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DYNAMICS OF THE 3D FRACTIONAL GINZBURG–LANDAU EQUATION WITH MULTIPLICATIVE NOISE ON AN UNBOUNDED DOMAIN*

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Abstract. We study a stochastic fractional complex Ginzburg–Landau equation with multiplicative noise in three spatial dimensions with particular interest in the asymptotic behavior of its solutions. We first transform our equation into a random equation whose solutions generate a random dynamical system. A priori estimates are derived when the nonlinearity satisfies certain growth conditions. Applying the estimates for far-field values of solutions and a cut-off technique, asymptotic compactness is proved. Furthermore, the existence of a random attractor in $H^1(\mathbb{R}^3)$ of the random dynamical system is established.

Key words. Stochastic fractional Ginzburg–Landau equation, asymptotic compactness, random attractor, pullback attractor.

AMS subject classifications. 37L55, 60H15, 35Q99.

1. Introduction

A fractional differential equation is an equation that contains fractional derivatives or fractional integrals. The fractional derivative and the fractional integral have a wide range of applications in physics, biology, chemistry, and other fields of science, such as kinetic theories of systems with chaotic dynamics (see [34, 41]), pseudochaotic dynamics (see [42]), dynamics in a complex or porous medium (see [13, 26, 35]), random walks with a memory and flights (see [24, 33, 40]), obstacle problems (see [6, 31]). Recently, some of the classical equations of mathematical physics have been postulated with fractional derivatives to better describe complex phenomena. Of particular interest are the fractional Schrödinger equation (see [12, 16, 17]), the fractional Landau–Lifshitz equation (see [19]), the fractional Landau–Lifshitz–Maxwell equation (see [28]) and the fractional Ginzburg–Landau equation (see [37]).

Small perturbations (such as molecular collisions in gases and liquids and electric fluctuations in resistors [15]) may be neglected during the derivation of these ideal models. However, the perturbations should be included to obtain a more realistic model and to better understand the dynamical behavior of the model.

One may represent the micro-effects by random perturbations in the dynamics of the macro observable through additive or multiplicative noise in the governing equation.

To study a stochastic partial differential equation, a key step is to examine the asymptotic behavior of the random dynamical systems generated by its solutions. Some nice works along these lines are, for example, by Crauel and Flandoli (see [7, 8]), who developed the theory of random attractors which closely parallels the deterministic case

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(see [36]), and by Debussche (see [11]) who proved that the Hausdorff dimension of the random attractor could be estimated by using global Lyapunov exponents.

The well-posedness of solutions of fractional partial differential equations has been studied to some extent (see [17, 19, 21, 28]). However, there are not many results for *stochastic* fractional partial differential equations. In this paper, we examine the asymptotic behavior of solutions of the fractional Ginzburg–Landau equation with multiplicative noise on an unbounded domain.

The fractional Ginzburg–Landau equation arises, for example, from the variational Euler–Lagrange equation for fractal media, which can be used to describe dynamical processes in a medium with fractal dispersion in [37]. In [29], the authors analyzed a one-dimensional fractional complex Ginzburg–Landau equation

$$u_t+(1+\mathrm{i}\nu)(-\triangle)^{\alpha}u+(1+\mathrm{i}\mu)|u|^{2\sigma}u=\rho u.$$

The well-posedness of solutions was obtained by applying the semigroup method under the condition

$$\frac{1}{2}\le\sigma\le\frac{1}{\sqrt{1+\mu^2}-1}.$$

The existence of a global attractor in L^2 was also proved when $\sigma=1$. In [23], the dynamics of a two-dimensional fractional complex Ginzburg–Landau equations is studied. A fractional Ginzburg–Landau equation on the line with special nonlinearity and multiplicative noise was analyzed in [22].

In this paper, we consider a general *three-dimensional* stochastic fractional Ginzburg–Landau equation with multiplicative noise of Stratonovich form defined in the entire space $\mathbb{R}^3$ given by

$$du+((1+\mathrm{i}\nu)(-\triangle)^{\alpha}u+\rho u)\,dt=f(x,u)dt+\beta u\circ dW(t),\qquad x\in\mathbb{R}^3,\quad t>0 \tag{1.1}$$

with the initial condition

$$u(x,0)=u_0(x),\qquad x\in\mathbb{R}^3, \tag{1.2}$$

where $u(x,t)$ is a complex-valued function on $\mathbb{R}^3\times[0,+\infty)$. In (1.1), i is the imaginary unit, ν is a real constants, $\rho>0$, $\alpha\in(1/2,1)$, and $f(x,u)$ is a nonlinear function, for instance, $f(x,u)=-(1+\mathrm{i}\mu)|\mathrm{u}|^{2\sigma}\mathrm{u}$ with $\mu\in\mathbb{R}$ and $\sigma>0$. For convenience, we sometimes write it as $f=f(x,u,\bar{u})$ or $f=f(u)$, and in the various lemmas that follow we assume f satisfies some of the following conditions:

$$\mathrm{Re}f(x,u)\bar{u}\le-\beta_1|u|^{2\sigma+2}+\gamma_1(x), \tag{1.3}$$

$$\mathrm{Re}f_u|\mathbf{V}|^2+\mathrm{Re}f_{\bar{u}}(\bar{\mathbf{V}})^2\le-\beta_\sigma|u|^{2\sigma}|\mathbf{V}|^2+|u|^{2\sigma-2}(\lambda_\sigma(u\bar{\mathbf{V}})^2+\bar{\lambda}_\sigma(\bar{u}\mathbf{V})^2), \tag{1.4}$$

$$\max\{|f_u|,|f_{\bar{u}}|\}\le\beta_2, \tag{1.5}$$

$$\left|\frac{\partial f(x,u)}{\partial x}\right|=|f_x|\le\gamma_2(x), \tag{1.6}$$

for $u\in\mathbb{C}$ and $\mathbf{V}\in\mathbb{C}^{\mathbf{n}}$, where $\sigma,\beta_i\,(i=1,2)$ are positive constants, β_σ is a positive constant depending on σ, λ_σ is a complex constant depending on σ, and $(\mathbf{V})^2=\mathbf{V}\cdot\mathbf{V}=\sum_{i=1}^n V_i^2$, (which is not an inner product on $\mathbb{C}^n$), and $\gamma_1(x)\in L^1(\mathbb{R}^3)$, $\gamma_2(x)\in L^2(\mathbb{R}^3)$. The white noise described by a two-sided Wiener process $W(t)$ on a complete probability space results from the fact that small irregularity has to be taken into account in some circumstances.

Most of the research with respect to random attractors is restricted to L^2. In this work, we obtain the existence of a pullback attractor in H^1 (actually, one can choose the space to be $H^\alpha, \alpha \in (0,1]$, but we prefer the stronger regularity of the random attractor in H^1).

The concept of pullback random attractor, which is an extension of global attractor in deterministic systems (see [2, 20, 30, 32, 36]) was introduced in [8, 14]. In the case of bounded domains, the existence of random attractors for stochastic partial differential equations has been investigated by many authors (see [1, 7, 8, 10, 14] and the references therein). However, the problem is more challenging in the case of unbounded domains. Recently, the existence of random attractors for systems on unbounded domains was studied in [3, 5, 38, 39], which provides guidance for this work.

It is well known that asymptotic compactness and the existence of a bounded absorbing set are sufficient to guarantee the existence of a random attractor for a continuous random dynamical system. However, Sobolev embeddings are not compact on an unbounded domain. In this paper, we employ a tail-estimates approach to prove the existence of a compact random attractor.

The paper is organized as follows. In Section 2, some preliminaries, notation, and random attractor theory for random dynamical systems are introduced. In Section 3, we define a continuous random dynamical system for the stochastic fractional complex Ginzburg–Landau equation. In Section 4, we derive uniform estimates for solutions, which include uniform estimates on far field values of solutions. In Section 5, we establish the asymptotic compactness of the solution operator, and then prove the existence of a pullback random attractor.

2. Preliminaries and notations

We first recall some basic concepts related to random attractors for stochastic dynamical systems (see [4, 8, 10] for more details).

Let $(X,||\cdot||_X)$ be a separable Hilbert space with Borel σ-algebra $\mathcal{B}(X)$, and let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space.

Definition 2.1. *$(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$ is called a measurable dynamical systems, if $\theta: \mathbb{R}\times\Omega\to\Omega$ is $(\mathcal{B}(\mathbb{R})\times\mathcal{F},\mathcal{F})$-measurable, $\theta_0=\mathbb{I}$, $\theta_{t+s}=\theta_t\circ\theta_s$ for all $t,s\in\mathbb{R}$, and $\theta_t A = A$ for all $t\in\mathbb{R}$ and $A\in\mathcal{F}$.*

Definition 2.2. *A stochastic process $\phi(t,\omega)$ is called a continuous random dynamical system (RDS) over $(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$ if ϕ is $(\mathcal{B}(\mathbb{R}^+)\times\mathcal{F}\times\mathcal{B}(X),\mathcal{B}(X))$-measurable, and for all $\omega\in\Omega$*

- *the mapping $\phi:\mathbb{R}^+\times\Omega\times X\to X$ is continuous;*
- *$\phi(0,\omega)=\mathbb{I}$ on X;*
- *$\phi(t+s,\omega,\chi)=\phi(t,\theta_s\omega,\phi(s,\omega,\chi))$ for all $t,s\geq 0$ and $\chi\in X$ (cocycle property).*

Definition 2.3. *A random bounded set $\{B(\omega)\}_{\omega\in\Omega}\subseteq X$ is called tempered with respect to $(\theta_t)_{t\in\mathbb{R}}$ if for P-a.e. $\omega\in\Omega$ and all $\epsilon>0$*

$$\lim_{t\to\infty} e^{-\epsilon t} d(B(\theta_{-t}\omega))=0.$$

where $d(B)=\sup_{\chi\in B}\|\chi\|_X$.

Consider a continuous random dynamical system $\phi(t,w)$ over $(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$ and let $\mathcal{D}$ be the collection of all tempered random set of X.

Definition 2.4. *$\mathcal{D}$ is called inclusion-closed if $D=\{D(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$ and $\tilde{D}=\{\tilde{D}(\omega)\subseteq X:\omega\in\Omega\}$ with $\tilde{D}(\omega)\subseteq D(\omega)$ for all $\omega\in\Omega$, then $\tilde{D}\in\mathcal{D}$.*

Definition 2.5. *Let $\mathcal{D}$ be a collection of random subsets of X and $\{K(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$. Then $\{K(\omega)\}_{\omega\in\Omega}$is called an absorbing set of ϕ in $\mathcal{D}$ if for all $B\in\mathcal{D}$ and P-a.e. $\omega\in\Omega$ there exist $t_B(\omega)>0$ such that*

$$\phi(t,\theta_{-t}\omega,B(\theta_{-t}\omega))\subseteq K(\omega),\qquad t\geq t_B(\omega).$$

Definition 2.6. *Let $\mathcal{D}$ be a collection of random subsets of X. Then ϕ is said to be $\mathcal{D}$-pullback asymptotically compact in X if for P-a.e. $\omega\in\Omega$, $\{\phi(t_n,\theta_{-t_n}\omega,\chi_n)\}_{n=1}^{\infty}$ has a convergent subsequence in X whenever $t_n\to\infty$, and $\chi_n\in B(\theta_{-t_n}\omega)$ with $\{B(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$.*

Definition 2.7. *Let $\mathcal{D}$ be a collection of random subsets of X and $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$. Then $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ is called a $\mathcal{D}$-random attractor (or $\mathcal{D}$-pullback attractor) for ϕ if the following conditions are satisfied, for P-a.e. $\omega\in\Omega$,*

- *$\mathcal{A}(\omega)$ is compact, and $\omega\to d(\chi,\mathcal{A}(\omega))$ is measurable for every $\chi\in X$;*
- *$\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ is strictly invariant, i.e., $\phi(t,\omega,\mathcal{A}(\omega))=\mathcal{A}(\theta_t\omega)$, for all $t\geq 0$ and for a.e. $\omega\in\Omega$;*
- *$\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ attracts all sets in $\mathcal{D}$, i.e., for all $B\in\mathcal{D}$ and a.e. $\omega\in\Omega$ we have*

$$\lim_{t\to\infty} d(\phi(t,\theta_{-t}\omega,B(\theta_{-t}\omega)),\mathcal{A}(\omega))=0,$$

where d is the Hausdorff semi-metric given by $d(Y,Z)=\sup_{y\in Y}\inf_{z\in Z}\|y-z\|_X$, for any $Y,Z\subseteq X$.

According to [9], we can infer the following result.

Proposition 2.8. *Let $\mathcal{D}$ be an inclusion-closed collection of random subsets of X and ϕ a continuous RDS on X over $(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$. Suppose that $\{K(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$ is a closed absorbing set of ϕ and ϕ is $\mathcal{D}$-pullback asymptotically compact in X. Then ϕ has a unique $\mathcal{D}$-random attractor which is given by $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ with*

$$\mathcal{A}(\omega)=\bigcap_{\kappa\geq 0}\overline{\bigcup_{t\geq\kappa}\phi(t,\theta_{-t}\omega,K(\theta_{-t}\omega))}.$$

For convenience, we recall some notation related to the fractional derivative and fractional Sobolev spaces. First, we present the definition and some properties of $(-\triangle)^{\alpha}$ through Fourier transforms (see [18]). The negative powers $(-\triangle)^{\frac{\beta}{2}}$ (that is, $(-\triangle)^{-\frac{\beta}{2}}$), $\mathrm{Re}\beta>0$, can be represented by Riesz potentials

$$(\mathcal{I}^{\beta}\varphi)(x)=\frac{1}{\gamma(\beta)}\int_{\mathbb{R}^3}|x-y|^{-3+\beta}\varphi(y)dy,$$

where $\gamma(\beta)=\pi^{3/2}2^{\beta}\Gamma(\frac{\beta}{2})/\Gamma(\frac{3}{2}-\frac{\beta}{2})$. We consider the Fourier transform

$$\Phi(\xi)=\int_{\mathbb{R}^3}\phi(x)e^{-\mathrm{i}(x\cdot\xi)}dz,$$

so $(-\triangle)^{\frac{\beta}{2}}$ can be defined as

$$\mathcal{F}\{(-\triangle)^{\frac{\beta}{2}}\varphi\}=|k|^{\beta}\Phi,$$

$$(-\triangle)^{\frac{\beta}{2}}\varphi=\mathcal{F}^{-1}\{|k|^{\beta}\Phi\}=\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}|k|^{\beta}\Phi e^{\mathrm{i}k\cdot x}dk,$$

where $\triangle=\partial^2/\partial x_1^2+\partial^2/\partial x_2^2+\partial^2/\partial x_3^2$.

Let $H^{2\alpha}(\mathbb{R}^3)$ denote the complete Sobolev space of order α under the norm:

$$\|u\|^2_{H^{2\alpha}(\mathbb{R}^3)}=\int_{\mathbb{R}^3}(1+|k|^{4\alpha})|\hat{u}(k)|^2dk.$$

By virtue of the definition of $(-\triangle)^{\alpha}$, we have the following formula for integration by parts.

LEMMA 2.9. *If $f,g\in H^{2\alpha}(\mathbb{R}^n)$, then the following equation holds.*

$$\int_{\mathbb{R}^n}(-\triangle)^{\alpha}f\cdot gdx=\int_{\mathbb{R}^n}(-\triangle)^{\alpha_1}f\cdot(-\triangle)^{\alpha_2}gdx, \tag{2.1}$$

where α_1,α_2 are nonnegative constant and satisfy $\alpha_1+\alpha_2=\alpha$.

Proof. By the definition of $(-\triangle)^{\alpha}$ and Parseval's formula, we have

$$\begin{aligned}\int_{\mathbb{R}^n}(-\triangle)^{\alpha}f\cdot gdx&=\int_{\mathbb{R}^n}\mathcal{F}^{-1}\{|k|^{2\alpha}\hat{f}\}\cdot gdx=\int_{\mathbb{R}^n}\mathcal{F}^{-1}\{|k|^{2\alpha}\hat{f}\}\cdot\mathcal{F}^{-1}\hat{g}dx\\&=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|k|^{2\alpha}\hat{f}\cdot\hat{g}dk=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|k|^{2\alpha_1}\hat{f}\cdot|k|^{2\alpha_2}\hat{g}dk\\&=\int_{\mathbb{R}^n}\mathcal{F}^{-1}\{|k|^{2\alpha_1}\hat{f}\}\cdot\mathcal{F}^{-1}\{|k|^{2\alpha_2}\hat{g}\}dx=\int_{\mathbb{R}^n}(-\triangle)^{\alpha_1}f\cdot(-\triangle)^{\alpha_2}gdx.\end{aligned}$$

□

In addition, the following Gagliardo–Nirenberg inequality (see [27]) is also frequently used.

LEMMA 2.10. *Let u belong to $L^q(\mathbb{R}^n)$ and its derivatives of order m, $D^m u$, belong to $L^r(\mathbb{R}^n)$, $1\le q,r\le\infty$. For the derivatives $D^j u$, $0\le j<m$, the following inequalities hold*

$$\|D^j u\|_{L^p}\le c\|D^m u\|^{\theta}_{L^r}\|u\|^{1-\theta}_{L^q}, \tag{2.2}$$

where

$$\frac{1}{p}=\frac{j}{n}+\theta(\frac{1}{r}-\frac{m}{n})+(1-\theta)\frac{1}{q},$$

for all θ in the interval

$$\frac{j}{m}\le\theta\le 1,$$

(the constant c depending only on n,m,j,q,r,θ), with the following exceptional case

1. If $j=0$, $rm<n$, $q=\infty$, then we make the additional assumption that either u tends to zero at infinite or $u\in L^{\tilde{q}}$ for some finite $\tilde{q}>0$.

2. If $1<r<\infty$, and $m-j-n/r$ is a nonnegative integer, then (2.2) *holds only for θ satisfying $j/m\le\theta<1$.*

In the forthcoming discussions, we denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the norm and the inner product in $L^2(\mathbb{R}^3)$ and use $\|\cdot\|_p$ to denote the norm in $L^p(\mathbb{R}^3)$. Otherwise, the letters c,c_j $(j=1,2,\ldots)$ are generic positive constants which may change their values from line to line or even in the same line.

3. Stochastic fractional complex Ginzburg–Landau equation

In the sequel, we consider the probability space $(\Omega,\mathcal{F},\mathbb{P})$ where

$$\Omega=\{\omega\in C(\mathbb{R},\mathbb{R}):\omega(0)=0\},$$

$\mathcal{F}$ is the Borel σ-algebra induced by the compact-open topology of Ω, and $\mathbb{P}$ the corresponding Wiener measure on $(\Omega,\mathcal{F})$. Define a shift on ω by

$$\theta_t\omega(\cdot)=\omega(\cdot+t)-\omega(t),\quad \omega\in\Omega,\ \ t\in\mathbb{R}.$$

Then $(\Omega,\mathcal{F},(\theta_t)_{t\in\mathbb{R}})$ is a metric dynamical system.

In this section, we discuss the existence of a continuous random dynamical system for the stochastic fractional complex Ginzburg–Landau equation perturbed by a multiplicative white noise in the Stratonovich sense. Thanks to the special linear multiplicative noise, the stochastic fractional Ginzburg–Landau equation can be reduced to an equation with random coefficients by a suitable change of variable. To this end, we consider the stationary process

$$z(t)=z(t,\omega)=z(\theta_t\omega)=-\int_{-\infty}^{0}e^{\tau}(\theta_t\omega)(\tau)d\tau,\quad t\in\mathbb{R},$$

satisfies the stochastic differential equation:

$$dz+zdt=dW(t).$$

Moreover, for any t,s,

$$z(t,\theta_s\omega)=z(t+s,\omega),\qquad P\text{-a.s..}$$

Here the exceptional set may depend a priori on t and s. In fact, we suppose that z has a continuous modification. Once this modification is chosen, the exceptional set is independent of t. It is known that the random variable $z(\omega)$ is tempered (see [1, 7, 14]), there exists a θ_t-invariant set $\tilde{\Omega}\subseteq\Omega$ of full $\mathbb{P}$ measure such that for every $\omega\in\tilde{\Omega}$, $z(\theta_t\omega)$ is continuous in t and

$$\lim_{t\to\pm\infty}\frac{|z(\theta_t\omega)|}{|t|}=0,\quad \text{for all } \omega\in\tilde{\Omega}, \tag{3.1}$$

and

$$\lim_{t\to\pm\infty}\frac{1}{t}\int_0^t z(\theta_t\omega)dt=0,\quad \text{for all } \omega\in\tilde{\Omega}. \tag{3.2}$$

We rewrite the unknown $v(t)$ as $v(t)=e^{-\beta z(\theta_t\omega)}u(t)$ to obtain the following random differential equation

$$v_t=-(1+\mathrm{i}\nu)(-\triangle)^{\alpha}v+e^{-\beta z(\theta_t\omega)}f(e^{\beta z(\theta_t\omega)}v)+(\beta z(\theta_t\omega)-\rho)v \tag{3.3}$$

with the initial data

$$v(x,0)=v_0(x)=e^{-\beta z(\omega)}u_0(x),\qquad\qquad x\in\mathbb{R}^3. \tag{3.4}$$

Next, we construct a random dynamical system modeling the stochastic fractional Ginzburg–Landau equation.

By the Galerkin method, one can show that if f satisfies (1.3)–(1.6), then in the case of a bounded domain with Dirichlet boundary conditions, for P-a.e. $\omega\in\Omega$, and for all $v_0\in H^1$, equation (3.3) has a unique solution $v(\cdot,\omega,v_0)\in C([0,\infty),H^1)\cap L^2((0,T);H^{1+\alpha})$ with $v(0,\omega,v_0)=v_0$ for every $T>0$. This is similar to [21]. Then, following the approach in [25], we take the domain to be a sequence of balls with radius approaching ∞ to deduce the existence of a weak solution of equation (3.3) on $\mathbb{R}^3$. Furthermore, we obtain that $v(t,\omega,v_0)$ is unique and continuous with respect to v_0 in $H^1(\mathbb{R}^3)$ for all $t\geq 0$. Let $u(t,\omega,u_0)=e^{\beta z(\theta_t\omega)}v(t,\omega,e^{-\beta z(\omega)}u_0))$. Then the process u is the solution of problem (1.1)–(1.2). We now define a mapping $\phi:\mathbb{R}^+\times\Omega\times H^1(\mathbb{R}^3)\to H^1(\mathbb{R}^3)$ by

$$\phi(t,\omega,u_0)=u(t,\omega,u_0)=e^{\beta z(\theta_t\omega)}v(t,\omega,e^{-\beta z(\omega)}u_0),$$

for $u_0\in H^1(\mathbb{R}^3)$, $t\geq 0$ and for all $\omega\in\Omega$. It is easy to check that ϕ satisfies the three conditions in Definition 2.2. Therefore, ϕ is a continuous random dynamical system associated with problem (3.3) on $H^1(\mathbb{R}^3)$.

Let

$$\varphi(t,\omega,v_0)=v(t,\omega,v_0)\quad\text{for}\quad v_0\in H^1(\mathbb{R}^3),\quad t\geq 0\quad\text{and for all}\quad \omega\in\Omega.$$

Then φ is a continuous random dynamical system associated with problem (1.1) on $H^1(\mathbb{R}^3)$. It is worth noticing that, the two random dynamical systems are equivalent. It is easy to check that ϕ has a random attractor provided φ possesses a random attractor. Then, we only need to consider the random dynamical system φ.

4. Uniform estimates of solutions

In this section, we deduce uniform estimates on the solutions of the stochastic fractional complex Ginzburg–Landau equation on $\mathbb{R}^3$ when $t\to\infty$. These estimates are necessary for proving the existence of bounded absorbing sets and the asymptotic compactness of the random dynamical system associated with the equation. In particular, we will show that the solutions for large space variables are uniformly small when time is sufficiently large.

From now on, we always suppose that $\mathcal{D}$ is the collection of all tempered random subsets of $H^1(\mathbb{R}^3)$. First, we derive the following uniform on v in $\mathcal{D}$.

Lemma 4.1. *Suppose that* (1.3) *holds. Let $B=\{B(\omega)\}\in\mathcal{D}$ and $v_0(\omega)\in B(\omega)$, and let $\varrho_0>0$ be fixed and $0<\delta<2\rho$. Then for P-a.e. $\omega\in\Omega$, there exists $T_{0_B}(\omega)>0$ such that for any $t\geq T_{0_B}(\omega)$, one has*

$$\delta\int_{-t}^{0}e^{2\beta\int_s^0 z(\theta_\tau\omega)d\tau+(2\rho-\delta)s}\|v(s+t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2ds+\|v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2\leq\varrho_0^2. \tag{4.1}$$

Proof. Taking the inner product in L^2 of (3.3) with v and taking the real part, we obtain

$$\frac{1}{2}\frac{d}{dt}\|v\|^2+\|(-\triangle)^{\frac{\alpha}{2}}v\|^2=e^{-\beta z(\theta_t\omega)}\mathrm{Re}\int_{\mathbb{R}^3}f(e^{\beta z(\theta_t\omega)}v)\bar{v}dx+(\beta z(\theta_t\omega)-\rho)\|v\|^2. \tag{4.2}$$

By condition (1.3), we have

$$e^{-\beta z(\theta_t\omega)}\mathrm{Re}\int_{\mathbb{R}^3}f(e^{\beta z(\theta_t\omega)}v)\bar{v}dx\leq-\beta_1e^{-2\beta z(\theta_t\omega)}\|e^{\beta z(\theta_t\omega)}v\|_{2\sigma+2}^{2\sigma+2}+e^{-2\beta z(\theta_t\omega)}\|\gamma_1(x)\|_{L^1}.$$

Then (4.2) can be rewritten as

$$\begin{aligned}&\frac{d}{dt}\|v\|^2+2\|(-\triangle)^{\frac{\alpha}{2}}v\|^2+2\beta_1 e^{-2\beta z(\theta_t\omega)}\|e^{\beta z(\theta_t\omega)}v\|_{2\sigma+2}^{2\sigma+2}\\ \le&2(\beta z(\theta_t\omega)-\rho)\|v\|^2+2e^{-2\beta z(\theta_t\omega)}\|\gamma_1(x)\|_1.\end{aligned}\tag{4.3}$$

Therefore,

$$\frac{d}{dt}\|v\|^2+\delta\|v\|^2\le(2\beta z(\theta_t\omega)-2\rho+\delta)\|v\|^2+2e^{-2\beta z(\theta_t\omega)}\|\gamma_1(x)\|_1.\tag{4.4}$$

Here, $\rho>0$, so there exists $\delta>0$ such that $2\rho>\delta>0$. Multiplying (4.4) by $e^{-2\beta\int_0^t z(\theta_s\omega)ds+(2\rho-\delta)t}$, and integrating over $(0,t)$, we infer that

$$\begin{aligned}&\|v(t,\omega,v_0(\omega))\|^2+\delta\int_0^t e^{2\beta\int_s^t z(\theta_\tau\omega)d\tau+(2\rho-\delta)(s-t)}\|v(s,\omega,v_0(\omega))\|^2ds\\ \le&e^{2\beta\int_0^t z(\theta_s\omega)ds+(\delta-2\rho)t}\|v_0(\omega)\|^2+2\int_0^t e^{2\beta\int_s^t z(\theta_\tau\omega)d\tau+(2\rho-\delta)(s-t)-2\beta z(\theta_s\omega)}\|\gamma_1(x)\|_1ds\\ \le&e^{2\beta\int_0^t z(\theta_s\omega)ds+(\delta-2\rho)t}\|v_0(\omega)\|^2+2c_1\int_0^t e^{2\beta\int_s^t z(\theta_\tau\omega)d\tau+(2\rho-\delta)(s-t)-2\beta z(\theta_s\omega)}ds.\end{aligned}\tag{4.5}$$

Substituting ω by $\theta_{-t}\omega$, then we deduce from (4.5),

$$\begin{aligned}&\delta\int_0^t e^{2\beta\int_s^t z(\theta_{\tau-t}\omega)d\tau+(2\rho-\delta)(s-t)}\|v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2ds+\|v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2\\ \le&e^{2\beta\int_0^t z(\theta_{s-t}\omega)ds+(\delta-2\rho)t}\|v_0(\theta_{-t}\omega)\|^2+2c_1\int_0^t e^{2\beta\int_s^t z(\theta_{\tau-t}\omega)d\tau+(2\rho-\delta)(s-t)-2\beta z(\theta_{s-t}\omega)}ds.\end{aligned}$$

Applying the transformation of variables, one has

$$\begin{aligned}&\delta\int_{-t}^0 e^{2\beta\int_s^0 z(\theta_\tau\omega)d\tau+(2\rho-\delta)s}\|v(s+t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2ds+\|v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2\\ \le&e^{2\beta\int_{-t}^0 z(\theta_s\omega)ds+(\delta-2\rho)t}\|v_0(\theta_{-t}\omega)\|^2+2c_1\int_{-t}^0 e^{2\beta\int_s^0 z(\theta_\tau\omega)d\tau+(2\rho-\delta)s-2\beta z(\theta_s\omega)}ds.\end{aligned}\tag{4.6}$$

$\{B(\omega)\}\in\mathcal{D}$ is tempered, so for any $v_0(\theta_{-t}\omega)\in B(\theta_{-t}\omega)$,

$$\lim_{t\to+\infty}e^{2\beta\int_{-t}^0 z(\theta_s\omega)ds+(\delta-2\rho)t}\|v_0(\theta_{-t}\omega)\|^2=\lim_{t\to+\infty}e^{2\beta\int_{-t}^0 z(\theta_s\omega)ds+(\delta-2\rho)t-2\beta z(\theta_t\omega)}=0.\tag{4.7}$$

Therefore, there exists $T_{0_B}(\omega)>0$ such that for any $t\ge T_{0_B}(\omega)$,

$$e^{2\beta\int_{-t}^0 z(\theta_s\omega)ds+(\delta-2\rho)t}\|v_0(\theta_{-t}\omega)\|^2+2c_1\int_{-t}^0 e^{2\beta\int_s^0 z(\theta_\tau\omega)d\tau+(2\rho-\delta)s-2\beta z(\theta_s\omega)}ds\le\varrho_0^2,\tag{4.8}$$

which along with (4.6) shows that, for any $t\ge T_{0_B}(\omega)$,

$$\delta\int_{-t}^0 e^{2\beta\int_s^0 z(\theta_\tau\omega)d\tau+(2\rho-\delta)s}\|v(s+t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2ds+\|v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2\le\varrho_0^2.\tag{4.9}$$

Thus completing the proof. □

LEMMA 4.2. *Suppose (1.4) and $\beta_\sigma \le 2|\lambda_\sigma|$. Let $B=\{B(\omega)\}\in\mathcal{D}$ and $v_0(\omega)\in B(\omega)$, let $\varrho_1>0$ be fixed. Then for P-a.e. $\omega\in\Omega$, there exists $T_{1_B}(\omega)>0$ such that for any $t\ge T_{1_B}(\omega)$, we have*

$$\begin{aligned}\int_{-t}^{0} & e^{2\beta\int_s^0 z(\theta_s\omega)d\tau+(2\rho-\delta)s}\|(-\triangle)^{\frac{\alpha+1}{2}}v(s+t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2ds\\ &+\frac{\delta}{2}\int_{-t}^{0}e^{2\beta\int_s^0 z(\theta_s\omega)d\tau+(2\rho-\delta)s}\|\nabla v(s+t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2ds\\ &+\|\nabla v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2\le\varrho_1^2. \end{aligned}\tag{4.10}$$

Proof. Taking the inner product in L^2 of (3.3) with $-\triangle v$ and taking the real part, we obtain

$$\begin{aligned}&\frac{d}{dt}\|\nabla v\|^2+2\|(-\triangle)^{\frac{\alpha+1}{2}}v\|^2\\ &=-2e^{-2\beta z(\theta_t\omega)}\mathrm{Re}\Big(f(e^{\beta z(\theta_t\omega)}v),\triangle(e^{\beta z(\theta_t\omega)}v)\Big)+2(\beta z(\theta_t\omega)-\rho)\|\nabla v\|^2.\end{aligned}\tag{4.11}$$

Now, we will estimate the first term on the right-hand side of (4.11). For convenience, we set $\psi=e^{\beta z(\theta_t\omega)}v$. Integrating by parts and using (1.4) and (1.6), then applying Young's inequality, we find

$$\begin{aligned}&-\mathrm{Re}\Big(f(e^{\beta z(\theta_t\omega)}v),\triangle(e^{\beta z(\theta_t\omega)}v)\Big)\\ &=-\mathrm{Re}(f(\psi),\triangle\psi)\\ &=\mathrm{Re}\int_{\mathbb{R}^3}\big(f_\psi(\psi)|\nabla\psi|^2+f_{\bar\psi}(\psi)\nabla\bar\psi\nabla\bar\psi\big)\,dx+\mathrm{Re}\int_{\mathbb{R}^3}f_x\nabla\bar\psi dx\\ &\le\int_{\mathbb{R}^3}\Big(-\beta_\sigma|\psi|^{2\sigma}|\nabla\psi|^2+|\psi|^{2(\sigma-1)}(\lambda_\sigma(\psi\nabla\bar\psi)^2+\bar\lambda_\sigma(\bar\psi\nabla\psi)^2)\Big)\,dx\\ &\quad+\int_{\mathbb{R}^3}|\gamma_2(x)||\nabla v|e^{\beta z(\theta_t\omega)}dx\\ &\le\int_{\mathbb{R}^3}|\psi|^{2(\sigma-1)}\big(-\beta_\sigma|\psi|^2|\nabla\psi|^2+\lambda_\sigma(\psi\nabla\bar\psi)^2+\bar\lambda_\sigma(\bar\psi\nabla\psi)^2\big)\,dx\\ &\quad+\frac{\delta}{4}\|\nabla v\|^2+c_2e^{2\beta z(\theta_t\omega)}\\ &=\int_{\mathbb{R}^3}|\psi|^{2(\sigma-1)}\mathrm{tr}(YMY^H)dx+\frac{\delta}{4}\|\nabla v\|^2+c_2e^{2\beta z(\theta_t\omega)},\end{aligned}\tag{4.12}$$

where

$$Y=\begin{pmatrix}\bar\psi\nabla\psi\\ \psi\nabla\bar\psi\end{pmatrix}^H,\ M=\begin{pmatrix}-\frac{\beta_\sigma}{2} & \lambda_\sigma\\ \bar\lambda_\sigma & -\frac{\beta_\sigma}{2}\end{pmatrix},$$

and Y^H is the conjugate transpose of the matrix Y. We observe that the condition $\beta_\sigma\le2|\lambda_\sigma|$ implies that the matrix M is nonpositive definite. One can rewrite (4.11) as

$$\frac{d}{dt}\|\nabla v\|^2+2\|(-\triangle)^{\frac{\alpha+1}{2}}v\|^2+\frac{\delta}{2}\|\nabla v\|^2\le(2\beta z(\theta_t\omega)-2\rho+\delta)\|\nabla v\|^2+2c_2e^{2\beta z(\theta_t\omega)}.\tag{4.13}$$

Multiplying (4.13) by $e^{-2\beta\int_0^t z(\theta_s\omega)ds+(2\rho-\delta)t}$ and integrating over $(0,t)$, we infer that

$$\begin{aligned}
&\|\nabla v(t,\omega,v_0(\omega))\|^2+\int_0^t e^{2\beta\int_s^t z(\theta_\tau\omega)d\tau+(2\rho-\delta)(s-t)}\|(-\triangle)^{\frac{\alpha+1}{2}}v(s,\omega,v_0(\omega))\|^2ds\\
&\quad+\frac{\delta}{2}\int_0^t e^{2\beta\int_s^t z(\theta_\tau\omega)d\tau+(2\rho-\delta)(s-t)}\|\nabla v(s,\omega,v_0(\omega))\|^2ds\\
&\le e^{2\beta\int_0^t z(\theta_s\omega)ds+(\delta-2\rho)t}\|\nabla v_0(\omega)\|^2+2c_2\int_0^t e^{2\beta\int_s^t z(\theta_\tau\omega)d\tau+(2\rho-\delta)(s-t)+2\beta z(\theta_s\omega)}ds.
\end{aligned}\tag{4.14}$$

Substituting $\theta_{-t}\omega$ for ω, we then deduce from (4.13) that,

$$\begin{aligned}
&\int_0^t e^{2\beta\int_s^t z(\theta_{\tau-t}\omega)d\tau+(2\rho-\delta)(s-t)}\|(-\triangle)^{\frac{\alpha+1}{2}}v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2ds\\
&\quad+\frac{\delta}{2}\int_0^t e^{2\beta\int_s^t z(\theta_{\tau-t}\omega)d\tau+(2\rho-\delta)(s-t)}\|\nabla v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2ds\\
&\quad+\|\nabla v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2\\
&\le e^{2\beta\int_0^t z(\theta_{s-t}\omega)ds+(\delta-2\rho)t}\|\nabla v_0(\theta_{-t}\omega)\|^2\\
&\quad+2c_2\int_0^t e^{2\beta\int_s^t z(\theta_{\tau-t}\omega)d\tau+(2\rho-\delta)(s-t)+2\beta z(\theta_{s-t}\omega)}ds.
\end{aligned}$$

Changing the variables in the integrals, one has

$$\begin{aligned}
&\int_{-t}^0 e^{2\beta\int_s^0 z(\theta_\tau\omega)d\tau+(2\rho-\delta)s}\|(-\triangle)^{\frac{\alpha+1}{2}}v(s+t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2ds\\
&\quad+\frac{\delta}{2}\int_{-t}^0 e^{2\beta\int_s^0 z(\theta_\tau\omega)d\tau+(2\rho-\delta)s}\|\nabla v(s+t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2ds\\
&\quad+\|\nabla v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2\\
&\le e^{2\beta\int_{-t}^0 z(\theta_s\omega)ds+(\delta-2\rho)t}\|\nabla v_0(\theta_{-t}\omega)\|^2\\
&\quad+2c_2\int_{-t}^0 e^{2\beta\int_s^0 z(\theta_\tau\omega)d\tau+(2\rho-\delta)s+2\beta z(\theta_s\omega)}ds.
\end{aligned}\tag{4.15}$$

$\{B(\omega)\}\in\mathcal{D}$ is tempered, so for any $v_0(\theta_{-t}\omega)\in B(\theta_{-t}\omega)$,

$$\lim_{t\to+\infty}e^{2\beta\int_{-t}^0 z(\theta_s\omega)ds+(\delta-2\rho)t}\|\nabla v_0(\theta_{-t}\omega)\|^2=\lim_{t\to+\infty}e^{2\beta\int_{-t}^0 z(\theta_s\omega)ds+(\delta-2\rho)t+2\beta z(\theta_t\omega)}=0.\tag{4.16}$$

Therefore, there exists $T_{1_B}(\omega)>0$ such that for any $t\ge T_{1_B}(\omega)$,

$$e^{2\beta\int_{-t}^0 z(\theta_s\omega)ds+(\delta-2\rho)t}\|\nabla v_0(\theta_{-t}\omega)\|^2+2c_2\int_{-t}^0 e^{2\beta\int_s^0 z(\theta_\tau\omega)d\tau+(2\rho-\delta)s+2\beta z(\theta_s\omega)}ds\le\varrho_1^2,\tag{4.17}$$

which along with (4.15) shows that, for any $t\ge T_{1_B}(\omega)$,

$$\int_{-t}^0 e^{2\beta\int_s^0 z(\theta_\tau\omega)d\tau+(2\rho-\delta)s}\|(-\triangle)^{\frac{\alpha+1}{2}}v(s+t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2ds$$

$$+\frac{\delta}{2}\int_{-t}^{0} e^{2\beta\int_s^0 z(\theta_\tau\omega)d\tau+(2\rho-\delta)s}\|\nabla v(s+t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2 ds$$
$$+\|\nabla v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2\leq \varrho_1^2.$$

Thus completing the proof. ∎

Lemma 4.3. *Suppose that (1.5) holds. Let $B=\{B(\omega)\}\in\mathcal{D}$ and $v_0(\omega)\in B(\omega)$. Then for P-a.e. $\omega\in\Omega$, there exists $T_{1_B}(\omega)>0$ such that for any $t\geq T_{1_B}(\omega)$, one has*

$$\|(-\triangle)^{\frac{1+\alpha}{2}}v(t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2\leq \varrho_1^2+r_0^2+r_1^2+r_2^2\triangleq\varrho_2^2. \tag{4.18}$$

Proof. Taking the inner product of (3.3) with $(-\triangle)^{1+\alpha}v$ and taking the real part, we obtain

$$\begin{aligned}&\frac{d}{dt}\|(-\triangle)^{\frac{1+\alpha}{2}}v\|^2+2\|(-\triangle)^{\alpha+\frac{1}{2}}v\|^2\\ =&-2(\rho-\beta z(\theta_t\omega))\|(-\triangle)^{\frac{1+\alpha}{2}}v\|^2+2e^{-\beta z(\theta_t\omega)}\mathrm{Re}\Big(f(e^{\beta z(\theta_t\omega)}v),(-\triangle)^{1+\alpha}v\Big).\end{aligned} \tag{4.19}$$

We estimate the second term of the right-hand side of (4.19). For convenience, we set $\psi=e^{\beta z(\theta_t\omega)}v$. Integrating by parts, applying (1.5) and (1.6), and using the Hölder and Young's inequalities, we obtain

$$\begin{aligned}&2e^{-\beta z(\theta_t\omega)}\mathrm{Re}\Big(f(e^{\beta z(\theta_t\omega)}v),(-\triangle)^{1+\alpha}v\Big)\\
&=2e^{-2\beta z(\theta_t\omega)}\mathrm{Re}\Big(f(e^{\beta z(\theta_t\omega)}v),(-\triangle)^{1+\alpha}(e^{\beta z(\theta_t\omega)}v)\Big)\\
&=2e^{-2\beta z(\theta_t\omega)}\mathrm{Re}\big(f(\psi),(-\triangle)^{1+\alpha}\psi\big)\\
&\leq 2e^{-2\beta z(\theta_t\omega)}|(f_\psi(\psi)\nabla\psi+f_{\bar\psi}(\psi)\nabla\bar\psi+f_x,(-\triangle)^{\frac{1}{2}+\alpha}\psi)|\\
&\leq 4\beta_2e^{-2\beta z(\theta_t\omega)}\int_{\mathbb{R}^3}|\nabla\psi||(-\triangle)^{\frac{1}{2}+\alpha}\psi|dx+2e^{-\beta z(\theta_t\omega)}\int_{\mathbb{R}^3}|f_x||(-\triangle)^{\frac{1}{2}+\alpha}v|dx\\
&\leq 4\beta_2e^{-2\beta z(\theta_t\omega)}\|(-\triangle)^{\frac{1}{2}+\alpha}\psi\|\|\nabla\psi\|+2e^{-\beta z(\theta_t\omega)}\|(-\triangle)^{\frac{1}{2}+\alpha}v\|\|\gamma_2(x)\|\\
&=4\beta_2\|(-\triangle)^{\frac{1}{2}+\alpha}v\|\|\nabla v\|+2e^{-\beta z(\theta_t\omega)}\|(-\triangle)^{\frac{1}{2}+\alpha}v\|\|\gamma_2(x)\|\\
&\leq\|(-\triangle)^{\frac{1}{2}+\alpha}v\|^2+8\beta_2^2\|\nabla v\|^2+c_3e^{-2\beta z(\theta_t\omega)}.\end{aligned} \tag{4.20}$$

Substituting (4.20) into (4.19), we deduce that

$$\begin{aligned}&\frac{d}{dt}\|(-\triangle)^{\frac{1+\alpha}{2}}v\|^2+2(\rho-\beta z(\theta_t\omega))\|(-\triangle)^{\frac{1+\alpha}{2}}v\|^2+\|(-\triangle)^{\frac{1}{2}+\alpha}v\|^2\\ &\leq 8\beta_2^2\|\nabla v\|^2+c_3e^{-2\beta z(\theta_t\omega)}.\end{aligned} \tag{4.21}$$

This implies that

$$\begin{aligned}&\frac{d}{dt}\|(-\triangle)^{\frac{1+\alpha}{2}}v\|^2+(2\rho-\delta-2\beta z(\theta_t\omega))\|(-\triangle)^{\frac{1+\alpha}{2}}v\|^2+\|(-\triangle)^{\frac{1}{2}+\alpha}v\|^2\\ &\leq 8\beta_2^2\|\nabla v\|^2+c_3e^{-2\beta z(\theta_t\omega)}.\end{aligned} \tag{4.22}$$

Taking $t\geq T_{1_B}(\omega)$ and $s\in(t,t+1)$, multiplying (4.22) by $e^{-2\beta\int_0^t z(\theta_s\omega)ds+(2\rho-\delta)t}$, and integrating (4.21) over $(s,t+1)$, we get

$$\|(-\triangle)^{\frac{1+\alpha}{2}}v(t+1,\omega,v_0(\omega))\|^2$$

$$
\begin{aligned}
&\quad +\int_s^{t+1} e^{2\beta\int_\tau^{t+1} z(\theta_{\tau_1}\omega)d\tau_1+(\delta-2\rho)(t+1-\tau)}\|(-\triangle)^{\frac{1}{2}+\alpha}v(\tau,\omega,v_0(\omega))\|^2 d\tau\\
&\le e^{2\beta\int_s^{t+1} z(\theta_\tau\omega)d\tau+(\delta-2\rho)(t+1-s)}\|(-\triangle)^{\frac{1+\alpha}{2}}v(s,\omega,v_0(\omega))\|^2\\
&\quad +8\beta_2^2\int_s^{t+1} e^{2\beta\int_\tau^{t+1} z(\theta_{\tau_1}\omega)d\tau_1+(\delta-2\rho)(t+1-\tau)}\|\nabla v(\tau,\omega,v_0(\omega))\|^2 d\tau\\
&\quad +c_3\int_s^{t+1} e^{2\beta\int_\tau^{t+1} z(\theta_{\tau_1}\omega)d\tau_1+(\delta-2\rho)(t+1-\tau)-2\beta z(\theta_\tau\omega)}d\tau.
\end{aligned}
\tag{4.23}
$$

Integrating (4.23) with respect to s over $(t,t+1)$, then applying Gagliardo–Nirenberg inequality, we obtain

$$
\begin{aligned}
&\|(-\triangle)^{\frac{1+\alpha}{2}}v(t+1,\omega,v_0(\omega))\|^2\\
&\quad +\int_t^{t+1} e^{2\beta\int_\tau^{t+1} z(\theta_{\tau_1}\omega)d\tau_1+(\delta-2\rho)(t+1-\tau)}\|(-\triangle)^{\frac{1}{2}+\alpha}v(\tau,\omega,v_0(\omega))\|^2 d\tau\\
&\le \int_t^{t+1} e^{2\beta\int_s^{t+1} z(\theta_\tau\omega)d\tau+(\delta-2\rho)(t+1-s)}\|(-\triangle)^{\frac{1+\alpha}{2}}v(s,\omega,v_0(\omega))\|^2 ds\\
&\quad +8\beta_2^2\int_t^{t+1} e^{2\beta\int_\tau^{t+1} z(\theta_{\tau_1}\omega)d\tau_1+(\delta-2\rho)(t+1-\tau)}\|\nabla v(\tau,\omega,v_0(\omega))\|^2 d\tau\\
&\quad +c_3\int_t^{t+1} e^{2\beta\int_\tau^{t+1} z(\theta_{\tau_1}\omega)d\tau_1+(\delta-2\rho)(t+1-\tau)-2\beta z(\theta_\tau\omega)}d\tau\\
&\le \frac{1}{2}\int_t^{t+1} e^{2\beta\int_s^{t+1} z(\theta_\tau\omega)d\tau+(\delta-2\rho)(t+1-s)}\|(-\triangle)^{\frac{1}{2}+\alpha}v(s,\omega,v_0(\omega))\|^2 ds\\
&\quad +c_4\int_t^{t+1} e^{2\beta\int_s^{t+1} z(\theta_\tau\omega)d\tau+(\delta-2\rho)(t+1-s)}\|v(s,\omega,v_0(\omega))\|^2 ds\\
&\quad +8\beta_2^2\int_t^{t+1} e^{2\beta\int_\tau^{t+1} z(\theta_{\tau_1}\omega)d\tau_1+(\delta-2\rho)(t+1-\tau)}\|\nabla v(\tau,\omega,v_0(\omega))\|^2 d\tau\\
&\quad +c_3\int_t^{t+1} e^{2\beta\int_\tau^{t+1} z(\theta_{\tau_1}\omega)d\tau_1+(\delta-2\rho)(t+1-\tau)-2\beta z(\theta_\tau\omega)}d\tau.
\end{aligned}
\tag{4.24}
$$

It follows that

$$
\begin{aligned}
&\left\|(-\triangle)^{\frac{1+\alpha}{2}}v(t+1,\omega,v_0(\omega))\right\|^2\\
&\quad +\frac{1}{2}\int_t^{t+1} e^{2\beta\int_\tau^{t+1} z(\theta_{\tau_1}\omega)d\tau_1+(\delta-2\rho)(t+1-\tau)}\left\|(-\triangle)^{\frac{1}{2}+\alpha}v(\tau,\omega,v_0(\omega))\right\|^2 d\tau\\
&\le c_4\int_t^{t+1} e^{2\beta\int_s^{t+1} z(\theta_\tau\omega)d\tau+(\delta-2\rho)(t+1-s)}\|v(s,\omega,v_0(\omega))\|^2 ds\\
&\quad +8\beta_2^2\int_t^{t+1} e^{2\beta\int_\tau^{t+1} z(\theta_{\tau_1}\omega)d\tau_1+(\delta-2\rho)(t+1-\tau)}\|\nabla v(\tau,\omega,v_0(\omega))\|^2 d\tau\\
&\quad +c_3\int_t^{t+1} e^{2\beta\int_\tau^{t+1} z(\theta_{\tau_1}\omega)d\tau_1+(\delta-2\rho)(t+1-\tau)-2\beta z(\theta_\tau\omega)}d\tau.
\end{aligned}
\tag{4.25}
$$

Replacing ω by $\theta_{-t-1}\omega$, we infer

$$
\|(-\triangle)^{\frac{1+\alpha}{2}}v(t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2
$$

$$
\begin{aligned}
&+\frac{1}{2}\int_t^{t+1} e^{2\beta\int_\tau^{t+1} z(\theta_{\tau_1-t-1}\omega)d\tau_1+(\delta-2\rho)(t+1-\tau)}\|(-\triangle)^{\frac{1}{2}+\alpha}v(\tau,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2 d\tau\\
\le& c_4\int_t^{t+1} e^{2\beta\int_s^{t+1} z(\theta_{\tau-t-1}\omega)d\tau+(\delta-2\rho)(t+1-s)}\|v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2 ds\\
&+8\beta_2^2\int_t^{t+1} e^{2\beta\int_\tau^{t+1} z(\theta_{\tau_1-t-1}\omega)d\tau_1+(\delta-2\rho)(t+1-\tau)}\|\nabla v(\tau,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2 d\tau\\
&+c_3\int_t^{t+1} e^{2\beta\int_\tau^{t+1} z(\theta_{\tau_1-t-1}\omega)d\tau_1+(\delta-2\rho)(t+1-\tau)-2\beta z(\theta_{\tau-t-1}\omega)}d\tau. \qquad (4.26)
\end{aligned}
$$

Now, we estimate the three terms on the right-hand side of (4.24). For the first term, by Lemma 4.1, for any $t\ge T_{1_B}(\omega)$, one has

$$
\begin{aligned}
&c_4\int_t^{t+1} e^{2\beta\int_s^{t+1} z(\theta_{\tau-t-1}\omega)d\tau+(\delta-2\rho)(t+1-s)}\|v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2 ds\\
\le& c_4\varrho_0^2\int_{-1}^{0} e^{2\beta\int_\tau^0 z(\theta_s\omega)ds+(2\rho-\delta)\tau}d\tau \triangleq r_0^2. \qquad (4.27)
\end{aligned}
$$

For the second term, by Lemma 4.2, for any $t\ge T_{1_B}(\omega)$, one has

$$
\begin{aligned}
&8\beta_2^2\int_t^{t+1} e^{2\beta\int_\tau^{t+1} z(\theta_{\tau_1-t-1}\omega)d\tau_1+(\delta-2\rho)(t+1-\tau)}\|\nabla v(\tau,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2 d\tau\\
\le& 8\beta_2^2\varrho_1^2\int_{-1}^{0} e^{2\beta\int_\tau^0 z(\theta_s\omega)ds+(2\rho-\delta)\tau}d\tau \triangleq r_1^2. \qquad (4.28)
\end{aligned}
$$

For the third term, we have

$$
\begin{aligned}
&c_3\int_t^{t+1} e^{2\beta\int_\tau^{t+1} z(\theta_{\tau_1-t-1}\omega)d\tau_1+(\delta-2\rho)(t+1-\tau)-2\beta z(\theta_{\tau-t-1}\omega)}d\tau\\
\le& c_3\int_{-1}^{0} e^{2\beta\int_\tau^0 z(\theta_s\omega)ds+(2\rho-\delta)\tau-2\beta z(\theta_\tau\omega)}d\tau \triangleq r_2^2. \qquad (4.29)
\end{aligned}
$$

Substituting (4.27), (4.28) and (4.29) into (4.24) gives

$$
\|(-\triangle)^{\frac{1+\alpha}{2}}v(t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2\le\varrho_1^2+r_0^2+r_1^2+r_2^2\triangleq\varrho_2^2, \qquad (4.30)
$$

which completes the proof. ∎

LEMMA 4.4. *Let $B=\{B(\omega)\}\in\mathcal{D}$ and $v_0(\omega)\in B(\omega)$. Then for P-a.e. $\omega\in\Omega$, there exist $T^*=T_B^*(\omega)>0$ and $R^*=R^*(\omega,\varepsilon)$ such that for any $t\ge T_B^*(\omega)$, one has*

$$
\int_{|x|\ge R^*}|v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2dx\le\varepsilon. \qquad (4.31)
$$

Proof. Take a smooth function χ such that $0\le\chi(s)\le 1$ for all $s\ge 0$ and

$$
\chi(s)=\begin{cases}0, & \text{if } 0\le s\le 1,\\ 1, & \text{if } s\ge 2.\end{cases} \qquad (4.32)
$$

There exists a positive constant c such that $|\chi'(s)|\le c$ for all $s\ge 0$. Taking the real part of the inner product of (3.3) with $\chi(\frac{x^2}{k^2})v$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\chi\left(\frac{x^2}{k^2}\right)|v|^2dx+(\rho-\beta z(\theta_t\omega))\int_{\mathbb{R}^3}\chi\left(\frac{x^2}{k^2}\right)|v|^2dx\\&=-\operatorname{Re}(1+\mathrm{i}\nu)\int_{\mathbb{R}^3}(-\triangle)^\alpha v\chi\left(\frac{x^2}{k^2}\right)\bar{v}dx+e^{-\beta z(\theta_t\omega)}\operatorname{Re}\int_{\mathbb{R}^3}f(e^{\beta z(\theta_t\omega)}v)\chi\left(\frac{x^2}{k^2}\right)\bar{v}dx.\end{aligned}\tag{4.33}$$

We estimate each term on the right-hand side of (4.32). For the first term, integrating by parts and applying the Hölder, Gagliardo–Nirenberg, and Young's inequalities, we have

$$\begin{aligned}&-\operatorname{Re}(1+\mathrm{i}\nu)\int_{\mathbb{R}^3}(-\triangle)^\alpha v\chi\left(\frac{x^2}{k^2}\right)\bar{v}dx\\&\le|1+\mathrm{i}\nu|\int_{\mathbb{R}^3}|(-\triangle)^{\alpha-\frac{1}{2}}v|\left(\chi\left(\frac{x^2}{k^2}\right)|\nabla v|+\chi'\left(\frac{x^2}{k^2}\right)\frac{2|x|}{k^2}|v|\right)dx\\&\le|1+\mathrm{i}\nu|\left(\|(-\triangle)^{\alpha-\frac{1}{2}}v\|\|\nabla v\|+\int_{k\le|x|\le\sqrt{2}k}|(-\triangle)^{\alpha-\frac{1}{2}}v|\left|\chi'\left(\frac{x^2}{k^2}\right)\right|\frac{2|x|}{k^2}|v|dx\right)\\&\le|1+\mathrm{i}\nu|\left(\|(-\triangle)^{\alpha-\frac{1}{2}}v\|\|\nabla v\|+\frac{2\sqrt{2}}{k}\int_{k\le|x|\le\sqrt{2}k}|(-\triangle)^{\alpha-\frac{1}{2}}v|\left|\chi'\left(\frac{x^2}{k^2}\right)\right||v|dx\right)\\&\le|1+\mathrm{i}\nu|\left(\|(-\triangle)^{\alpha-\frac{1}{2}}v\|\|\nabla v\|+\frac{c}{k}\int_{k\le|x|\le\sqrt{2}k}|(-\triangle)^{\alpha-\frac{1}{2}}v||v|dx\right)\\&\le c\left(\|v\|^2+\|\nabla v\|^2\right)+\frac{c}{k}\left(\|\nabla v\|^2+\|v\|^2\right).\end{aligned}\tag{4.34}$$

For the second term, applying (1.3), one has

$$\begin{aligned}&e^{-\beta z(\theta_t\omega)}\operatorname{Re}\int_{\mathbb{R}^3}f(e^{\beta z(\theta_t\omega)}v)\chi\left(\frac{x^2}{k^2}\right)\bar{v}dx\le e^{-2\beta z(\theta_t\omega)}\int_{\mathbb{R}^3}\gamma_1(x)\chi\left(\frac{x^2}{k^2}\right)dx\\&\quad-\beta_1e^{-2\beta z(\theta_t\omega)}\int_{\mathbb{R}^3}|e^{\beta z(\theta_t\omega)}v|^{2\sigma+2}\chi\left(\frac{x^2}{k^2}\right)dx\\&\le e^{-2\beta z(\theta_t\omega)}\int_{\mathbb{R}^3}|\gamma_1(x)|\chi\left(\frac{x^2}{k^2}\right)dx.\end{aligned}\tag{4.35}$$

Using (4.34) and (4.35), (4.32) can be rewritten as

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^3}\chi\left(\frac{x^2}{k^2}\right)|v|^2dx+(2\rho-\delta-2\beta z(\theta_t\omega))\int_{\mathbb{R}^3}\chi\left(\frac{x^2}{k^2}\right)|v|^2dx\\&\le c\left(\|v\|^2+\|\nabla v\|^2\right)+\frac{c}{k}\left(\|\nabla v\|^2+\|v\|^2\right)+e^{-2\beta z(\theta_t\omega)}\int_{\mathbb{R}^3}|\gamma_1(x)|\chi\left(\frac{x^2}{k^2}\right)dx.\end{aligned}\tag{4.36}$$

Multiplying (4.36) by $e^{-2\beta\int_0^tz(\theta_s\omega)ds+(2\rho-\delta)t}$ and integrating over (T_1,t), we have

$$\begin{aligned}&\int_{\mathbb{R}^3}\chi\left(\frac{x^2}{k^2}\right)|v(t,\omega,v_0(\omega))|^2dx\\&\le e^{2\beta\int_{T_1}^tz(\theta_s\omega)ds+(\delta-2\rho)(t-T_1)}\int_{\mathbb{R}^3}\chi\left(\frac{x^2}{k^2}\right)|v(T_1,\omega,v_0(\omega))|^2dx\\&\quad+\int_{T_1}^te^{2\beta\int_s^tz(\theta_\tau\omega)d\tau+(\delta-2\rho)(t-s)-2\beta z(\theta_s\omega)}\int_{\mathbb{R}^3}|\gamma_1(x)|\chi\left(\frac{x^2}{k^2}\right)dxds\end{aligned}$$

$$+c\int_{T_1}^{t} e^{2\beta\int_s^t z(\theta_\tau\omega)d\tau+(\delta-2\rho)(t-s)}\left(\|v(s,\omega,v_0(\omega))\|^2+\|\nabla v(s,\omega,v_0(\omega))\|^2\right)ds$$
$$+\frac{c}{k}\int_{T_1}^{t} e^{2\beta\int_s^t z(\theta_\tau\omega)d\tau+(\delta-2\rho)(t-s)}\left(\|\nabla v(s,\omega,v_0(\omega))\|^2+\|v(s,\omega,v_0(\omega))\|^2\right)ds. \quad (4.37)$$

Replacing ω by $\theta_{-t}\omega$, in (4.37), we deduce that for all $t\geq T_1$,

$$\begin{aligned}&\int_{\mathbb{R}^3}\chi\left(\frac{x^2}{k^2}\right)|v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2dx\\ \leq& e^{2\beta\int_{T_1}^t z(\theta_{s-t}\omega)ds+(\delta-2\rho)(t-T_1)}\int_{\mathbb{R}^3}\chi\left(\frac{x^2}{k^2}\right)|v(T_1,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2dx\\ &+\int_{T_1}^t e^{2\beta\int_s^t z(\theta_{\tau-t}\omega)d\tau+(\delta-2\rho)(t-s)-2\beta z(\theta_{s-t}\omega)}\int_{\mathbb{R}^3}|\gamma_1(x)|\chi\left(\frac{x^2}{k^2}\right)dxds\\ &+c\int_{T_1}^t e^{2\beta\int_s^t z(\theta_{\tau-t}\omega)d\tau+(\delta-2\rho)(t-s)}\mathcal{W}(s)ds\\ &+\frac{c}{k}\int_{T_1}^t e^{2\beta\int_s^t z(\theta_{\tau-t}\omega)d\tau+(\delta-2\rho)(t-s)}\mathcal{W}(s)ds,\end{aligned} \quad (4.38)$$

where

$$\mathcal{W}(x)=\|v(x,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2+\|\nabla v(x,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2.$$

In what follows, we estimate each term on the right-hand side of (4.38). For the first term, replacing t by T_1 and ω by $\theta_{-t}\omega$ in (4.5), we have

$$\begin{aligned}&e^{2\beta\int_{T_1}^t z(\theta_{s-t}\omega)ds+(\delta-2\rho)(t-T_1)}\int_{\mathbb{R}^3}\chi\left(\frac{x^2}{k^2}\right)|v(T_1,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2dx\\ \leq& e^{2\beta\int_{T_1}^t z(\theta_{s-t}\omega)ds+(\delta-2\rho)(t-T_1)}\int_{\mathbb{R}^3}|v(T_1,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2dx\\ \leq& e^{2\beta\int_{T_1}^t z(\theta_{s-t}\omega)ds+(\delta-2\rho)(t-T_1)}e^{2\beta\int_0^{T_1} z(\theta_{s-t}\omega)ds+(1-2\rho)T_1}\|v_0(\theta_{-t}\omega)\|^2\\ =&e^{2\beta\int_0^t z(\theta_{s-t}\omega)ds+(\delta-2\rho)t}\|v_0(\theta_{-t}\omega)\|^2\\ =&e^{2\beta\int_{-t}^0 z(\theta_s\omega)ds+(\delta-2\rho)t}\|v_0(\theta_{-t}\omega)\|^2.\end{aligned} \quad (4.39)$$

We find that, given $\varepsilon>0$, there exists $T_2=T_2(B,\omega,\varepsilon)>T_1$ such that for all $t\geq T_2$,

$$e^{2\beta\int_{T_1}^t z(\theta_{s-t}\omega)ds+(\delta-2\rho)(t-T_1)}\int_{\mathbb{R}^3}\chi\left(\frac{x^2}{k^2}\right)|v(T_1,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2dx\leq\frac{\varepsilon}{4}. \quad (4.40)$$

For the second term, note that $\gamma_1(x)\in L^1(\mathbb{R}^3)$, so there exists $R_1=R_1(\varepsilon)$ such that for all $k\geq R_1$, we have

$$\int_{|x|\geq k}|\gamma_1(x)|\chi\left(\frac{x^2}{k^2}\right)dx\leq c\varepsilon. \quad (4.41)$$

Given $\varepsilon_0>0$, there exists $T_3=T_3(\omega)>0$ such that for $s<-T_3$, we have

$$\int_{T_1}^t e^{2\beta\int_s^t z(\theta_{\tau-t}\omega)d\tau+(\delta-2\rho)(t-s)-2\beta z(\theta_{s-t}\omega)}ds$$

$$
\begin{aligned}
&= \int_{T_1-t}^{0} e^{2\beta\int_s^0 z(\theta_\tau\omega)d\tau+(2\rho-\delta)s-2\beta z(\theta_s\omega)}ds \\
&\le \int_{T_1-T_3}^{0} e^{2\beta\int_s^0 z(\theta_\tau\omega)d\tau+(2\rho-\delta)s-2\beta z(\theta_s\omega)}ds + \int_{T_1-t}^{T_1-T_3} e^{s(2\rho-\delta+\varepsilon_0)}ds \\
&\le c(\omega)+c_1(\omega).
\end{aligned}
\tag{4.42}
$$

So there exists $R_1 = R_1(\varepsilon,\omega)$ such that for all $t \ge T_3$ and $k \ge R_1$,

$$
\int_{T_1}^{t} e^{2\beta\int_s^t z(\theta_{\tau-t}\omega)d\tau+(\delta-2\rho)(t-s)-2\beta z(\theta_{s-t}\omega)} \int_{\mathbb{R}^3} |\gamma_1(x)|\chi\left(\frac{x^2}{k^2}\right)dxds \le \frac{\varepsilon}{4}. \tag{4.43}
$$

For the third term, by (4.6) and (4.15), one has

$$
\begin{aligned}
&c\int_{T_1}^{t} e^{2\beta\int_s^t z(\theta_{\tau-t}\omega)d\tau+(\delta-2\rho)(t-s)}\mathcal{W}(s)ds \\
&\le c\int_{T_1-t}^{0} e^{2\beta\int_s^0 z(\theta_\tau\omega)d\tau+(2\rho-\delta)s}\mathcal{W}(s+t)ds \\
&\le c e^{2\beta\int_{T_1-t}^0 z(\theta_s\omega)ds+(2\rho-\delta)(T_1-t)}(\|v_0(\theta_{-t}\omega)\|^2+\|\nabla v_0(\theta_{-t}\omega)\|^2).
\end{aligned}
\tag{4.44}
$$

Since $\{B(\omega)\} \in \mathcal{D}$ is tempered, for any $v_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$,

$$
\lim_{t\to+\infty} e^{2\beta\int_{T_1-t}^0 z(\theta_s\omega)ds+(2\rho-\delta)(T_1-t)}(\|v_0(\theta_{-t}\omega)\|^2+\|\nabla v_0(\theta_{-t}\omega)\|^2)=0. \tag{4.45}
$$

Therefore, there exists $T_4 = T_4(B,\omega,\varepsilon) > T_1$ such that for any $t \ge T_4$,

$$
\begin{aligned}
&c\int_{T_1}^{t} e^{2\beta\int_s^t z(\theta_{\tau-t}\omega)d\tau+(\delta-2\rho)(t-s)}\mathcal{W}(s)ds \\
&\le e^{2\beta\int_{T_1-t}^0 z(\theta_s\omega)ds+(\delta-2\rho)(T_1-t)}(\|v_0(\theta_{-t}\omega)\|^2+\|\nabla v_0(\theta_{-t}\omega)\|^2) \le \frac{\varepsilon}{4}.
\end{aligned}
\tag{4.46}
$$

Similarly, there exists $R_2 = R_2(\omega,\varepsilon)$ such that for all $t \ge T_4$ and $k \ge R_2$,

$$
\frac{c}{k}\int_{T_1}^{t} e^{2\beta\int_s^t z(\theta_{\tau-t}\omega)d\tau+(\delta-2\rho)(t-s)}\mathcal{W}(s)ds \le \frac{\varepsilon}{4}. \tag{4.47}
$$

Let $T^* = T^*(B,\omega,\varepsilon) = \max\{T_1,T_2,T_3,T_4\}$. Then by (4.40), (4.46) and (4.47), for all $t \ge T^*$ and $k \ge R^* = \max\{R_1,R_2\}$, one has

$$
\int_{\mathbb{R}^3} \chi\left(\frac{x^2}{k^2}\right)|v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2dx \le \varepsilon. \tag{4.48}
$$

This implies that for all $t \ge T^*$ and $k \ge R^*$, we have

$$
\int_{|x|\ge k} |v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2dx \le \int_{\mathbb{R}^3} \chi\left(\frac{x^2}{k^2}\right)|v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2dx \le \varepsilon. \tag{4.49}
$$

Thus completing the proof. □

LEMMA 4.5. *Let $B = \{B(\omega)\} \in \mathcal{D}$ and $v_0(\omega) \in B(\omega)$. Then for P-a.e. $\omega \in \Omega$, there exists $T^{**} = T_B^{**}(\omega) > 0$ such that for any $t \ge T_B^{**}(\omega)$, one has*

$$
\int_{|x|\ge k} |\nabla v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2dx \le \varepsilon. \tag{4.50}
$$

Proof. Differentiating (3.3) with respect to $x=(x_1,x_2,x_3)$, then taking the real part of the inner product with $\chi(\frac{x^2}{k^2})\nabla v$ gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\chi\left(\frac{x^2}{k^2}\right)|\nabla v|^2dx+(\rho-\beta z(\theta_t\omega))\int_{\mathbb{R}^3}\chi\left(\frac{x^2}{k^2}\right)|\nabla v|^2dx\\&=-\mathrm{Re}(1+\mathrm{i}\nu)\int_{\mathbb{R}^3}((-\triangle)^{\alpha}(\nabla v))\chi\left(\frac{x^2}{k^2}\right)\nabla\bar{v}dx\\&\quad+e^{-\beta z(\theta_t\omega)}\mathrm{Re}\int_{\mathbb{R}^3}\nabla f(e^{\beta z(\theta_t\omega)}v)\chi\left(\frac{x^2}{k^2}\right)\nabla\bar{v}dx. \end{aligned}\tag{4.51}$$

Now, we estimate the right-hand side of (4.51). For the first term, we have

$$\begin{aligned}-\mathrm{Re}(1+\mathrm{i}\nu)\int_{\mathbb{R}^3}((-\triangle)^{\alpha}(\nabla v))\chi\left(\frac{x^2}{k^2}\right)\nabla\bar{v}dx&\le|1+\mathrm{i}\nu|\|(-\triangle)^{\alpha+\frac{1}{2}}v\|\|\nabla v\|\\&\le c\left(\|(-\triangle)^{\alpha+\frac{1}{2}}v\|^2+\|\nabla v\|^2\right).\end{aligned}\tag{4.52}$$

For the second term, one has

$$\begin{aligned}&e^{-\beta z(\theta_t\omega)}\mathrm{Re}\int_{\mathbb{R}^3}\nabla f(e^{\beta z(\theta_t\omega)}v)\chi\left(\frac{x^2}{k^2}\right)\nabla\bar{v}dx\\&\le2\beta_2\int_{\mathbb{R}^3}|\nabla v|^2\chi\left(\frac{x^2}{k^2}\right)dx+e^{-\beta z(\theta_t\omega)}\int_{\mathbb{R}^3}|\gamma_2(x)||\nabla v|\chi\left(\frac{x^2}{k^2}\right)dx\\&\le(2\beta_2+\frac{1}{2})\|\nabla v\|^2+\frac{1}{2}e^{-2\beta z(\theta_t\omega)}\int_{\mathbb{R}^3}|\gamma_2(x)|^2|\chi^2\left(\frac{x^2}{k^2}\right)dx.\end{aligned}\tag{4.53}$$

Substituting (4.52) and (4.53) into (4.51), we deduce that

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^3}\chi\left(\frac{x^2}{k^2}\right)|\nabla v|^2dx+(2\rho-\delta-2\beta z(\theta_t\omega))\int_{\mathbb{R}^3}\chi\left(\frac{x^2}{k^2}\right)|\nabla v|^2dx\\&\le c\left(\|(-\triangle)^{\alpha+\frac{1}{2}}v\|^2+\|\nabla v\|^2\right)+\frac{1}{2}e^{-2\beta z(\theta_t\omega)}\int_{\mathbb{R}^3}|\gamma_2(x)|^2|\chi^2\left(\frac{x^2}{k^2}\right)dx.\end{aligned}\tag{4.54}$$

Multiplying (4.54) by $e^{-2\beta\int_0^t z(\theta_s\omega)ds+(2\rho-\delta)t}$ and integrating over (T_1,t) gives for all $t\ge T_1$,

$$\begin{aligned}&\int_{\mathbb{R}^3}\chi\left(\frac{x^2}{k^2}\right)|\nabla v(t,\omega,v_0(\omega))|^2dx\\&\le e^{2\beta\int_{T_1}^t z(\theta_s\omega)ds+(\delta-2\rho)(t-T_1)}\int_{\mathbb{R}^3}\chi\left(\frac{x^2}{k^2}\right)|\nabla v(T_1,\omega,v_0(\omega))|^2dx\\&\quad+c\int_{T_1}^t e^{2\beta\int_s^t z(\theta_\tau\omega)d\tau+(\delta-2\rho)(t-s)}\left(\|(-\triangle)^{\alpha+\frac{1}{2}}v(s,\omega,v_0(\omega))\|^2+\|\nabla v(s,\omega,v_0(\omega))\|^2\right)ds\\&\quad+\frac{1}{2}\int_{T_1}^t e^{2\beta\int_s^t z(\theta_\tau\omega)d\tau+(\delta-2\rho)(t-s)-2\beta z(\theta_s\omega)}\int_{\mathbb{R}^3}|\gamma_2(x)|^2|\chi^2\left(\frac{x^2}{k^2}\right)dxds.\end{aligned}\tag{4.55}$$

Replacing ω by $\theta_{-t}\omega$ and applying (4.37), for all $t\ge T_1$, one has

$$\int_{\mathbb{R}^3}\chi\left(\frac{x^2}{k^2}\right)|\nabla v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2dx$$

$$
\begin{aligned}
\le & e^{2\beta\int_{T_1}^{t} z(\theta_{s-t}\omega)ds+(\delta-2\rho)(t-T_1)}\int_{\mathbb{R}^3}|\nabla v(T_1,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2dx\\
&+c\int_{T_1}^{t} e^{2\beta\int_s^t z(\theta_{\tau-t}\omega)d\tau+(\delta-2\rho)(t-s)}\|(-\triangle)^{\alpha+\frac{1}{2}}v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2ds\\
&+c\int_{T_1}^{t} e^{2\beta\int_s^t z(\theta_{\tau-t}\omega)d\tau+(\delta-2\rho)(t-s)}\|\nabla v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2ds\\
&+\frac{1}{2}\int_{T_1}^{t} e^{2\beta\int_s^t z(\theta_{\tau-t}\omega)d\tau+(\delta-2\rho)(t-s)-2\beta z(\theta_{s-t}\omega)}\int_{\mathbb{R}^3}|\gamma_2(x)|^2\chi^2\left(\frac{x^2}{k^2}\right)dxds. \qquad (4.56)
\end{aligned}
$$

We estimate each term on the right-hand side of (4.56). For the first term, replacing t by T_1 and ω by $\theta_{-t}\omega$ in (4.14), we have

$$
\begin{aligned}
& e^{2\beta\int_{T_1}^{t} z(\theta_{s-t}\omega)ds+(\delta-2\rho)(t-T_1)}\int_{\mathbb{R}^3}|\nabla v(T_1,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2dx\\
\le & e^{2\beta\int_{T_1}^{t} z(\theta_{s-t}\omega)ds+(\delta-2\rho)(t-T_1)}e^{2\beta\int_0^{T_1} z(\theta_{s-t}\omega)ds+(\delta-2\rho)T_1}\|\nabla v_0(\theta_{-t}\omega)\|^2\\
= & e^{2\beta\int_0^{t} z(\theta_{s-t}\omega)ds+(\delta-2\rho)t}\|\nabla v_0(\theta_{-t}\omega)\|^2\\
= & e^{2\beta\int_{-t}^{0} z(\theta_{s}\omega)ds+(\delta-2\rho)t}\|\nabla v_0(\theta_{-t}\omega)\|^2. \qquad (4.57)
\end{aligned}
$$

Since $\{B(\omega)\}\in\mathcal{D}$ is tempered, for any $v_0(\theta_{-t}\omega)\in B(\theta_{-t}\omega)$,

$$
\lim_{t\to+\infty} e^{2\beta\int_{-t}^{0} z(\theta_{s}\omega)ds+(\delta-2\rho)t}\|\nabla v_0(\theta_{-t-1}\omega)\|^2=0. \qquad (4.58)
$$

Therefore, given $\varepsilon>0$, there exists $T_5=T_5(B,\omega,\varepsilon)>T_1$ such that for all $t\ge T_5$,

$$
e^{2\beta\int_{T_1}^{t} z(\theta_{s-t}\omega)ds+(\delta-2\rho)(t-T_1)}\int_{\mathbb{R}^3}|\nabla v(T_1,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2dx\le\frac{\varepsilon}{4}. \qquad (4.59)
$$

For the second term, one has

$$
\begin{aligned}
& c\int_{T_1}^{t} e^{2\beta\int_s^t z(\theta_{\tau-t}\omega)d\tau+(\delta-2\rho)(t-s)}\|(-\triangle)^{\alpha+\frac{1}{2}}v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2ds\\
\le & c\int_{T_1-t}^{0} e^{2\beta\int_s^0 z(\theta_{\tau}\omega)d\tau+(2\rho-\delta)s}\|(-\triangle)^{\alpha+\frac{1}{2}}v(s+t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2ds\\
\le & c\int_{-t}^{0} e^{2\beta\int_s^0 z(\theta_{\tau}\omega)d\tau+(2\rho-\delta)s}\|(-\triangle)^{\alpha+\frac{1}{2}}v(s+t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2ds. \qquad (4.60)
\end{aligned}
$$

Replacing ω by $\theta_{-t-1}\omega$ in (4.23), dropping the first term on the left-hand side, and integrating with respect to s over $(T_1,t+1)$, we obtain

$$
\begin{aligned}
& \int_{T_1}^{t+1} e^{2\beta\int_\tau^{t+1} z(\theta_{\tau_1-t-1}\omega)d\tau_1+(\delta-2\rho)(t+1-\tau)}\|(-\triangle)^{\frac{1}{2}+\alpha}v(\tau,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2d\tau\\
\le & \int_{T_1}^{t+1} e^{2\beta\int_s^{t+1} z(\theta_{\tau-t-1}\omega)d\tau+(\delta-2\rho)(t+1-s)}\|(-\triangle)^{\frac{1+\alpha}{2}}v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2ds\\
&+8\beta_2^2\int_{T_1}^{t+1} e^{2\beta\int_s^{t+1} z(\theta_{\tau-t-1}\omega)d\tau+(\delta-2\rho)(t+1-s)}\|\nabla v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2ds\\
= & \int_{T_1-t-1}^{0} e^{2\beta\int_s^0 z(\theta_{\tau}\omega)d\tau+(2\rho-\delta)s}\|(-\triangle)^{\frac{1+\alpha}{2}}v(s+t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2ds
\end{aligned}
$$

$$
\begin{aligned}
&+8\beta_2^2\int_{T_1-t-1}^0 e^{2\beta\int_s^0 z(\theta_\tau\omega)d\tau+(2\rho-\delta)s}\|\nabla v(s+t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2 ds\\
\le&\int_{-t-1}^0 e^{2\beta\int_s^0 z(\theta_\tau\omega)d\tau+(2\rho-\delta)s}\|(-\triangle)^{\frac{1+\alpha}{2}}v(s+t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2 ds\\
&+8\beta_2^2\int_{-t-1}^0 e^{2\beta\int_s^0 z(\theta_\tau\omega)d\tau+(2\rho-\delta)s}\|\nabla v(s+t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2 ds. \quad (4.61)
\end{aligned}
$$

By (4.15), for all $t\ge T_{1_B}(\omega)-1$, we have

$$
\begin{aligned}
&\int_{-t-1}^0 e^{2\beta\int_s^0 z(\theta_s\omega)d\tau+(2\rho-\delta)s}\|(-\triangle)^{\frac{\alpha+1}{2}}v(s+t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2 ds\\
&\quad+\delta\int_{-t-1}^0 e^{2\beta\int_s^0 z(\theta_s\omega)d\tau+(2\rho-\delta)s}\|\nabla v(s+t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2 ds\\
&\le e^{2\beta\int_{-t-1}^0 z(\theta_s\omega)ds+(\delta-2\rho)(t+1)}\|\nabla v_0(\theta_{-t-1}\omega)\|^2. \quad (4.62)
\end{aligned}
$$

Substituting (4.62) into (4.61), one has

$$
\begin{aligned}
&\int_{T_1}^{t+1} e^{2\beta\int_\tau^{t+1} z(\theta_{\tau_1-t-1}\omega)d\tau_1+(\delta-2\rho)(t+1-\tau)}\|(-\triangle)^{\frac{1}{2}+\alpha}v(\tau,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2 d\tau\\
\le&\left(1+\frac{8\beta_2^2}{\delta}\right)e^{2\beta\int_{-t-1}^0 z(\theta_s\omega)ds+(\delta-2\rho)(t+1)}\|\nabla v_0(\theta_{-t-1}\omega)\|^2. \quad (4.63)
\end{aligned}
$$

Again, since $\{B(\omega)\}\in\mathcal{D}$ is tempered, by a similar argument, there exists $T_6=T_6(B,\omega,\varepsilon)>T_{1_B}(\omega)$ such that for any $t\ge T_6$,

$$
\left(1+\frac{8\beta_2^2}{\delta}\right)e^{2\beta\int_{T_1-t}^0 z(\theta_s\omega)ds+(\delta-2\rho)(T_1-t)}(\|v_0(\theta_{-t}\omega)\|^2+\|\nabla v_0(\theta_{-t}\omega)\|^2)\le\frac{\varepsilon}{4}. \quad (4.64)
$$

So, we infer that

$$
c\int_{T_1}^t e^{2\beta\int_s^t z(\theta_{\tau-t}\omega)d\tau+(\delta-2\rho)(t-s)}\|(-\triangle)^{\alpha+\frac{1}{2}}v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2 ds\le\frac{\varepsilon}{4}. \quad (4.65)
$$

For the third term, by (4.46), there exists $T_4=T_4(B,\omega,\varepsilon)>T_1$ such that for any $t\ge T_4$,

$$
c\int_{T_1}^t e^{2\beta\int_s^t z(\theta_{\tau-t}\omega)d\tau+(\delta-2\rho)(t-s)}\|\nabla v(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2 ds\le\frac{\varepsilon}{4}. \quad (4.66)
$$

For the last term, note that $\gamma_2(x)\in L^2(\mathbb{R}^3)$. In a manner similar to the argument for (4.43), there exists $R_1^*=R_1^*(\varepsilon)$ such that for all $t\ge T_3$ and $k\ge R_1^*$,

$$
\int_{T_1}^t e^{2\beta\int_s^t z(\theta_{\tau-t}\omega)d\tau+(\delta-2\rho)(t-s)-2\beta z(\theta_{s-t}\omega)}\int_{\mathbb{R}^3}|\gamma_2(x)|^2\chi^2\left(\frac{x^2}{k^2}\right)dxds\le\frac{\varepsilon}{4}. \quad (4.67)
$$

Let $T^{**}=T^{**}(B,\omega,\varepsilon)=\max\{T_3,T_4,T_5,T_6\}$. Then by (4.59), (4.65) and (4.66), for all $t\ge T^{**}$ and $k\ge R_1^*$, one has

$$
\int_{\mathbb{R}^3}\chi\left(\frac{x^2}{k^2}\right)|\nabla v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2 dx\le\varepsilon. \quad (4.68)
$$

This implies that for all $t \geq T^{**}$ and $k \geq R_1^*$,

$$\int_{|x|\geq k} |\nabla v(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega))|^2 dx \leq \int_{\mathbb{R}^3} \chi\left(\frac{x^2}{k^2}\right) |\nabla v(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega))|^2 dx \leq \varepsilon. \quad (4.69)$$

This completes the proof. □

By Lemmas 4.4 and 4.5, we have

COROLLARY 4.6. *Let $B = \{B(\omega)\} \in \mathcal{D}$ and $v_0(\omega) \in B(\omega)$. Then for P-a.e. $\omega \in \Omega$, there exists $T_B^\star = \max\{T_B^*(\omega), T_B^{**}(\omega)\}$ and $R^* = R^*(\omega,\varepsilon)$ such that for any $t \geq T_B^\star(\omega)$, one has*

$$\|v(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega))\|^2_{H^1(|x|\geq R^*)} \leq \varepsilon. \quad (4.70)$$

5. Random attractor

In this section, we prove the existence of a random attractor for the random dynamical system generated by (3.3) on $\mathbb{R}^3$. From Lemma 4.2, φ has a closed random absorbing set in $\mathcal{D}$. The $\mathcal{D}$-pullback asymptotic compactness of φ is demonstrated below using the uniform estimates obtained in the previous sections.

LEMMA 5.1. *Assume (1.3)–(1.5) and $\beta_\sigma \leq 2|\lambda_\sigma|$. Then the random dynamical system φ is $\mathcal{D}$-pullback asymptotically compact in $H^1(\mathbb{R}^3)$; that is, for P-a.e. $\omega \in \Omega$, the sequence $\varphi(t_n, \theta_{-t_n}\omega, v_{0,n}(\theta_{-t_n}\omega))$ has a convergent subsequence in $H^1(\mathbb{R}^3)$ provided $t_n \to \infty$, $B = \{B(\omega)\} \in \mathcal{D}$ and $v_{0,n}(\theta_{-t_n}\omega) \in B(\theta_{-t_n}\omega)$.*

Proof. Let $t_n \to \infty$, $B = \{B(\omega)\} \in \mathcal{D}$ and $v_{0,n}(\theta_{-t_n}\omega) \in B(\theta_{-t_n}\omega)$. Applying Lemmas 4.1 and 4.2, for P-a.e. $\omega \in \Omega$, we have

$$\{\varphi(t_n, \theta_{-t_n}\omega, v_{0,n}(\theta_{-t_n}\omega))\}_{n=1}^\infty \quad \text{is bounded in } H^1(\mathbb{R}^3).$$

Therefore, there exists $\eta(\omega) \in H^1(\mathbb{R}^3)$ and a subsequence, for convenience, still denoted by $\{\varphi(t_n, \theta_{-t_n}\omega, v_{0,n}(\theta_{-t_n}\omega))\}$, such that

$$\varphi(t_n, \theta_{-t_n}\omega, v_{0,n}(\theta_{-t_n}\omega)) \to \eta \quad \text{weakly in } H^1(\mathbb{R}^3). \quad (5.1)$$

Given $\varepsilon > 0$, by Corollary 4.6, there is $T_B^\star = \max\{T_B^*(\omega), T_B^{**}(\omega)\}$ and $R^* = R^*(\omega,\varepsilon)$ such that for any $t \geq T_B^\star(\omega)$,

$$\|\varphi(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega))\|^2_{H^1(|x|\geq R^*)} \leq \varepsilon. \quad (5.2)$$

Since $t_n \to \infty$, there exists $N_1 = N_1(B,\omega,\varepsilon)$ such that $t_n \geq T_B^\star$ for all $n \geq N_1$. Then, by (5.2), we have for all $n \geq N_1$,

$$\|\varphi(t_n, \theta_{-t_n}\omega, v_{0,n}(\theta_{-t_n}\omega))\|^2_{H^1(|x|\geq R^*)} \leq \varepsilon, \quad (5.3)$$

and hence,

$$||\eta||^2_{H^1(|x|\geq R^*)} \leq \varepsilon. \quad (5.4)$$

Applying Lemmas 4.1 and 4.3, there exists $T_{2_B} = \max\{T_{0_B}(\omega), T_{1_B}(\omega)\}$ such that for all $t \geq T_{2_B}$,

$$\|\varphi(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega))\|^2_{H^{1+\alpha}(\mathbb{R}^3)} \leq \varrho_0^2 + \varrho_2^2 \triangleq \varrho_3^2. \quad (5.5)$$

Let $N_2 = N_2(B,\omega)$ be large enough such that $t_n \geq T_{2_B}$ for $n \geq N_2$. It follows from (5.5) that, for all $n \geq N_2$,

$$\|\varphi(t_n,\theta_{-t_n}\omega,v_{0,n}(\theta_{-t_n}\omega))\|^2_{H^{1+\alpha}(\mathbb{R}^3)} \leq \varrho_3^2. \tag{5.6}$$

Let $B_{R^*} = \{x \in \mathbb{R}^3 : |x| \leq R^*\}$ be a ball. By the compactness of the embedding $H^{1+\alpha}(B_{R^*}) \hookrightarrow H^1(B_{R^*})$, from (5.6), we deduce that, up to a subsequence depending on R^*, $\varphi(t_n,\theta_{-t_n}\omega,v_{0,n}(\theta_{-t_n}\omega)) \to \eta$ strongly in $H^1(\hat{B}_{R^*})$, which implies that there exists $N_3 = N_3(B,\omega,\varepsilon) \geq N_2$ such that for all $n \geq N_3$,

$$\|\varphi(t_n,\theta_{-t_n}\omega,v_{0,n}(\theta_{-t_n}\omega)) - \eta\|^2_{H^1(B_{R^*})} \leq \varepsilon.$$

Let $N^\star = \max\{N_1,N_3\}$. Then, from (5.2), (5.3) and (5.4), we have for all $n \geq N^\star$,

$$\begin{aligned}
&\|\varphi(t_n,\theta_{-t_n}\omega,v_{0,n}(\theta_{-t_n}\omega)) - \eta\|^2_{H^1(\mathbb{R}^3)} \\
\leq &\|\varphi(t_n,\theta_{-t_n}\omega,v_{0,n}(\theta_{-t_n}\omega)) - \eta\|^2_{|x|\leq R^*} + \|\varphi(t_n,\theta_{-t_n}\omega,v_{0,n}(\theta_{-t_n}\omega))\|^2_{|x|\geq R^*} + \|\eta\|^2_{|x|\geq R^*} \\
\leq &5\varepsilon,
\end{aligned}$$

which implies that

$$\varphi(t_n,\theta_{-t_n}\omega,v_{0,n}(\theta_{-t_n}\omega)) \to \eta \quad \text{strongly in } H^1(\mathbb{R}^3).$$

This completes the proof. □

By Proposition 2.8, we have

Theorem 5.2. *Assume* (1.3)–(1.5) *and* $\beta_\sigma \leq 2|\lambda_\sigma|$. *Then the random dynamical system* φ *associated with the fractional Ginzburg–Landau equation with multiplicative noise* (1.1) *has a unique* $\mathcal{D}$*-random attractor in* $H^1(\mathbb{R}^3)$.

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