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GRAVITY WATER FLOWS WITH DISCONTINUOUS VORTICITY AND STAGNATION POINTS*

CALIN IULIAN MARTIN† AND BOGDAN–VASILE MATIOC‡

Abstract. We construct small-amplitude steady periodic gravity water waves arising as the free surface of water flows that contain stagnation points and possess a discontinuous distribution of vorticity in the sense that the flows consist of two layers of constant but different vorticities. We also describe the streamline pattern in the moving frame for the constructed flows.

Key words. Irregular vorticity, stagnation points, gravity waves.

AMS subject classifications. 35J60, 76B03, 76B15, 47J15.

1. Introduction

We present here a study of steady periodic traveling water waves that propagate at the free surface of a two-dimensional inviscid and incompressible fluid of finite depth allowing for stagnation points and for a discontinuous distribution of vorticity. More precisely, we consider water waves interacting with two vertically superposed currents of different constant vorticities.

Confined first to the investigation of waves of small amplitude, which can be satisfactorily approximated by sinusoidal curves within the linear theory, the examination of periodic traveling water waves arising as the free surface of an irrotational flow with a flat bed originates in the beginning of the 19^{th} century. The description of waves that are flatter near the trough and have steeper elevations near the crest necessitates a nonlinear approach, which was in fact conducted within the last few decades and led to the first rigorous results concerning the existence of wave trains in irrotational flow, see, for instance, the case of Stokes waves [34] and the flow beneath them (particle trajectories and behavior of the pressure) cf. [2, 3, 6, 8].

To go beyond irrotational flows and to treat wave current interactions one needs to incorporate vorticity into the problem, cf. [4, 16, 32]. However, the difficulties generated by the presence of the vorticity have prevented a rigorous mathematical development, which appeared only relatively recently in [7], where the existence of small and large amplitude steady periodic gravity water waves with a general (continuous) vorticity distribution was proved.

Of high significance is the investigation of steady periodic rotational waves interacting with currents that possess a rough – that is discontinuous or unbounded – vorticity. Discontinuous vorticities model sudden changes in the underlying current: numerical simulations of such flows being quite recent [18, 19]. Unbounded vorticities on the other hand can describe turbulent flows in channels (see the empirical law on [1, p. 106]) and are also relevant in the setting of wind generated waves that possess a thin layer of high vorticity adjacent to the wave surface [30, 31]. The discontinuous vorticity distribution was considered in the groundbreaking paper [9], where the existence of steady two-dimensional periodic gravity water waves of small and large amplitude on water flows

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†Institut für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, 1090 Wien, Österreich (calin.martin@univie.ac.at).

‡Institut für Angewandte Mathematik, Leibniz Universität Hannover, Deutschland (matioc@ifam.uni-hannover.de).

with an arbitrary bounded (but discontinuous) vorticity was proved. Small amplitude capillary–gravity waves with discontinuous but bounded vorticity were constructed in [25, 29]. Waves with unbounded vorticity were first shown to exist in [26] but only when allowing for surface tension as a restoring force. This situation appears in many physical settings one of which being that of wind blowing over a still fluid surface and giving rise to two-dimensional small amplitude wave trains driven by capillarity [17] which grow larger and turn into capillary–gravity waves.

Another striking occurrence in water flows is the presence of stagnation points, that is points where the steady velocity field vanishes, thus making the analysis more intricate, since the usual Dubreil-Jacotin transform which converts the original free boundary problem into a problem in a fixed domain, is no longer applicable. There is a short list of papers dealing with existence of water flows allowing for stagnation points and for a non-vanishing continuous vorticity, cf. [10, 13, 14, 20, 35] for gravity waves and [22, 23, 24, 28] for waves with capillarity. Under consideration in this paper is a more involved setting where, in addition to permitting stagnation points (whose existence in the fluid is proven), we also allow for a discontinuous distribution of the vorticity. To our best knowledge the incorporation of both stagnation points and of a discontinuous vorticity is a feature that was not rigorously analyzed before.

The governing equations are the Euler equations of motion, together with boundary conditions on the free surface and on the flat bed of the water flow. The discontinuous vorticity that we consider here is of the following type: we assume that the flow has a layer of constant vorticity γ_2 adjacent to the free surface above another layer of constant vorticity γ_1 neighboring the flat bed. Of course, the interesting situation (that we pursue here) is when $\gamma_1 \neq \gamma_2$. The unknowns here are the free surface, the interface separating the regions of different vorticities (which can be seen as an internal wave due to the discontinuity in vorticity), the velocity field and the pressure function. In a first step we reduce the number of unknowns by means of the stream function whose utilization converts the problem into a transmission problem along the line of discontinuity of vorticity with fewer unknowns. The second step that we undertake is to consider a flattening transformation which has the advantage that changes the free boundary value problem into a problem in a fixed domain, thus making it more tractable for the analysis. For studying the latter resulting problem we employ the Crandall–Rabinowitz theorem on bifurcation from simple eigenvalues.

The dispersion relation that we obtain – which is a formula giving the speed at the free surface of the bifurcation inducing laminar flows in terms of the two vorticities γ_1, γ_2, the thickness of the two rotational layers and the wavelength – generalizes the one in [10] obtained in the case of a water flow with constant vorticity and allowing for stagnation points. The intricacy of the dispersion relation – a third order algebraic equation – allows us to prove existence of water waves of small wavelength arising as the free surface of water flows with rotational layers of different constant vorticities and containing stagnation points, cf. Theorems 3.4–3.6, 4.2, 4.3. We present also the streamline pattern -in the moving frame for the solutions, cf. figures 3.1-3.3. Our results especially show that the ratio of the amplitudes of the surface wave and that of the internal wave – and the fact that the surface wave and the internal wave are in phase or anti-phase – is highly influenced by the vorticities of the currents and by the speed at the free surface of the bifurcation inducing laminar flows.

We briefly outline the content of the paper. We present in Section 2 the governing equations together with the analytic setting we work within. Moreover, we also find the dispersion relation whose analysis is undertaken in Section 3 for the case $\gamma_2 > 0$, while the more singular case $\gamma_2=0$ is treated in Section 4. The appendix contains several technical lemmas.

2. The governing equations

Under consideration is a two-dimensional steady periodic flow, moving under the influence of gravity, such that the surface waves propagate in the positive x-direction. The water flow occupies the domain Ω bounded below by the flat bed $y=-d$, with $d>0$, and above by the free surface $y=h(x)$, which is a small perturbation of the flat free surface $y=0$. In a reference frame moving with the wave speed $c>0$, the equations of motion in Ω are Euler's equations

$$\begin{cases} (u-c)u_x+vu_y = -P_x, \\ (u-c)v_x+vv_y = -P_y-g, \\ u_x+v_y = 0, \end{cases} \tag{2.1a}$$

where (u,v) denotes the velocity field, P stands for pressure and g is the gravity constant. The equations of motion are supplemented by the boundary conditions, which, ignoring surface tension effects, are

$$\begin{cases} P = P_0 & \text{on } y=h(x), \\ v = (u-c)h' & \text{on } y=h(x), \\ v=0 & \text{on } y=-d, \end{cases} \tag{2.1b}$$

with P_0 being the constant atmospheric pressure.

We are interested in solutions to the problem (2.1), for which the vorticity $\omega:=u_y-v_x$ of the flow presents discontinuities of the following type: we assume that, adjacent to the free surface, the water flow possesses a layer

$$\Omega(f,h):=\{(x,y): x\in\mathbb{R}, -d_2+f(x)<y<h(x)\},$$

of constant vorticity γ_2, situated above another layer

$$\Omega(f):=\{(x,y): x\in\mathbb{R}, -d<y<-d_2+f(x)\},$$

which is adjacent to the flat bed and is of constant vorticity γ_1, that is

$$\omega:=\begin{cases} \gamma_1, & \text{in } \Omega(f), \\ \gamma_2, & \text{in } \Omega(f,h). \end{cases} \tag{2.1c}$$

We assume throughout the text that $\gamma_1\neq\gamma_2$ and that $d_2>0$, $d-d_2=:d_1>0$. We note that, additionally to (u,v,P,h), we have a further unknown: the function f whose graph separates the two currents of different vorticities. By Helmholtz's law, the vorticity is constant along streamlines of the steady flow, and as a consequence of this $y=-d_2+f$ has to be a streamline of the flow. This streamline can be viewed as an internal wave due to the jump in vorticity.

With the help of the stream function ψ, introduced (up to an additive constant) via the relation $\nabla\psi=(-v,u-c)$ we can reformulate (2.1a)–(2.1c) as the free-boundary problem

$$\begin{cases} \Delta\psi_2=\gamma_2 & \text{in } \Omega(f,h), \\ \Delta\psi_1=\gamma_1 & \text{in } \Omega(f), \\ \psi_2=0 & \text{on } y=h(x), \\ \psi_2=\psi_1 & \text{on } y=-d_2+f(x), \\ \psi_1=m & \text{on } y=-d, \end{cases} \tag{2.2a}$$

subjected to the conditions

$$\begin{cases} \partial_y \psi_2 = \partial_y \psi_1 & \text{on } y = -d_2 + f(x), \\ |\nabla \psi_2|^2 + 2g(d+h) = Q & \text{on } y = h(x), \end{cases} \tag{2.2b}$$

where the constant $-m$ represents the relative mass flux and $Q \in \mathbb{R}$ is related to the hydraulic head. Moreover, $\psi_1 := \psi\big|_{\Omega(f)}$ and $\psi_2 := \psi\big|_{\Omega(f,h)}$, so that from the fourth equation of (2.2a) and the first equation of (2.2b) we see that the $\nabla\psi$ (hence also the velocity field) is continuous across the interface $y = -d_2 + f(x)$.

Given $\alpha \in (0,1)$, it is not difficult to see that any solution

$$((f,h),\psi_1,\psi_2) \in \left(C^{3+\alpha}_{per}(\mathbb{R})\right)^2 \times C^{3+\alpha}_{per}\left(\overline{\Omega(f)}\right) \times C^{3+\alpha}_{per}\left(\overline{\Omega(f,h)}\right)$$

of (2.2) defines a solution

$$\begin{aligned} &(u,v,P,(f,h)) \in \left(C^{1-}_{per}(\overline{\Omega})\right)^3 \times \left(C^{3+\alpha}_{per}(\mathbb{R})\right)^2 \\ &\left((u,v)\big|_{\Omega(f)}, (u,v)\big|_{\Omega(f,h)}\right) \in \left(C^{2+\alpha}_{per}\left(\overline{\Omega(f)}\right)\right)^2 \times \left(C^{2+\alpha}_{per}\left(\overline{\Omega(f,h)}\right)\right)^2 \\ &\left(P\big|_{\Omega(f)}, P\big|_{\Omega(f,h)}\right) \in C^{2+\alpha}_{per}\left(\overline{\Omega(f)}\right) \times C^{2+\alpha}_{per}\left(\overline{\Omega(f,h)}\right) \end{aligned}$$

of (2.1). The subscript *per* stands for functions that are periodic in the horizontal variable, meaning that all the functions considered above are L-periodic with respect to x, with $L > 0$ being fixed.

We first determine laminar flow solutions to problem (2.2), that is water flows with a flat free surface and parallel streamlines, meaning that they present no x-dependence. Of interest are laminar flows that contain stagnation points, more precisely laminar flows that contain streamlines consisting entirely of stagnation points. Then we study when non-laminar solutions bifurcate from the laminar flows and describe the qualitative picture of the streamline pattern for the bifurcating solutions.

Laminar flow solutions. Because the stream function is constant along the streamline $y = -d_2 + f(x)$, we use the value of the stream function

$$\psi_1 = \psi_2 = \lambda \qquad \text{on } y = -d_2 + f(x), \tag{2.3}$$

to parametrize a family of laminar solutions to (2.2a). Setting $f \equiv h \equiv 0$ we obtain from (2.2a) that the stream function $\psi^0 := (\psi_1^0, \psi_2^0)$ satisfies

$$\begin{aligned} \psi_1^0(y) &= \frac{\gamma_1 y^2}{2} + \left(\frac{\gamma_1(d+d_2)}{2} + \frac{\lambda - m}{d_1}\right) y + \frac{\lambda d}{d_1} + \frac{\gamma_1 d d_2}{2} - \frac{m d_2}{d_1}, \quad y \in [-d, -d_2] \\ \psi_2^0(y) &= \frac{\gamma_2 y^2}{2} + \left(\frac{\gamma_2 d_2}{2} - \frac{\lambda}{d_2}\right) y, \quad y \in [-d_2, 0]. \end{aligned}$$

The equations of (2.2b) are equivalent to

$$m = \frac{\lambda d}{d_2} + d_1 \frac{\gamma_1 d_1 + \gamma_2 d_2}{2}, \qquad Q = \left(\frac{\gamma_2 d_2}{2} - \frac{\lambda}{d_2}\right)^2 + 2gd. \tag{2.4}$$

In the following we choose the constants m and Q in (2.2a) and (2.2b) to be given by (2.4), the constant λ introduced via (2.3) being left as a parameter. Hence, each $\lambda \in \mathbb{R}$ determines a unique laminar solution $((f,h),\psi_1,\psi_2) := (0, \psi_1^0, \psi_2^0)$ of (2.2) when m and Q are defined by (2.4). These are the laminar solutions, from which we study bifurcation.

Conditions for stagnation. We note that the laminar flows determined above possess stagnation points – that is water particles that travel horizontally with the wave speed – if and only if

$$\partial_y\psi_2^0(-d_2)\cdot\partial_y\psi_2^0(0)\le 0 \qquad \text{or} \qquad \partial_y\psi_1^0(-d)\cdot\partial_y\psi_1^0(-d_2)\le 0. \tag{2.5}$$

If (2.5) holds true, then there exists $y_0\in[-d,0]$ such that

$$\partial_y\psi_i^0(y_0)=0 \quad \text{for} \quad i=1 \,\text{or}\, 2.$$

The streamline $y=y_0$ consists only of stagnation points, and we expect that the solutions to (2.1) that bifurcate from these laminar solutions possess stagnation points too, cf. [12, 35]. The first inequality ensures stagnation in the layer adjacent to the wave surface, and is equivalent to

$$\Lambda(\Lambda-\gamma_2 d_2)\le 0, \tag{2.6}$$

respectively the condition for stagnation in the bottom layer is

$$(\Lambda-\gamma_2 d_2)(\Lambda-\gamma_1 d_1-\gamma_2 d_2)\le 0. \tag{2.7}$$

Hereby, we set

$$\Lambda:=\frac{\gamma_2 d_2}{2}-\frac{\lambda}{d_2}. \tag{2.8}$$

The constant Λ has a physical interpretation: it is the relative horizontal speed at the free surface for the laminar flow determined by λ, that is $\Lambda=\partial_y\psi_2^0\big|_{y=0}$. For this reason we define λ via (2.8) and use Λ as parameter.

The functional analytic setting. With Λ as parameter, we are left to seek special values of Λ such that branches of non-laminar solutions to (2.2) bifurcate from the curve of laminar flows. For this, we need to recast (2.2) in a suitable analytic setting.

In the following $\alpha\in(0,1)$ is a fixed Hölder exponent. Because the equations of (2.2a) and (2.2b) are posed on manifolds that depend on the unknown functions (f,h), it is suitable to transform the problem (2.2) on fixed manifolds. For this, we set $\Omega_1:=\Omega(0)$, $\Omega_2:=\Omega(0,0)$ and define the mappings

$$\Phi_f:\Omega_1\to\Omega(f), \quad \Phi_f(x,y)=\left(x,\frac{d_1+f(x)}{d_1}y+\frac{d}{d_1}f(x)\right),$$
$$\Phi_{(f,h)}:\Omega_2\to\Omega(f,h), \quad \Phi_{(f,h)}(x,y)=\left(x,\frac{h(x)-f(x)+d_2}{d_2}y+h(x)\right).$$

It is easy to see that Φ_f and $\Phi_{(f,h)}$ are $C^{3+\alpha}$-diffeormorphisms for each $(f,h)\in\mathcal{O}$, by which

$$\mathcal{O}:=\{(f,h)\in\left(C^{3+\alpha}_{e,per}(\mathbb{R})\right)^2: -d<-d_2+f<h\},$$

the subscript e referring to the fact that we consider only even function in x. Using these diffeomorphisms, we define the linear elliptic operators

$$\mathcal{A}(f):C^{3+\alpha}_{e,per}(\overline{\Omega}_1)\to C^{1+\alpha}_{e,per}(\overline{\Omega}_1), \quad \mathcal{A}(f)w_1:=\Delta(w_1\circ\Phi_f^{-1})\circ\Phi_f,$$
$$\mathcal{A}(f,h):C^{3+\alpha}_{e,per}(\overline{\Omega}_2)\to C^{1+\alpha}_{e,per}(\overline{\Omega}_2), \quad \mathcal{A}(f,h)w_2:=\Delta(w_2\circ\Phi_{(f,h)}^{-1})\circ\Phi_{(f,h)},$$

and the boundary operators

$$\mathcal{B}_1 : \mathbb{R}\times\mathcal{O}\times C^{3+\alpha}_{e,per}(\overline{\Omega}_2)\to C^{2+\alpha}_{e,per}(\mathbb{R}),$$
$$\mathcal{B}_2 : \mathbb{R}\times\mathcal{O}\times C^{3+\alpha}_{e,per}(\overline{\Omega}_1)\times C^{3+\alpha}_{e,per}(\overline{\Omega}_2)\to C^{2+\alpha}_{e,per}(\mathbb{R}),$$

respectively through

$$\mathcal{B}_1(\Lambda,(f,h),w_2) := \Big(|\nabla(w_2\circ\Phi^{-1}_{(f,h)})|^2\circ\Phi_{(f,h)} + 2g(d+h) - Q\Big)\Big|_{y=0},$$
$$\mathcal{B}_2(\Lambda,(f,h))[w_1,w_2] := \Big[\big(\partial_y(w_2\circ\Phi^{-1}_{(f,h)})\big)\circ\Phi_{(f,h)} - \big(\partial_y(w_1\circ\Phi^{-1}_{f})\big)\circ\Phi_{f}\Big]\Big|_{y=-d_2}.$$

REMARK 2.1. Let $\Lambda\in\mathbb{R}$, $((f,h),\psi_1,\psi_2)\in\mathcal{O}\times C^{3+\alpha}_{per}\big(\overline{\Omega(f)}\big)\times C^{3+\alpha}_{per}\big(\overline{\Omega(f,h)}\big)$, and assume that λ,m,Q are defined by (2.3), (2.4), and (2.8). Then, the tuple $((f,h),\psi_1,\psi_2)$ solves problem (2.2) if and only if

(*i*) $w_1 := \psi_1\circ\Phi_f \in C^{3+\alpha}_{e,per}(\overline{\Omega}_1)$ is the unique solution of the Dirichlet problem

$$\begin{cases} \mathcal{A}(f)w_1 = \gamma_1 & \text{in } \Omega_1,\\ w_1 = \lambda & \text{on } y=-d_2,\\ w_1 = m & \text{on } y=-d. \end{cases} \tag{2.9}$$

(*ii*) $w_2 := \psi_2\circ\Phi_{(f,h)} \in C^{3+\alpha}_{e,per}(\overline{\Omega}_2)$ is the unique solution of the Dirichlet problem

$$\begin{cases} \mathcal{A}(f,h)w_2 = \gamma_2 & \text{in } \Omega_2,\\ w_2 = 0 & \text{on } y=0,\\ w_2 = \lambda & \text{on } y=-d_2. \end{cases} \tag{2.10}$$

(*iii*) $\mathcal{B}_1(\Lambda,(f,h),w_2) = \mathcal{B}_2(\Lambda,(f,h))[w_1,w_2] = 0$ in $C^{2+\alpha}_{e,per}(\mathbb{R})$.

Thanks to Remark 2.1 we can recast the problem (2.2) as a nonlinear and nonlocal equation with $(\Lambda,(f,h))$ as unknown. In order to proceed, we first establish the following result.

LEMMA 2.2. *Given* $(\Lambda,(f,h))\in\mathbb{R}\times\mathcal{O}$, *we let* $w_1 := w_1(\Lambda,(f,h))$ *and* $w_2 := w_2(\Lambda,(f,h))$ *denote the unique solution of* (2.9) *and* (2.10), *respectively, with* λ *given by* (2.8). *Then, we have* $w_i\in C^\omega\big(\mathbb{R}\times\mathcal{O}, C^{3+\alpha}_{e,per}(\overline{\Omega}_i)\big), i=1,2.$

Proof. We prove just the real-analyticity of the solution operator w_1, the claim for w_2 following similarly. By elliptic theory, see [15], we see that $w_1 : \mathbb{R}\times\mathcal{O}\to C^{3+\alpha}_{e,per}(\overline{\Omega}_1)$ is well-defined. Moreover, we have that

$$\mathcal{F}(\Lambda,(f,h),w_1(\Lambda,(f,h))) = 0 \qquad \text{for all } (\Lambda,(f,h))\in\mathbb{R}\times\mathcal{O},$$

thus $\mathcal{F}\in C^\omega\big(\mathbb{R}\times\mathcal{O}\times C^{3+\alpha}_{e,per}(\overline{\Omega}_1), C^{1+\alpha}_{e,per}(\overline{\Omega}_1)\times(C^{3+\alpha}_{e,per}(\mathbb{R}))^2\big)$ is the operator defined by

$$\mathcal{F}(\Lambda,(f,h),w_1) := (\mathcal{A}(f)w_1-\gamma_1, w_1\big|_{y=-d_2}, w_1\big|_{y=-d}).$$

Taking into account that Fréchet derivative

$$\partial_{w_1}\mathcal{F}(\Lambda,(f,h),w_1((\Lambda,(f,h))))[z] := (\mathcal{A}(f)z, z\big|_{y=-d_2}, z\big|_{y=-d})$$

is an isomorphism, the assertion follows from the implicit function theorem. ∎

Because $\mathcal{B}_i, i=1,2$, depend real-analytically on their arguments too, we obtain from Lemma 2.2 and Remark 2.1 that the problem (2.2) is equivalent to the nonlinear and nonlocal equation

$$\Phi(\Lambda,(f,h))=0, \tag{2.11}$$

by which $\Phi:=(\Phi_1,\Phi_2)\in C^\omega\big(\mathbb{R}\times\mathcal{O},\big(C^{2+\alpha}_{e,per}(\mathbb{R})\big)^2\big)$ is the operator defined by

$$\Phi(\Lambda,(f,h)):=(\mathcal{B}_1(\Lambda,(f,h),w_2(\Lambda,(f,h))),\mathcal{B}_2(\Lambda,(f,h))[w_1(\Lambda,(f,h)),w_2(\Lambda,(f,h))]). \tag{2.12}$$

The laminar flow solutions to (2.2) correspond to the trivial solutions $(\Lambda,0)\in\mathbb{R}\times\mathcal{O}$ of (2.11). In order to find other solutions, we use the theorem on bifurcations from simple eigenvalues due to Crandall and Rabinowitz [11].

Theorem 2.3 (Crandall and Rabinowitz). *Let $\mathbb{X},\mathbb{Y}$ be real Banach spaces and let the mapping $\Phi\in C^\omega(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ satisfy:*

(a) $\Phi(\Lambda,0)=0$ for all $\Lambda\in\mathbb{R}$;

(b) There exists $\Lambda_*\in\mathbb{R}$ such that Fréchet derivative $\partial_x\Phi(\Lambda_*,0)$ is a Fredholm operator of index zero with a one-dimensional kernel and

$$\operatorname{Ker}\partial_x\Phi(\Lambda_*,0)=\operatorname{span}\{x_0\} \qquad \text{with } 0\neq x_0\in\mathbb{X};$$

(c) The transversality condition

$$\partial_{\Lambda x}\Phi(\Lambda_*,0)[x_0]\notin\operatorname{Im}\partial_x\Phi(\Lambda_*,0).$$

Then, $(\Lambda_*,0)$ is a bifurcation point in the sense that there exists $\varepsilon>0$ and a real-analytic curve $(\Lambda,x):(-\varepsilon,\varepsilon)\to\mathbb{R}\times\mathbb{X}$ consisting only of solutions to the equation $\Phi(\Lambda,x)=0$. Moreover, as $s\to0$, we have that

$$\Lambda(s)=\Lambda_*+O(s) \qquad \text{and} \qquad x(s)=sx_0+O(s^2).$$

Furthermore, there exists an open set $U\subset\mathbb{R}\times\mathbb{X}$ with $(\Lambda_*,0)\in U$ and

$$\{(\Lambda,x)\in U:\Phi(\Lambda,x)=0,x\neq0\}=\{(\Lambda(s),x(s)):0<|s|<\varepsilon\}.$$

In order to apply this abstract bifurcation result, we need to compute the Fréchet derivative of the operator Φ. To this end we state the following lemma.

Lemma 2.4. *Let $\Lambda\in\mathbb{R}$ be given. The Fréchet derivative $\partial_{(f,h)}\Phi(\Lambda,0)$ is the matrix operator*

$$\partial_{(f,h)}\Phi(\Lambda,0)=\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}\in\mathcal{L}\big(\big(C^{3+\alpha}_{e,per}(\mathbb{R})\big)^2,\big(C^{2+\alpha}_{e,per}(\mathbb{R})\big)^2\big).$$

Given $1\le i,j\le2$, the operator $A_{ij}\in\mathcal{L}\big(C^{3+\alpha}_{e,per}(\mathbb{R}),C^{2+\alpha}_{e,per}(\mathbb{R})\big)$ is the Fourier multiplier with symbol $m^{ij}(\Lambda):=(m^{ij}_k(\Lambda))_{k\in\mathbb{N}}$ defined by

$$m^{11}_k(\Lambda)=-2\Lambda(\gamma_2d_2-\Lambda)\frac{R_k}{\sinh(R_kd_2)}, \tag{2.13}$$

$$m^{12}_k(\Lambda)=2\Big[g+\gamma_2\Lambda-\Lambda^2\frac{R_k}{\tanh(R_kd_2)}\Big], \tag{2.14}$$

$$m_k^{21}(\Lambda)=\gamma_2-\gamma_1+(\Lambda-\gamma_2 d_2)\Big[\frac{R_k}{\tanh(R_k d_1)}+\frac{R_k}{\tanh(R_k d_2)}\Big], \tag{2.15}$$

$$m_k^{22}(\Lambda)=-\Lambda\frac{R_k}{\sinh(R_k d_2)} \tag{2.16}$$

for $k\in\mathbb{N}$, *by which* $R_k:=2k\pi/L$. *For* $k=0$ *the right-hand side of* (2.13)-(2.16) *should be understood as the limit of the expressions when letting* $R_k\to 0$.

Proof. See Appendix. ∎

With the help of Lemma 2.4 we are now able to determine when the Fréchet derivative $\partial_{(f,h)}\Phi(\Lambda,0)$ is a Fredholm operator.

LEMMA 2.5. *Let* $\Lambda\in\mathbb{R}$ *be given. We have:*

(*i*) *If* $\Lambda\in\{0,\gamma_2 d_2\}$, *then* $\partial_{(f,h)}\Phi(\Lambda,0)$ *is not a Fredholm operator.*

(*ii*) *If* $\Lambda\notin\{0,\gamma_2 d_2\}$, *then* $\partial_{(f,h)}\Phi(\Lambda,0)$ *is a Fredholm operator of index zero.*

Proof. In order to prove (*i*), we infer from (2.13) and (2.14) that for $\Lambda=0$ we have

$$\partial_{(f,h)}\Phi_1(\Lambda,0)[(f,h)]=2gh \qquad \text{for all } (f,h)\in\big(C^{3+\alpha}_{e,per}(\mathbb{R})\big)^2,$$

meaning that $\operatorname{Im}\partial_{(f,h)}\Phi_1(\Lambda,0)=C^{3+\alpha}_{e,per}(\mathbb{R})$. Since $C^{3+\alpha}_{e,per}(\mathbb{R})$ is not a closed subspace of $C^{2+\alpha}_{e,per}(\mathbb{R})$, the assertion is evident. Furthermore, if $\Lambda=\gamma_2 d_2\neq 0$, then

$$\partial_{(f,h)}\Phi_2(\Lambda,0)[(f,h)]=(\gamma_2-\gamma_1)f+\mathcal{K}h \qquad \text{for all } (f,h)\in\big(C^{3+\alpha}_{e,per}(\mathbb{R})\big)^2,$$

by which

$$\mathcal{K}\sum_{k\in\mathbb{N}}b_k\cos(R_k x)=-\Lambda\sum_{k\in\mathbb{N}}\frac{R_k}{\sinh(R_k d_2)}b_k\cos(R_k x).$$

It is easy to see that $\mathcal{K}\big(C^{3+\alpha}_{e,per}(\mathbb{R})\big)\subset C^{\infty}_{e,per}(\mathbb{R})$, hence $\operatorname{Im}\partial_{(f,h)}\Phi_2(\Lambda,0)=C^{3+\alpha}_{e,per}(\mathbb{R})$. Therefore, $\operatorname{Im}\partial_{(f,h)}\Phi(\Lambda,0)$ is not a closed subspace of $\big(C^{2+\alpha}_{e,per}(\mathbb{R})\big)^2$. This proves (*i*).

To prove (*ii*), we choose $\Lambda\notin\{0,\gamma_2 d_2\}$ and set

$$D(k,\Lambda):=m_k^{11}(\Lambda)m_k^{22}(\Lambda)-m_k^{12}(\Lambda)m_k^{21}(\Lambda), \qquad k\in\mathbb{N}.$$

From (2.13)–(2.16) it is clear that there exists $k_0\in\mathbb{N}$ such that $D(k,\Lambda)\neq 0$ for all $k\geq k_0$. Defining the symbols $\widetilde{m}^{ij}(\Lambda)$ by $\widetilde{m}^{ij}_k(\Lambda)=m^{ij}_k(\Lambda)$ for $k\geq k_0$ and $1\leq i,j\leq 2$ and

$$\widetilde{m}^{11}_k(\Lambda)=\widetilde{m}^{22}_k(\Lambda)=1, \quad \widetilde{m}^{12}_k(\Lambda)=\widetilde{m}^{21}_k(\Lambda)=0 \qquad \text{for } 0\leq k\leq k_0-1,$$

we see that $\partial_{(f,h)}\Phi(\Lambda,0)$ is a compact perturbation of the operator

$$T:=\begin{pmatrix}\widetilde{A}_{11} & \widetilde{A}_{12}\\ \widetilde{A}_{21} & \widetilde{A}_{22}\end{pmatrix}\in\mathcal{L}\Big(\big(C^{3+\alpha}_{e,per}(\mathbb{R})\big)^2,\big(C^{2+\alpha}_{e,per}(\mathbb{R})\big)^2\Big),$$

where $\widetilde{A}_{ij}\in\mathcal{L}\big(C^{3+\alpha}_{e,per}(\mathbb{R}),C^{2+\alpha}_{e,per}(\mathbb{R})\big)$ is the Fourier multiplier with $\widetilde{m}^{ij}(\Lambda)$, $1\leq i,j\leq 2$. Because $\widetilde{D}(k,\Lambda):=\widetilde{m}^{11}_k(\Lambda)\widetilde{m}^{22}_k(\Lambda)-\widetilde{m}^{12}_k(\Lambda)\widetilde{m}^{21}_k(\Lambda)\neq 0$ for all $k\in\mathbb{N}$, we can define the formal inverse of T by

$$S:=\begin{pmatrix}\widetilde{B}_{11} & \widetilde{B}_{12}\\ \widetilde{B}_{21} & \widetilde{B}_{22}\end{pmatrix}.$$

Here, $\widetilde{B}_{ij}$ is the Fourier multiplier corresponding to the symbol b^{ij}, $1\le i,j\le 2$, by which

$$b_k^{11} := \frac{\widetilde{m}_k^{22}(\Lambda)}{\widetilde{D}(k,\Lambda)}, \quad b_k^{12} := -\frac{\widetilde{m}_k^{12}(\Lambda)}{\widetilde{D}(k,\Lambda)}, \quad b_k^{21} := -\frac{\widetilde{m}_k^{21}(\Lambda)}{\widetilde{D}(k,\Lambda)}, \quad b_k^{22} := \frac{\widetilde{m}_k^{11}(\Lambda)}{\widetilde{D}(k,\Lambda)} \qquad \text{for } k\in\mathbb{N}.$$

Using now [21, Theorem 2.1], we see that a Fourier multiplier

$$\sum_{k\in\mathbb{N}} \alpha_k \cos(R_k x) \to \sum_{k\in\mathbb{N}} \lambda_k \alpha_k \cos(R_k x)$$

belongs to $\mathcal{L}\big(C^{2+\alpha}_{e,per}(\mathbb{R}), C^{3+\alpha}_{e,per}(\mathbb{R})\big)$ if

$$\sup_{k\in\mathbb{N}} |k\lambda_k| < \infty \qquad \text{and} \qquad \sup_{k\in\mathbb{N}} k^2|\lambda_{k+1}-\lambda_k| < \infty.$$

Because of this, it is a matter of direct computation to see that the operators $\widetilde{B}_{ij}$ belong to $\mathcal{L}\big(C^{2+\alpha}_{e,per}(\mathbb{R}), C^{3+\alpha}_{e,per}(\mathbb{R})\big)$ for all $1\le i,j\le 2$. Hence, T is an isomorphism, and therefore $\partial_{(f,h)}\Phi(\Lambda,0)$ is a Fredholm operator of index zero. ∎

Because of Lemma 2.5 (i) it is clear that we cannot apply the Crandall–Rabinowitz bifurcation theorem at $(\Lambda,0)$ with $\Lambda\in\{0,\gamma_2 d_2\}$. As a consequence of this, the laminar flows from which we show that non-laminar waves bifurcate will not possess stagnation points at the wave surface or on the interface separating the two layers of constant vorticity, cf. (2.6), (2.7), but only inside the layers. This is different from the case of internal waves propagating between two layers of constant but different density, where in the presence of capillarity stagnation points may be located also on the internal wave, cf. [27].

It is now evident that potential bifurcation values for $\Lambda\notin\{0,\gamma_2 d_2\}$ are to be looked for among the solutions to

$$D(k,\Lambda)=0 \tag{2.17}$$

for some integer $k\ge 1$. Since in Theorem 2.3 the Fréchet derivative $\partial_{(f,h)}\Phi(\Lambda,0)$ needs to be a Fredholm operator of index zero with a one-dimensional kernel, we need to find Λ such that (2.17) has exactly one root $1\le k\in\mathbb{N}$. Plugging the expressions (2.13)-(2.16) in (2.17), we rediscover the dispersion relation

$$\begin{aligned}
\Lambda^3 &- \frac{1}{R_k}\Big[\gamma_2\Big(R_k d_2 + \frac{\sinh(R_k d_2)\cosh(R_k d_1)}{\cosh(R_k d)}\Big) + \gamma_1 \frac{\sinh(R_k d_1)\cosh(R_k d_2)}{\cosh(R_k d)}\Big]\Lambda^2 \\
&+ \tanh(R_k d)\Big[\frac{\gamma_2^2 d_2 - g}{R_k} + \gamma_2(\gamma_1-\gamma_2)\frac{\sinh(R_k d_1)\sinh(R_k d_2)}{R_k^2\sinh(R_k d)}\Big]\Lambda \\
&+ g\frac{\tanh(R_k d)}{R_k^2}\Big[(\gamma_1-\gamma_2)\frac{\sinh(R_k d_1)\sinh(R_k d_2)}{\sinh(R_k d)} + \gamma_2 d_2 R_k\Big] = 0,
\end{aligned} \tag{2.18}$$

found also in [25, Equation (5.11)] (with $\sigma=0$). This relation has been analyzed in the setting of flows without stagnation points in [5] for $\gamma_1\neq 0=\gamma_2$ and in [9] for $\gamma_1=0\neq\gamma_2$. Herein, we assume only that $\gamma_1\neq\gamma_2$ and restrict the analysis to the complementary case when stagnation points are included.

In studying the dispersion relation (2.18) we will make use of the following remark, which allows us to restrict our attention to a few of relevant cases, the remaining ones being analogous.

REMARK 2.6. Note that (2.18) possesses the following symmetry property: k is a solution to (2.18) for some $\Lambda \notin \{0, \gamma_2 d_2\}$ and $(\gamma_1, \gamma_2) \in \mathbb{R}^2$ if and only if k is a solution of (2.18) for $-\Lambda \notin \{0, -\gamma_2 d_2\}$ and $(-\gamma_1, -\gamma_2) \in \mathbb{R}^2$. Because additionally the inequalities (2.6) and (2.7) are invariant under the transformation $(\Lambda, (\gamma_1, \gamma_2)) \mapsto (-\Lambda, (-\gamma_1, -\gamma_2))$, we are left only with the two cases:

(*i*) $\gamma_2 > 0$ and $\gamma_1 \neq \gamma_2$;

(*ii*) $\gamma_2 = 0$ and $\gamma_1 < 0$.

3. Analysis of the dispersion relation: the case $\gamma_2 > 0$ and $\gamma_1 \neq \gamma_2$

Because the dispersion relation is highly nonlinear in k, the study of the roots of (2.18) when keeping Λ fixed seems to be very difficult. Therefore, we consider the inverse problem of determining the zeros Λ_1, Λ_2, Λ_3 of this cubic equation when keeping k fixed, and then study the properties of the mappings $k \mapsto \Lambda_i(k)$, $1 \le i \le 3$. We will do this for small wavelength L, because then we can use asymptotic expansions and Cardano's formula in order to determine the roots Λ_i of (2.18). This small wavelength regime corresponds to the setting $t \to \infty$, where

$$t := R_k = \frac{2\pi k}{L} \in \mathbb{R}.$$

Plugging in t for R_k, the equation (2.18) can be written in the more concise form

$$\Lambda^3 + A(t)\Lambda^2 + B(t)\Lambda + C(t) = 0. \tag{3.1}$$

We will show in the sequel that Equation (3.1) has three real roots when t is sufficiently large. To this end, we first note that the coefficient functions $A = A(t)$, $B = B(t)$, and $C = C(t)$ and their first derivatives have the following asymptotic expansions for $t \to \infty$:

$$\begin{aligned}
A &= -\gamma_2 d_2 - \frac{\gamma_1+\gamma_2}{2t} + o\Big(\frac{1}{t^2}\Big), & A' &= \frac{\gamma_1+\gamma_2}{2} \cdot \frac{1}{t^2} + o\Big(\frac{1}{t^3}\Big), \\
B &= \frac{\gamma_2^2 d_2 - g}{t} + \frac{\gamma_2(\gamma_1-\gamma_2)}{2t^2} + o\Big(\frac{1}{t^3}\Big), & B' &= -\frac{\gamma_2^2 d_2 - g}{t^2} - \frac{\gamma_2(\gamma_1-\gamma_2)}{t^3} + o\Big(\frac{1}{t^4}\Big), \\
C &= \frac{g\gamma_2 d_2}{t} + \frac{g(\gamma_1-\gamma_2)}{2t^2} + o\Big(\frac{1}{t^3}\Big), & C' &= -\frac{g\gamma_2 d_2}{t^2} - \frac{g(\gamma_1-\gamma_2)}{t^3} + o\Big(\frac{1}{t^4}\Big).
\end{aligned} \tag{3.2}$$

Letting $z := \Lambda + A/3$, we find that z solves the depressed cubic equation

$$z^3 + pz + q = 0, \tag{3.3}$$

with

$$\frac{p}{3} = \frac{B}{3} - \frac{A^2}{9} = -\frac{(\gamma_2 d_2)^2}{9} + \frac{\gamma_2 d_2(2\gamma_2 - \gamma_1) - 3g}{9t} + \frac{4\gamma_1\gamma_2 - 7\gamma_2^2 - \gamma_1^2}{36t^2} + o\Big(\frac{1}{t^3}\Big)$$

and

$$\begin{aligned}
\frac{q}{2} &= \frac{A^3}{27} - \frac{AB}{6} + \frac{C}{2} \\
&= -\frac{(\gamma_2 d_2)^3}{27} + \gamma_2 d_2 \frac{6g + \gamma_2 d_2(2\gamma_2 - \gamma_1)}{18t} - \Big[\gamma_2 d_2 \frac{\gamma_1^2 - \gamma_1\gamma_2 + \gamma_2^2}{36} + \frac{g(2\gamma_2 - \gamma_1)}{3}\Big]\frac{1}{t^2} \\
&\quad + \frac{9\gamma_2(\gamma_1^2 - \gamma_2^2) - (\gamma_1+\gamma_2)^3}{216}\frac{1}{t^3} + o\Big(\frac{1}{t^4}\Big).
\end{aligned}$$

Observe that the discriminant for (3.3) is

$$D := \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 = -\frac{9g(\gamma_2 d_2)^4}{243t} + O\left(\frac{1}{t^2}\right) < 0 \qquad \text{for } t \to \infty,$$

property which implies, cf. [33], that (3.3), and hence also (3.1), has three real roots. They are given by the relation $z = r\cos(\beta)$, which implies

$$r = \sqrt{\frac{-4p}{3}} = \frac{2\gamma_2 d_2}{3} - \frac{\gamma_2 d_2(2\gamma_2 - \gamma_1) - 3g}{3\gamma_2 d_2 t} + O\left(\frac{1}{t^2}\right) \tag{3.4}$$

and β is one of the solution of

$$\cos(3\beta) = -\frac{q}{2}\sqrt{-\frac{27}{p^3}} = 1 - \frac{3^3 g}{2(\gamma_2 d_2)^2 t} + O\left(\frac{1}{t^2}\right).$$

Thus, choosing $\beta := 3^{-1}\arccos\left(-(q/2)\sqrt{-27/p^3}\right)$ we see that $\beta(t) \searrow_{t\to\infty} 0$ and the roots of (3.1) are

$$\begin{aligned}
\Lambda_1 &= r\cos(\beta) - \frac{A}{3},\\
\Lambda_2 &= r\cos\left(\beta - \frac{2\pi}{3}\right) - \frac{A}{3} = -r\cos\left(\beta + \frac{\pi}{3}\right) - \frac{A}{3},\\
\Lambda_3 &= r\cos\left(\beta + \frac{2\pi}{3}\right) - \frac{A}{3} = -r\cos\left(\beta - \frac{\pi}{3}\right) - \frac{A}{3}.
\end{aligned} \tag{3.5}$$

Together with (3.2) and (3.4), it follows at once that for $t \to \infty$ we have

$$\Lambda_1(t) \to \gamma_2 d_2, \qquad \Lambda_2(t) \to 0, \qquad \Lambda_3(t) \to 0. \tag{3.6}$$

Let us also observe that since $C(t) > 0$ for $t \to \infty$, it must hold that $\Lambda_2(t)\Lambda_3(t) < 0$ for $t \to \infty$. Moreover, it is clear from (3.5) that $\Lambda_2(t) > \Lambda_3(t)$, hence

$$\Lambda_2(t) > 0 > \Lambda_3(t) \qquad \text{for } t \to \infty.$$

3.1. Existence of water flows bifurcating from Λ_1. In order to consider the bifurcation problem for (2.11), we need to first study the properties of the mapping $[t \mapsto \Lambda_1(t)]$.

LEMMA 3.1. *There is a constant $t_0 \geq 0$ such that the function*

$$[[t_0,\infty) \ni t \mapsto \Lambda_1(t) \in (0,\infty)]$$

is strictly monotone.

Proof. Let $\phi(t,\Lambda) := \Lambda^3 + A(t)\Lambda^2 + B(t)\Lambda + C(t)$ for $\Lambda \in \mathbb{R}$ and $t \geq 0$. Since for $t \to \infty$ we have

$$\phi_\Lambda(t,\Lambda_1(t)) = (\Lambda_1(t) - \Lambda_2(t))(\Lambda_1(t) - \Lambda_3(t)) > 0, \tag{3.7}$$

we conclude that Λ_1 is differentiable with respect to t. On the other hand

$$t^2\Phi_t(t,\Lambda_1(t)) = t^2\left(\Lambda_1^2(t)A'(t) + \Lambda_1(t)B'(t) + C'(t)\right) \xrightarrow[t\to\infty]{} \frac{(\gamma_2 d_2)^2(\gamma_1 - \gamma_2)}{2}.$$

Since $\Lambda_1'(t) = -\phi_t(t,\Lambda_1(t))/\phi_\Lambda(t,\Lambda_1(t))$, we see that Λ_1' has the same sign as $\gamma_2 - \gamma_1$. The constant t_0 is defined as $t_0 := \inf\{t > 0 : |\Lambda_1'| > 0 \text{ on } (t,\infty)\}$. ∎

From Lemma 3.1 it follows at once that

- if $\gamma_1 < \gamma_2$, then $\Lambda_1(t)$ satisfies (2.6) for $t \geq t_0$;
- if $\gamma_1 > \gamma_2$, then $\Lambda_1(t)$ satisfies (2.7) for $t \geq t_0$.

We look now for bifurcation solutions when choosing Λ_1 as the bifurcation point. Therefore, we choose $t_0 > 0$ in Lemma 3.1 large enough to guarantee additionally that

$$\begin{aligned} &\inf_{[t_0,\infty)} \Lambda_1^2 > \sup_{[t_0,\infty)} \left(\Lambda_2^2 + \Lambda_3^2\right), \\ &D(0,\Lambda_1(t)) \neq 0 \qquad \text{for all } t \geq t_0. \end{aligned} \tag{3.8}$$

Let

$$L_0 := 2\pi/t_0, \tag{3.9}$$

fix $L \leq L_0$, and set $\Lambda_1 := \Lambda_1(2\pi/L)$. Then, since $\phi(2\pi/L,\Lambda_1) = 0$, we get $D(1,\Lambda_1) = 0$. Due to the choice of t_0, the equation $D(\cdot,\Lambda_1) = 0$ has no solutions $k \in \mathbb{N}$ other than $k = 1$. Consequently, since $\Lambda_1 \notin \{0, \gamma_2 d_2\}$, the derivative $\partial_{(f,h)}\Phi_1(\Lambda_1,0)$ is a Fredholm operator with a one-dimensional kernel

$$\operatorname{Ker} \partial_{(f,h)}\Phi_1(\Lambda_1,0) = \operatorname{span}\left\{\left(m_1^{22}(\Lambda_1), -m_1^{21}(\Lambda_1)\right)\cos(2\pi x/L)\right\}. \tag{3.10}$$

In order to apply Theorem 2.3 to this particular setting, it remains to study whether the transversality condition is satisfied. To this end, we obtain the following characterization of $\operatorname{Im}\partial_{(f,h)}\Phi_1(\Lambda_1,0)$.

LEMMA 3.2. *Let L_0 be given by* (3.9), *$L \leq L_0$, and set $\Lambda_1 := \Lambda_1(2\pi/L)$. Then, we have*

$$\operatorname{Im}\partial_{(f,h)}\Phi(\Lambda_1,0) = \left\{(\xi,\eta) = \Big(\sum_{k\in\mathbb{N}} \xi_k \cos(R_k x), \sum_{k\in\mathbb{N}} \eta_k \cos(R_k x)\Big) : \xi_1 = \frac{m_1^{11}(\Lambda_1)}{m_1^{21}(\Lambda_1)}\eta_1\right\}. \tag{3.11}$$

Proof. To prove the claim, let $(f,h) = \left(\sum_{k\in\mathbb{N}} a_k \cos(R_k x), \sum_{k\in\mathbb{N}} b_k \cos(R_k x)\right)$ be such that $\partial_{(f,h)}\Phi(\Lambda_1,0)(f,h) = (\xi,\eta)$. Then, obviously

$$\begin{cases} m_1^{11}(\Lambda_1)a_1 + m_1^{12}(\Lambda_1)b_1 = \gamma_1, \\ m_1^{21}(\Lambda_1)a_1 + m_1^{22}(\Lambda_1)b_1 = \eta_1. \end{cases}$$

Because $D(1,\Lambda_1) = 0$, we find from (2.13)–(2.16) that

$$\frac{m_1^{11}(\Lambda_1)}{m_1^{21}(\Lambda_1)} = \frac{m_1^{12}(\Lambda_1)}{m_1^{22}(\Lambda_1)} =: \mu \neq 0.$$

Hence, (ξ,η) is an element of the set defined by the right-hand side of (3.11). Because the latter set is a closed subspace of $\left(C^{2+\alpha}_{e,per}(\mathbb{R})\right)^2$ of codimension one that contains $\operatorname{Im}\partial_{(f,h)}\Phi(\Lambda_1,0)$, the conclusion follows from Lemma 2.5. ∎

We are now at the point of showing the transversality condition (c) from Theorem 2.3.

LEMMA 3.3. *We have that*

$$\partial_{\Lambda(f,h)}\Phi(\Lambda_1,0)\left[\left(m_1^{22}(\Lambda_1), -m_1^{21}(\Lambda_1)\right)\cos(2\pi x/L)\right] \notin \operatorname{Im}\partial_{(f,h)}\Phi(\Lambda_1,0).$$

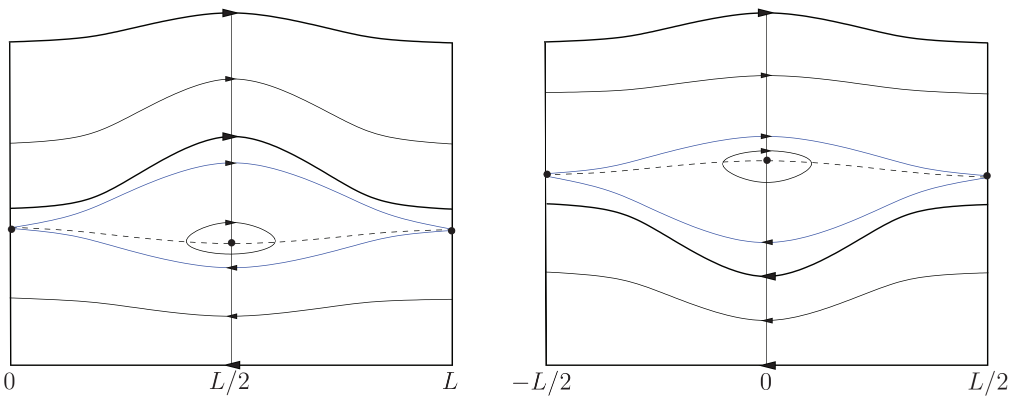

FIG. 3.1. *This figure illustrates the streamlines in the moving frame for the solutions found in Theorem 3.4 for $\gamma_1>\gamma_2>0$ (left) and $\gamma_1<\gamma_2,\gamma_2>0$ (right), cf. Lemmas A.1–A.2. The thick streamlines represent the wave surface, the internal wave, and the flat bad, respectively. The blue streamlines are separatrices which bound the critical layer and the dashed line consists of points where the y-derivative of the stream function vanishes. This line contains in both cases exactly three stagnation points: two located at $x=0$ and $x=L$, and a third one inside the critical layer at $x=L/2$.*

Proof. Since for $a,b\in\mathbb{R}$, it holds that

$$\begin{aligned}&\partial_{\Lambda(f,h)}\Phi(\Lambda_1,0)\big[(a,b)\cos(2\pi x/L)\big]\\&=\big(am^{11}_{1,\Lambda}(\Lambda_1)+bm^{12}_{1,\Lambda}(\Lambda_1),am^{21}_{1,\Lambda}(\Lambda_1)+bm^{22}_{1,\Lambda}(\Lambda_1)\big)\cos(2\pi x/L),\end{aligned}$$

we are left to show that

$$\begin{aligned}&m^{22}_1(\Lambda_1)m^{11}_{1,\Lambda}(\Lambda_1)-m^{21}_1(\Lambda_1)m^{12}_{1,\Lambda}(\Lambda_1)\\&\neq\frac{m^{11}_1(\Lambda_1)}{m^{21}_1(\Lambda_1)}\big(m^{22}_1(\Lambda_1)m^{21}_{1,\Lambda}(\Lambda_1)-m^{21}_1(\Lambda_1)m^{22}_{1,\Lambda}(\Lambda_1)\big),\end{aligned}$$

or equivalently that

$$m^{22}_1(\Lambda_1)m^{11}_{1,\Lambda}(\Lambda_1)-m^{21}_1(\Lambda_1)m^{12}_{1,\Lambda}(\Lambda_1)\neq m^{12}_1(\Lambda_1)m^{21}_{1,\Lambda}(\Lambda_1)-m^{11}_1(\Lambda_1)m^{22}_{1,\Lambda}(\Lambda_1).$$

Hence, we need to show that $D_\Lambda(1,\Lambda_1)\neq0$. Recalling the definition of the mapping ϕ from the proof of Lemma 3.1, we have that $D(1,\Lambda)=\phi(2\pi/L,\Lambda)$, and therefore

$$D_\Lambda(1,\Lambda_1)=\Phi_\Lambda(2\pi/L,\Lambda_1(2\pi/L))>0,$$

which is the desired property. ∎

THEOREM 3.4 (Bifurcation from Λ_1). *Let $\gamma_2>0$, $\gamma_1\neq\gamma_2$ and let $\alpha\in(0,1)$ be given. Furthermore, let L_0 be the constant defined by* (3.9) *and $L\leq L_0$. Then, there exists a real-analytic curve $(\Lambda,(f,h)):(-\varepsilon,\varepsilon)\to(0,\infty)\times\mathcal{O}$ consisting only of solutions to the problem* (2.11)*. This curve contains exactly one trivial solution of* (2.11)*, and for $s\to0$ we have that*

$$\Lambda(s)=\Lambda_1+O(s),\qquad(f,h)(s)=s\big(m^{22}_1(\Lambda_1),-m^{21}_1(\Lambda_1)\big)\cos(2\pi x/L)+O(s^2),$$

which implies $\Lambda_1:=\Lambda_1(2\pi/L)$. The flow determined by $(\Lambda(s),(f,h)(s)),s\in(-\varepsilon,\varepsilon)$, contains a critical layer consisting of closed streamlines very close to the internal wave (*i*) *in the layer adjacent to the wave surface if* $\gamma_1<\gamma_2$, *or*

(*ii*) *in the bottom layer if* $\gamma_1>\gamma_2$.

Moreover, the amplitude of the internal wave is much larger than that of the surface wave, cf. Figure 3.1.

Proof. It remains only to show that the amplitude of the internal wave is much larger than that of the surface wave. To this end, we note that due to $D(1,\Lambda_1)=0$, we have

$$-\frac{m_1^{22}(\Lambda_1)}{m_1^{21}(\Lambda_1)}=-\frac{m_1^{12}(\Lambda_1)}{m_1^{11}(\Lambda_1)}\xrightarrow[L\to 0]{}\operatorname{sign}(\gamma_1-\gamma_2)\infty,$$

since

$$\lim_{L\to 0}\frac{m_1^{12}(\Lambda_1)}{m_1^{11}(\Lambda_1)}=\lim_{t\to\infty}\frac{g+\gamma_2\Lambda_1(t)-\Lambda_1^2(t)\dfrac{t}{\tanh(td_2)}}{-\Lambda_1(t)(\gamma_2 d_2-\Lambda_1(t))\dfrac{t}{\sinh(td_2)}}=\operatorname{sign}(\gamma_2-\gamma_1)\infty.$$

This concludes the proof. ∎

3.2. Existence of water flows bifurcating from Λ_2. For $t\to\infty$ we have that

- $\Lambda_2(t)$ satisfies (2.6);
- if $\gamma_1 d_1+\gamma_2 d_2\le 0$, then $\Lambda_2(t)$ satisfies also (2.7).

Letting $\phi=\phi(t,\Lambda)$ be the function defined in the proof of Lemma 3.1, we note that for large t we have

$$\phi_\Lambda(t,\Lambda_2(t))=(\Lambda_2(t)-\Lambda_1(t))(\Lambda_2(t)-\Lambda_3(t))<0.$$

Hence, Λ_2 is differentiable with respect to t. Moreover, it follows from (3.2) that

$$t^2\phi_t(t,\Lambda_2(t))\xrightarrow[t\to\infty]{}-g\gamma_2 d_2,$$

and therefore $\Lambda_2'(t)=-\phi_t(t,\Lambda_2(t))/\phi_\Lambda(t,\Lambda_2(t))<0$ when t is large. Defining

$$t_0:=\inf\{t>0:\Lambda_2'<0 \text{ on } (t,\infty)\},$$

we see that $[[t_0,\infty)\ni t\mapsto\Lambda_2(t)\in(0,\infty)]$ is decreasing. In view of this property, we can choose $t_0>0$ large enough to guarantee that

$$\begin{aligned}&\sup_{[t_0,\infty)}\Lambda_2<\inf_{[t_0,\infty)}\Lambda_1,\\ &D(0,\Lambda_2(t))\neq 0\qquad\text{for all } t\ge t_0.\end{aligned}\tag{3.12}$$

Then, we set

$$L_0:=2\pi/t_0,\tag{3.13}$$

we fix $L\le L_0$, and define $\Lambda_2:=\Lambda_2(2\pi/L)$. Recalling that $\phi(2\pi/L,\Lambda_2)=0$, we obtain that $D(1,\Lambda_2)=0$. In fact, the equation $D(\cdot,\Lambda_2)=0$ has $k=1$ as the only integer solution, cf. (3.12). Because of $\Lambda_2\in(0,\gamma_2 d_2)$, $\partial_{(f,h)}\Phi_1(\Lambda_2,0)$ is a Fredholm operator with a one-dimensional kernel

$$\operatorname{Ker}\partial_{(f,h)}\Phi_1(\Lambda_2,0)=\operatorname{span}\left\{\left(m_1^{22}(\Lambda_2),-m_1^{21}(\Lambda_2)\right)\cos(2\pi x/L)\right\}.$$

Using the same arguments as in the proof of UH Lemma 3.2, we see that

$$\mathrm{Im}\,\partial_{(f,h)}\Phi(\Lambda_2,0)=\Big\{(\xi,\eta)=\Big(\sum_{k\in\mathbb{N}}\xi_k\cos(R_kx),\sum_{k\in\mathbb{N}}\eta_k\cos(R_kx)\Big):\xi_1=\frac{m_1^{11}(\Lambda_2)}{m_1^{21}(\Lambda_2)}\eta_1\Big\}.$$

Moreover, the transversality condition

$$\partial_{\Lambda(f,h)}\Phi(\Lambda_2,0)\big[\big(m_1^{22}(\Lambda_2),-m_1^{21}(\Lambda_2)\big)\cos(2\pi x/L)\big]\notin\mathrm{Im}\,\partial_{(f,h)}\Phi(\Lambda_2,0)$$

reduces to showing that $D_\Lambda(1,\Lambda_2)=\phi_\Lambda(2\pi/L,\Lambda_2(2\pi/L))\neq0$, relation which holds true. We conclude with the following result.

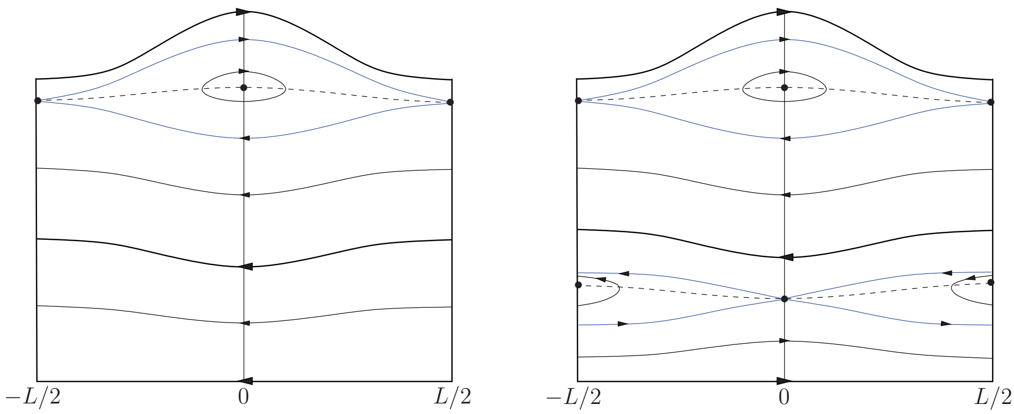

FIG. 3.2. *This figure illustrates the streamlines in the moving frame for the solutions found in Theorem 3.5 for $\gamma_1d_1+\gamma_2d_2>0$ (left) and $\gamma_1d_1+\gamma_2d_2\le0$ (right), cf. Lemmas A.3–A.4.*

THEOREM 3.5 (Bifurcation from Λ_2). *Let $\gamma_2>0$, $\gamma_1\neq\gamma_2$ and let $\alpha\in(0,1)$ be given. Furthermore, let L_0 be the constant defined by* (3.13) *and $L\le L_0$. Then, there exists a real-analytic curve $(\Lambda,(f,h)):(-\varepsilon,\varepsilon)\to(0,\infty)\times\mathcal{O}$ consisting only of solutions to the problem* (2.11). *This curve contains exactly one trivial solution of* (2.11), *and for $s\to0$ we have that*

$$\Lambda(s)=\Lambda_2+O(s),\qquad (f,h)(s)=s\big(m_1^{22}(\Lambda_2),-m_1^{21}(\Lambda_2)\big)\cos(2\pi x/L)+O(s^2),$$

by which $\Lambda_2:=\Lambda_2(2\pi/L)$. The flow determined by $(\Lambda(s),(f,h)(s)),s\in(-\varepsilon,\varepsilon)$, contains a critical layer consisting of closed streamlines

(*i*) *in the layer adjacent to the wave surface if $\gamma_1d_1+\gamma_2d_2>0$;*

(*ii*) *in each of the layers if $\gamma_1d_1+\gamma_2d_2\le0$.*

The vortex in the top layer is located right beneath the wave surface. Moreover, the amplitude of the internal wave between the two layers is much smaller than that of the surface wave, cf. Figure 3.2.

Proof. It remains only to show that the amplitude of the surface wave is much larger than that of the internal wave. This follows from (2.15),(2.16), as we have

$$-\frac{m_1^{21}(\Lambda_2)}{m_1^{22}(\Lambda_2)}\xrightarrow[L\to0]{}-\infty,$$

and the proof is completed. ∎

3.3. Existence of water flows bifurcating from Λ_3. Since $0>\Lambda_3(t)\xrightarrow[t\to\infty]{}0$ we see that $\Lambda_3(t)$ satisfies (2.7) provided that $\gamma_1 d_1+\gamma_2 d_2<0$. Because for large t

$$\phi_\Lambda(t,\Lambda_3(t))=(\Lambda_3(t)-\Lambda_1(t))(\Lambda_3(t)-\Lambda_2(t))>0,$$

the function Λ_3 is differentiable with respect to t. Since $t^2\phi_t(t,\Lambda_3(t))\xrightarrow[t\to\infty]{}-g\gamma_2 d_2$, we conclude that $\Lambda_3'(t)>0$ when t is large. Defining $t_0:=\inf\{t>0:\Lambda_3'>0 \text{ on } (t,\infty)\}$, we see that $[[t_0,\infty)\ni t\mapsto\Lambda_3(t)\in(-\infty,0)]$ is increasing. In view of this property, we can choose $t_0>0$ large enough to guarantee that

$$D(0,\Lambda_3(t))\neq 0 \qquad \text{for all } t\geq t_0. \tag{3.14}$$

Let

$$L_0:=2\pi/t_0, \tag{3.15}$$

choose $L\leq L_0$, and define $\Lambda_3:=\Lambda_3(2\pi/L)$. Since $\phi(2\pi/L,\Lambda_3)=0$, we get that $k\in\mathbb{N}$ solves $D(k,\Lambda_3)=0$ if and only if $k=1$. Moreover, since $\Lambda_3<0$, $\partial_{(f,h)}\Phi_1(\Lambda_3,0)$ is a Fredholm operator with a one-dimensional kernel

$$\operatorname{Ker}\partial_{(f,h)}\Phi_1(\Lambda_3,0)=\operatorname{span}\left\{\left(m_1^{22}(\Lambda_3),-m_1^{21}(\Lambda_3)\right)\cos(2\pi x/L)\right\}.$$

As in Lemma 3.2, we find that

$$\operatorname{Im}\partial_{(f,h)}\Phi(\Lambda_3,0)=\left\{(\xi,\eta)=\Big(\sum_{k\in\mathbb{N}}\xi_k\cos(R_k x),\sum_{k\in\mathbb{N}}\eta_k\cos(R_k x)\Big):\xi_1=\frac{m_1^{11}(\Lambda_3)}{m_1^{21}(\Lambda_3)}\eta_1\right\},$$

the transversality condition

$$\partial_{\Lambda(f,h)}\Phi(\Lambda_3,0)\left[\left(m_1^{22}(\Lambda_3),-m_1^{21}(\Lambda_3)\right)\cos(2\pi x/L)\right]\notin\operatorname{Im}\partial_{(f,h)}\Phi(\Lambda_3,0)$$

being equivalent to $D_\Lambda(1,\Lambda_3)=\phi_\Lambda(2\pi/L,\Lambda_3(2\pi/L))\neq 0$. This shows that all the assumptions of Theorem 2.3 are satisfied. Consequently, we have the following result.

THEOREM 3.6 (Bifurcation from Λ_3). *Let $\gamma_2>0$, $\alpha\in(0,1)$, and assume $\gamma_1 d_1+\gamma_2 d_2<0$. Furthermore, let L_0 be the constant defined by (3.15) and $L\leq L_0$. Then, there exists a real-analytic curve $(\Lambda,(f,h)):(-\varepsilon,\varepsilon)\to(0,\infty)\times\mathcal{O}$ consisting only of solutions to the problem (2.11). This curve contains exactly one trivial solution of (2.11), and for $s\to 0$ we have that*

$$\Lambda(s)=\Lambda_3+O(s),\qquad (f,h)(s)=s\left(m_1^{22}(\Lambda_3),-m_1^{21}(\Lambda_3)\right)\cos(2\pi x/L)+O(s^2),$$

by which $\Lambda_3:=\Lambda_3(2\pi/L)$. The flow determined by $(\Lambda(s),(f,h)(s)),s\in(-\varepsilon,\varepsilon)$, contains a critical layer consisting of closed streamlines in the layer adjacent to the bed. Moreover, the amplitude of the internal wave between the two layers is much smaller than that of the surface wave, cf. Figure 3.3.

Proof. The claim concerning the amplitude of the surface and internal waves follows from (2.15),(2.16), as we have

$$-\frac{m_1^{21}(\Lambda_2)}{m_1^{22}(\Lambda_2)}\xrightarrow[L\to 0]{}\infty.$$

This completes the proof. ∎

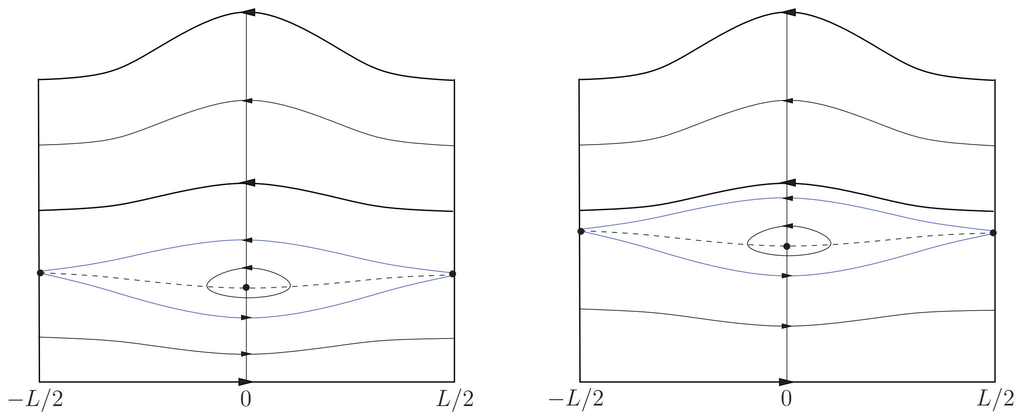

FIG. 3.3. *This figure illustrates the streamlines in the moving frame for the solutions found in Theorem 3.6 (left) and Theorems 4.2 and 4.3 (right), cf. Lemmas A.5–A.6.*

4. Analysis of the dispersion relation: the case $\gamma_2 = 0$ and $\gamma_1 < 0$

Because of $\gamma_2 = 0$, the inequality (2.6) reduces to $\Lambda = 0$ situation when $\partial_{(f,h)}\Phi(\Lambda,0)$ is not even a Fredholm operator, cf. Lemma 2.5. For this reason we consider the bifurcation problem for (2.11) just for values of Λ which satisfy (2.7). Hence, the flows that we construct will have stagnation points only in the bottom layer.

With the notation from Section 3, we determine for the depressed cubic equation (3.3) that

$$D = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 = -\frac{g^3}{27}t^{-3} + O\left(t^{-4}\right) < 0 \qquad \text{for } t \to \infty,$$

hence (3.1) has again three positive roots. They are given by the relation $z = r\cos(\beta)$, from which

$$r = \sqrt{\frac{-4p}{3}} = 2\sqrt{\frac{g}{3}}t^{-1/2} + \frac{\gamma_1^2}{36}\sqrt{\frac{3}{g}}t^{-3/2} + O\left(t^{-5/2}\right), \tag{4.1}$$

and β is one of the solutions to

$$\cos(3\beta) = -\frac{q}{2}\sqrt{-\frac{27}{p^3}} = -\gamma_1\sqrt{\frac{3}{g}}t^{-1/2} + \frac{10\gamma_1^3}{72g}\sqrt{\frac{3}{g}}t^{-3/2} + O\left(t^{-5/2}\right).$$

Setting $\beta := 3^{-1}\arccos\left(-(q/2)\sqrt{-27/p^3}\right)$, we see that $\beta(t) \xrightarrow[t\to\infty]{} \pi/6$ and the roots of (3.1) are

$$\begin{aligned}
\Lambda_1 &= r\cos(\beta) - \frac{A}{3}, \\
\Lambda_2 &= r\cos\left(\beta - \frac{2\pi}{3}\right) - \frac{A}{3} = -r\cos\left(\beta + \frac{\pi}{3}\right) - \frac{A}{3}, \\
\Lambda_3 &= r\cos\left(\beta + \frac{2\pi}{3}\right) - \frac{A}{3} = -r\cos\left(\beta - \frac{\pi}{3}\right) - \frac{A}{3}.
\end{aligned} \tag{4.2}$$

It now easily follows from (3.2), (4.1), and (4.2) that $\Lambda_i \xrightarrow[t\to\infty]{} 0$ for $i \in \{1,2,3\}$ and that $\Lambda_3 < \Lambda_2 < 0 < \Lambda_1$ for $t \to \infty$. Thus, we can find $t_0 > 0$ such that

$$\begin{aligned}
&\gamma_1 d_1 < \Lambda_3 < \Lambda_2 < 0 < \Lambda_1 \qquad \text{on } [t_0,\infty), \\
&D(0,\Lambda_i(t)) \neq 0 \qquad \text{for all } t \geq t_0,\ i = 2,3.
\end{aligned} \tag{4.3}$$

In view of $\gamma_1<0$, the relation (2.7) is equivalent to $\Lambda\in(\gamma_1 d_1,0)$ and therefore just flows bifurcating from negative Λ may contain stagnation points. For this reason, we only investigate in the following the functions Λ_2 and Λ_3.

LEMMA 4.1. *There exists $t_0>0$ such that* (4.3) *holds and $\Lambda_i:[t_0,\infty)\to(-\infty,0)$, $i=2,3$, are both increasing functions.*

Proof. Note first that $\phi_\Lambda(t,\Lambda_2(t))<0$ and $\phi_\Lambda(t,\Lambda_3(t))>0$ for $t\geq t_0$. Therefore, the functions Λ_i, $i=2,3$, are differentiable on $[t_0,\infty)$. Moreover, it is easy to see from (4.1) and (3.2) that

$$\lim_{t\to\infty} t^{5/2}\phi_t(t,\Lambda_3(t))=\lim_{t\to\infty} t^{5/2}\Lambda_3(t)B'(t)=-g^{3/2}.$$

Hence, we may chose t_0 large to ensure the assertion for the mapping Λ_3.

This argument does no longer work for Λ_2 as $\cos(\beta+\pi/3)\xrightarrow[t\to\infty]{}0$. Hence, we have to determine an expansion for $\cos(\beta+\pi/3)$. Let

$$z_0:=\frac{\sqrt{3}}{2}-\frac{\gamma_1}{6}\sqrt{\frac{3}{g}}t^{-1/2},$$

and observe that

$$\begin{aligned}|\cos(\beta)-z_0||\cos^2(\beta)+z_0\cos(\beta)+z_0^2|&=|4\cos^3(\beta)-3\cos(\beta)-4z_0^3+3z_0|\\ &=\left|\cos(3\beta)+\gamma_1\sqrt{\frac{3}{g}}t^{-1/2}\right|+O(t^{-1})=O(t^{-1}).\end{aligned}$$

Hence $\cos(\beta)=z_0+O(t^{-1})$ for $t\to\infty$. It is now easy to see that for $t\to\infty$ we have

$$\sin(\beta)=\frac{1}{2}+\frac{\gamma_1}{2\sqrt{g}}t^{-1/2}+O(t^{-1}),$$

from which it follows easily that

$$\Lambda_2(t):=\frac{5\gamma_1}{6}t^{-1}+O(t^{-3/2})\qquad\text{and}\qquad\Lambda_3(t):=-\sqrt{g}t^{-1/2}-\frac{\gamma_1}{6}t^{-1}+O(t^{-3/2}).\quad(4.4)$$

The expansion (4.4) combined with (3.2) shows that

$$\lim_{t\to\infty} t^3\phi_t(t,\Lambda_2(t))=\lim_{t\to\infty} t^3\big(\Lambda_2(t)B'(t)+C'(t)\big)=-g\gamma_1/6>0,$$

relation which proves the claim. ∎

THEOREM 4.2 (Bifurcation from Λ_3). *Let $\gamma_2=0$, $\gamma_1<0$, and $\alpha\in(0,1)$. Furthermore, let $L_0:=2\pi/t_0$ and $L\leq L_0$. Then, there exists a real-analytic curve $(\Lambda,(f,h)):(-\varepsilon,\varepsilon)\to(0,\infty)\times\mathcal{O}$ consisting only of solutions to problem* (2.11)*. This curve contains exactly one trivial solution of* (2.11)*, and for $s\to0$ we have that*

$$\Lambda(s)=\Lambda_3+O(s),\qquad(f,h)(s)=s\big(m_1^{22}(\Lambda_3),-m_1^{21}(\Lambda_3)\big)\cos(2\pi x/L)+O(s^2),$$

which implies $\Lambda_3:=\Lambda_3(2\pi/L)$. The flow determined by $(\Lambda(s),(f,h)(s)),s\in(-\varepsilon,\varepsilon)$, contains a critical layer consisting of closed streamlines in the layer adjacent to the bed just below the internal wave. Moreover, the amplitude of the internal wave between the two layers is much smaller than that of the surface wave, cf. Figure 3.3.

Proof. Because $t:=2\pi/L\geq t_0$, we know from (4.3) and Lemma 2.5 that $\partial_{(f,h)}\Phi_1(\Lambda_3,0)$ is a Fredholm operator. To determine its kernel we need to solve $D(k,\Lambda_3)=\phi(kt,\Lambda_3)=0$. As $\Lambda_3=\Lambda_3(t)$, we see that $D(1,\Lambda_3)=0$, while (4.3) ensures that $D(0,\Lambda_3)=0$. Recalling Lemma 4.1, we see that $D(k,\Lambda_3)\neq 0$ for all $k\geq 2$. Indeed, for $k\geq 2$, $\Lambda_3(kt)>\Lambda_3$, and if $\Lambda_3=\Lambda_2(kt)$, then $\Lambda_3=\Lambda_2(kt)>\Lambda_2(t)>\Lambda_3$, a contradiction. Hence, $\partial_{(f,h)}\Phi_1(\Lambda_3,0)$ is a Fredholm operator with a one-dimensional kernel

$$\operatorname{Ker}\partial_{(f,h)}\Phi_1(\Lambda_3,0)=\operatorname{span}\left\{\left(m_1^{22}(\Lambda_3),-m_1^{21}(\Lambda_3)\right)\cos(2\pi x/L)\right\}.$$

Similarly, as before we have

$$\operatorname{Im}\partial_{(f,h)}\Phi(\Lambda_3,0)=\left\{(\xi,\eta)=\Big(\sum_{k\in\mathbb{N}}\xi_k\cos(R_kx),\sum_{k\in\mathbb{N}}\eta_k\cos(R_kx)\Big):\xi_1=\frac{m_1^{11}(\Lambda_3)}{m_1^{21}(\Lambda_3)}\eta_1\right\},$$

and one can verify that the transversality condition

$$\partial_{\Lambda(f,h)}\Phi(\Lambda_3,0)\left[\left(m_1^{22}(\Lambda_3),-m_1^{21}(\Lambda_3)\right)\cos(2\pi x/L)\right]\notin\operatorname{Im}\partial_{(f,h)}\Phi(\Lambda_3,0)$$

is also satisfied. We are thus in a position to apply Theorem 2.3. Gathering (2.15), (2.16), and (4.4), we infer that

$$-\frac{m_1^{21}(\Lambda_3)}{m_1^{22}(\Lambda_3)}\xrightarrow[L\to 0]{}\infty,$$

which finishes the proof. □

When considering bifurcation from Λ_2 the situation is more complicated because the derivative $\partial_{(f,h)}\Phi_1(\Lambda_2,0)$ may possess a two-dimensional kernel if $\Lambda_3(2\pi k/L)=\Lambda_2(2\pi/L)$ for some $L\geq L_0$ and some integer $k\geq 2$. When this happens, the integer k is unique, cf. Lemma 4.1, so that we can conclude the existence of a curve of bifurcating solutions from Theorem 4.2. When $\Lambda_3(2\pi k/L)\neq\Lambda_2(2\pi/L)$ for all $k\geq 2$, we can again apply Theorem 2.3.

Theorem 4.3 (Bifurcation from Λ_2). *Let $\gamma_2=0$, $\gamma_1<0$, and let $\alpha\in(0,1)$. Furthermore, let $L_0:=2\pi/t_0$ and $L\leq L_0$.*

(*i*) *Assume that $\Lambda_3(2\pi k/L)\neq\Lambda_2(2\pi/L)$ for all integers $k\geq 2$. Then, there exists a real-analytic curve $(\Lambda,(f,h)):(-\varepsilon,\varepsilon)\to(0,\infty)\times\mathcal{O}$ consisting only of solutions to the problem* (2.11). *This curve contains exactly one trivial solution of* (2.11), *and for $s\to 0$ we have that*

$$\Lambda(s)=\Lambda_2+O(s),\qquad (f,h)(s)=s\left(m_1^{22}(\Lambda_2),-m_1^{21}(\Lambda_2)\right)\cos(2\pi x/L)+O(s^2),$$

by which $\Lambda_2:=\Lambda_2(2\pi/L)$.

(*ii*) *Assume that $\Lambda_3(2\pi k/L)=\Lambda_2$ for some integer $k\geq 2$. Then the assertion of Theorem 4.2 holds true, but with L replaced by L/k.*

The flow determined by $(\Lambda(s),(f,h)(s)),s\in(-\varepsilon,\varepsilon)$, contains a critical layer consisting of closed streamlines in the layer adjacent to the bed just beneath the internal wave. Moreover, the amplitude of the internal wave between the two layers is much smaller than that of the surface wave, cf. Figure 3.3.

Proof. Setting $t:=2\pi/L\geq t_0$, we know from (4.3) and Lemma 2.5 that $\partial_{(f,h)}\Phi_1(\Lambda_2,0)$ is a Fredholm operator. To determine its kernel we need to solve

$D(k,\Lambda_2)=\Phi(kt,\Lambda_2)=0$. A solution of this equation is $k=1$ as $\Lambda_2=\Lambda_2(t)$. Equation (4.3) ensures additionally that $D(0,\Lambda_2)=0$. Because Λ_3 is increasing to zero, there may exist a (unique) integer $k\geq 2$ such that $\Lambda_3(2\pi k/L)=\Lambda_2(2\pi/L)$, hence $D(k,\Lambda_2)=0$. In this case we are in the situation *(ii)* and the proof is obvious. If $\Lambda_3(2\pi k/L)\neq\Lambda_2(2\pi/L)$ for all integers $k\geq 2$, then we are in the case *(i)* and the proof is similar to that of Theorem 4.2. ∎

Remark 4.4. Since the properties of the functions $\Lambda_i(t)$, for $i=1,2,3$ were essential in finding the branches of solutions to the water wave problem, we summarize them in the Table 4.1 below.

<table>
<tr><th>–</th><th>$\gamma_1>0$</th><th>$\gamma_1=0$</th><th>$\gamma_1<0$</th></tr>
<tr><td>$\gamma_2>0$</td><td colspan="3">$\Lambda_3<0<\Lambda_2<\Lambda_1$
$\Lambda_1\to\gamma_2 d_2,\quad \Lambda_i\to 0, i\in\{2,3\}$</td></tr>
<tr><td>$\gamma_2=0$</td><td>$\Lambda_3<0<\Lambda_2<\Lambda_1<\gamma_1 d_1$
$\Lambda_i\to 0,\quad i\in\{1,2,3\}$</td><td>–</td><td>$\gamma_1 d_1<\Lambda_3<\Lambda_2<0<\Lambda_1$
$\Lambda_i\to 0,\quad i\in\{1,2,3\}$</td></tr>
<tr><td>$\gamma_2<0$</td><td colspan="3">$\Lambda_3<\Lambda_2<0<\Lambda_1$
$\Lambda_3\to\gamma_2 d_2,\quad \Lambda_i\to 0, i\in\{1,2\}$</td></tr>
</table>

Table 4.1. *Properties of the roots $\Lambda_i, i\in\{1,2,3\}$, of the dispersion relation* (2.18) *in dependence of the vorticity constants $\gamma_i, i\in\{1,2\}$ for $\gamma_1\neq\gamma_2$ and for large $R_k=2\pi k/L$. Our analysis is dedicated to the cases:* (i) *$\gamma_2>0$ and $\gamma_1\neq\gamma_2$; and* (ii) *$\gamma_2=0$ and $\gamma_1<0$. The analysis in the other two cases:* (iii) *$\gamma_2<0$ and $\gamma_1\neq\gamma_2$; and* (iv) *$\gamma_2=0$ and $\gamma_1>0$ is similar to that for* (i) *and* (ii)*, respectively (see Remark 2.6).*

Appendix A. We present herein the proof of Lemma 2.4 and additionally we rigorously prove that the streamline pattern for the solutions that we found is as shown in Figures 3.1–3.3, respectively. To this end, we first determine explicit expressions for the elliptic and boundary operators introduced right before Remark 2.1. Given $(f,h)\in\mathcal{O}$, it is easy to see that

$$\mathcal{A}(f)=\partial_{xx}-2\frac{d+y}{d_1+f}f'\partial_{xy}+\frac{|(d+y)f'|^2+d_1^2}{(d_1+f)^2}\partial_{yy}-(d+y)\frac{(d_1+f)f''-2f'^2}{(d_1+f)^2}\partial_y, \tag{A.1}$$

$$\begin{aligned}\mathcal{A}(f,h)=&\partial_{xx}-2\frac{d_2h'+(h-f)'y}{h-f+d_2}\partial_{xy}+\frac{|d_2h'+(h-f)'y|^2+d_2^2}{(h-f+d_2)^2}\partial_{yy}\\&-\Big[\frac{d_2h''+(h-f)''y}{h-f+d_2}-2\frac{d_2h'(h-f)'+(h'-f')^2y}{(h-f+d_2)^2}\Big]\partial_y,\end{aligned} \tag{A.2}$$

respectively, given $(w_1,w_2)\in C^{3+\alpha}_{e,per}(\overline{\Omega}_1)\times C^{3+\alpha}_{e,per}(\overline{\Omega}_2)$ and $\Lambda\in\mathbb{R}$, we have that

$$\begin{aligned}\mathcal{B}_1(\Lambda,(f,h),w_2)=&\Big[|\partial_x w_2|^2-\frac{2d_2h'}{h-f+d_2}\partial_x w_2\partial_y w_2+\frac{d_2^2(1+h'^2)}{(h-f+d_2)^2}|\partial_y w_2|^2\Big]\Big|_{y=0}\\&+2g(d+h)-Q(\Lambda),\end{aligned} \tag{A.3}$$

$$\mathcal{B}_2(\Lambda,(f,h))[w_1,w_2]=\frac{d_2}{h-f+d_2}\partial_y w_2\Big|_{y=-d_2}-\frac{d_1}{d_1+f}\partial_y w_1\Big|_{y=-d_2}. \tag{A.4}$$

Proof. (Proof of Lemma 2.4.) Since

$$\partial_{(f,h)}\Phi(\Lambda,0)[(f,h)] = \begin{pmatrix} \partial_f\Phi_1(\Lambda,0)[f] & \partial_h\Phi_1(\Lambda,0)[h] \\ \partial_f\Phi_2(\Lambda,0)[f] & \partial_h\Phi_2(\Lambda,0)[h] \end{pmatrix}$$

we only need to determine the entries in the matrix $\partial_{(f,h)}\Phi(\Lambda,0)$.

The derivative $\partial_f\Phi_1(\Lambda,0)$: Using the definition of Φ_1, we see that

$$\partial_f\Phi_1(\Lambda,0)[f] = 2\Big[\frac{f}{d_2}|\partial_y\psi_2^0|^2 + \partial_y\psi_2^0\partial_y(\partial_f w_2(\Lambda,0)[f])\Big]\Big|_{y=0}, \tag{A.5}$$

by which $\partial_y\psi_2^0\big|_{y=0} = \Lambda$ and $z := \partial_f w_2(\Lambda,0)[f]$ is, in view of (2.10), the solution of the Dirichlet problem

$$\begin{cases} \Delta z = -\partial_f\mathcal{A}(0,0)[f]\psi_2^0 & \text{in } \Omega_2, \\ z = 0 & \text{on } \partial\Omega_2. \end{cases} \tag{A.6}$$

A routine calculation shows now that

$$\partial_f\mathcal{A}(0,0)[f]\psi_2^0 = \frac{2\gamma_2 f}{d_2} + \Big(\frac{\gamma_2}{d_2}y^2 + \frac{\Lambda}{d_2}y\Big)f''.$$

Expanding f and $z(y), y\in[-d_2,0]$, by their Fourier series, we have

$$f = \sum_{k\in\mathbb{N}} a_k\cos(R_k x) \qquad \text{and} \qquad z(y) = \sum_{k\in\mathbb{N}} a_k A_k(y)\cos(R_k x).$$

The coefficients A_k solve, in view of (A.6), the following boundary value problem

$$\begin{cases} A_k'' - R_k^2 A_k = -\frac{2\gamma_2}{d_2} + R_k^2\Big(\frac{\gamma_2}{d_2}y^2 + \frac{\Lambda}{d_2}y\Big) & \text{in } (-d_2,0), \\ A_k(-d_2) = A_k(0) = 0, \end{cases}$$

and therefore

$$A_k(y) = (\Lambda - \gamma_2 d_2)\frac{\sinh(R_k y)}{\sinh(R_k d_2)} - \Big(\frac{\gamma_2}{d_2}y^2 + \frac{\Lambda}{d_2}y\Big).$$

Using the relation (A.5) we obtain now that

$$\partial_f\Phi_1(\Lambda,0)[f] = \sum_{k\in\mathbb{N}} m_k^{11} a_k\cos(R_k x),$$

by which $(m_k^{11})_{k\in\mathbb{N}}$ is defined by (2.13).

The derivative $\partial_h\Phi_1(\Lambda,0)$: We have that

$$\partial_h\Phi_1(\Lambda,0)[h] = 2\Big[-\frac{h}{d_2}|\partial_y\psi_2^0|^2 + \partial_y\psi_2^0\partial_y(\partial_h w_2(\Lambda,0)[h])\Big]\Big|_{y=0} + 2gh,$$

with $z := \partial_h w_2(\Lambda,0)[h])$ solving the Dirichlet problem

$$\begin{cases} \Delta z = -\partial_h\mathcal{A}(0,0)[h]\psi_2^0 & \text{in } \Omega_2, \\ z = 0 & \text{on } \partial\Omega_2, \end{cases}$$

cf. (2.10). Recalling (A.2), we compute that

$$\partial_h\mathcal{A}(0,0)[h]\psi_2^0=-\frac{2\gamma_2}{d_2}h-\Big(\frac{\gamma_2}{d_2}y^2+\frac{\gamma_2 d_2+\Lambda}{d_2}y+\Lambda\Big)h''.$$

Using Fourier expansions as before, that is

$$h=\sum_{k\in\mathbb{N}}b_k\cos(R_kx)\qquad\text{and}\qquad z(y)=\sum_{k\in\mathbb{N}}b_kB_k(y)\cos(R_kx)\quad\text{for } y\in[-d_2,0],$$

we obtain that the coefficients B_k satisfy

$$\begin{cases} B_k''-R_k^2B_k=\frac{2\gamma_2}{d_2}-R_k^2\Big(\frac{\gamma_2}{d_2}y^2+\frac{\gamma_2d_2+\Lambda}{d_2}y+\Lambda\Big) \text{ in } (-d_2,0),\\ B_k(-d_2)=B_k(0)=0.\end{cases}$$

The solution of this boundary value problem is

$$B_k(y)=-\Lambda\Big(\frac{\sinh(R_ky)}{\tanh(R_kd_2)}+\cosh(R_ky)\Big)+\frac{\gamma_2}{d_2}y^2+\frac{\gamma_2d_2+\Lambda}{d_2}y+\Lambda,$$

and the desired representation for the derivative $\partial_h\Phi_1(\Lambda,0)$ follows at once.

The derivative $\partial_f\Phi_2(\Lambda,0)$: From the definition of Φ_2 we obtain that

$$\partial_f\Phi_2(\Lambda,0)[f]=\Big[\frac{f}{d_2}\partial_y\psi_2^0+\frac{f}{d_1}\partial_y\psi_1^0+\partial_y\big(\partial_fw_2(\Lambda,0)[f]-\partial_fw_1(\Lambda,0)[f]\big)\Big]\Big|_{y=-d_2},$$

by which, in the equality above, $z:=\partial_fw_1(\Lambda,0)[f]$ solves the Dirichlet problem

$$\begin{cases}\Delta z=-\partial_f\mathcal{A}(0)[f]\psi_1^0 & \text{in } \Omega_1,\\ z=0 & \text{on } \partial\Omega_1.\end{cases}$$

In view of (A.1), we compute that

$$\partial_f\mathcal{A}(0)[f]\psi_1^0=-\frac{2\gamma_1f}{d_1}-\Big[\frac{\gamma_1}{d_1}y^2+\Big(\frac{d+d_2}{d_1}\gamma_1+\frac{\Lambda-\gamma_2d_2}{d_1}\Big)y+\frac{d\Lambda}{d_1}+\frac{dd_2(\gamma_1-\gamma_2)}{d_1}\Big]f''.$$

Expanding f and $z(y), y\in[-d,-d_2]$, by their Fourier series

$$f=\sum_{k\in\mathbb{N}}a_k\cos(R_kx)\qquad\text{and}\qquad z(y)=\sum_{k\in\mathbb{N}}a_kC_k(y)\cos(R_kx),$$

we find that the coefficient C_k is the solution of

$$\begin{cases} C_k''-R_k^2C_k=\frac{2\gamma_1}{d_1}-R_k^2\Big[\frac{\gamma_1}{d_1}y^2+\Big(\frac{d+d_2}{d_1}\gamma_1+\frac{\Lambda-\gamma_2d_2}{d_1}\Big)y+\frac{d\Lambda}{d_1}+\frac{dd_2(\gamma_1-\gamma_2)}{d_1}\Big] \text{ in } (-d,-d_2),\\ C_k(-d)=C_k(-d_2)=0,\end{cases}$$

whence

$$C_k(y)=\Lambda\frac{\sinh((d+y)R_k)}{\sinh(R_kd_1)}+\frac{\gamma_1}{d_1}y^2+\Big(\frac{d+d_2}{d_1}\gamma_1+\frac{\Lambda-\gamma_2d_2}{d_1}\Big)y+\frac{d\Lambda}{d_1}+\frac{dd_2(\gamma_1-\gamma_2)}{d_1}.$$

The representation of $\partial_f\Phi_2(\Lambda,0)$ as a Fourier multiplier now easily follows.

The derivative $\partial_h\Phi_2(\Lambda,0)$: Observing that

$$\partial_h\Phi_2(\Lambda,0)[h]=\frac{\gamma_2 d_2-\Lambda}{d_2}h+\partial_y(\partial_h w_2(\Lambda,0)[h])|_{y=-d_2},$$

the desired representation for $\partial_h\Phi_2(\Lambda,0)$ follows by using the expression for $\partial_h w_2(\Lambda,0)[h]$ determined in the second part of this proof. ∎

In the remaining part we establish the Lemmas A.1–A.6 that provide the justification for the streamlines pattern, as seen from a reference frame moving with the wave, as shown in Figures 3.1–3.3. Because the proofs of Lemmas A.1–A.6 use similar arguments, we present herein only the proof for Lemma A.1. For this, it is important to note that because there is no time dependence in problem (2.1) (or (2.2)), the particle trajectories and the streamlines corresponding to the solutions found in Theorems 3.4–3.6, 4.2, and 4.3 coincide with the level curves of the corresponding stream function. They are parametrized by solutions to the system of ordinary differential equations

$$\begin{cases} x'=u-c=\psi_y,\\ y'=v=-\psi_x,\end{cases} \tag{A.7}$$

stagnation points of the flows corresponding to equilibria of (A.7). Hence, our task is to determine the level curves of the stream function. The direction of motion of the particles along the level curves is determined by the sign of $u-c$ or v.

LEMMA A.1. *Assume that* $\gamma_1>\gamma_2>0$ *and let*

$$((f,h),\psi_1,\psi_2)\in\left(C^{3+\alpha}_{per}(\mathbb{R})\right)^2\times C^{3+\alpha}_{per}\left(\overline{\Omega(f)}\right)\times C^{3+\alpha}_{per}\left(\overline{\Omega(f,h)}\right)$$

be a solution of (2.2) *that is determined by a point* $(\Lambda(s),(f,h)(s))$ *on the bifurcation curve found in Theorem 3.4. Provided that* s *is small enough, the following assertions are true:*

- (i) $f'>0$ *and* $h'>0$ *on* $(0,L/2)$;
- (ii) $\partial_x\psi_2<0$ *in* $\{(x,y)\in\Omega(f,h):x\in(0,L/2)\}$ *and* $\partial_y\psi_2>0$ *in* $\Omega(f,h)$;
- (iii) $\partial_x\psi_1<0$ *in* $\{(x,y)\in\Omega(f):x\in(0,L/2)\}$;
- (iv) *There is a smooth curve* $\{(x,\xi(x)):x\in[0,L/2]\}$ *with* $-d<\xi(x)<-d_2+f(x)$ *for all* $x\in[0,L/2]$ *and additionally satisfying:*
  - (a) *Given* $x\in[0,L/2]$, *it holds that:* $\partial_y\psi_1(x,\xi(x))=0$, $\partial_y\psi_1(x,y)<0$ *for all* $y\in[-d,\xi(x))$, *and* $\partial_y\psi_1(x,y)>0$ *for all* $y\in(\xi(x),-d_2+f(x)]$;
  - (b) ξ *is strictly decreasing on* $[0,L/2]$;
  - (c) *The function* $[x\mapsto\psi_1(x,\xi(x))]$ *is strictly decreasing on* $[0,L/2]$.

Proof. Since $\Lambda(0)=\Lambda_1\in(\gamma_2 d_2,\gamma_1 d_1+\gamma_2 d_2)$, for small s it holds $\Lambda(s)\in(\gamma_2 d_2,\gamma_1 d_1+\gamma_2 d_2)$. Recalling that

$$f(s)=sm_1^{22}(\Lambda_1)\cos\left(\frac{2\pi}{L}x\right)+O(s^2),\quad h(s)=-sm_1^{21}(\Lambda_1)\cos\left(\frac{2\pi}{L}x\right)+O(s^2), \tag{A.8}$$

with $m_1^{22}(\Lambda_1)<0$ and $m_1^{21}(\Lambda_1)>0$, the claim (i) follows by using the same arguments as in the proof of [35, Lemma 4.2].

For (ii), we see first that $\partial_y\psi_2^0=\gamma_2 y+\Lambda_1>\gamma_2 y+\gamma_2 d_2>0$ in $\overline{\Omega}_2$. Therefore, $\partial_y\psi_2>0$ in $\Omega(f,h)$ provided that s is small. Using now (i) and the fact that ψ is constant on

$\partial\Omega(f,h)$, and even with respect to x, it is easy to see that $\partial_x\psi_2 \le 0$ on the boundary of the set $\{(x,y)\in\Omega(f,h) : x\in(0,L/2)\}$. Observing that $\partial_x\psi_2(x,h(x))<0$ for all $x\in(0,L/2)$ and $\Delta\psi_x=0$ in $\Omega(f,h)$, elliptic maximum principles ensure that $\partial_x\psi_2<0$ in $\{(x,y)\in\Omega(f,h) : x\in(0,L/2)\}$. The claim (iii) is obtained in a similar manner.

For (iv), we remark that

$$\partial_y\psi_1^0\big|_{y=-d}<0, \qquad \partial_y\psi_1^0\big|_{y=-d_2}>0, \qquad \partial_{yy}\psi_1^0>0 \text{ in } \overline{\Omega}_1.$$

Hence, for small s the function ψ_1 satisfies the similar inequalities

$$\partial_y\psi_1\big|_{y=-d}<0, \qquad \partial_y\psi_1\big|_{y=-d_2+f}>0, \qquad \partial_{yy}\psi_1>0 \text{ in } \overline{\Omega}(f). \tag{A.9}$$

Hence, for each $x\in[0,L/2]$, there exists a unique $\xi(x)\in(-d,-d_2+f(x))$ such that $\partial_y\psi_1(x,\xi(x))=0$. Due to the third inequality in (A.9) we conclude from the implicit function theorem that ξ is smooth and

$$\partial_{xy}\psi_1(x,\xi(x))+\xi'(x)\partial_{yy}\psi_1(x,\xi(x))=0 \qquad \text{for all } x\in[0,L/2]. \tag{A.10}$$

We are going to determine now the sign of $\partial_{xy}\psi_1$. To this end note that $\psi_1=w_1\circ\Phi_f^{-1}$ where $w_1\in C^{3+\alpha}_{e,per}(\overline{\Omega}_1)$ is the unique solution of the problem (2.9), that is $w_1:=w_1(\Lambda(s),(f,h)(s))$. By the chain rule we get

$$\begin{aligned}\partial_{xy}\psi_1=&-\frac{d_1f'}{(d_1+f)^2}\partial_y w_1\circ\Phi_f^{-1}+\frac{d_1}{d_1+f}\partial_{xy}w_1\circ\Phi_f^{-1}-\frac{d_1^2f'y}{(d_1+f)^3}\partial_{yy}w_1\circ\Phi_f^{-1}\\&+\frac{dd_1ff'}{(d_1+f)^3}\partial_{yy}w_1\circ\Phi_f^{-1}-\frac{dd_1f'}{(d_1+f)^2}\partial_{yy}w_1\circ\Phi_f^{-1}.\end{aligned}$$

On the other hand we have the following expansion

$$w_1(\Lambda(s),(f,h)s))=w_1(\Lambda_1,0)+\partial_\Lambda w_1(\Lambda_1,0)[\Lambda(s)-\Lambda_1]+\partial_f w_1(\Lambda_1,0)[f(s)]+O(s^2)$$

in $C^{3+\alpha}_{e,per}(\overline{\Omega}_1)$. Observing that

$$\left.\begin{aligned}&\partial_y w_1\circ\Phi_f^{-1}=\partial_y\psi_1^0+O(s),\\&\partial_{xy}w_1\circ\Phi_f^{-1}=\partial_{xy}(\partial_f w_1(\Lambda_1,0)[f])+O(s^2),\\&\partial_{yy}w_1\circ\Phi_f^{-1}=\gamma_1+O(s),\end{aligned}\right\}\qquad \text{in } C^{2+\alpha}_{e,per}\big(\overline{\Omega(f)}\big),$$

a lengthy calculation leads us to

$$\partial_{xy}\psi_1=-sm_1^{22}(\Lambda_1)L_1\Lambda_1\frac{\cosh(L_1(d+y))}{\sinh(L_1d_1)}\sin(L_1x)+O(s^2) \qquad \text{in } C^{1+\alpha}_{e,per}\big(\overline{\Omega(f)}\big).$$

A similar argument to the one used in (i) shows that $\partial_{xy}\psi_1>0$ in $\Omega(f)$ if $s>0$ is sufficiently small. The latter property together with (A.9) and (A.10) implies that $\xi'<0$ in $x\in(0,L/2)$. This proves the claim in (b). Since (c) is an obvious consequence of (iii) we have completed the proof. ∎

It follows now readily from Theorem 3.4 and Lemma A.1 that the streamline pattern in the moving frame for the non-laminar solutions found in Theorem 3.4 for $\gamma_1>\gamma_2$ is as in Figure 3.1 (left image). The next lemma justifies the right image of Figure 3.1.

LEMMA A.2. *Assume that* $\gamma_1<\gamma_2,\gamma_2>0$ *and let*

$$((f,h),\psi_1,\psi_2)\in\big(C^{3+\alpha}_{per}(\mathbb{R})\big)^2\times C^{3+\alpha}_{per}\big(\overline{\Omega(f)}\big)\times C^{3+\alpha}_{per}\big(\overline{\Omega(f,h)}\big)$$

be a solution of (2.2) *that is determined by a point* $(\Lambda(s),(f,h)(s))$ *on the bifurcation curve found in Theorem 3.4. Provided that* s *is small enough, the following assertions are true:*

(*i*) $f'>0$ *and* $h'<0$ *on* $(0,L/2)$*;*

(*ii*) $\partial_x\psi_1>0$ *in* $\{(x,y)\in\Omega(f):x\in(0,L/2)\}$ *and* $\partial_y\psi_1<0$ *in* $\Omega(f)$*;*

(*iii*) $\partial_x\psi_2>0$ *in in* $\{(x,y)\in\Omega(f,h):x\in(0,L/2)\}$*;*

(*iv*) *There is a smooth curve* $\{(x,\xi(x)):x\in[0,L/2]\}$ *with* $-d_2+f(x)<\xi(x)<h(x)$ *for all* $x\in[0,L/2]$ *and additionally satisfying:*

(*a*) *Given* $x\in[0,L/2]$*, it holds that:* $\partial_y\psi_2(x,\xi(x))=0$*,* $\partial_y\psi_2(x,y)<0$ *for all* $y\in[-d_2+f(x),\xi(x))$*, and* $\partial_y\psi_2(x,y)>0$ *for all* $y\in(\xi(x),h(x)]$*;*

(*b*) ξ *is strictly decreasing on* $[0,L/2]$*;*

(*c*) *The function* $[x\mapsto\psi_2(x,\xi(x))]$ *is strictly increasing on* $[0,L/2]$*.*

The next lemma provides a justification for the left image of Figure 3.2.

LEMMA A.3. *Assume that* $\gamma_2>0,\gamma_1d_1+\gamma_2d_2>0$ *and let*

$$((f,h),\psi_1,\psi_2)\in\left(C^{3+\alpha}_{per}(\mathbb{R})\right)^2\times C^{3+\alpha}_{per}\left(\overline{\Omega(f)}\right)\times C^{3+\alpha}_{per}\left(\overline{\Omega(f,h)}\right)$$

be a solution of (2.2) *that is determined by a point* $(\Lambda(s),(f,h)(s))$ *on the bifurcation curve found in Theorem 3.5. Provided that* s *is small enough, the following assertions are true:*

(*i*) $f'>0$ *and* $h'<0$ *on* $(0,L/2)$*;*

(*ii*) $\partial_x\psi_1>0$ *in* $\{(x,y)\in\Omega(f):x\in(0,L/2)\}$ *and* $\partial_y\psi_1<0$ *in* $\Omega(f)$*;*

(*iii*) $\partial_x\psi_2>0$ *in in* $\{(x,y)\in\Omega(f,h):x\in(0,L/2)\}$*;*

(*iv*) *There is a smooth curve* $\{(x,\xi(x)):x\in[0,L/2]\}$ *with* $-d_2+f(x)<\xi(x)<h(x)$ *for all* $x\in[0,L/2]$ *and additionally satisfying:*

(*a*) *Given* $x\in[0,L/2]$*, it holds that:* $\partial_y\psi_2(x,\xi(x))=0$*,* $\partial_y\psi_2(x,y)<0$ *for all* $y\in[-d_2+f(x),\xi(x))$*, and* $\partial_y\psi_2(x,y)>0$ *for all* $y\in(\xi(x),h(x)]$*;*

(*b*) ξ *is strictly decreasing on* $[0,L/2]$*;*

(*c*) *The function* $[x\mapsto\psi_2(x,\xi(x))]$ *is strictly increasing on* $[0,L/2]$*.*

We provide now a justification for the right image of Figure 3.2.

LEMMA A.4. *Assume that* $\gamma_2>0,\gamma_1d_1+\gamma_2d_2\le0$ *and let*

$$((f,h),\psi_1,\psi_2)\in\left(C^{3+\alpha}_{per}(\mathbb{R})\right)^2\times C^{3+\alpha}_{per}\left(\overline{\Omega(f)}\right)\times C^{3+\alpha}_{per}\left(\overline{\Omega(f,h)}\right)$$

be a solution of (2.2) *that is determined by a point* $(\Lambda(s),(f,h)(s))$ *on the bifurcation curve found in Theorem 3.5. Provided that* s *is small enough, the following assertions are true:*

(*i*) $f'>0$ *and* $h'<0$ *on* $(0,L/2)$*;*

(*ii*) $\partial_x\psi_1>0$ *in* $\{(x,y)\in\Omega(f):x\in(0,L/2)\}$*;*

(*iii*) $\partial_x\psi_2>0$ *in in* $\{(x,y)\in\Omega(f,h):x\in(0,L/2)\}$*;*

(*iv*) *There is a smooth curve* $\{(x,\xi_1(x)):x\in[0,L/2]\}$ *with* $-d<\xi_1(x)<-d_2+f(x)$ *for all* $x\in[0,L/2]$ *and additionally satisfying:*

(*a*) *Given* $x\in[0,L/2]$*, it holds that:* $\partial_y\psi_1(x,\xi_1(x))=0$*,* $\partial_y\psi_1(x,y)>0$ *for all* $y\in[-d,\xi_1(x))$*, and* $\partial_y\psi_1(x,y)<0$ *for all* $y\in(\xi_1(x),-d_2+f(x)]$*;*

(*b*) ξ_1 *is strictly increasing on* $[0,L/2]$*;*

(*c*) *The function* $[x\mapsto\psi_1(x,\xi_1(x))]$ *is strictly increasing on* $[0,L/2]$*.*

(*v*) *There is a smooth curve* $\{(x,\xi_2(x)) : x\in[0,L/2]\}$ *with* $-d_2+f(x)<\xi_2(x)<h(x)$ *for all* $x\in[0,L/2]$ *and additionally satisfying:*

- (*a*) *Given* $x\in[0,L/2]$, *it holds that:* $\partial_y\psi_2(x,\xi_2(x))=0$, $\partial_y\psi_2(x,y)<0$ *for all* $y\in[-d_2+f(x),\xi_2(x))$, *and* $\partial_y\psi_2(x,y)>0$ *for all* $y\in(\xi_2(x),h(x)]$;
- (*b*) ξ_2 *is strictly decreasing on* $[0,L/2]$;
- (*c*) *The function* $[x\mapsto\psi_2(x,\xi_2(x))]$ *is strictly increasing on* $[0,L/2]$.

We consider now the non-laminar flows corresponding to the bifurcation solutions found in Theorem 3.6 and prove the following result which justifies the left image of Figure 3.3.

Lemma A.5. *Assume that* $\gamma_2>0,\gamma_1 d_1+\gamma_2 d_2<0$ *and let*

$$((f,h),\psi_1,\psi_2)\in\left(C^{3+\alpha}_{per}(\mathbb{R})\right)^2\times C^{3+\alpha}_{per}\left(\overline{\Omega(f)}\right)\times C^{3+\alpha}_{per}\left(\overline{\Omega(f,h)}\right)$$

be a solution of (2.2) *that is determined by a point* $(\Lambda(s),(f,h)(s))$ *on the bifurcation curve found in Theorem 3.6. Provided that s is small enough, the following assertions are true:*

(*i*) $f'<0$ *and* $h'<0$ *on* $(0,L/2)$;

(*ii*) $\partial_x\psi_2<0$ *in* $\{(x,y)\in\Omega(f,h) : x\in(0,L/2)\}$ *and* $\partial_y\psi_2<0$ *in* $\Omega(f,h)$;

(*iii*) $\partial_x\psi_1<0$ *in* $\{(x,y)\in\Omega(f) : x\in(0,L/2)\}$;

(*iv*) *There is a smooth curve* $\{(x,\xi(x)) : x\in[0,L/2]\}$ *with* $-d<\xi(x)<-d_2+f(x)$ *for all* $x\in[0,L/2]$ *and additionally satisfying:*

- (*a*) *Given* $x\in[0,L/2]$, *it holds that:* $\partial_y\psi_1(x,\xi(x))=0$, $\partial_y\psi_1(x,y)>0$ *for all* $y\in[-d,\xi(x))$, *and* $\partial_y\psi_1(x,y)<0$ *for all* $y\in(\xi(x),-d_2+f(x)]$;
- (*b*) ξ *is strictly increasing on* $[0,L/2]$;
- (*c*) *The function* $[x\mapsto\psi_1(x,\xi(x))]$ *is strictly decreasing on* $[0,L/2]$.

Finally, we have the following result which justifies the right image of Figure 3.3.

Lemma A.6. *Assume that* $\gamma_2=0$, $\gamma_1<0$, *and let*

$$((f,h),\psi_1,\psi_2)\in\left(C^{3+\alpha}_{per}(\mathbb{R})\right)^2\times C^{3+\alpha}_{per}\left(\overline{\Omega(f)}\right)\times C^{3+\alpha}_{per}\left(\overline{\Omega(f,h)}\right)$$

be a solution of (2.2) *that is determined by a point* $(\Lambda(s),(f,h)(s))$ *on one of the bifurcation curves found in Theorems 4.2 and 4.3. Then, the assertions from Lemma A.5 hold verbatim.*

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