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ZERO RELAXATION LIMIT TO RAREFACTION WAVES FOR GENERAL 2×2 HYPERBOLIC SYSTEMS WITH RELAXATION*

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Abstract. For the general 2×2 hyperbolic conservation laws with relaxation, the convergence to the rarefaction wave of the equilibrium equation as the relaxation parameter tends to zero is proved, and the convergence rate is given.

Key words. hyperbolic systems with relaxation, zero relaxation limit, rarefaction waves.

AMS subject classifications. 35L60, 35B25.

1. Introduction

The purpose of this paper is to study the zero relaxation limit towards the rarefaction wave of the equilibrium equation for the following general 2×2 genuinely nonlinear strictly hyperbolic conservation laws with relaxation

$$u_t + f(u,v)_x = 0, \quad v_t + g(u,v)_x = \epsilon^{-1}\Phi(u,v), \tag{1.1}$$

where u and v are real functions of the time variable $t \geq 0$ and the spatial variable $x \in \mathbb{R}$, f, g, and Φ are real functions of u and v, $\epsilon > 0$ is the small relaxation parameter. The relaxation term is assumed to satisfy

$$\Phi_v(u,v) < 0, \quad \Phi(u, v_*(u)) = 0, \tag{1.2}$$

for all (u,v) under consideration. Here and thereafter $A_u(u,v)$ and $A_v(u,v)$ are denoted as the partial derivatives of the function $A(u,v)$ with respect to the first and second independent variables. Let $\lambda_1(u,v)$ and $\lambda_2(u,v)$ be the distinct characteristic wave speeds of the corresponding homogeneous system of (1.1):

$$u_t + f(u,v)_x = 0, \quad v_t + g(u,v)_x = 0. \tag{1.3}$$

We will seek global smooth solutions to (1.1) with the initial value of the form

$$(u(x,0), v(x,0)) = (u_0(x), v_0(x)) \to (u_\pm, v_\pm) \quad \text{as} \quad x \to \pm\infty, \tag{1.4}$$

where $u_\pm$ and $v_\pm$ are given constants.

Formally, as ϵ tends to zero, the system is in equilibrium and the equilibrium equation corresponding to (1.1) is given by

$$u_t + f_*(u)_x = 0, \quad v = v_*(u), \quad \text{where} \quad f_*(u) = f(u, v_*(u)). \tag{1.5}$$

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Assume that $f_*(u)$ is strictly convex and $u_- < u_+$. The Riemann problem of the equilibrium equation for u with the initial value

$$u(x,0) = u_-, \quad x < 0, \quad u(x,0) = u_+, \quad x > 0, \tag{1.6}$$

admits a centered rarefaction wave solution $u^r(x/t)$. Denote $\lambda_*(u) = f'_*(u)$. Without loss of generality, we assume

$$\lambda_*(u_-) < 0 < \lambda_*(u_+). \tag{1.7}$$

Indeed, this can be achieved by the following transformation:

$$t \to t \quad \text{and} \quad x \to x - \frac{\lambda_*(u_-) + \lambda_*(u_+)}{2} t.$$

In the present work, we will justify this relaxation limit from the solution of (1.1) and (1.4) to the equilibrium solution $(u^r, v_*(u^r))$, under the following sub-characteristic condition (cf. [26])

$$\lambda_1(u,v) < \lambda_*(u) < \lambda_2(u,v), \tag{1.8}$$

for all (u,v) under consideration.

Relaxation phenomena arise and are important in many physical situations such as kinetic theory of gases, gas flow with thermal nonequilibrium, water waves, elasticity with memory and traffic flows (cf. [6, 7, 33, 37, 43]). The zero relaxation limit problem is always a challenging physics motivated problem, in particular in the presence of initial layers, shock layers and boundary layers. Unlike vanishing viscosity problems of viscous conservation laws (in particular the compressible Navier–Stokes equations) for which important progress has been made and a satisfactory theory has been established (cf. [3–5, 10, 12–14, 16–18, 20, 21, 36, 45, 48]), not much results are available for zero relaxation problems. Compared with the viscosity, the relaxation is more stiffed and less dissipative. The study of zero relaxation limit problems have been restricted to various specific physical models (cf. [1, 8, 9, 15, 23, 30, 31, 39, 42, 44]) or the semi-linear Jin–Xin model proposed in [22] for the numerical computation of hyperbolic conservation laws (cf. [2, 11, 34, 38, 41, 47, 50]). For the general nonlinear hyperbolic systems, much less results of zero relaxation limits are available.

For the general 2×2 hyperbolic conservation laws with relaxation (1.1) which was first studied in [26], the zero relaxation limit was studied in [24] by assuming that there is a uniform L^∞-bounds for the solutions of the relaxation systems by the method of compensated compactness, and in [49] by establishing the uniform BV bounds for a class of initial data. To the best of our knowledge, there has been no results on the zero relaxation limit for the general 2×2 relaxation system (1.1) for general initial data without assuming a priori uniform bounds on solutions. For the equilibrium equation (1.5), the basic nonlinear waves are shock waves and rarefaction waves. The purpose of this paper is to give a rigorous justification of the zero relaxation limit for the general 2×2 nonlinear hyperbolic conservation laws with relaxation towards the rarefaction waves of (1.5). The basic ideas and techniques are energy estimates and rescaling arguments as in [45]. However, due to its full generality of the nonlinearity of the left-hand side of (1.1), it is quite technically involved to prove such a zero relaxation limit.

Before stating our main result, we review some previous results closely related to this work, besides the above mentioned results. The long time convergence to rarefaction waves for some relaxation systems are studied in [28, 29, 35, 51, 52] for fixed relaxation parameters. For the nonlinear Boltzmann equation, the long time convergence and zero dissipation limit to rarefaction waves are investigated in [27] and [25, 46], respectively. The convergence of solutions for the Boltzmann equation to the Riemann solutions for compressible Euler equations is given in [19] as the mean free path tends to zero. In the present work, we prove that when the solution to equilibrium equation (1.5) is a weak centered rarefaction wave and the relaxation parameter is sufficiently small, the Cauchy problem of system (1.1) with the well-prepared initial data admits a global-in-time smooth solution that converges towards the rarefaction wave with detailed convergence rates uniformly away from $t=0$ as the relaxation parameter tends to zero. It should be remarked that in the study of zero dissipation limit problems, most of previous results are also based on the analysis of the well-prepared initial data (which may depend on the dissipation parameter) by ignoring initial layers. The main novelty of this paper is that our results are for the general 2×2 strictly hyperbolic system with a general relaxation system under natural assumptions, which includes some frequently discussed models in the literature, such as p-systems and Jin–Xin models (see the references mentioned above).

The main result of this paper is as follows.

THEOREM 1.1. *Suppose that* (1.2) *and* (1.8) *are satisfied,* $f_v(u,v)\neq 0$ *and* $f_*(u)$ *is strictly convex. Assume that* (1.7) *holds and* $v_\pm = v_*(u_\pm)$. *Let* $u^r(x/t)$ *be the centered rarefaction wave solution to* (1.5) *and* (1.6). *Then there exist small positive constants* η_0 *and* ϵ_0 *such that for any fixed* $\epsilon\in(0,\epsilon_0]$, *we can construct a global smooth solution* $(u,v)(x,t)$ *with initial value* (3.4) *to the relaxation system* (1.1) *satisfying*

$$\begin{aligned} &(u-u^r, v-v_*(u^r))\in C^0((0,+\infty);L^2(\mathbb{R})),\\ &(u_x,u_t,v_x,v_t)\in C^0([0,+\infty);H^1(\mathbb{R})) \quad \text{and} \quad (u_{tt},v_{tt})\in C^0([0,+\infty);L^2(\mathbb{R})), \end{aligned} \tag{1.9}$$

provided that

$$0 < u_+ - u_- \le \eta_0. \tag{1.10}$$

The solution also satisfies that $(u,v)(x,t)$ *converges to* $(u^r, v_*(u^r))(x/t)$ *pointwise except at* $(0,0)$, *as* $\epsilon\to 0$. *Furthermore, for any given positive constant* h, *there is a constant* $C_h>0$, *independent of* ϵ, *so that*

$$\sup_{t\ge h}\|(u-u^r,\ v-v_*(u^r))(\cdot,t)\|_{L^\infty(\mathbb{R})}\le C_h\epsilon^{1/5}|\ln\epsilon|. \tag{1.11}$$

REMARK 1.2. The initial value (3.4) is a small perturbation of $(u^r, v_*(u^r))$ and in equilibrium (i.e., $\Phi(u(x,0),v(x,0))=0$). Indeed, we can deal with the non-equilibrium initial value by introducing a correction term due to the decay properties of the relaxation term (i.e., $\Phi_v<0$).

REMARK 1.3. In the study of the zero dissipation limit to rarefaction waves, the condition on the strength of waves, (1.10), can be removed for specific models, such as Navier–Stokes equations, the Boltzmann equation, etc., which satisfy some good specific structure conditions. The model we consider is a general hyperbolic system, so condition (1.10) is needed.

REMARK 1.4. The convergence rate shown in (1.11) is the same as that obtained in [46] for the study of the Boltzmann equation, which was improved by [25] using a different scaling. In our case, no matter which scaling is used the convergence rate cannot vary due to the generality of the system (1.1) (see Remark 3.5 for details).

The rest of the paper is organized as follows. In Section 2, we construct a smooth rarefaction wave which approximates the centered rarefaction wave based on the inviscid Burgers' equation, reformulate problem (1.1) with respect to the perturbation around the approximate wave and linearize the reformulated problem. Section 3 is devoted to the proof of the main result.

2. Preliminaries

Throughout the rest of paper, we use the following notation.

1) C will denote a positive constant which does not depend on the data. They are referred as universal and can change from one inequality to another one. Also we use C_ς to denote a certain positive constant depending on quantity ς.

2) We will employ the notation $a \lesssim b$ to denote $a \le Cb$, where C is the universal constant as defined above.

3) We will use the notation

$$\int = \int_{\mathbb{R}}, \quad \|\cdot\| = \|\cdot\|_{L^2(\mathbb{R})} \quad \text{and} \quad \|\cdot\|_{L^\infty} = \|\cdot\|_{L^\infty(\mathbb{R})}.$$

2.1. Approximate rarefaction waves.

In this subsection, we construct smooth rarefaction waves which approximate centered rarefaction waves. Consider

$$\begin{cases} w_t + w w_x = 0, \\ w(x,0) = \begin{cases} w_-, & x<0, \\ w_+, & x>0. \end{cases} \end{cases} \tag{2.1}$$

If $w_- < w_+$, then (2.1) has the centered rarefaction wave solution $w^r(x,t) = w^r(\frac{x}{t})$ given by

$$w^r(x,t) = \begin{cases} w_-, & \frac{x}{t} < w_-, \\ \frac{x}{t}, & w_- \le \frac{x}{t} \le w_+, \\ w_+, & \frac{x}{t} > w_+. \end{cases}$$

To construct a smooth rarefaction wave solution of the Burgers' equation which approximates the centered rarefaction wave, we set for each $\delta > 0$,

$$w_\delta(x) = w(x/\delta) \equiv (w_+ + w_-)/2 + \tanh(x/\delta)(w_+ - w_-)/2$$

and solve the following initial value problem

$$w_t + w w_x = 0, \quad w(x,0) = w_\delta(x). \tag{2.2}$$

Next, we state certain properties for the smooth rarefaction wave (see, for instance [45], for the proof).

LEMMA 2.1. *For each $\delta > 0$, (2.2) has a unique global smooth solution $w^r_\delta(x,t)$, such that the following hold:*

(a) $w_- < w^r_\delta(x,t) < w_+$ and $0 < \partial_x w^r_\delta(x,t) \le C(w_+ - w_-)\delta^{-1}$ for $x \in \mathbb{R}, t \ge 0, \delta > 0$.

(b) The following estimates hold for all $t>0$, $\delta>0$, *and* $p\in[1,\infty]$*:*

$$\|\partial_x w^r_\delta(\cdot,t)\|_{L^p}\le C(w_+-w_-)^{1/p}(\delta+t)^{-1+1/p},$$
$$\|\partial_x^2 w^r_\delta(\cdot,t)\|_{L^p}\le C(\delta+t)^{-1}\delta^{-1+1/p},\quad |\partial_x^2 w^r_\delta(x,t)|\le 4\delta^{-1}\partial_x w^r_\delta(x,t),$$
$$\|\partial_x^3 w^r_\delta(\cdot,t)\|_{L^p}\le C(\delta+t)^{-1}\delta^{-2+1/p}.$$

(c) There exist a constant $\delta_0\in(0,1)$*, such that for* $\delta\in(0,\delta_0]$, $t>0$,

$$\|w^r_\delta(\cdot,t)-w^r(\cdot/t)\|_{L^\infty}\le Ct^{-1}\delta(\ln(1+t)+|\ln\delta|).$$

Set $w_\pm=\lambda_*(u_\pm)$, we define the smooth approximation $u^r_\delta(x,t)$ of the centered rarefaction wave $u^r(x/t)$ by

$$\lambda_*(u^r_\delta(x,t))=w^r_\delta(x,t). \tag{2.3}$$

Clearly, it holds that

$$\partial_t\lambda_*(u^r_\delta)+\lambda_*(u^r_\delta)\partial_x\lambda_*(u^r_\delta)=0\quad\text{and}\quad \partial_t u^r_\delta+\partial_x f_*(u^r_\delta)=0, \tag{2.4}$$

because of the strict convexity of $f_*(u)$. Due to Lemma 2.1, the following lemma holds.

LEMMA 2.2. *The function* $u^r_\delta(x,t)$ *constructed by* (2.3) *has the following properties:*
(a) $u_-<u^r_\delta(x,t)<u_+$ *and* $0<\partial_x u^r_\delta(x,t)\le C(u_+-u_-)\delta^{-1}$ *for* $x\in\mathbb{R}$, $t\ge0$, $\delta>0$.
(b) The following estimates hold for all $t>0$, $\delta>0$, *and* $p\in[1,\infty]$*:*

$$\|\partial_x u^r_\delta(\cdot,t)\|_{L^p}\le C(u_+-u_-)^{1/p}(\delta+t)^{-1+1/p},$$
$$\|\partial_x^2 u^r_\delta(\cdot,t)\|_{L^p}\le C(\delta+t)^{-1}\delta^{-1+1/p},\quad |\partial_x^2 u^r_\delta(x,t)|\le C\delta^{-1}\partial_x u^r_\delta(x,t)$$
$$\|\partial_x^3 u^r_\delta(\cdot,t)\|_{L^p}\le C(\delta+t)^{-1}\delta^{-2+1/p}.$$

(c) There exist a constant $\delta_0\in(0,1)$ *such that for* $\delta\in(0,\delta_0]$, $t>0$,

$$\|u^r_\delta(\cdot,t)-u^r(\cdot/t)\|_{L^\infty}\le Ct^{-1}\delta(\ln(1+t)+|\ln\delta|).$$

According to Lemma 2.2, we can derive from (1.7) and (1.10) that

$$|\lambda_*(u^r_\delta)|\le C\eta_0. \tag{2.5}$$

This, together with the sub-characteristic condition (1.8) and smallness of η_0, gives

$$\lambda_1(u^r_\delta,v_*(u^r_\delta))\lambda_2(u^r_\delta,v_*(u^r_\delta))<0. \tag{2.6}$$

2.2. Reformulation of the problem. To prove Theorem 1.1, we construct the solution to (1.1) as the perturbation around the approximate rarefaction wave $(u^r_\delta,v_*(u^r_\delta))$. Consider the Cauchy problem of (1.1) with the following smooth initial data

$$(u,\ v)(x,t=0)=(u^r_\delta,\ v^r_\delta)(x,0),\quad\text{where}\quad v^r_\delta=v_*(u^r_\delta). \tag{2.7}$$

Set the perturbation

$$(z,m)(y,\tau)=(u-u^r_\delta,v-v^r_\delta)(x,t)\quad\text{with}\quad (y,\tau)=\epsilon^{-1}(x,t). \tag{2.8}$$

Then, it follows from (1.1), (2.4), and (2.7) that

$$\begin{aligned} &z_\tau + [f(u_\delta^r + z, v_\delta^r + m) - f(u_\delta^r, v_\delta^r)]_y = 0, \\ &m_\tau + [g(u_\delta^r + z, v_\delta^r + m) - g(u_\delta^r, v_\delta^r)]_y = [\Phi(u_\delta^r + z, v_\delta^r + m) - \Phi(u_\delta^r, v_\delta^r)] + e_1, \end{aligned} \tag{2.9}$$

$$(z,\ m)(y, \tau = 0) = (0,\ 0), \tag{2.10}$$

where

$$e_1 = v_*'(u_\delta^r) f(u_\delta^r, v_\delta^r)_y - g(u_\delta^r, v_\delta^r)_y. \tag{2.11}$$

For any smooth function $\beta(u,v)$, we denote

$$\begin{aligned} &Q_1(\beta) = \beta(u_\delta^r + z, v_\delta^r + m) - \beta(u_\delta^r, v_\delta^r), \quad Q_2(\beta) = Q_1(\beta) - (\beta_u z + \beta_v m), \\ &Q_3(\beta) = Q_2(\beta) - \frac{1}{2}\left(\beta_{uu} z^2 + 2\beta_{uv} zm + \beta_{vv} m^2\right); \end{aligned}$$

which satisfies

$$|Q_1| \lesssim |z| + |m|, \qquad |Q_2| \lesssim |z|^2 + |m|^2, \qquad \text{and} \qquad |Q_3| \lesssim |z|^3 + |m|^3. \tag{2.12}$$

Here and thereafter, all the quantities appearing in the coefficients are evaluated at (u_δ^r, v_δ^r). So, system (2.9) can be rewritten as

$$\begin{aligned} &z_\tau + (f_u z + f_v m)_y = J_1, \\ &m_\tau + (g_u z + g_v m)_y - (\Phi_u z + \Phi_v m) = J_2, \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} J_1 &= -[Q_1(f_u) z_y + Q_1(f_v) m_y] - [Q_2(f_u) + Q_2(f_v) v_*'(u_\delta^r)] u_{\delta y}^r \\ &= -\frac{1}{2}\left[f_{uu}(z^2)_y + 2 f_{uv}(zm)_y + f_{vv}(m^2)_y\right] - [Q_2(f_u) z_y + Q_2(f_v) m_y] \\ &\quad - [Q_2(f_u) + Q_2(f_v) v_*'(u_\delta^r)] u_{\delta y}^r \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} J_2 =& Q_2(\Phi) - [Q_1(g_u) z_y + Q_1(g_v) m_y] - [Q_2(g_u) + Q_2(g_v) v_*'(u_\delta^r)] u_{\delta y}^r + e_1 \\ =& \frac{1}{2}\left[\Phi_{uu} z^2 + 2\Phi_{uv} zm + \Phi_{vv} m^2\right] + Q_3(\Phi) \\ & - [Q_1(g_u) z_y + Q_1(g_v) m_y] - [Q_2(g_u) + Q_2(g_v) v_*'(u_\delta^r)] u_{\delta y}^r + e_1. \end{aligned} \tag{2.15}$$

It follows from $(2.13)_{1\tau} - \Phi_v \left(f_v \Phi_v^{-1} (2.13)_2\right)_y$ that

$$\begin{aligned} & z_{\tau\tau} + (\lambda_1 + \lambda_2) z_{y\tau} + \lambda_1 \lambda_2 z_{yy} - \Phi_v \left[z_\tau + (\lambda_* z)_y\right] \\ =& J_{1\tau} + g_v J_{1y} - \Phi_v J_1 - \Phi_v \left(\Phi_v^{-1} f_v J_2\right)_y + J_3 + J_4 + J_5, \end{aligned} \tag{2.16}$$

where

$$J_3 = \left[-f_{uy\tau} - f_{uyy} g_v + \Phi_v (\Phi_v^{-1} f_v g_{uy})_y\right] z$$

$$
\begin{aligned}
&+\left[-f_{vy\tau}+(f_v g_{vy})_y-f_{vyy}g_v+\Phi_v(\Phi_v^{-1})_y f_v g_{vy}\right]m,\\
J_4=&\left[-f_{u\tau}-2f_{uy}g_v+f_{vy}g_u+2f_v g_{uy}+\Phi_v(\Phi_v^{-1})_y f_v g_u\right]z_y\\
&+\left[-f_{v\tau}+2f_v g_{vy}-f_{vy}g_v+\Phi_v(\Phi_v^{-1})_y f_v g_v\right]m_y,\\
J_5=&-f_{uy}z_\tau+\Phi_v\left(\Phi_v^{-1}\right)_y f_v m_\tau.
\end{aligned}
\tag{2.17}
$$

We will work on the Cauchy problem of equation (2.16), which is a wave equation for z, with the initial data

$$z(y,0)=0 \quad\text{and}\quad z_\tau(y,0)=0$$

to derive the estimates on z, by viewing m and its derivatives as error terms. To deal with m, we rewrite (2.9) as

$$
\begin{aligned}
m_y=&-f_v^{-1}(u_\delta^r+z,v_\delta^r+m)\left[z_\tau+f_u(u_\delta^r+z,v_\delta^r+m)z_y\right]\\
&\qquad -f_v^{-1}(u_\delta^r+z,v_\delta^r+m)\left[Q_1(f_u)+Q_1(f_v)v_*'(u_\delta^r)\right]u_{\delta y}^r,\\
m=&\Phi_v^{-1}\left[m_\tau+(g_u z+g_v m)_y-J_2\right]-\Phi_v^{-1}\Phi_u z.
\end{aligned}
\tag{2.18}
$$

3. Proof of Theorem 1.1

Proof. First, we seek a global (in time) solution (z,m) to the reformulated problem (2.9) and (2.10) in the space defined as

$$
\begin{aligned}
X(0,\tau_1)=\big\{&(z,m)|(z,m)\in C^0([0,\tau_1];H^2(\mathbb{R})),\ (z_\tau,m_\tau)\in C^0([0,\tau_1];H^1(\mathbb{R})),\\
&(z_{\tau\tau},m_{\tau\tau})\in C^0([0,\tau_1];L^2(\mathbb{R})),\ (z_y,m_y,z_\tau,m_\tau)\in L^2(0,\tau_1;H^1(\mathbb{R})),\\
&(z_{\tau\tau},m_{\tau\tau})\in L^2(0,\tau_1;L^2(\mathbb{R}))\big\}
\end{aligned}
$$

with $0\le\tau_1<+\infty$. The local existence and uniqueness of smooth solutions to (2.9) and (2.10) can be obtained as in [32], while the global existence will follow from the following a priori estimates.

PROPOSITION 3.1 (a priori estimates). *Suppose that problem* (2.9) *and* (2.10) *has a solution $(z,m)\in X(0,\tau_1)$ for some $\tau_1>0$. There exist positive constants $\epsilon_1\le 1$, $\kappa\le 1/20$, $\mathcal{E}_0$ and C, independent of ϵ, δ and τ_1, such that if*

$$0<\epsilon\le\epsilon_1,\quad \epsilon^{1-2\kappa}\le\delta\le\epsilon^{2\kappa}, \tag{3.1}$$

$$\sup_{\tau\in[0,\tau_1]}\|(m,z,m_y,z_y,z_\tau)(\cdot,\tau)\|_{L^\infty}\le\mathcal{E}_0, \tag{3.2}$$

for small ϵ_1 and $\mathcal{E}_0$, then

$$
\begin{aligned}
&\sup_{\tau\in[0,\tau_1]}\|(z,z_y,z_\tau,z_{\tau\tau},z_{\tau y},z_{yy},m,m_y,m_\tau,m_{\tau\tau},m_{\tau y},m_{yy})(\cdot,\tau)\|^2\\
&+\int_0^{\tau_1}\int u_{\delta y}^r(z^2+m^2)dyd\tau+\int_0^{\tau_1}\|(z_y,m_y,z_\tau,m_\tau)(\cdot,\tau)\|^2d\tau\\
&+\int_0^{\tau_1}\|(z_{\tau\tau},z_{\tau y},z_{yy},m_{\tau\tau},m_{\tau y},m_{yy})(\cdot,\tau)\|^2d\tau\le C(\epsilon/\delta)^{1/2}.
\end{aligned}
\tag{3.3}
$$

REMARK 3.2 (the choice of δ). One can derive from (3.3) and the fact $\|\cdot\|_{L^\infty}\le\|\cdot\|_{H^1}$ that

$$\|(u-u_\delta^r,\ v-v_\delta^r)(\cdot,t)\|_{L^\infty}=\|(z,\ m)(\cdot,\tau)\|_{L^\infty}\le C(\epsilon/\delta)^{1/4}.$$

This, together with (c) of Lemma 2.2, gives

$$\begin{aligned}&\|(u-u^r,\ v-v_*(u^r))(\cdot,t)\|_{L^\infty}\le\|(u-u^r_\delta,\ v-v^r_\delta)(\cdot,t)\|_{L^\infty}\\&+\|(u^r-u^r_\delta,\ v_*(u^r)-v^r_\delta)(\cdot,t)\|_{L^\infty}\le Ct^{-1}\delta(\ln(1+t)+|\ln\delta|)+C(\epsilon/\delta)^{1/4}.\end{aligned}$$

Here C is a positive constant independent of ϵ. To obtain the best rate of convergence with the method used in this paper, one has to choose $\delta=\epsilon^{1/5}$.

Once Proposition 3.1 is proved, one can take $\delta=\epsilon^{1/5}$, so that (3.3) and the fact $\|\cdot\|_{L^\infty}\le\|\cdot\|_{H^1}$ imply that there exists a positive constant C independent of ϵ such that

$$\|(z,z_y,z_\tau,m,m_y,m_\tau)(\cdot,\tau)\|_{L^\infty}\le C\epsilon^{1/5},\quad \tau\ge 0.$$

(Indeed, this verifies the a prior assumption (3.2).) Therefore, there exists a global solution $(u,v)(x,t)$ $(t\ge 0)$ to the Cauchy problem of (1.1) with the initial data

$$(u,\ v)(x,t=0)=(u^r_\delta,\ v_*(u^r_\delta))(x,0),\quad \delta=\epsilon^{1/5}; \tag{3.4}$$

which satisfies (1.9) and

$$\|(u-u^r_\delta,v-v^r_\delta)(\cdot,t)\|_{L^\infty}\le C\epsilon^{1/5},\quad t\ge 0, \tag{3.5}$$

where $\delta=\epsilon^{1/5}$ and C is a certain constant independent of ϵ. With the help of (c) of Lemma 2.2 and (3.5), one obtains (1.11) and finishes the proof of Theorem 1.1. ◻

REMARK 3.3 (bounds for (1.9)). For each fixed small positive ϵ, it follows from (3.3), (2.8), Lemma 2.2, and the definitions of u^r and u^r_δ that

$$\begin{aligned}&\|(u-u^r,\ v-v_*(u^r))(\cdot,t)\|\le C\epsilon^{1/10}\left(1+t^{-1/2}\ln(1+t)\right),\quad t>0,\\&\|(u_x,u_t,u_{xx},u_{xt},u_{tt},v_x,v_t,v_{xx},v_{xt},v_{tt})(\cdot,t)\|\le C_\epsilon,\quad t\ge 0,\end{aligned}$$

where C is a positive constant and C_ϵ is also a positive constant depending on ϵ which may goes to infinity as ϵ tends to zero. For instance, for all $t\ge 0$,

$$\begin{aligned}\|u_x(x,t)\|_{L^2_x}&\le\|u^r_{\delta x}(x,t)\|_{L^2_x}+\epsilon^{-1/2}\|z_y(y,\epsilon^{-1}t)\|_{L^2_y}\\&\le C(\epsilon^{1/5}+t)^{-1}\epsilon^{-1/10}+C\epsilon^{-3/10}\le C\epsilon^{-3/10}.\end{aligned}$$

To prove Proposition 3.1, which can be derived from Lemmas 3.7 and 3.8, we notice the following facts. It follows from the initial value (2.10) and the system (2.9) that

$$(z,z_\tau,z_y)(y,0)=(0,0,0)\ \text{ and }\ (m,m_y,m_\tau)(y,0)=(0,0,e_1(y,0)), \tag{3.6}$$

where e_1 is defined by (2.11). It follows from Lemma 2.2, (3.1), (2.4) and (2.5) that

$$0<u^r_{\delta y}\le C\epsilon^{2\kappa},\quad |u^r_{\delta\tau}|\le C\eta_0u^r_{\delta y}\le C\epsilon^{2\kappa}\ \text{ and }\ |u^r_{\delta yy}|\le C\epsilon^{2\kappa}u^r_{\delta y}. \tag{3.7}$$

Moreover, it follows from the strict convexity of $f_*(u)$ that

$$C^{-1}u^r_{\delta y}\le(\lambda_*(u^r_\delta))_y\le Cu^r_{\delta y}. \tag{3.8}$$

3.1. Lower-order estimates.
LEMMA 3.4. *Suppose that the assumptions in Proposition 3.1 hold. Then for any $\varsigma>0$,*

$$\frac{1}{2}\int\left(|\Phi_v|z^2+2z_\tau z-2J_1z\right)(y,\tau)dy+\frac{1}{8}\int_0^\tau\int|\lambda_1\lambda_2|z_y^2dyd\tau+\frac{1}{8}\int_0^\tau\int|\Phi_v|\lambda_{*y}z^2dyd\tau$$
$$\lesssim_\varsigma \sup_{s\in[0,\tau]}\int z^2(y,s)dy+\int_0^\tau\int z_\tau^2dyd\tau+(\mathcal{E}_0+\epsilon^\kappa)\int_0^\tau\int u_{\delta y}^r m^2dyd\tau$$
$$+(\mathcal{E}_0+\epsilon^\kappa)\int_0^\tau\int\left(m_\tau^2+m_y^2\right)dyd\tau+C_\varsigma\epsilon^{1/2}\delta^{-1/2},\quad \tau\in[0,\tau_1]. \tag{3.9}$$

Proof. Multiply equation (2.16) by z, integrate the resulting equation with respect to the spatial and temporal variables and use (1.2), (2.6), and (3.6) to get

$$\frac{1}{2}\int\left(|\Phi_v|z^2+2z_\tau z-2J_1z\right)(y,\tau)dy+\int_0^\tau\int|\lambda_1\lambda_2|z_y^2dyd\tau+\frac{1}{2}\int_0^\tau\int|\Phi_v|\lambda_{*y}z^2dyd\tau$$
$$=\int_0^\tau\int\left[z_\tau^2+(\lambda_1+\lambda_2)z_yz_\tau+(\lambda_1+\lambda_2)_yzz_\tau+(\lambda_1\lambda_2)_yzz_y\right]dyd\tau$$
$$-\int_0^\tau\int\left[J_1z_\tau+J_1(g_vz)_y-\Phi_v^{-1}f_vJ_2(\Phi_vz)_y-(\Phi_vJ_1-J_3-J_4-J_5)z\right]dyd\tau,$$

due to $\Phi_{v\tau}+\lambda_*\Phi_{vy}=0$; which implies, with the aid of the Cauchy inequality, the smallness of ϵ, (3.7), and (3.8), that

$$\frac{1}{2}\int\left(|\Phi_v|z^2+2z_\tau z-2J_1z\right)(y,\tau)dy+\frac{1}{2}\int_0^\tau\int|\lambda_1\lambda_2|z_y^2dyd\tau$$
$$+\frac{1}{4}\int_0^\tau\int|\Phi_v|\lambda_{*y}z^2dyd\tau$$
$$\lesssim\int_0^\tau\int\left(z_\tau^2+J_1^2+|(J_3+J_4+J_5)z|\right)dyd\tau+\left|\int_0^\tau\int\Phi_v^{-1}f_vJ_2(\Phi_vz)_ydyd\tau\right|$$
$$+\left|\int_0^\tau\int\Phi_vJ_1zdyd\tau\right|=P_1+P_2+P_3. \tag{3.10}$$

We want to bound P_1. It follows from (2.14) and (2.12) that

$$|J_1|\lesssim(|z|+|m|)(|z_y|+|m_y|)+u_{\delta y}^r(z^2+m^2), \tag{3.11}$$

which, together with (3.2) and (3.7), gives

$$\int J_1^2dy\lesssim\mathcal{E}_0^2\int\left[z_y^2+m_y^2+u_{\delta y}^r(z^2+m^2)\right]dy.$$

It follows from (2.17) that

$$|J_3|+|J_4|+|J_5|\lesssim(|u_{\delta y}^r|^2+|u_{\delta yy}^r|)(|m|+|z|)+u_{\delta y}^r(|z_y|+|m_y|+|z_\tau|+|m_\tau|), \tag{3.12}$$

which implies, using (3.7) and (3.2), that

$$\int|(J_3+J_4+J_5)z|dy$$
$$\lesssim\mathcal{E}_0\int(|u_{\delta y}^r|^2+|u_{\delta yy}^r|)|z|dy+\epsilon^\kappa\left(\int(z_y^2+m_y^2+z_\tau^2+m_\tau^2)dy+\int u_{\delta y}^rz^2dy\right).$$

Then, we have

$$P_1 \lesssim \int_0^\tau \int \left[z_\tau^2 + \mathcal{E}_0(|u_{\delta y}^r|^2 + |u_{\delta yy}^r|)|z|\right] dyd\tau \\ + \left(\mathcal{E}_0^2 + \epsilon^\kappa\right) \int_0^\tau \int \left[u_{\delta y}^r(m^2+z^2) + \left(z_y^2 + m_y^2 + m_\tau^2\right)\right] dyd\tau. \tag{3.13}$$

We are to bound P_2. Note that

$$\int \Phi_v^{-1} f_v J_2 (\Phi_v z)_y dy = \int \Phi_v^{-1} f_v J_2 \Phi_{vy} z dy + \int f_v J_2 z_y dy = P_{21} + P_{22}. \tag{3.14}$$

It follows from (2.15) that

$$|J_2| \lesssim (1+u_{\delta y}^r)(z^2+m^2) + (|z|+|m|)(|z_y|+|m_y|) + u_{\delta y}^r, \tag{3.15}$$

which gives, using (3.2) and (3.7), that

$$|P_{21}| \lesssim \int u_{\delta y}^r |J_2||z| dy \lesssim \mathcal{E}_0 \int \left[u_{\delta y}^r(z^2+m^2) + z_y^2 + m_y^2\right] + \int |u_{\delta y}^r|^2 |z| dy. \tag{3.16}$$

In view of (2.15), we rewrite P_{22} as

$$P_{22} = \frac{1}{2} \int \left[\Phi_{uu} z^2 + 2\Phi_{uv} zm + \Phi_{vv} m^2\right] f_v z_y dy + \int Q_3(\Phi) f_v z_y dy \\ - \int \left[[Q_1(g_u) z_y + Q_1(g_v) m_y] + [Q_2(g_u) + Q_2(g_v) v_*'(u_\delta^r)] u_{\delta y}^r\right] f_v z_y dy \\ + \int e_1 f_v z_y dy = P_{221} + P_{222} + P_{223} + P_{224}. \tag{3.17}$$

Easily, P_{223} and P_{224} can be bounded by

$$|P_{223}| \lesssim \mathcal{E}_0 \int \left[u_{\delta y}^r(z^2+m^2) + z_y^2 + m_y^2\right], \tag{3.18}$$

$$|P_{224}| = \left|\int e_1 f_v z_y dy\right| = \left|\int (e_1 f_v)_y z dy\right| \lesssim \int \left(|u_{\delta y}^r|^2 + |u_{\delta yy}^r|\right) |z| dy. \tag{3.19}$$

It follows from the fact $\|\beta\|_{L^\infty}^2 \le \|\beta\| \|\beta_y\|$ and (3.2) that

$$|P_{222}| \lesssim \int \left(|z|^3 + |m|^3\right) |z_y| dy \lesssim \left(\|z\|_{L^\infty}^2 + \|m\|_{L^\infty}^2\right) \|z_y\| \|z\| \\ \lesssim (\|z\|\|z_y\| + \|m\|\|m_y\|) \|z_y\| \|z\| \lesssim \left(\|z\|^2 + \|m\|^2\right)\left(\|z_y\|^2 + \|m_y\|^2\right) \\ \lesssim \mathcal{E}_0^2 \left(\|z_y\|^2 + \|m_y\|^2\right) = \mathcal{E}_0^2 \int (z_y^2 + m_y^2) dy. \tag{3.20}$$

For P_{221}, we first note that

$$\left|\int \Phi_{uu} z^2 f_v z_y dy\right| = \left|\frac{1}{3} \int (\Phi_{uu} f_v)_y z^3 dy\right| \lesssim \mathcal{E}_0 \int u_{\delta y}^r z^2 dy. \tag{3.21}$$

It follows from $(2.18)_2$, (3.7), (3.20), and (3.21) that

$$\left|\int \Phi_{uv} zm f_v z_y dy\right| \\ \lesssim \left|\int \Phi_{uv} f_v \Phi_v^{-1} \Phi_u z^2 z_y dy\right| + \int \left[|m_\tau| + |z_y| + |m_y| + (z^2+m^2) + u_{\delta y}^r\right] |z z_y| dy \\ \lesssim (\mathcal{E}_0 + \epsilon^\kappa) \int u_{\delta y}^r z^2 dy + (\mathcal{E}_0 + \epsilon^\kappa) \int \left(m_\tau^2 + z_y^2 + m_y^2\right) dy.$$

Here we have used the following estimate

$$\begin{aligned}\left|m+\Phi_v^{-1}\Phi_u z\right|=&\left|\Phi_v^{-1}\left[m_\tau+(g_u z+g_v m)_y-J_2\right]\right|\\ &\lesssim\left[|m_\tau|+|z_y|+|m_y|+(z^2+m^2)+u^r_{\delta y}\right],\end{aligned}$$

due to $(2.18)_2$, (3.15), and (3.2). Similarly,

$$\begin{aligned}&\left|\int\Phi_{vv}m^2 f_v z_y dy\right|\\ \lesssim&\left|\int\Phi_{vv}f_v(\Phi_v^{-1}\Phi_u z)^2 z_y dy\right|+\int\left[|m_\tau|+|z_y|+|m_y|+(z^2+m^2)+u^r_{\delta y}\right]|zz_y|dy\\ &+\int\left[|m_\tau|+|z_y|+|m_y|+(z^2+m^2)+u^r_{\delta y}\right]^2|z_y|dy\\ \lesssim&(\mathcal{E}_0+\epsilon^\kappa)\int u^r_{\delta y}z^2dy+(\mathcal{E}_0+\epsilon^\kappa)\int\left(m_\tau^2+z_y^2+m_y^2\right)dy+\int|u^r_{\delta y}|^2|z_y|dy.\end{aligned}$$

Then, we have

$$|P_{221}|\lesssim(\mathcal{E}_0+\epsilon^\kappa)\int u^r_{\delta y}z^2dy+(\mathcal{E}_0+\epsilon^\kappa)\int\left(m_\tau^2+z_y^2+m_y^2\right)dy+\int|u^r_{\delta y}|^2|z_y|dy. \quad (3.22)$$

This, together with (3.17)-(3.20), gives

$$\begin{aligned}|P_{22}|\lesssim&(\mathcal{E}_0+\epsilon^\kappa)\int u^r_{\delta y}(z^2+m^2)dy+(\mathcal{E}_0+\epsilon^\kappa)\int\left(m_\tau^2+z_y^2+m_y^2\right)dy\\ &+\int|u^r_{\delta y}|^2|z_y|dy+\int\left(|u^r_{\delta y}|^2+|u^r_{\delta yy}|\right)|z|dy;\end{aligned}$$

which implies, with the aid of (3.14) and (3.16), that

$$\begin{aligned}P_2\lesssim&(\mathcal{E}_0+\epsilon^\kappa)\int_0^\tau\int u^r_{\delta y}(z^2+m^2)dyd\tau+\int_0^\tau\int\left(|u^r_{\delta y}|^2+|u^r_{\delta yy}|\right)|z|dyd\tau\\ &+(\mathcal{E}_0+\epsilon^\kappa)\int_0^\tau\int\left(m_\tau^2+z_y^2+m_y^2\right)dyd\tau+\int_0^\tau\int|u^r_{\delta y}|^2|z_y|dyd\tau. \quad (3.23)\end{aligned}$$

For P_3, it follows from (2.14) that

$$\begin{aligned}\int\Phi_v J_1 z dy=&-\frac{1}{2}\int\Phi_v\left[f_{uu}(z^2)_y+2f_{uv}(zm)_y+f_{vv}(m^2)_y\right]zdy\\ &-\int\Phi_v\left[Q_2(f_u)z_y+Q_2(f_v)m_y\right]zdy-\int\Phi_v\left[Q_2(f_u)+Q_2(f_v)v_*'(u^r_\delta)\right]u^r_{\delta y}zdy\\ =&\frac{1}{2}\int\Phi_v\left[f_{uu}z^2+2f_{uv}zm+f_{vv}m^2\right]z_y dy\\ &+\frac{1}{2}\int\left[(\Phi_v f_{uu})_y z^2+2(\Phi_v f_{uv})_y zm+(\Phi_v f_{vv})_y m^2\right]zdy\\ &-\int\Phi_v\left[Q_2(f_u)z_y+Q_2(f_v)m_y\right]zdy-\int\Phi_v\left[Q_2(f_u)+Q_2(f_v)v_*'(u^r_\delta)\right]u^r_{\delta y}zdy.\end{aligned}$$

In a similar way to the derivation of (3.22), we have

$$P_3\lesssim(\mathcal{E}_0+\epsilon^\kappa)\int_0^\tau\int u^r_{\delta y}(z^2+m^2)dyd\tau+\int_0^\tau\int|u^r_{\delta y}|^2|z_y|dyd\tau$$

$$+(\mathcal{E}_0+\epsilon^\kappa)\int_0^\tau\int\left(m_\tau^2+z_y^2+m_y^2\right)dyd\tau. \tag{3.24}$$

Finally, it follows from (3.8), (3.10), (3.13), (3.23), (3.24), the Cauchy inequality, and the smallness of $\mathcal{E}_0$ and ϵ that

$$\begin{aligned}&\frac{1}{2}\int\left(|\Phi_v|z^2+2z_\tau z-2J_1z\right)(y,\tau)dy+\frac{1}{4}\int_0^\tau\int|\lambda_1\lambda_2|z_y^2dyd\tau+\frac{1}{8}\int_0^\tau\int|\Phi_v|\lambda_{*y}z^2dyd\tau\\&\lesssim\int_0^\tau\int z_\tau^2dyd\tau+(\mathcal{E}_0+\epsilon^\kappa)\int_0^\tau\int u^r_{\delta y}m^2dyd\tau+\int_0^\tau\int|u^r_{\delta y}|^4dyd\tau\\&\quad+(\mathcal{E}_0+\epsilon^\kappa)\int_0^\tau\int\left(m_\tau^2+m_y^2\right)dyd\tau+\int_0^\tau\int\left(|u^r_{\delta y}|^2+|u^r_{\delta yy}|\right)|z|dyd\tau.\end{aligned} \tag{3.25}$$

It follows from Lemma 2.2, the fact $\|\beta\|^2_{L^\infty}\le\|\beta\|\|\beta_y\|$ and Young's inequality that

$$\int_0^\tau\int|u^r_{\delta y}|^4dyd\tau=\int_0^\tau\left(\epsilon^3\int_{\mathbb{R}}|u^r_{\delta x}|^4dx\right)d\tau\lesssim\int_0^\infty\epsilon^2(\delta+t)^{-3}dt\lesssim\epsilon^2\delta^{-2}, \tag{3.26}$$

$$\begin{aligned}&\int_0^\tau\int\left(|u^r_{\delta y}|^2+|u^r_{\delta yy}|\right)|z|dyd\tau\\&\le\int_0^\tau\|z\|_{L^\infty}\left(\int\left(|u^r_{\delta y}|^2+|u^r_{\delta yy}|\right)dy\right)d\tau\\&\lesssim\epsilon\int_0^\tau\|z\|^{1/2}\|z_y\|^{1/2}(\delta+\epsilon\tau)^{-1}d\tau\\&\lesssim\varsigma\int_0^\tau\|z_y\|^2d\tau+C_\varsigma\epsilon^{4/3}\left(\sup_{[0,\tau]}\|z\|^{2/3}\right)\int_0^\tau(\delta+\epsilon\tau)^{-4/3}d\tau\\&\lesssim\varsigma\left(\int_0^\tau\|z_y\|^2d\tau+\sup_{[0,\tau]}\|z\|^2\right)+C_\varsigma\epsilon^{1/2}\delta^{-1/2},\end{aligned} \tag{3.27}$$

for any positive constant ς. Substituting (3.26) and (3.27) into (3.25) gives (3.9). This finishes the proof of Lemma 3.4. ∎

Remark 3.5. If we use the scaling $(y,\tau)=\epsilon^{-\nu}(x,t)$ with $\nu\in(0,1]$ constant, instead of $(y,\tau)=\epsilon^{-1}(x,t)$ as chosen in (2.8), we can obtain, using a similar method to the derivation of (3.9), that for any $\varsigma>0$,

$$\begin{aligned}&\frac{1}{2}\int\left(\epsilon^{\nu-1}|\Phi_v|z^2+2z_\tau z-2J_1z\right)(y,\tau)dy+\frac{1}{8}\int_0^\tau\int|\lambda_1\lambda_2|z_y^2dyd\tau\\&\quad+\frac{1}{8}\epsilon^{\nu-1}\int_0^\tau\int|\Phi_v|\lambda_{*y}z^2dyd\tau\\&\lesssim\varsigma\sup_{[0,\tau]}\int\epsilon^{\nu-1}z^2dy+\int_0^\tau\int z_\tau^2dyd\tau+\left(\mathcal{E}_0\epsilon^{\nu-1}+\epsilon^\kappa\right)\int_0^\tau\int u^r_{\delta y}m^2dyd\tau\\&\quad+(\mathcal{E}_0+\epsilon^\kappa)\int_0^\tau\int\left(m_\tau^2+m_y^2\right)dyd\tau+C_\varsigma\epsilon^{1/2}\delta^{-1/2},\quad\tau\in[0,\tau_1],\end{aligned}$$

provided that for all $\tau\in[0,\tau_1]$,

$$\epsilon^{\nu-1}\|(z,m)(\cdot,\tau)\|^2+\|(m,z)(\cdot,\tau)\|^2_{L^\infty}+\epsilon^{2(1-\nu)}\|z_y(\cdot,\tau)\|^2_{L^\infty}\le\mathcal{E}_0^2.$$

This implies that the basic energy estimate is bounded by $\epsilon^{1/2}\delta^{-1/2}$, no matter which scaling is chosen. And the other estimates are based on this one. So, the bounds of the norm of (z,m) cannot be improved by using different scaling arguments.

LEMMA 3.6. *Suppose that the assumptions in Proposition 3.1 hold. Then for all* $\tau\in[0,\tau_1]$,

$$\begin{aligned}&\int\left(z^2+z_y^2+z_\tau^2\right)(y,\tau)dy+\int_0^\tau\int\left(u_{\delta y}^r z^2+z_\tau^2+z_y^2\right)dyd\tau\\ \lesssim&\mathcal{E}_0\int m^2(y,\tau)dy+(\mathcal{E}_0+\epsilon^\kappa)\int_0^\tau\int\left(u_{\delta y}^r m^2+m_\tau^2+m_y^2\right)dyd\tau+(\epsilon/\delta)^{1/2}.\end{aligned}\tag{3.28}$$

Proof. Multiply Equation (2.16) by $-\Phi_v^{-1}z_\tau$, integrate the resulting equation with respect to the spatial and temporal variables and use (1.2), (2.6), and (3.6) to get

$$\begin{aligned}&\frac{1}{2}\int|\Phi_v^{-1}|\left(z_\tau^2+|\lambda_1\lambda_2|z_y^2\right)(y,\tau)dy+\int_0^\tau\int z_\tau^2 dyd\tau\\ =&\frac{1}{2}\int_0^\tau\int\Big[(\Phi_v^{-1}\lambda_1\lambda_2)_\tau z_y^2-\Big[\left(\Phi_v^{-1}(\lambda_1+\lambda_2)\right)_y+(\Phi_v^{-1})_\tau\Big]z_\tau^2\\ &-2(\Phi_v^{-1}\lambda_1\lambda_2)_y z_\tau z_y\Big]dyd\tau-\int_0^\tau\int(\lambda_{*y}z+\lambda_* z_y)z_\tau dyd\tau-\int_0^\tau\int\Phi_v^{-1}\big[J_{1\tau}\\ &+g_vJ_{1y}-\Phi_vJ_1-\Phi_v\left(\Phi_v^{-1}f_vJ_2\right)_y+J_3+J_4+J_5\Big]z_\tau dyd\tau,\end{aligned}$$

which implies, using (3.7), the Cauchy inequality, and smallness of ϵ, that

$$\begin{aligned}&\frac{1}{2}\int|\Phi_v^{-1}|\left(z_\tau^2+|\lambda_1\lambda_2|z_y^2\right)(y,\tau)dy+\frac{1}{2}\int_0^\tau\int z_\tau^2 dyd\tau\\ \lesssim&\left[\epsilon^{2\kappa}\int_0^\tau\int\left(z_y^2+\lambda_{*y}z^2\right)dyd\tau+\int_0^\tau\int\lambda_*^2 z_y^2 dyd\tau\right]\\ &+\int_0^\tau\int\left[|J_1|^2+|J_3+J_4+J_5|^2\right]dyd\tau+\left|\int_0^\tau\int\Phi_v^{-1}J_{1\tau}z_\tau dyd\tau\right|\\ &+\left|\int_0^\tau\int\Phi_v^{-1}g_vJ_{1y}z_\tau dyd\tau\right|+\left|\int_0^\tau\int\left(\Phi_v^{-1}f_vJ_2\right)_y z_\tau dyd\tau\right|=\sum_{i=1}^5 Q_i.\end{aligned}\tag{3.29}$$

For Q_2, it follows from (3.11), (3.12), and (3.7) that

$$Q_2\lesssim\left(\epsilon^{2\kappa}+\mathcal{E}_0^2\right)\int_0^\tau\int\left[z_y^2+m_y^2+z_\tau^2+m_\tau^2+u_{\delta y}^r(z^2+m^2)\right]dyd\tau.$$

For Q_3, it follows from (2.14) that

$$\begin{aligned}Q_3\lesssim&\left|\int_0^\tau\int\Phi_v^{-1}Q_1(f_u)z_{\tau y}z_\tau dyd\tau\right|+\left|\int_0^\tau\int\Phi_v^{-1}Q_1(f_v)m_{y\tau}z_\tau dyd\tau\right|\\ &+\int_0^\tau\int|(Q_1(f_u))_\tau z_y+(Q_1(f_v))_\tau m_y||z_\tau|dyd\tau\\ &+\int_0^\tau\int\left|\left[\left[Q_2(f_u)+Q_2(f_v)v_*'(u_\delta^r)\right]u_{\delta y}^r\right]_\tau\right||z_\tau|dyd\tau=\sum_{i=1}^4 Q_{3i}.\end{aligned}$$

In view of (3.7) and (3.2), we see that

$$\begin{aligned}
&Q_{31}\leq\left|\int_0^\tau\int\left(\Phi_v^{-1}Q_1(f_u)\right)_y z_\tau^2dyd\tau\right|\lesssim\mathcal{E}_0\int_0^\tau\int z_\tau^2dyd\tau,\\
&Q_{33}\lesssim\mathcal{E}_0\int_0^\tau\int\left(z_\tau^2+m_\tau^2+z_y^2+m_y^2\right)dyd\tau,\\
&Q_{34}\lesssim\mathcal{E}_0\int_0^\tau\int\left[u_{\delta y}^r\left(m^2+z^2\right)+z_\tau^2+m_\tau^2\right]dyd\tau.
\end{aligned}$$

It follows from $(2.18)_1$ that

$$\begin{aligned}
m_{y\tau}=&-f_v^{-1}(u_\delta^r+z,v_\delta^r+m)z_{\tau\tau}-(f_v^{-1}f_u)(u_\delta^r+z,v_\delta^r+m)z_{y\tau}\\
&-\left[f_v^{-1}(u_\delta^r+z,v_\delta^r+m)\right]_\tau z_\tau-\left[(f_v^{-1}f_u)(u_\delta^r+z,v_\delta^r+m)\right]_\tau z_y\\
&-\left[f_v^{-1}(u_\delta^r+z,v_\delta^r+m)\left[Q_1(f_u)+Q_1(f_v)v_*'(u_\delta^r)\right]u_{\delta y}^r\right]_\tau,
\end{aligned}\tag{3.30}$$

which implies

$$\begin{aligned}
Q_{32}\lesssim&\left|\int_0^\tau\int\Phi_v^{-1}Q_1(f_v)f_v^{-1}(u_\delta^r+z,v_\delta^r+m)z_{\tau\tau}z_\tau dyd\tau\right|\\
&+\left|\int_0^\tau\int\Phi_v^{-1}Q_1(f_v)(f_v^{-1}f_u)(u_\delta^r+z,v_\delta^r+m)z_{y\tau}z_\tau dyd\tau\right|\\
&+\left(\epsilon^{2\kappa}+\mathcal{E}_0\right)\int_0^\tau\int\left[u_{\delta y}^r\left(m^2+z^2\right)+z_\tau^2+m_\tau^2+z_y^2\right]dyd\tau\\
\lesssim&\mathcal{E}_0\int z_\tau^2(y,\tau)dy+\left(\epsilon^{2\kappa}+\mathcal{E}_0\right)\int_0^\tau\int\left[u_{\delta y}^r\left(m^2+z^2\right)+z_\tau^2+m_\tau^2+z_y^2\right]dyd\tau.
\end{aligned}\tag{3.31}$$

Here we have used the following estimates:

$$\begin{aligned}
&\int_0^\tau\int\Phi_v^{-1}Q_1(f_v)f_v^{-1}(u_\delta^r+z,v_\delta^r+m)z_{\tau\tau}z_\tau dyd\tau\\
=&\frac{1}{2}\int\left[\Phi_v^{-1}Q_1(f_v)f_v^{-1}(u_\delta^r+z,v_\delta^r+m)z_\tau^2\right](y,\tau)dyd\tau\\
&-\frac{1}{2}\int_0^\tau\int\left[\Phi_v^{-1}Q_1(f_v)f_v^{-1}(u_\delta^r+z,v_\delta^r+m)\right]_\tau z_\tau^2dyd\tau,
\end{aligned}$$

$$\begin{aligned}
&\int_0^\tau\int\Phi_v^{-1}Q_1(f_v)(f_v^{-1}f_u)(u_\delta^r+z,v_\delta^r+m)z_{y\tau}z_\tau dyd\tau\\
=&-\frac{1}{2}\int\int\left[\Phi_v^{-1}Q_1(f_v)(f_v^{-1}f_u)(u_\delta^r+z,v_\delta^r+m)\right]_y z_\tau^2dyd\tau.
\end{aligned}$$

Then, we have

$$Q_3\lesssim\mathcal{E}_0\int z_\tau^2(y,\tau)dy+\left(\epsilon^{2\kappa}+\mathcal{E}_0\right)\int_0^\tau\int\left[u_{\delta y}^r\left(m^2+z^2\right)+z_\tau^2+m_\tau^2+z_y^2+m_y^2\right]dyd\tau.$$

Similarly,

$$\begin{aligned}
Q_4\lesssim&\mathcal{E}_0\int z_y^2(y,\tau)dy+\left(\epsilon^{2\kappa}+\mathcal{E}_0\right)\int_0^\tau\int\left[u_{\delta y}^r\left(m^2+z^2\right)+z_\tau^2+m_\tau^2+z_y^2+m_y^2\right]dyd\tau,\\
Q_5\lesssim&\mathcal{E}_0\int z_y^2(y,\tau)dy+\left(\epsilon^{2\kappa}+\mathcal{E}_0\right)\int_0^\tau\int\left[u_{\delta y}^r\left(m^2+z^2\right)+z_\tau^2+m_\tau^2+z_y^2+m_y^2\right]dyd\tau\\
&+\int_0^\tau\int\left(|u_{\delta y}^r|^2+|u_{\delta yy}^r|\right)|z_\tau|dyd\tau.
\end{aligned}$$

Therefore, it follows from (3.29) that

$$
\begin{aligned}
&\frac{1}{2}\int|\Phi_v^{-1}|\left(z_\tau^2+|\lambda_1\lambda_2|z_y^2\right)(y,\tau)dy+\frac{1}{2}\int_0^\tau\int z_\tau^2dyd\tau\\
\lesssim&\mathcal{E}_0\int(z_\tau^2+z_y^2)(y,\tau)dy+\int_0^\tau\int\lambda_*^2z_y^2dyd\tau+\int_0^\tau\int\left(|u_{\delta y}^r|^2+|u_{\delta yy}^r|\right)|z_\tau|dyd\tau\\
&+\left(\epsilon^{2\kappa}+\mathcal{E}_0\right)\int_0^\tau\int\left[u_{\delta y}^r\left(m^2+z^2\right)+z_\tau^2+m_\tau^2+z_y^2+m_y^2\right]dyd\tau,
\end{aligned}
$$

which, together with (3.26), the Cauchy inequality, and smallness of ϵ and $\mathcal{E}_0$, implies

$$
\begin{aligned}
&\frac{1}{4}\int|\Phi_v^{-1}|\left(z_\tau^2+|\lambda_1\lambda_2|z_y^2\right)(y,\tau)dy+\frac{1}{4}\int_0^\tau\int z_\tau^2dyd\tau\\
\lesssim&\int_0^\tau\int\lambda_*^2z_y^2dyd\tau+\epsilon^2\delta^{-2}+\left(\epsilon^{2\kappa}+\mathcal{E}_0\right)\int_0^\tau\int\left[u_{\delta y}^r\left(m^2+z^2\right)+m_\tau^2+z_y^2+m_y^2\right]dyd\tau.
\end{aligned}
\tag{3.32}
$$

It follows from (3.9)+K(3.32) with suitably large constant K that for any $\varsigma>0$,

$$
\begin{aligned}
&\int\left(z^2+z_y^2+z_\tau^2\right)(y,\tau)dy+\int_0^\tau\int\left(u_{\delta y}^rz^2+z_\tau^2+z_y^2\right)dyd\tau\\
\lesssim&\int|(J_1z)(y,\tau)|dy+\varsigma\sup_{[0,\tau]}\int z^2dy+\int_0^\tau\int\lambda_*^2z_y^2dyd\tau\\
&+(\mathcal{E}_0+\epsilon^\kappa)\int_0^\tau\int\left(u_{\delta y}^rm^2+m_\tau^2+m_y^2\right)dyd\tau+C_\varsigma\epsilon^{1/2}\delta^{-1/2},
\end{aligned}
$$

due to the smallness of ϵ and $\mathcal{E}_0$. In view of (3.11), we see that

$$
\int|(J_1z)(y,\tau)|dy\lesssim\mathcal{E}_0\int(z^2+m^2)(y,\tau)dy,
$$

which gives (3.28), by using (2.5) and the smallness of η_0 and choosing suitably small ς. This finishes the proof of Lemma 3.6. □

LEMMA 3.7. *Suppose that the assumptions in Proposition 3.1 hold. Then,*

$$
\begin{aligned}
&\|(z,m,z_y,m_y,z_\tau,m_\tau)(\cdot,\tau)\|^2+\int_0^\tau\int u_{\delta y}^r(z^2+m^2)dyd\tau\\
&+\int_0^\tau\|(z_y,m_y,z_\tau,m_\tau)(\cdot,\tau)\|^2d\tau\lesssim(\epsilon/\delta)^{1/2},\quad\tau\in[0,\tau_1].
\end{aligned}
\tag{3.33}
$$

Proof. It follows from $(2.13)_{2\tau}$ that

$$
m_{\tau\tau}-\Phi_vm_\tau=(\Phi_uz)_\tau+\Phi_{v\tau}m-(g_uz+g_vm)_{y\tau}+J_{2\tau}.
\tag{3.34}
$$

Multiply the equation above by m_τ and integrate the resulting equation with respect to the spatial and temporal variables and use (1.2) to get

$$
\begin{aligned}
&\frac{1}{2}\int m_\tau^2(y,\tau)dy+\int_0^\tau\int|\Phi_v|m_\tau^2dyd\tau\\
=&\frac{1}{2}\int m_\tau^2(y,0)dy+\int_0^\tau\int\left[(\Phi_uz)_\tau+\Phi_{v\tau}m\right]m_\tau dyd\tau
\end{aligned}
$$

$$+\int_0^\tau\int (g_u z+g_v m)_\tau m_{y\tau}dyd\tau+\int_0^\tau\int J_{2\tau}m_\tau dyd\tau=\sum_{i=1}^4 R_i. \tag{3.35}$$

For R_1, it follows from (3.6) and Lemma 2.2 that

$$|R_1|\lesssim\int |u^r_{\delta y}(y,0)|^2dy\lesssim \epsilon\delta^{-1}.$$

For R_2, it follows from (3.7) and the Cauchy inequality that for any $\varsigma>0$,

$$|R_2|\lesssim \epsilon^\kappa\int_0^\tau\int \left[u^r_{\delta y}(m^2+z^2)+m_\tau^2\right]dyd\tau+\int_0^\tau\int\left(\varsigma m_\tau^2+C_\varsigma z_\tau^2\right)dyd\tau.$$

For R_3, note that

$$\begin{aligned}R_3=&\int_0^\tau\int g_u z_\tau m_{y\tau}dyd\tau-\int_0^\tau\int\left[(g_{u\tau}z+g_{v\tau}m)_y m_\tau+\frac{1}{2}g_{vy}m_\tau^2\right]dyd\tau\\ =&R_{31}+R_{32}\end{aligned}$$

In view of (3.30), we can obtain, suing a similar way to the derivation of (3.31), that

$$|R_{31}|\lesssim\int z_\tau^2(y,\tau)dy+\left(\epsilon^{2\kappa}+\mathcal{E}_0\right)\int_0^\tau\int\left[u^r_{\delta y}\left(m^2+z^2\right)+z_\tau^2+m_\tau^2+z_y^2\right]dyd\tau.$$

It follows from the Cauchy inequality and (3.7) that

$$|R_{32}|\lesssim\epsilon^{2\kappa}\int_0^\tau\int\left[u^r_{\delta y}\left(m^2+z^2\right)+z_y^2+m_\tau^2+m_y^2\right]dyd\tau.$$

Thus,

$$|R_3|\lesssim\int z_\tau^2(y,\tau)dy+\left(\epsilon^{2\kappa}+\mathcal{E}_0\right)\int_0^\tau\int\left[u^r_{\delta y}\left(m^2+z^2\right)+z_\tau^2+m_\tau^2+z_y^2+m_y^2\right]dyd\tau.$$

For R_4, notice from (2.15) that

$$\begin{aligned}R_4=&\int_0^\tau\int\left[Q_2(\Phi)-\left[Q_2(g_u)+Q_2(g_v)v_*'(u^r_\delta)\right]u^r_{\delta y}+e_1\right]_\tau m_\tau dyd\tau.\\ &-\int_0^\tau\left[Q_1(g_u)z_y+Q_1(g_v)m_y\right]_\tau m_\tau dyd\tau=R_{41}+R_{42}.\end{aligned}$$

It follows from (3.2), (3.7), and the Cauchy inequality that for any $\varsigma>0$,

$$\begin{aligned}|R_{41}|\lesssim&\mathcal{E}_0\int_0^\tau\int\left[u^r_{\delta y}\left(m^2+z^2\right)+z_\tau^2+m_\tau^2\right]dyd\tau\\ &+\varsigma\int_0^\tau\int m_\tau^2dyd\tau+C_\varsigma\int_0^\tau\int\left(|u^r_{\delta y}|^4+|u^r_{\delta yy}|^2\right)dyd\tau.\end{aligned}$$

Similar to deriving (3.31), we have

$$\begin{aligned}|R_{42}|\le&\left|\int_0^\tau\int\left\{\left[(Q_1(g_u))_\tau z_y+(Q_1(g_v))_\tau m_y\right]m_\tau-\frac{1}{2}(Q_1(g_v))_y m_\tau^2\right.\right.\\ &\left.\left.-(Q_1(g_u))_y z_\tau m_\tau\right\}dyd\tau\right|+\left|\int_0^\tau\int Q_1(g_u)z_\tau m_{y\tau}dyd\tau\right|\\ \lesssim&\mathcal{E}_0\int z_\tau^2(y,\tau)dy+\left(\epsilon^{2\kappa}+\mathcal{E}_0\right)\int_0^\tau\int\left[u^r_{\delta y}\left(m^2+z^2\right)+z_\tau^2+m_\tau^2+z_y^2\right]dyd\tau.\end{aligned}$$

Then, it holds that for any $\varsigma > 0$,

$$|R_4| \lesssim \mathcal{E}_0 \int z_\tau^2(y,\tau)dy + \varsigma \int_0^\tau \int m_\tau^2 dyd\tau + C_\varsigma \epsilon^2 \delta^{-2} + \left(\epsilon^{2\kappa} + \mathcal{E}_0\right) \int_0^\tau \int \left[u_{\delta y}^r \left(m^2 + z^2\right) + z_\tau^2 + m_\tau^2 + z_y^2\right] dyd\tau.$$

Substituting these calculations into (3.35) to give

$$\int m_\tau^2(y,\tau)dy + \int_0^\tau \int m_\tau^2 dyd\tau \lesssim \int z_\tau^2(y,\tau)dy + \int_0^\tau \int z_\tau^2 dyd\tau + (\epsilon^\kappa + \mathcal{E}_0) \int_0^\tau \int \left[u_{\delta y}^r \left(m^2 + z^2\right) + z_y^2 + m_y^2\right] dyd\tau + \epsilon\delta^{-1}. \tag{3.36}$$

With the estimates for m_τ, we can bound m and m_τ as follows. It follows from (2.18), (3.7), and (3.2) that

$$|m_y| \lesssim |z_\tau| + |z_y| + u_{\delta y}^r (|m| + |z|), \quad |m| \lesssim |m_\tau| + |z_y| + |m_y| + |z| + u_{\delta y}^r + \left(\epsilon^{2\kappa} + \mathcal{E}_0\right) |m|;$$

which implies, with the aid of the smallness of ϵ and $\mathcal{E}_0$, that

$$|m_y| \lesssim |z_\tau| + |z_y| + u_{\delta y}^r |z| + \epsilon^{2\kappa} |m_\tau| + \left(u_{\delta y}^r\right)^2, \quad |m| \lesssim |m_\tau| + |z_\tau| + |z_y| + |z| + u_{\delta y}^r.$$

Clearly,

$$\int_0^\tau \int \left(m_y^2 + u_{\delta y}^r m^2\right) dyd\tau \lesssim \int_0^\tau \int \left[z_y^2 + z_\tau^2 + u_{\delta y}^r z^2 + \epsilon^{2\kappa} m_\tau^2 + \left(u_{\delta y}^r\right)^3\right] dyd\tau,$$
$$\int \left(m_y^2 + m^2\right)(y,\tau)dy \lesssim \int \left[z_y^2 + z_\tau^2 + z^2 + m_\tau^2 + \left(u_{\delta y}^r\right)^2\right](y,\tau)dy. \tag{3.37}$$

Put $(3.37)_1$ into (3.36) to yield

$$\int m_\tau^2(y,\tau)dy + \int_0^\tau \int \left(u_{\delta y}^r m^2 + m_\tau^2 + m_y^2\right) dyd\tau \lesssim \int z_\tau^2(y,\tau)dy + \int_0^\tau \int \left(u_{\delta y}^r z^2 + z_\tau^2 + z_y^2\right) dyd\tau + \epsilon\delta^{-1},$$

which, together with $(3.37)_2$, implies

$$\int \left(m^2 + m_\tau^2 + m_y^2\right)(y,\tau)dy + \int_0^\tau \int \left(u_{\delta y}^r m^2 + m_\tau^2 + m_y^2\right) dyd\tau \lesssim \int \left(z^2 + z_\tau^2 + z_y^2\right)(y,\tau)dy + \int_0^\tau \int \left(u_{\delta y}^r z^2 + z_\tau^2 + z_y^2\right) dyd\tau + \epsilon\delta^{-1}, \tag{3.38}$$

due to

$$\int_0^\tau \int |u_{\delta y}^r|^3 dyd\tau \lesssim \epsilon\delta^{-1} \quad \text{and} \quad \int |u_{\delta y}^r(y,\tau)|^2 dy \lesssim \epsilon\delta^{-1}, \quad \tau \geq 0.$$

So, we prove (3.33) by a suitable linear combination of (3.28) and (3.38). This finishes the proof of Lemma 3.7. □

3.2. Higher-order estimates. In order to verify the a priori assumption (3.2), we need to estimate the higher-order derivatives. For this purpose, we apply ∂_τ to equation (2.16) to obtain

$$z_{\tau\tau\tau}+\lambda_1\lambda_2 z_{\tau yy}-\Phi_v z_{\tau\tau}=-(\lambda_1\lambda_2)_\tau z_{yy}+\Phi_{v\tau}z_\tau+\Big[\Phi_v(\lambda_* z)_y-(\lambda_1+\lambda_2)z_{y\tau}\Big]_\tau$$
$$+\Big[J_{1\tau}+g_v J_{1y}-\Phi_v J_1-\Phi_v\left(\Phi_v^{-1}f_v J_2\right)_y+J_3+J_4+J_5\Big]_\tau=(\text{RHS}). \tag{3.39}$$

We use this equation to derive the following estimates.

Lemma 3.8. *Suppose that the assumptions in Proposition 3.1 hold. Then,*

$$\|(z_{\tau\tau},z_{\tau y},z_{yy},m_{\tau\tau},m_{\tau y},m_{yy})(\cdot,\tau)\|^2$$
$$+\int_0^\tau\|(z_{\tau\tau},z_{\tau y},z_{yy},m_{\tau\tau},m_{\tau y},m_{yy})(\cdot,\tau)\|^2 d\tau\lesssim(\epsilon/\delta)^{1/2},\quad \tau\in[0,\tau_1]. \tag{3.40}$$

Proof. Multiply equation (3.39) by $z_{\tau\tau}$, integrate the resulting equation with respect to the spatial and temporal variables and use (1.2) and (2.6) to get

$$\frac{1}{2}\int\left(z_{\tau\tau}^2+|\lambda_1\lambda_2|z_{\tau y}^2\right)(y,s)dy\Big|_{s=0}^{\tau}+\int_0^\tau\int|\Phi_v|z_{\tau\tau}^2 dyd\tau$$
$$=\int_0^\tau\int\Big[(\lambda_1\lambda_2)_y z_{\tau\tau}z_{\tau y}-\frac{1}{2}(\lambda_1\lambda_2)_\tau z_{\tau y}^2+(\text{RHS})z_{\tau\tau}\Big]dyd\tau.$$

With the aid of (3.33), we apply the techniques used in the proof of Lemma 3.6 to obtain

$$\int\left(z_{\tau\tau}^2+z_{\tau y}^2\right)(y,\tau)dy+\int_0^\tau\int z_{\tau\tau}^2 dyd\tau$$
$$\lesssim(\epsilon/\delta)^{1/2}+(\epsilon^\kappa+\mathcal{E}_0)\int_0^\tau\|(z_{\tau y},z_{yy},m_{\tau\tau},m_{\tau y})(\cdot,\tau)\|^2 d\tau. \tag{3.41}$$

Multiply Equation (3.39) by z_τ, integrate the resulting equation with respect to the spatial and temporal variables and use (1.2), (2.6) and (3.6) to get

$$\int\left(\frac{1}{2}|\Phi_v|z_\tau^2+zz_{\tau\tau}\right)(y,\tau)dy+\int_0^\tau\int|\lambda_1\lambda_2|z_{\tau y}^2 dyd\tau$$
$$=\int_0^\tau\int\Big[z_{\tau\tau}^2+(\lambda_1\lambda_2)_y z_\tau z_{\tau y}-\frac{1}{2}\Phi_{v\tau}z_\tau^2+(\text{RHS})z_\tau\Big]dyd\tau.$$

With the aid of (3.33) and (3.41), we can show

$$\int\left(z_{\tau\tau}^2+z_{\tau y}^2\right)(y,\tau)dy+\int_0^\tau\int\left(z_{\tau\tau}^2+z_{\tau y}^2\right)dyd\tau$$
$$\lesssim\mathcal{E}_0\|(z_{yy},m_{\tau y},m_{yy})(\cdot,\tau)\|^2+(\epsilon/\delta)^{1/2}+(\epsilon^\kappa+\mathcal{E}_0)\int_0^\tau\|(z_{yy},m_{\tau\tau},m_{\tau y},m_{yy})(\cdot,\tau)\|^2 d\tau. \tag{3.42}$$

It follows from (2.16) and (2.6) that

$$z_{yy}=(\lambda_1\lambda_2)^{-1}\Big\{\Phi_v\Big[z_\tau+(\lambda_* z)_y\Big]-[z_{\tau\tau}+(\lambda_1+\lambda_2)z_{y\tau}]\Big\}$$
$$+(\lambda_1\lambda_2)^{-1}\Big[J_{1\tau}+g_v J_{1y}-\Phi_v J_1-\Phi_v\left(\Phi_v^{-1}f_v J_2\right)_y+J_3+J_4+J_5\Big].$$

This, together with (3.42) and (3.33), implies

$$
\begin{aligned}
&\|(z_{yy},z_{\tau y},z_{\tau\tau})(\cdot,\tau)\|^2+\int_0^\tau \|(z_{yy},z_{\tau y},z_{\tau\tau})(\cdot,\tau)\|^2 d\tau\\
\lesssim&\mathcal{E}_0\|(m_{\tau y},m_{yy})(\cdot,\tau)\|^2 \qquad +(\epsilon/\delta)^{1/2}+(\epsilon^\kappa+\mathcal{E}_0)\int_0^\tau \|(m_{\tau\tau},m_{\tau y},m_{yy})(\cdot,\tau)\|^2 d\tau.
\end{aligned}
$$

Similarly, we can get the estimates for m_{yy} and $m_{y\tau}$ by taking ∂_y and ∂_τ on $(2.18)_1$, respectively. With all these estimates at hand, we can bound $m_{\tau\tau}$ by use of (3.34). Finally, we obtain (3.40) and finish the proof of Lemma 3.8. ∎

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