb{S}} u_x^3(x,t)dx<-\sqrt{2E_1(0)\varkappa}.$$

The standard argument on the Riccati type inequality and the initial hypothesis ensure that there exists a finite time T, such that

$$\lim_{t\to T}\int_{\mathbb{S}} u_x^3(x,t)dx=-\infty.$$

Since

$$\int_{\mathbb{S}} u_x^3dx\ge\inf u_x(x,t)\int_{\mathbb{S}} u_x^2dx>\inf u_x(x,t)E_1(0).$$

This implies that

$$\lim_{t\to T}\inf u_x(x,t)=-\infty.$$

Then it contradicts the assumption $u_x(x,t)>-\mathcal{M}^*$. By Theorem 2.3, we know that the solution must blow up in finite time. ∎

Remark 3.1. Compared with Theorem 3 in [18], we in the present case are able to remove the restriction on the boundedness of the component $\rho(x,t)$ in the condition due to the inner structure of (1.1).

THEOREM 3.7. *Let U be an open neighborhood containing $(0,[0])$ in $H^s\times H^{s-1}/\mathbb{R}$, $s>5/2$, and $X_0=(u_0,\pi(\rho_0))\in U$. Assume that there exists some suitable positive constant κ_1 such that the initial energy $E_1(0)>\kappa_1$ and u_0 is not zero equivalently, satisfying*

$$\int_{\mathbb{S}} u_0(x)dx=0.$$

Then the corresponding solution to initial data X_0 of (1.1) *blows up in finite time.*

Proof. Differentiating both sides of the first equation of (2.1) with respect to variable x, we obtain

$$u_{xt}+u_x^2+uu_{xx}+\partial_x^2\left(G*(u^2+\frac{1}{2}u_x^2+\frac{1}{2}\pi(\rho)^2)\right)=0. \tag{3.8}$$

Applying the relation $\partial_x^2(G*f)=G*f-f$ to (3.8), it follows that

$$u_{xt}=-\frac{1}{2}u_x^2-uu_{xx}-G*\left(u^2+\frac{1}{2}u_x^2+\frac{1}{2}\pi(\rho)^2\right)+u^2+\frac{1}{2}\pi(\rho)^2. \tag{3.9}$$

Multiplying by u_x^2 on both sides of (3.9) and integrating by parts with respect to x, one obtains

$$\begin{aligned}\frac{1}{3}\frac{d}{dt}\int_{\mathbb{S}}u_x^3dx=&-\frac{1}{6}\int_{\mathbb{S}}u_x^4dx+\int_{\mathbb{S}}u^2u_x^2dx+\frac{1}{2}\int_{\mathbb{S}}u_x^2\pi(\rho)^2dx\\&-\int_{\mathbb{S}}u_x^2\left(G*(u^2+\frac{1}{2}u_x^2+\frac{1}{2}\pi(\rho)^2)\right)dx,\end{aligned}$$

where we have used the following identity

$$\int_{\mathbb{S}}u_x^2uu_{xx}dx=-\frac{1}{3}\int_{\mathbb{S}}u_x^4dx.$$

Now it is easy to show that $\int_{\mathbb{S}}u(x,t)dx=0$ in view of the hypothesis, and the following inequality holds

$$\frac{1}{2\sinh(1/2)}\le G(x)\le\frac{\cosh(1/2)}{2\sinh(1/2)}, \tag{3.10}$$

where

$$\frac{\cosh(1/2)}{2\sinh(1/2)}\approx 1.082>1.$$

Using Lemma 3.5 and (3.10), we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{S}}u_x^3dx=&-\frac{1}{2}\int_{\mathbb{S}}u_x^4dx+3\int_{\mathbb{S}}u^2u_x^2dx+\frac{3}{2}\int_{\mathbb{S}}u_x^2\pi(\rho)^2dx\\&-3\int_{\mathbb{S}}u_x^2\left(G*(u^2+\frac{1}{2}u_x^2+\frac{1}{2}\pi(\rho)^2)\right)dx\\\le&-\frac{1}{2}\int_{\mathbb{S}}u_x^4dx+\frac{1}{4}\left(\int_{\mathbb{S}}u_x^2dx\right)^2-\frac{3}{2}\int_{\mathbb{S}}u_x^2\left(G*u_x^2\right)dx+\frac{3}{2}\int_{\mathbb{S}}u_x^2\pi(\rho)^2dx\\\le&-\frac{1}{2}\int_{\mathbb{S}}u_x^4dx+\left(\frac{1}{4}-\frac{3}{4\sinh(1/2)}\right)\left(\int_{\mathbb{S}}u_x^2dx\right)^2+\frac{3}{2}\int_{\mathbb{S}}u_x^2\pi(\rho)^2dx.\end{aligned}$$

From similar arguments to (3.6) in Theorem 3.6, we obtain that

$$\frac{d}{dt}\int_{\mathbb{S}} u_x^3 dx \leq -\frac{1}{2}\int_{\mathbb{S}} u_x^4 dx + \left(\frac{1}{4} - \frac{3}{4\sinh(1/2)}\right)\left(\int_{\mathbb{S}} u_x^2 dx\right)^2 + \frac{3}{2}\mathcal{M}_1^2 E_1(0).$$

Since $E_1(0) > \kappa_1$, there is some $\delta > 0$ such that

$$\frac{3}{2}\mathcal{M}_1^2 E_1(0) \leq \delta\left(\int_{\mathbb{S}} u_x^2 dx\right)^2.$$

and

$$\frac{3}{4\sinh(1/2)} - \frac{1}{4} - \delta > 0.$$

However, Lemma 3.5 also implies that

$$\int_{\mathbb{S}} u_x^2(x) dx \geq \frac{12}{13}||u(x)||^2_{H^1(\mathbb{S})}.$$

We note that $||u(x)||^2_{H^1(\mathbb{S})}$ is bounded. Hence

$$\frac{d}{dt}\int_{\mathbb{S}} u_x^3 dx \leq -\frac{1}{2}\int_{\mathbb{S}} u_x^4 dx - \frac{144}{169}\left(\frac{3}{4\sinh(1/2)} - \frac{1}{4} - \delta\right)||u||^4_{H^1(\mathbb{S})},$$

On the other hand, in view of Hölder's inequality, there holds

$$\int_{\mathbb{S}} u_x^4 dx \geq \left(\int_{\mathbb{S}} u_x^3 dx\right)^{\frac{4}{3}}.$$

For simplicity of notations, we denote by $\varphi(t)$ and $\mu > 0$ the following quantities

$$\int_{\mathbb{S}} u_x^3 dx \quad \text{and} \quad \frac{144}{169}\left(\frac{3}{4\sinh(1/2)} - \frac{1}{4} - \delta\right),$$

respectively. Therefore we have

$$\frac{d\varphi(t)}{dt} \leq -\frac{1}{2}\varphi^{\frac{4}{3}}(t) - \mu||u||^4_{H^1(\mathbb{S})}.$$

First, since $\varphi(t) \leq \varphi(0) - \mu||u||^4_{H^1(\mathbb{S})} t$, it is not difficult to find that there exists a time t_0 such that $\varphi(t_0) < 0$. Then for all $t > t_0$, we have

$$\frac{d\varphi(t)}{dt} \leq -\frac{1}{2}\varphi^{\frac{4}{3}}(t), \quad \text{with } \varphi(t_0) < 0.$$

Solving this inequality yields

$$\varphi(t) \leq \left(\varphi^{-\frac{1}{3}}(t_0) + \frac{1}{6}(t - t_0)\right)^{-3},$$

which goes to $-\infty$ as t tends to $-6\varphi^{-\frac{1}{3}}(t_0) + t_0$, i.e., there exists a time $T \leq -6\varphi^{-\frac{1}{3}}(t_0) + t_0$ such that

$$\lim_{t \to T}\int_{\mathbb{S}} u_x^3 dx = -\infty.$$

Since

$$\int_{\mathbb{S}} u_x^3 dx \geq \inf u_x(x,t) \int_{\mathbb{S}} u_x^2 dx \geq \inf u_x(x,t) ||u||^2_{H^1(\mathbb{S})},$$

which shows that

$$\lim_{t\to T} \inf_{x\in\mathbb{S}} u_x(x,t) = -\infty.$$

Then it contradicts the assumption $u_x(x,t) > -\mathcal{M}^*$. By Theorem 2.3, we know that the solution must blow up in finite time. This finishes the proof. ∎

As we know, different estimates will lead to different results for some partial differential equation problems. So it is important to give different estimates for some quantity. To be precise, we show

THEOREM 3.8. *Let U be an open neighborhood containing* $(0,[0])$ *in* $H^s \times H^{s-1}/\mathbb{R}$, $s > 5/2$, *and* $X_0 = (u_0, \pi(\rho_0)) \in U$. *Suppose the initial energy* $E_1(0) > \kappa_2$ *for some suitable positive constant* κ_2, *and* u_0 *is not zero equivalently, satisfying*

$$\int_{\mathbb{S}} \left(u_0^3 + u_0 u_{0x}^2 + u_0 \pi(\rho_0)^2\right) dx = 0.$$

Then existence time of the corresponding solution to (1.1) *is finite.*

Proof. Since E_2 is an invariant with respect to time, it is trivial to get

$$\int_{\mathbb{S}} u\left(u^2 + u_x^2 + \pi(\rho)^2\right) dx = \int_{\mathbb{S}} \left(u^3 + u u_x^2 + u\pi(\rho)^2\right) dx = 0.$$

Consequently, $u(x,t)$ must change its sign on $\mathbb{S}$, so there must exist at least one zero point. Then for $t \in [0,T)$, suppose that there is a $\xi_t \in [0,1]$ such that $u(\xi_t, t) = 0$. Now we have

$$u^2(x,t) = \left(\int_{\xi_t}^x u_x dz\right)^2 \leq (x - \xi_t) \int_{\xi_t}^x u_x^2 dz, \quad x \in [\xi_t, \xi_t + \frac{1}{2}]. \tag{3.11}$$

Thus, the relation above and integration by parts give

$$\begin{aligned}\int_{\xi_t}^{\xi_t+\frac{1}{2}} u^2 u_x^2 dx &\leq \int_{\xi_t}^{\xi_t+\frac{1}{2}} (x-\xi_t)\left(\int_{\xi_t}^x u_x^2 dz\right) u_x^2 dx \\ &= \int_{\xi_t}^{\xi_t+\frac{1}{2}} (x-\xi_t)\left(\int_{\xi_t}^x u_x^2 dz\right) d\left(\int_{\xi_t}^x u_x^2 dz\right) \\ &= \frac{1}{4}\left(\int_{\xi_t}^{\xi_t+\frac{1}{2}} u_x^2 dz\right)^2 - \frac{1}{2}\int_{\xi_t}^{\xi_t+\frac{1}{2}} \left(\int_{\xi_t}^x u_x^2 dz\right)^2 dx \\ &\leq \frac{1}{4}\left(\int_{\xi_t}^{\xi_t+\frac{1}{2}} u_x^2 dz\right)^2.\end{aligned}$$

Doing the similar estimate on $[\xi_t + \frac{1}{2}, \xi_t + 1]$, we obtain

$$\int_{\mathbb{S}} u^2 u_x^2 dx \leq \frac{1}{4}\left(\int_{\mathbb{S}} u_x^2 dx\right)^2, \tag{3.12}$$

and in view of (3.11) that

$$\|u(x,t)\|_{L^\infty(\mathbb{S})}^2 \le \frac{1}{2}\int_{\mathbb{S}} u_x^2 dx. \tag{3.13}$$

We observe that (3.12) and (3.13) actually provide two basic estimates instead of the ones in Lemma 3.5, once the new estimates are prepared, it is possible to establish different wave breaking criterion. There holds by similar steps as above that

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{S}} u_x^3 dx &= -\frac{1}{2}\int_{\mathbb{S}} u_x^4 dx + 3\int_{\mathbb{S}} u^2u_x^2 dx + \frac{3}{2}\int_{\mathbb{S}} u_x^2 \pi(\rho)^2 dx \\ &\quad -3\int_{\mathbb{S}} u_x^2 \left(G*(u^2+\frac{1}{2}u_x^2+\frac{1}{2}\pi(\rho)^2)\right) dx \\ &\le -\frac{1}{2}\int_{\mathbb{S}} u_x^4 dx + \left(\frac{3}{4}-\frac{3}{4\sinh(1/2)}\right)\left(\int_{\mathbb{S}} u_x^2 dx\right)^2 + \frac{3}{2}\int_{\mathbb{S}} u_x^2\pi(\rho)^2 dx.\end{aligned}$$

Similarly, there exists a constant $\delta^\star > 0$ such that

$$\frac{3}{2}\mathcal{M}_1^2 \int_{\mathbb{S}} u_x^2 dx \le \delta^\star \left(\int_{\mathbb{S}} u_x^2 dx\right)^2,$$

and

$$\frac{3}{4\sinh(1/2)} - \frac{3}{4} - \delta^\star > 0,$$

where we used the same notation $\mathcal{M}_1$ as the bound of $\pi(\rho)$. Notice that (3.13) implies

$$\int_{\mathbb{S}} u_x^2 dx \ge \frac{2}{3}\|u\|_{H^1(\mathbb{S})}^2.$$

We still use the notation $\varphi(t)$ as in the above theorem to derive

$$\begin{aligned}\frac{d\varphi(t)}{dt} &\le -\frac{1}{2}\int_{\mathbb{S}} u_x^4 dx + \frac{1}{9}\left(3 - \frac{3}{\sinh(1/2)} + 4\delta^\star\right)\|u\|_{H^1(\mathbb{S})}^4 \\ &\le -\frac{1}{2}\varphi^{\frac{4}{3}}(t) - \frac{1}{9}\left(\frac{3}{\sinh(1/2)} - 3 - 4\delta^\star\right)\|u\|_{H^1(\mathbb{S})}^4,\end{aligned}$$

where

$$\frac{3}{\sinh(1/2)} - 3 - 4\delta^\star > 0, \text{ for some } \delta^\star > 0.$$

The remaining part is very close to the proof of Theorem 3.7, we omit it. □

REMARK 3.2. Scrutinizing the whole proof of Theorem 3.8, we find that the condition guarantees that $u(x,t)$ has at least one zero point. So in this sense Theorem 3.8 is still true once the condition is replaced by $\int_{\mathbb{S}} u_0(x)dx = 0$ or $\int_{\mathbb{S}} y_0(x)dx = 0$. We would also like to point out that the condition in Theorem 3.7 can still lead to the inequalities (3.12) and (3.13) where the crucial point is the existence of one zero point. However, we cannot get any information from the condition of Theorem 3.7 on the quantity $E_2(0)$, this implies that Theorem 3.8 and Theorem 3.7 are essentially different from each other. On the other hand, Constantin and Ivanov [8] claimed that global solutions may exist provided that the initial E_1 is small, while Theorem 3.7 and Theorem 3.8 actually give an answer what would happen if initial E_1 is larger.

Wave breaking may also occur while condition is added on the initial slope of u at some point, we state some criteria as following.

THEOREM 3.9. *Let U be an open neighborhood containing $(0,[0])$ in $H^s\times H^{s-1}/\mathbb{R}$, $s>5/2$, and $X_0=(u_0,\pi(\rho_0))\in U$. $X=(u,\pi(\rho))$ is the solution to system (1.1) with initial data X_0. If there is some point $x_0\in\mathbb{S}$ such that*

$$u_0'(x_0)<-\sqrt{2(1-C_0)C_1E_1(0)},$$

where the constants C_0 and C_1 are given in Lemma 3.2 and Lemma 3.3. Then the solution X must blow up in finite time.

Proof. Differentiating the first equation in system (2.1) with respect to x, and noticing that $\partial_x^2 G*f=G*f-f$, we have

$$u_{xt}+u_x^2+uu_{xx}=u^2+\frac{1}{2}u_x^2+\frac{1}{2}\pi(\rho)^2-G*\left(u^2+\frac{1}{2}u_x^2+\frac{1}{2}\pi(\rho)^2\right).$$

This equation, in combination with (2.2), yields

$$\begin{aligned}\frac{d}{dt}u_x(q(x,t),t)&=(u_{xt}+uu_{xx})(q,t)\\&\le\left(-\frac{1}{2}u_x^2+(1-C_0)u^2+\frac{1}{2}\pi(\rho)^2\right)(q,t),\end{aligned}$$

where we have used Lemma 3.2. Note that

$$\int_{\mathbb{S}}\pi(\rho)dx=0.$$

We deduce that there exists at least one point x_0 such that $\pi(\rho(q(x_0,t),t))=0$ for $t\in[0,T)$. Let us consider this problem at $(q(x_0,t),t)$, and for sake of convenience, we denote $u_x(q(x_0,t),t)=m(t)$. Then we have

$$\begin{aligned}\frac{dm(t)}{dt}&\le-\frac{1}{2}m^2(t)+(1-C_0)u^2\\&\le-\frac{1}{2}m^2(t)+(1-C_0)C_1E_1(0).\end{aligned}\tag{3.14}$$

Using the notation $\varrho=2(1-C_0)C_1E_1(0)$, we have

$$\frac{dm(t)}{dt}\le-\frac{1}{2}\left(m^2(t)-\varrho\right).\tag{3.15}$$

In view of the initial condition, it is not difficult to obtain

$$\frac{dm(t)}{dt}\le\frac{\delta-1}{2}m^2(t),$$

with $0<\delta<1$ determined by $\delta m^2(0)=\varrho$. Then, by using the standard arguments for this type of inequality and our hypothesis, it is easy to conclude that the lifespan of the solution is finite, i.e., blow-up phenomenon occurs. ∎

THEOREM 3.10. *Let U be an open neighborhood containing $(0,[0])$ in $H^s\times H^{s-1}/\mathbb{R}$, $s>5/2$, and $X_0=(u_0,\pi(\rho_0))\in U$. If there holds that for some x_0*

$$u_0'(x_0)<-\sqrt{2(1-C_0)K(0)},$$

where C_0 is the best constant given by Lemma 3.2 and

$$K(0)=\left(\int_{\mathbb{S}}u_0(x)dx\right)^2+\int_{\mathbb{S}}\left(u_0^2+u_{0x}^2+\pi(\rho_0)^2\right)dx.$$

Let T be the maximal existence time of the corresponding solution to (1.1) with the initial data X_0. Then T is finite.

Proof. This result differs the estimate on $u(x,t)$ from Theorem 3.9. We easily know that $\int_{\mathbb{S}}u(x,t)dx$ is also an invariant with respect to time. Thus Theorem 3.10 can be proved by using Lemma 3.4 with

$$\|u(x)\|^2_{L^\infty(\mathbb{S})}\le\left(\int_{\mathbb{S}}u(x)dx\right)^2+\|u(x)\|^2_{H^1(\mathbb{S})}<K(0)$$

in (3.14) instead of $\|u(x,t)\|^2_{L^\infty(\mathbb{S})}\le C_1\|u(x,t)\|^2_{H^1(\mathbb{S})}\le C_1E_1(0)$. □

When we study the blow-up problems for differential equations, according to [23], the basic questions includes when, where, and how. Theorems 3.6–3.10 give answers to the first two questions. The following theorem answers one aspect of the third question, the rate of blow-up. In the remaining part the deep phenomenon is examined while the solution blows up in finite time.

THEOREM 3.11. *Let U be an open neighborhood containing $(0,[0])$ in $H^s\times H^{s-1}/\mathbb{R}$, $s>5/2$, and $X_0=(u_0,\pi(\rho_0))\in U$. $X(x,t)$ is the corresponding solution. If there holds the condition of Theorem 3.9, then we have the following description*

$$\lim_{t\to T}\{(T-t)m(t)\}=-2,$$

where $m(t)$ is defined in Theorem 3.9.

Proof. The conclusion follows from the theory of ordinary differential equations to inequality (3.15). Indeed, we have by (3.14)

$$\frac{dm(t)}{dt}\le-\frac{1}{2}m^2(t)+(1-C_0)u^2.$$

In view of Lemma 3.3 and the conservation of E_1, there holds for all $t\in[0,T)$ that

$$\left|\frac{dm(t)}{dt}+\frac{1}{2}m^2(t)\right|\le\frac{\varrho}{2}.$$

It follows that

$$-\frac{\varrho}{2}\le\frac{dm(t)}{dt}+\frac{1}{2}m^2(t)\le\frac{\varrho}{2},i\quad a.e.\text{ on }(0,T).$$

Since $\lim_{t\to T}m(t)=-\infty$ by Theorem 3.9, it implies that for any $\varepsilon\in(0,1/2)$ there exists a t_0 such that $m^2(t)>\varrho/2\varepsilon$ for all $t\in[t_0,T)$. Therefore,

$$-m^2(t)\varepsilon\le\frac{dm(t)}{dt}+\frac{1}{2}m^2(t)\le m^2(t)\varepsilon,$$

it follows that

$$-\frac{1}{2}-\varepsilon\leq\frac{1}{m^2(t)}\frac{dm(t)}{dt}\leq-\frac{1}{2}+\varepsilon.$$

Direct integration from t to T gives

$$-\frac{1}{2}-\varepsilon\leq\frac{1}{(T-t)m(t)}\leq-\frac{1}{2}+\varepsilon,$$

the arbitrariness of ε leads to our result. □

REMARK 3.3. The first investigation concerning the blow-up rate for the Camassa–Holm equation can be found in [5], the authors have proved that the blow-up rate for Camassa–Holm equation is -2. Actually, we find that the blow-up rate largely relies on the coefficient of higher order term $m^2(t)$ in the present problem. Moreover, we can summarize that for a class of nonlinear nonlocal evolution equations the blow-up rate is a constant which is determined by the coefficients of the leading term $u_x^2(x,t)$ in some sense when blow-up occurs. We also noticed that in [5] the blow-up set was studied, it gave an answer to the second basic question, where blow-up occurs. However, in our case the point $q(x_0,t)$ in Theorem 3.9 can be regarded as the blow-up set.

4. Conclusion and remarks

We presented in this work some new blow-up criteria where some special initial data play very important role for a new π-CH2 system. Particularly, wave breaking phenomenon was investigated via the associated conservation laws. The blow-up rate has shown more insight into the wave breaking phenomenon. We remark here that Theorem 3.9 and Theorem 3.10 are interesting and different themselves. Examples can be given to show their applicability, the readers who are interested in it, please see [19] for the details. However, there are still some interesting problems to be solved. We are trying to establish some sufficient conditions added on the second component $\rho(x,t)$ to show wave breaking. Furthermore, the condition which guarantees the global existence of solutions is worthy of being investigated. We are going to introduce a new method to discuss it. As a matter of fact, we are also concerned about its application of our method to other related models. We therefore hope these problems can be shown in the forthcoming work.

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