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ZERO MACH NUMBER LIMIT OF THE COMPRESSIBLE NAVIER–STOKES–KORTEWEG EQUATIONS*

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Abstract. In this paper, we investigate the zero Mach number limit for the three-dimensional compressible Navier–Stokes–Korteweg equations in the regime of smooth solutions. Based on the local existence theory of the compressible Navier–Stokes–Korteweg equations, we establish a convergence-stability principle. Then we show that, when the Mach number is sufficiently small, the initial value problem of the compressible Navier–Stokes–Korteweg equations has a unique smooth solution in the time interval where the corresponding incompressible Navier–Stokes equations have a smooth solution. It is important to remark that when the incompressible Navier–Stokes equations have a global smooth solution, the existence time of the solution for the compressible Navier–Stokes–Korteweg equations tends to infinity as the Mach number goes to zero. Moreover, we obtain the convergence of smooth solutions for the compressible Navier–Stokes–Korteweg equations towards those for the incompressible Navier–Stokes equations with a convergence rate. As we know, it is the first result about zero Mach number limit of the compressible Navier–Stokes–Korteweg equations.

Key words. Compressible Navier–Stokes–Korteweg equations, Mach number limit, convergence-stability principle, incompressible Navier–Stokes equations, energy-type error estimates.

AMS subject classifications. 76W05, 35B40.

1. Introduction

In this paper, we are concerned with the three-dimensional compressible Navier–Stokes–Korteweg system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) = \mu \Delta u + (\mu+\nu)\nabla \operatorname{div} u + \kappa \rho \nabla \Delta \rho \end{cases} \tag{1.1}$$

for $(x,t) \in \Omega \times [0,+\infty)$. Throughout this paper, Ω is assumed to be $\mathbf{R}^3$ or the 3-dimensional torus. Here the unknown functions are the density ρ and the velocity $u \in R^3$, $p(\rho)$ is a given pressure function, μ and ν are viscosity coefficients satisfying $\mu > 0$ and $2\mu + 3\nu \geq 0$, and κ is the Weber number. This compressible Navier–Stokes–Korteweg system can be used to model the capillarity effect of materials, see the pioneering works by Dunn and Serrin in [6] and also in [2, 5]. Note that, when $\kappa = 0$, (1.1) reduces to the compressible Navier–Stokes equations.

It is well known that the incompressible limit of compressible fluid dynamical equations is an important and challenging mathematical problem. Klainerman and Majda [18] firstly studied the incompressible limit of the compressible Euler equations. Lions and Masmoudi [20] investigated the incompressible limit of the compressible isentropic Navier–Stokes equations for the whole space R^d and the periodic domain. These results have been extended or improved by many others, i.e., the authors of the references [1, 3, 7, 8, 9, 10, 11, 12, 13, 16, 19, 21, 24] and so on.

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In this paper, we analyze the incompressible limit of smooth solutions for the compressible Navier–Stokes–Korteweg equation (1.1). To have some intuition, we introduce the scaling

$$\rho(x,t)=\rho^\varepsilon(x,\varepsilon t),\quad u(x,t)=\varepsilon u^\varepsilon(x,\varepsilon t)$$

and assume that the viscosity coefficients μ and ν are small and scaled like

$$\mu=\varepsilon\mu',\quad \nu=\varepsilon\nu'\quad \kappa=\varepsilon\kappa'$$

with $\varepsilon\in(0,1)$ a small parameter. With such scalings, the compressible Navier–Stokes–Korteweg equation (1.1) take the form

$$\begin{cases}\partial_t\rho^\varepsilon+\mathrm{div}\,(\rho^\varepsilon u^\varepsilon)=0,\\ \partial_t(\rho^\varepsilon u^\varepsilon)+\mathrm{div}\,(\rho^\varepsilon u^\varepsilon\otimes u^\varepsilon)+\frac{\nabla p(\rho^\varepsilon)}{\varepsilon^2}=\mu'\Delta u^\varepsilon+(\mu'+\nu')\nabla\mathrm{div}\,u^\varepsilon+\kappa'\frac{\rho^\varepsilon\nabla\Delta\rho^\varepsilon}{\varepsilon}.\end{cases}\tag{1.2}$$

Formally, letting $\varepsilon\to0$ we obtain from the momentum equation $(1.2)_2$ that ρ^ε converges to some function $\rho^*(t)$ depending only on t. Here, we expect it to be the positive constant ρ^*. Without loss of generality, let us assume that $\rho^*=1$. Then passing to the limit in the mass conservation equation of (1.2), we get $\mathrm{div}\,u^\varepsilon=0$. Therefore by denoting the formal limits of $\frac{\nabla p(\rho^\varepsilon)-\kappa'\varepsilon\rho^\varepsilon\nabla\Delta\rho^\varepsilon}{\varepsilon^2}$ and u^ε by ∇p^0 and u^0, respectively, we can formally obtain the incompressible Navier–Stokes equations:

$$\begin{cases}\partial_t u^0+u^0\cdot\nabla u^0+\nabla p^0=\mu'\Delta u^0,\\ \mathrm{div}\,u^0=0.\end{cases}\tag{1.3}$$

The goal of this paper is to justify the above formal procedure and the main result can be stated as follows.

Theorem 1.1. *Suppose $p(\rho^\varepsilon)$ is smooth function with $p'(\rho^\varepsilon)>0$ for $\rho^\varepsilon>0$, $u_0(x)\in H^6(\Omega)$ is divergence-free. Denote by $T_0>0$ the life-span of the unique classical solution $u^0(t,x)\in C([0,T_0],H^6(\Omega))$ to the incompressible Navier–Stokes equation (1.3) with initial data $u_0(x)$. If $T_0<\infty$, then, for ε sufficiently small, the compressible Navier–Stokes–Korteweg equation (1.2) with initial data*

$$\rho^\varepsilon(x,0)=1,\qquad u^\varepsilon(x,0)=u_0(x)$$

has a unique solution $(\rho^\varepsilon,u^\varepsilon)(t,x)$ satisfying

$$\rho^\varepsilon-1\in C([0,T_0],H^5(\Omega)),\qquad u^\varepsilon\in C([0,T_0],H^4(\Omega)).$$

Moreover, there exists a constant $K>0$, independent of ε but dependent on T_0, such that

$$\sup_{t\in[0,T_0]}(\|\frac{\rho^\varepsilon(\cdot,t)-1}{\varepsilon}\|_4+\sum_{|\alpha|=5}\|\varepsilon^{\frac12}\partial_x^\alpha(\rho^\varepsilon-1)(\cdot,t)\|+\|(u^\varepsilon-u^0)(\cdot,t)\|_4)\le K\varepsilon.\tag{1.4}$$

In case $T_0=\infty$, the maximal existence time T_ε of $(\rho^\varepsilon,u^\varepsilon)(t,x)$ tends to infinity as ε goes to zero.

Remark 1.2. The initial data

$$\rho^\varepsilon(x,0)=1,\quad u^\varepsilon(x,0)=u_0(x)$$

can be relaxed as

$$\rho^\varepsilon(x,0)=1+O(\varepsilon^2),\quad u^\varepsilon(x,0)=u_0(x)+O(\varepsilon)$$

without changing our arguments.

Remark 1.3. If the unique classical solution $u^0(t,x)$ to the incompressible Navier–Stokes equation (1.3) has more regularity, i.e., $u^0(t,x)\in C([0,T_0],H^8(\Omega))$ with initial data $u_0(x)\in H^8(\Omega)$, we can obtain

$$\rho^\varepsilon-1\in C([0,T_0],H^6(\Omega)),\qquad u^\varepsilon\in C([0,T_0],H^5(\Omega)),$$

and

$$\sup_{t\in[0,T_0]}\Big(\|\frac{\rho^\varepsilon(\cdot,t)-1}{\varepsilon}\|_4+\sum_{|\alpha|=5}\|\partial_x^\alpha(\rho^\varepsilon-1)(\cdot,t)\|+\|(u^\varepsilon-u^0)(\cdot,t)\|_4\Big)\le K\varepsilon.$$

Indeed, using a bootstrapping idea [15], the energy method, and uniform stability [18], we can show this result.

Moreover, we do not know if the convergence rate in (1.4) is optimal, in particular, the velocity convergence rate. Regard this topic, we thank the anonymous referees for telling us the paper [4], and we will try to address it in the future study.

Remark 1.4. We believe that the methods developed in [12, 13, 18, 19] etc. for studying the zero-Mach number limit of compressible Navier–Stokes equations, can also be applied to that of the compressible Navier–Stokes–Korteweg systems. However, with the method here, we can obtain the sharp convergence rate (1.4), and no smallness condition on the initial data is required.

Remark 1.5. Here we only consider the zero-Mach limit of the smooth solutions for the compressible Navier–Stokes–Korteweg equations with well-prepared initial data. It is more interesting to consider the similar problem of the compressible Navier–Stokes–Korteweg equations for general initial data (ill-prepared initial data). That is, we should take account of acoustic waves which propagate with the high speed $\frac{1}{\varepsilon}$ in the space domain, as in [7, 8, 9, 20, 21, 22]. Moreover, we also hope that similar results are right for the limit of the compressible Navier–Stokes–Korteweg equations in critical space. These are what our effort should aim at in the forthcoming future.

Let us outlined the idea of proof as follows. We first reformulate the original equations in terms of the pressure variable p^ε and the velocity u^ε (see, for example, [27]). Next, on the basis of a local-in-time existence theory due to Hattori–Li [14, 15] for (1.2) we establish a convergence-stability principle, which are similar to those developed in [25, 26] for singular limit problems of symmetrizable hyperbolic systems. Thus, instead of deriving ε-uniform *a priori* estimates, we only need to make the error estimate (1.4) in the common time interval $[0,\min\{T_0,T_\varepsilon\})$, where both solutions $(\rho^\varepsilon,u^\varepsilon)$ and (ρ^0,u^0) are regular. Due to the third-order term in (1.1) or (1.2), deriving the error estimate requires some elaborated treatments thereof. See Section 4 for the treatments of terms I_5 and H_4.

The rest of this paper is organized as follows. In Section 2 we recall a local-in-time existence theory for (1.2) and reformulate the original equations in terms of the pressure variable p^ε and the velocity u^ε. Then, we present our main ideas in Section 3. All required (error) estimates are obtained in Section 4.

NOTATION. $|U|$ denotes some norm of a vector or matrix U. For a nonnegative integer k, $H^k = H^k(\Omega)$ denotes the usual L^2-type Sobolev space of order k. We write $\|\cdot\|_k$ for the standard norm of H^k and $\|\cdot\|$ for $\|\cdot\|_0$. When U is a function of another variable t as well as $x \in \Omega$, we write $\|U(\cdot,t)\|$ to recall that the norm is taken with respect to x while t is viewed as a parameter. In addition, we denote by $C([0,T],\mathbf{X})$ (resp. $L^2([0,T],\mathbf{X})$) the space of continuous (resp. square integrable) functions on $[0,T]$ with values in a Banach space $\mathbf{X}$.

2. Preliminaries

In this section we recall a local-in-time existence theory due to Hattori–Li [14, 15] for (1.2) and reformulate the original equations in terms of the pressure variable p and the velocity u. We start with the local-in-time existence of the classical solution to the compressible Navier–Stokes–Korteweg equation (1.2).

LEMMA 2.1. *Let $p(\rho^\varepsilon)$ be a smooth function with $p'(\rho^\varepsilon) > 0$ for $\rho^\varepsilon > 0$. Assume that $\bar{u}(x) \in H^4$ and $\bar{\rho}(x) - 1 \in H^5$ with $\inf \bar{\rho}(x) > 0$. Then there exists a positive constant T such that equation (1.2) with initial data $(\bar{\rho},\bar{u})(x)$ have a unique solution $(\rho^\varepsilon,u^\varepsilon) = (\rho^\varepsilon,u^\varepsilon)(x,t)$, satisfying $\rho^\varepsilon(t,x) > 0$ for all $(x,t) \in \Omega \times [0,T]$ and*

$$\begin{aligned}&\rho^\varepsilon - 1 \in C([0,T],H^5) \cap L^2([0,T],H^6),\\ &u^\varepsilon \in C([0,T],H^4) \cap L^2([0,T],H^5).\end{aligned}$$

Next, we follow [27] and reformulate the compressible Navier–Stokes–Korteweg equation (1.2) in terms of the pressure variable $p^\varepsilon = p(\rho^\varepsilon)$ and the velocity u^ε. Since $p(\rho^\varepsilon)$ is strictly increasing, it has an inverse $\rho^\varepsilon = \rho(p^\varepsilon)$. Set $q(p^\varepsilon) = [\rho(p^\varepsilon)p'(\rho(p^\varepsilon))]^{-1}$. Then the compressible Navier–Stokes–Korteweg equation (1.2) for smooth solution can be rewritten as

$$\begin{cases} q(p^\varepsilon)(p^\varepsilon_t + u^\varepsilon\cdot\nabla p^\varepsilon) + \operatorname{div} u^\varepsilon = 0,\\ \rho(p^\varepsilon)(u^\varepsilon_t + u^\varepsilon\cdot\nabla u^\varepsilon) + \varepsilon^{-2}\nabla p^\varepsilon = \mu'\Delta u^\varepsilon + (\mu'+\nu')\nabla\operatorname{div} u^\varepsilon + \varepsilon^{-1}\kappa'\rho(p^\varepsilon)\nabla\Delta\rho(p^\varepsilon). \end{cases} \tag{2.1}$$

Further, we introduce

$$\tilde{p}^\varepsilon = (p^\varepsilon - p_0)/\varepsilon, \qquad \tilde{u}^\varepsilon = u^\varepsilon$$

with $p_0 = p(1) > 0$. Then (2.1) can be rewritten as

$$\begin{cases} q(p_0+\varepsilon\tilde{p}^\varepsilon)(\tilde{p}^\varepsilon_t + \tilde{u}^\varepsilon\cdot\nabla\tilde{p}^\varepsilon) + \varepsilon^{-1}\operatorname{div}\tilde{u}^\varepsilon = 0,\\ \rho(p_0+\varepsilon\tilde{p}^\varepsilon)(\tilde{u}^\varepsilon_t + \tilde{u}^\varepsilon\cdot\nabla\tilde{u}^\varepsilon) + \varepsilon^{-1}\nabla\tilde{p}^\varepsilon = \mu'\Delta\tilde{u}^\varepsilon + (\mu'+\nu')\nabla\operatorname{div}\tilde{u}^\varepsilon\\ \qquad\qquad + \varepsilon^{-1}\kappa'\rho(p_0+\varepsilon\tilde{p}^\varepsilon)\nabla\Delta\rho(p_0+\varepsilon\tilde{p}^\varepsilon). \end{cases} \tag{2.2}$$

For this system, we see immediately from Lemma 2.1 that

COROLLARY 2.2. *Under the assumptions of Lemma 2.1, there exists a positive constant $T_\varepsilon > 0$ such that the equation (2.2) with initial data $(\frac{p(\bar{\rho})-p_0}{\varepsilon},\bar{u})$ have a unique classical solution $(\tilde{p}^\varepsilon,\tilde{u}^\varepsilon) = (\tilde{p}^\varepsilon,\tilde{u}^\varepsilon)(t,x)$, satisfying $\varepsilon\tilde{p}^\varepsilon(x,t) + p_0 > 0$ for all $(x,t) \in \Omega \times [0,T_\varepsilon]$ and*

$$\begin{aligned}&\tilde{p}^\varepsilon(t,x) \in C([0,T_\varepsilon],H^5) \cap L^2([0,T_\varepsilon],H^6),\\ &\tilde{u}^\varepsilon(t,x) \in C([0,T_\varepsilon],H^4) \cap L^2([0,T_\varepsilon],H^5).\end{aligned}$$

Thus, Theorem 1.1 can be stated as follows.

THEOREM 2.3. *Under the assumptions of Theorem 1.1, for ε sufficiently small the equation (2.2) with initial data*

$$\tilde{p}^\varepsilon(x,0)=0, \qquad \tilde{u}^\varepsilon(x,0)=u_0$$

have a unique solution $(p^\varepsilon,u^\varepsilon)(t,x)$ satisfying

$$\tilde{p}^\varepsilon(t,x)\in C([0,T_0],H^5), \qquad \tilde{u}^\varepsilon(t,x)\in C([0,T_0],H^4).$$

Moreover, there exists a constant $K>0$, independent of ε but dependent on T_0, such that

$$\sup_{t\in[0,T_0]}(\|\tilde{p}^\varepsilon-\varepsilon p^0\|_4+\sum_{|\alpha|=5}\|\varepsilon^{\frac{1}{2}}\partial_x^\alpha(\tilde{p}^\varepsilon-\varepsilon p^0)\|+\|\tilde{u}^\varepsilon-u^0\|_4)\le K\varepsilon. \tag{2.3}$$

In case $T_0=\infty$, the maximal existence time T_ε of $(\tilde{p}^\varepsilon,\tilde{u}^\varepsilon)(t,x)$ tends to infinity as ε goes to zero. Here $(p^0,u^0)(t,x)$ is given in Lemma 3.1 and satisfies (3.3).

In Sections 3 and 4 we focus on the proof of Theorem 2.3.

3. Main ideas

Our proof of Theorem 2.3 is guided by the spirit of the convergence-stability principle developed in [25, 26] for singular limit problems of symmetrizable hyperbolic systems. Fix $\varepsilon>0$ in (2.2). According to Corollary 2.2, there is a time interval $[0,T]$ such that the equation (2.2) with initial data $(\bar{p},\bar{u})(x,\varepsilon)$ have a unique solution $(\tilde{p}^\varepsilon,\tilde{u}^\varepsilon)$ satisfying $\varepsilon\tilde{p}^\varepsilon+p_0>0$ for all $(x,t)\in\Omega\times[0,T]$ and

$$\tilde{p}^\varepsilon(t,x)\in C([0,T],H^5), \qquad \tilde{u}^\varepsilon(t,x)\in C([0,T],H^4).$$

Define

$$\begin{aligned} T_\varepsilon=\sup\Big\{&T>0:\tilde{p}^\varepsilon(t,x)\in C([0,T],H^5),\tilde{u}^\varepsilon(t,x)\in C([0,T],H^4);\\ &-\tfrac{1}{2}p_0\le\tilde{p}^\varepsilon(x,t)\le 2p_0, \quad \forall(x,t)\in\Omega\times[0,T]\Big\}. \end{aligned} \tag{3.1}$$

(Here the 2 can be replaced with any positive number larger than 1.) Namely, $[0,T_\varepsilon)$ is the maximal time interval of $H^5\times H^4$-existence. Note that T_ε may tend to 0 as ε goes to 0.

In order to show that $\lim_{\varepsilon\to 0}T_\varepsilon>0$, we follow the convergence-stability principle [25] and seek a formal approximation of $(p^\varepsilon,u^\varepsilon)(t,x)$. To this end, we consider the initial-value problem of the incompressible Navier–Stokes equations:

$$\begin{cases} \partial_t u^0+u^0\cdot\nabla u^0+\nabla p^0=\mu\Delta u^0,\\ \operatorname{div}u^0=0, \qquad u^0(x,0)=u_0(x). \end{cases} \tag{3.2}$$

Since $u_0(x)\in H^6$ and $\operatorname{div}u_0=0$, we know from [17, 23] that

LEMMA 3.1. *There exists $T_0\in(0,+\infty)$ such that the initial-value problem (3.2) has a unique smooth solution $(u^0,p^0)(t,x)\in C([0,T_0],H^6)\times C([0,T_0],H^7)$ satisfying*

$$M_0=:\sup_{0\le t\le T_0}(\|u^0(t)\|_6+\|\partial_t u^0(t)\|_4+\|p^0(t)\|_7+\|\partial_t p^0(t)\|_5)<\infty. \tag{3.3}$$

Here p^0 in the torus Ω is unique up to a shift constant.

In the next section we will prove the following theorem.

THEOREM 3.2. *Under the conditions of Theorem 1.1, there exist constants $K=K(T_0)$ and $\varepsilon_0=\varepsilon_0(T_0)$ such that for all $\varepsilon\le\varepsilon_0$,*

$$\|(\tilde{p}^\varepsilon-\varepsilon p^0)(\cdot,t)\|_4+\sum_{|\alpha|=5}\|\varepsilon^{\frac{1}{2}}\partial_x^\alpha(\tilde{p}^\varepsilon-\varepsilon p^0)(\cdot,t)\|+\|(\tilde{u}^\varepsilon-u^0)(\cdot,t)\|_4\le K\varepsilon$$

for $t\in[0,\min\{T_0,T_\varepsilon\})$.

Having this theorem, we slightly modify the arguments in [26] to prove

THEOREM 3.3. *Under the conditions of Theorem 1.1, there exists a constant $\varepsilon_0=\varepsilon_0(T_0)$ such that for all $\varepsilon\le\varepsilon_0$,*

$$T_\varepsilon>T_0.$$

Proof. Otherwise, there is a sequence $\{\varepsilon_k\}_{k\ge1}$ such that $\lim_{k\to\infty}\varepsilon_k=0$ and $T_{\varepsilon_k}\le T_0$. Thanks to the error estimate in Theorem 3.2, there exists a k such that $4\tilde{p}^{\varepsilon_k}(x,t)\in(-3p_0,5p_0)$ for all x and t. Next we deduce from

$$\begin{aligned}\|\tilde{p}^{\varepsilon_k}(\cdot,t)\|_5+\|\tilde{u}^{\varepsilon_k}(\cdot,t)\|_4\le{}&\|\tilde{p}^{\varepsilon_k}(\cdot,t)-\varepsilon_k p^0(\cdot,t)\|_5+\|\varepsilon_k p^0(\cdot,t)\|_5\\&+\|\tilde{u}^{\varepsilon_k}(\cdot,t)-u^0(\cdot,t)\|_4+\|u^0(\cdot,t)\|_4,\end{aligned}$$

and Theorem 3.2 that $\|\tilde{p}^{\varepsilon_k}(\cdot,t)\|_5+\|\tilde{u}^{\varepsilon_k}(\cdot,t)\|_4$ is bounded uniformly with respect to $t\in[0,T_{\varepsilon_k})$. Now we could apply Corollary 2.2, beginning at a time t less than T_{ε_k}(k is fixed here), to continue this solution beyond T_{ε_k}. This contradicts the definition of T_{ε_k} in (3.1).

Finally, Theorem 2.3 is proved by combining Theorems 3.2 and 3.3. □

We conclude this section with the following interesting remark, which is a by-product of our approach.

REMARK 3.4. The proof of Theorem 3.2 requires that $T_0<\infty$. However, when the initial-value problem (3.2) of the incompressible Navier–Stokes equations has a global-in-time regular solution, T_0 can be any positive number. Hence we have an almost global-in-time existence result for (2.2):

$$\lim_{\varepsilon\to0}T_\varepsilon=+\infty.$$

4. Error estimate

In this section we prove the error estimate in Theorem 3.2. For this purpose, we need the following classical calculus inequalities in Sobolev spaces [22].

LEMMA 4.1.
(i) For $s\ge2$, $H^s=H^s(\Omega)$ is an algebra. Namely, for all multi-indices α with $|\alpha|\le s$ and $f,g\in H^s(\Omega)$, it holds that $\partial_x^\alpha(fg)\in L^2(\Omega)$ and

$$\|\partial_x^\alpha(fg)\|\le C_s\|f\|_s\|g\|_s.$$

(ii) For $s\ge3$, let $f\in H^s(\Omega)$ and $g\in H^{s-1}(\Omega)$. Then for all multi-indices α with $|\alpha|\le s$, it holds that the commutator $[\partial_x^\alpha,f]g\in L^2(\Omega)$ and

$$\|[\partial_x^\alpha,f]g\|\le C_s\|\nabla f\|_{s-1}\|g\|_{s-1}.$$

(iii) Assume $g(u)$ is a smooth function on G, $u(x)$ is a continuous function with $u(x)\in G_1, \bar{G}_1 \subset\subset G$, and $u(x)\in L^\infty\cap H^s$. Then for $s\geq 1$,

$$\|D^s g(u)\| \leq C_s |\frac{\partial g}{\partial u}|_{s-1,\bar{G}_1} \|u\|_{L^\infty}^{s-1} \|D^s u\|.$$

Here $|\cdot|_{r,\bar{G}_1}$ is the C^r-norm on the set $\bar{G}_1$ and C_s is a generic constant depending only on s.

Next we notice that, with u^0 and p^0 constructed in Lemma 3.1,

$$(p_\varepsilon, u_\varepsilon) := (\varepsilon p^0, u^0)$$

satisfies

$$\begin{cases} q(p_0+\varepsilon p_\varepsilon)(p_{\varepsilon t}+u_\varepsilon\cdot\nabla p_\varepsilon)+\varepsilon^{-1}\mathrm{div}\, u_\varepsilon=\varepsilon R_1, \\ \rho(p_0+\varepsilon p_\varepsilon)(u_{\varepsilon t}+u_\varepsilon\cdot\nabla u_\varepsilon)+\varepsilon^{-1}\nabla p_\varepsilon=\mu'\Delta u_\varepsilon+(\mu'+\nu')\nabla\mathrm{div}\, u_\varepsilon+R_2 \end{cases} \tag{4.1}$$

with

$$\begin{aligned} R_1 &= q(p_0+\varepsilon^2 p^0)(p_t^0+u^0\cdot\nabla p^0), \\ R_2 &= (\rho(p_0+\varepsilon^2 p^0)-\rho(p_0))(u_t^0+u^0\cdot\nabla u^0). \end{aligned}$$

Note that Lemma 3.1 and Sobolev inequality imply $-\frac{1}{2}p_0\leq \varepsilon p_\varepsilon (=\varepsilon^2 p^0)\leq p_0$ for $\varepsilon\ll 1$, which yields $\frac{1}{2}p^0\leq p_0+\varepsilon p_\varepsilon\leq 2p^0$. Further, if $\rho(p_0+\varepsilon p_\varepsilon)$ is strictly increasing, then we have

$$\rho(\frac{1}{2}p_0)\leq \rho(p_0+\varepsilon p_\varepsilon)\leq \rho(2p_0). \tag{4.2}$$

From the definition of $q(p_0+\varepsilon p_\varepsilon)$, we also have

$$\frac{1}{2p'(2)}\leq q(p_0+\varepsilon p_\varepsilon)\leq \frac{2}{p'(\frac{1}{2})}. \tag{4.3}$$

Set

$$P=\tilde{p}^\varepsilon - p_\varepsilon, \qquad U=\tilde{u}^\varepsilon-u_\varepsilon.$$

Then we deduce from (2.2) and (4.1) that

$$P_t+\tilde{u}^\varepsilon\cdot\nabla P+U\cdot\nabla p_\varepsilon+\varepsilon^{-1}q^{-1}(p_0+\varepsilon\tilde{p}^\varepsilon)\mathrm{div}\, U=f_1, \tag{4.4}$$

and

$$\begin{aligned} U_t+\tilde{u}^\varepsilon\cdot\nabla U+U\cdot\nabla u_\varepsilon+\varepsilon^{-1}\rho^{-1}(p_0+\varepsilon\tilde{p}^\varepsilon)\nabla P \\ -\rho^{-1}(p_0+\varepsilon\tilde{p}^\varepsilon)(\mu'\Delta U+(\mu'+\nu')\nabla\mathrm{div}\, U)=\varepsilon^{-1}\kappa'\nabla\Delta\tilde{\rho}^\varepsilon+f_2. \end{aligned} \tag{4.5}$$

Here we have used $\mathrm{div}\, u_\varepsilon=0$ and $\tilde{\rho}^\varepsilon=\rho(p_0+\varepsilon\tilde{p}^\varepsilon)$, and

$$\begin{aligned} f_1 &= -q^{-1}(p_0+\varepsilon p_\varepsilon)\varepsilon R_1, \\ f_2 &= -\rho^{-1}(p_0+\varepsilon p_\varepsilon)R_2-\Big(\rho^{-1}(p_0+\varepsilon\tilde{p}^\varepsilon)-\rho^{-1}(p_0+\varepsilon p_\varepsilon)\Big)(\nabla p^0-\mu'\Delta u^0). \end{aligned}$$

From Lemmas 3.1 and 4.1 it follows that, for $t\in[0,T_0]$,

$$\|f_1\|_4\leq C(M_0)\varepsilon, \qquad \|f_2\|\leq C(M_0)(\varepsilon^2+\|P\|_4^2). \tag{4.6}$$

Here and in the subsequent, $C>0$ is the generic constant and $C(\cdot)>0$ stands for the generic constant depending on "$\cdot$".

Let α be a multi-index with $|\alpha|\le 4$. Differentiating the two sides of the equations in (4.4) and (4.5) with ∂_x^α and setting

$$P_\alpha=\partial_x^\alpha P, \quad U_\alpha=\partial_x^\alpha U, \quad \tilde{\rho}_\alpha^\varepsilon=\partial_x^\alpha \tilde{\rho}^\varepsilon,$$

we obtain

$$\begin{aligned}&\partial_t P_\alpha+\tilde{u}^\varepsilon\cdot\nabla P_\alpha+\varepsilon^{-1}q^{-1}(p_0+\varepsilon\tilde{p}^\varepsilon)\operatorname{div} U_\alpha\\ &=-[\partial_x^\alpha,\tilde{u}^\varepsilon]\nabla P-\partial_x^\alpha(U\cdot\nabla p_\varepsilon)-\varepsilon^{-1}[\partial_x^\alpha,q^{-1}(p_0+\varepsilon\tilde{p}^\varepsilon)]\operatorname{div} U+\partial_x^\alpha f_1,\end{aligned}\tag{4.7}$$

and

$$\begin{aligned}&\partial_t U_\alpha+\tilde{u}^\varepsilon\cdot\nabla U_\alpha+\varepsilon^{-1}\rho^{-1}(p_0+\varepsilon\tilde{p}^\varepsilon)\nabla P_\alpha-\rho^{-1}(p_0+\varepsilon\tilde{p}^\varepsilon)(\mu'\Delta U_\alpha+(\mu'+\nu')\nabla\operatorname{div} U_\alpha)\\ &=-[\partial_x^\alpha,\tilde{u}^\varepsilon]\nabla U-\partial_x^\alpha(U\cdot\nabla u_\varepsilon)-\varepsilon^{-1}[\partial_x^\alpha,\rho^{-1}(p_0+\varepsilon\tilde{p}^\varepsilon)]\nabla P\\ &\quad+[\partial_x^\alpha,\rho^{-1}(p_0+\varepsilon\tilde{p}^\varepsilon)](\mu'\Delta U+(\mu'+\nu')\nabla\operatorname{div} U)+\varepsilon^{-1}\kappa'\nabla\Delta\tilde{\rho}_\alpha^\varepsilon+\partial_x^\alpha f_2.\end{aligned}\tag{4.8}$$

Taking the inner product of (4.7) and (4.8) with $q(p_0+\varepsilon\tilde{p}^\varepsilon)P_\alpha$ and $\rho(p_0+\varepsilon\tilde{p}^\varepsilon)U_\alpha$, respectively, and summing up the two resultant equalities gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega\Big(q^\varepsilon P_\alpha^2+\tilde{\rho}^\varepsilon|U_\alpha|^2\Big)dx+\int_\Omega\Big(\mu'|\nabla U_\alpha|^2+(\mu'+\nu')|\operatorname{div} U_\alpha|^2\Big)dx\\ &=\int_\Omega\Big(\frac{1}{2}q_t^\varepsilon P_\alpha^2+\frac{1}{2}\rho_t^\varepsilon|U_\alpha|^2-q^\varepsilon P_\alpha(u^\varepsilon\cdot\nabla)P_\alpha-\tilde{\rho}^\varepsilon U_\alpha(u^\varepsilon\cdot\nabla)U_\alpha\Big)dx\\ &\quad+\int_\Omega\Big(-([\partial_x^\alpha,\tilde{u}^\varepsilon]\nabla P+\partial_x^\alpha(U\cdot\nabla p_\varepsilon))q^\varepsilon P_\alpha-(\partial_x^\alpha(U\cdot\nabla u_\varepsilon)+[\partial_x^\alpha,\tilde{u}^\varepsilon]\nabla U)\rho^\varepsilon U_\alpha\Big)dx\\ &\quad-\varepsilon^{-1}\int_\Omega\Big(q^\varepsilon P_\alpha[\partial_x^\alpha,q^{-1}(p_0+\varepsilon\tilde{p}^\varepsilon)]\operatorname{div} U+\rho^\varepsilon U_\alpha[\partial_x^\alpha,\rho^{-1}(p_0+\varepsilon\tilde{p}^\varepsilon)]\nabla P\Big)dx\\ &\quad+\int_\Omega\tilde{\rho}^\varepsilon U_\alpha[\partial_x^\alpha,\rho^{-1}(p_0+\varepsilon\tilde{p}^\varepsilon)](\mu'\Delta U+(\mu'+\nu')\nabla\operatorname{div} U)dx\\ &\quad+\varepsilon^{-1}\kappa'\int_\Omega\tilde{\rho}^\varepsilon U_\alpha\nabla\Delta\tilde{\rho}_\alpha^\varepsilon dx+\int_\Omega\Big(q^\varepsilon\partial_x^\alpha f_1 P_\alpha+\tilde{\rho}^\varepsilon\partial_x^\alpha f_2 U_\alpha\Big)dx\\ &=:I_1+I_2+I_3+I_4+I_5+I_6.\end{aligned}\tag{4.9}$$

Here and below, we often use

$$\tilde{\rho}^\varepsilon=\rho(p_0+\varepsilon\tilde{p}^\varepsilon), \qquad q^\varepsilon=q(p_0+\varepsilon\tilde{p}^\varepsilon).$$

To estimate the I_i's, we first have the bound of $\rho'(p_0+\varepsilon\tilde{p}^\varepsilon)$ and $q'(p_0+\varepsilon\tilde{p}^\varepsilon)$ as follows.

Lemma 4.2. *It holds that*

$$c_1\le\rho'(p_0+\varepsilon\tilde{p}^\varepsilon)\le c_2,\ c_3\le q'(p_0+\varepsilon\tilde{p}^\varepsilon)\le c_4,$$

where and in the subsequent, $c_i(i=1,2,3,4)$ are positive constants independent of ε.

Proof. From the smoothness of ρ and $\frac{1}{2}p^0\le p_0+\varepsilon\tilde{p}^\varepsilon\le 2p^0$, it is easy to get

$$\rho'(\frac{1}{2}p^0)\le\rho'(p_0+\varepsilon\tilde{p}^\varepsilon)\le\rho'(2p^0).$$

Moreover, from the definition of q, similarly, we have

$$c_3 \le q'(p_0+\varepsilon\tilde{p}^\varepsilon) \le c_4.$$

This completes the proof. ∎

Next, we follow [25, 26] and formulate the following lemma.

LEMMA 4.3. *Set*

$$D=D(t)=\|P(\cdot,t)\|_4+\sum_{|\beta|=5}\|\varepsilon^{\frac{1}{2}}\partial_x^\beta P(\cdot,t)\|+\|U(\cdot,t)\|_4$$

for $t\in[0,\min\{T_0,T_\varepsilon\})$. *Then for multi-indices* γ *satisfying* $|\gamma|\le 2$ *it holds that*

$$\|\partial_x^\gamma \tilde{p}^\varepsilon\|_{L^\infty}+\|\partial_x^\gamma \tilde{u}^\varepsilon\|_{L^\infty}\le C(M_0)(1+D).$$

Proof. It is obvious from Lemma 3.1 and the Sobolev inequality that

$$\|\partial_x^\gamma \tilde{u}^\varepsilon\|_{L^\infty}\le\|\partial_x^\gamma(\tilde{u}^\varepsilon-u_\varepsilon)\|_{L^\infty}+\|\partial_x^\gamma u^0\|_{L^\infty}\le C(M_0)(1+D).$$

The other estimates can be showed similarly. This completes the proof.

Now we turn to estimate the I_i's in (4.9). Using integration by parts, and Lemmas 4.2 and 4.3, we deduce that

$$\begin{aligned}
I_1=&\frac{1}{2}\int_\Omega\Big(q'(p_0+\varepsilon\tilde{p}^\varepsilon)\varepsilon\tilde{p}_t^\varepsilon+q^\varepsilon\operatorname{div}\tilde{u}^\varepsilon+\tilde{u}^\varepsilon\cdot\nabla q^\varepsilon\Big)P_\alpha^2dx\\
&+\frac{1}{2}\int_\Omega\Big(\rho'(p_0+\varepsilon\tilde{p}^\varepsilon)\varepsilon\tilde{p}_t^\varepsilon+\tilde{\rho}^\varepsilon\operatorname{div}\tilde{u}^\varepsilon+\tilde{u}^\varepsilon\cdot\nabla\tilde{\rho}^\varepsilon\Big)|U_\alpha|^2dx\\
\le& C(\|\operatorname{div}\tilde{u}^\varepsilon\|_{L^\infty}+\|\tilde{u}^\varepsilon\|_{L^\infty}\|\nabla\tilde{p}^\varepsilon\|_{L^\infty})(\|P_\alpha\|^2+\|U_\alpha\|^2)\\
\le& C(M_0)(1+D^2)(\|P_\alpha\|^2+\|U_\alpha\|^2),
\end{aligned}$$

with the help of $(2.2)_1$.

Thanks to Lemma 4.1, I_2 can be simply treated as

$$\begin{aligned}
I_2\le& C\|P_\alpha\|(\|\partial_x^\alpha(U\cdot\nabla p_\varepsilon)\|+\|[\partial_x^\alpha,\tilde{u}^\varepsilon]\nabla P\|)\\
&+C\|U_\alpha\|(\|\partial_x^\alpha(U\cdot\nabla u_\varepsilon)\|+\|[\partial_x^\alpha,\tilde{u}^\varepsilon]\nabla U\|)\\
\le& C\|P_\alpha\|(\|\nabla p_\varepsilon\|_4\|U\|_4+\|\nabla\tilde{u}^\varepsilon\|_3\|\nabla P\|_3)\\
&+C\|U_\alpha\|(\|\nabla u_\varepsilon\|_4\|U\|_4+\|\nabla\tilde{u}^\varepsilon\|_3\|\nabla U\|_3)\\
\le& C(M_0)(1+D)(\|U\|_4^2+\|P\|_4^2).
\end{aligned}$$

For I_3 and I_4, from Lemma 4.3, we first compute that

$$\|\nabla\rho^{-1}(p_0+\varepsilon\tilde{p}^\varepsilon)\|_3,\quad\|\nabla q^{-1}(p_0+\varepsilon\tilde{p}^\varepsilon)\|_3\le C(M_0)\varepsilon(1+D^4). \tag{4.10}$$

Then we have

$$\begin{aligned}
I_3\le&\frac{1}{\varepsilon}\Big(\|[\partial_x^\alpha,q^{-1}(p_0+\varepsilon\tilde{p}^\varepsilon)]\operatorname{div}U\|\|q^\varepsilon P_\alpha\|+\|[\partial_x^\alpha,\rho^{-1}(p_0+\varepsilon\tilde{p}^\varepsilon)]\nabla P\|\|\tilde{\rho}^\varepsilon U_\alpha\|\Big)\\
\le&\frac{C}{\varepsilon}(\|\nabla q^{-1}(p_0+\varepsilon\tilde{p}^\varepsilon)\|_3\|\operatorname{div}U\|_3\|P_\alpha\|+\|\nabla\rho^{-1}(p_0+\varepsilon\tilde{p}^\varepsilon)\|_3\|\nabla P\|_3\|U_\alpha\|)
\end{aligned}$$

$$\begin{aligned}&\le C(M_0)(1+D^4)(\|\operatorname{div} U\|_3\|P_\alpha\|+\|\nabla P\|_3\|U_\alpha\|)\\&\le C(M_0)(1+D^4)(\|U\|_4^2+\|P\|_4^2),\end{aligned}$$

and

$$\begin{aligned}I_4&\le C\|\nabla\rho^{-1}(p_0+\varepsilon\tilde p^\varepsilon)\|_3\|\mu'\Delta U+(\mu'+\nu')\nabla\operatorname{div} U\|_3\|U_\alpha\|\\&\le\delta\|\nabla U\|_4^2+C(M_0)\varepsilon(1+D^8)\|U_\alpha\|^2,\end{aligned}$$

here and in the subsequent δ is a proper positive constant, which is determined.

In order to estimate I_5, we firstly use integration by parts to obtain

$$\begin{aligned}I_5&=-\varepsilon^{-1}\kappa'\int_\Omega\Delta\tilde\rho_\alpha^\varepsilon\Big(\varepsilon\rho'(p_0+\varepsilon\tilde p^\varepsilon)U_\alpha\nabla\tilde p^\varepsilon+\tilde\rho^\varepsilon\operatorname{div} U_\alpha\Big)dx\\&\le C(M_0)(1+D)\|\Delta\tilde\rho_\alpha^\varepsilon\|\|U_\alpha\|-\varepsilon^{-1}\kappa'\int_\Omega\tilde\rho^\varepsilon\Delta\tilde\rho_\alpha^\varepsilon\operatorname{div} U_\alpha dx.\end{aligned}\tag{4.11}$$

As

$$\tilde\rho^\varepsilon\operatorname{div} U_\alpha=\tilde\rho^\varepsilon\operatorname{div}\partial_x^\alpha\tilde u^\varepsilon=-(\partial_t\tilde\rho_\alpha^\varepsilon+\partial_x^\alpha(\tilde u^\varepsilon\cdot\nabla\tilde\rho^\varepsilon)+[\partial_x^\alpha,\tilde\rho^\varepsilon]\operatorname{div}\tilde u^\varepsilon),$$

the second term in the right-hand side of (4.11) can be estimated as

$$\begin{aligned}&-\varepsilon^{-1}\kappa'\int_\Omega\tilde\rho^\varepsilon\Delta\tilde\rho_\alpha^\varepsilon\operatorname{div} U_\alpha dx\\&=\varepsilon^{-1}\kappa'\int_\Omega\Delta\tilde\rho_\alpha^\varepsilon\partial_t\tilde\rho_\alpha^\varepsilon dx+\varepsilon^{-1}\kappa'\int_\Omega\Delta\tilde\rho_\alpha^\varepsilon\Big(\partial_x^\alpha(\tilde u^\varepsilon\cdot\nabla\tilde\rho^\varepsilon)+[\partial_x^\alpha,\tilde\rho^\varepsilon]\operatorname{div}\tilde u^\varepsilon\Big)dx\\&\le-\frac{\kappa'}{2\varepsilon}\frac{d}{dt}\int_\Omega|\nabla\tilde\rho_\alpha^\varepsilon|^2dx+C\varepsilon^{-1}\|\Delta\tilde\rho_\alpha^\varepsilon\|\|\tilde u^\varepsilon\|_4\|\nabla\tilde\rho^\varepsilon\|_4.\end{aligned}$$

Since

$$\nabla\tilde\rho^\varepsilon=\varepsilon\rho'(p_0+\varepsilon\tilde p^\varepsilon)\nabla\tilde p^\varepsilon=\varepsilon\rho'(p_0+\varepsilon\tilde p^\varepsilon)(\nabla P+\nabla p_\varepsilon),$$

and

$$\begin{aligned}\Delta\tilde\rho_\alpha^\varepsilon=&\varepsilon\rho'(p_0+\varepsilon\tilde p^\varepsilon)\Delta P_\alpha+\varepsilon[\partial_x^\alpha,\rho'(p_0+\varepsilon\tilde p^\varepsilon)]\Delta P+\varepsilon\partial_x^\alpha(\rho'(p_0+\varepsilon\tilde p^\varepsilon)\Delta p_\varepsilon)\\&+\varepsilon^2\partial_x^\alpha(\rho''(p_0+\varepsilon\tilde p^\varepsilon)(\nabla\tilde p^\varepsilon)^2),\end{aligned}\tag{4.12}$$

we have

$$\|\nabla\tilde\rho^\varepsilon\|_4\le C(M_0)\Big(\varepsilon\sum_{|\alpha|=4}\|\nabla P_\alpha\|+\varepsilon(1+D^4)\|\nabla P\|_3+\varepsilon^2\Big),$$

and

$$\|\Delta\tilde\rho_\alpha^\varepsilon\|\le C(M_0)\Big(\varepsilon\|\Delta P_\alpha\|+\varepsilon(1+D^4)\|\Delta P\|_3+\varepsilon(1+D^5)\|P\|_4+\varepsilon^2\Big).\tag{4.13}$$

Moreover, due to

$$\begin{aligned}\nabla\tilde\rho_\alpha^\varepsilon&=\varepsilon\partial_x^\alpha(\rho'(p_0+\varepsilon\tilde p^\varepsilon)\nabla\tilde p^\varepsilon)\\&=\varepsilon\rho'(p_0+\varepsilon\tilde p^\varepsilon)\nabla P_\alpha+\varepsilon[\partial_x^\alpha,\rho'(p_0+\varepsilon\tilde p^\varepsilon)]\nabla P+\varepsilon\partial_x^\alpha(\rho'(p_0+\varepsilon\tilde p^\varepsilon)\nabla p_\varepsilon),\end{aligned}$$

a straightforward calculation yields

$$
\begin{aligned}
&-\frac{d}{dt}\int_\Omega |\nabla\tilde\rho^\varepsilon_\alpha|^2dx\\
&=-\varepsilon^2\frac{d}{dt}\int_\Omega |\rho'(p_0+\varepsilon\tilde p^\varepsilon)\nabla P_\alpha|^2dx-2\varepsilon^2\int_\Omega\Big(\partial_x^\alpha(\rho'(p_0+\varepsilon\tilde p^\varepsilon)\nabla p_\varepsilon)+[\partial_x^\alpha,\rho'(p_0+\varepsilon\tilde p^\varepsilon)]\nabla P\\
&\quad+\rho'(p_0+\varepsilon\tilde p^\varepsilon)\nabla P_\alpha\Big)\cdot\partial_x^\alpha\Big(\rho'(p_0+\varepsilon\tilde p^\varepsilon)\nabla p_{\varepsilon t}+\varepsilon\rho''(p_0+\varepsilon\tilde p^\varepsilon)\tilde p^\varepsilon_t\nabla p_\varepsilon\Big)dx\\
&\quad-2\varepsilon^2\int_\Omega\Big([\partial_x^\alpha,\rho'(p_0+\varepsilon\tilde p^\varepsilon)]\nabla P+\rho'(p_0+\varepsilon\tilde p^\varepsilon)\nabla P_\alpha+\partial_x^\alpha(\rho'(p_0+\varepsilon\tilde p^\varepsilon)\nabla p_\varepsilon)\Big)\\
&\quad\cdot\Big([\partial_x^\alpha,\rho'(p_0+\varepsilon\tilde p^\varepsilon)]\nabla P_t+[\partial_x^\alpha,\varepsilon\rho''(p_0+\varepsilon\tilde p^\varepsilon)\tilde p^\varepsilon_t]\nabla P\Big)dx-2\varepsilon^2\int_\Omega\Big(\rho'(p_0+\varepsilon\tilde p^\varepsilon)\nabla P_{\alpha t}\\
&\quad+\varepsilon\rho''(p_0+\varepsilon\tilde p^\varepsilon)\tilde p^\varepsilon_t\nabla P_\alpha\Big)\cdot\Big([\partial_x^\alpha,\rho'(p_0+\varepsilon\tilde p^\varepsilon)]\nabla P+\partial_x^\alpha(\rho'(p_0+\varepsilon\tilde p^\varepsilon)\nabla p_\varepsilon)\Big)dx\\
&=:-\varepsilon^2\frac{d}{dt}\int_\Omega |\rho'(p_0+\varepsilon\tilde p^\varepsilon)\nabla P_\alpha|^2dx+I_{51}+I_{52}+I_{53}.
\end{aligned}
$$

Using Lemma 3.1, and Lemmas 4.1–4.3, it is easy to get

$$I_{51}\le C(M_0)(1+D^8)(\|P\|_4^2+\|\varepsilon^{\frac12}\nabla P_\alpha\|^2+\|U\|_4^2)+C(M_0)\varepsilon^2.$$

Moreover, with the help of $(4.1)_1$, and using Lemma 3.1, and Lemmas 4.1–4.3, we also have

$$I_{52}\le C(M_0)(1+D^{10})(\|P\|_4^2+\|\varepsilon^{\frac12}\nabla P_\alpha\|^2+\|U\|_4^2)+\delta\|\nabla U\|_4^2+C(M_0)\varepsilon^2.$$

Similarly, using (4.4), Lemma 3.1, and Lemmas 4.1–4.3, we can obtain

$$I_{53}\le C(M_0)(1+D^{10})(\|P\|_4^2+\|\varepsilon^{\frac12}\nabla P_\alpha\|^2+\|U\|_4^2)+\delta\|\nabla U\|_4^2+\delta\|\varepsilon^{\frac12}\Delta P_\alpha\|^2+C(M_0)\varepsilon^2.$$

Therefore, substitution the above inequalities into (4.11) yields

$$
\begin{aligned}
I_5\le&-\frac{\kappa'\varepsilon}{2}\frac{d}{dt}\int_\Omega|\rho'(p_0+\varepsilon\tilde p^\varepsilon)\nabla P_\alpha|^2dx+\delta\|\varepsilon^{\frac12}\Delta P_\alpha\|^2+2\delta\|\nabla U\|_4^2\\
&+C(M_0)(1+D^{10})(\|P\|_4^2+\|\varepsilon^{\frac12}\nabla P_\alpha\|^2+\|U\|_4^2)+C(M_0)\varepsilon^2.
\end{aligned}
$$

Finally, from (4.6) we deduce that

$$I_6\le C(M_0)\varepsilon^2+C(M_0)(1+D)(\|P\|_4^2+\|U_\alpha\|^2).$$

Hence, putting the estimates of $I_i(i=1,2,\dots,6)$ into (4.9), we have

$$
\begin{aligned}
&\frac12\frac{d}{dt}\int_\Omega\Big(q^\varepsilon P_\alpha^2+\tilde\rho^\varepsilon|U_\alpha|^2+\kappa'\varepsilon|\rho'(p_0+\varepsilon\tilde p^\varepsilon)\nabla P_\alpha|^2\Big)dx+\mu'\|\nabla U_\alpha\|^2+(\mu'+\nu')\|\mathrm{div}\,U_\alpha\|^2\\
&\le C(M_0)(\varepsilon^2+(1+D^{10})(\|\varepsilon^{\frac12}\nabla P_\alpha\|+\|P\|_4^2+\|U\|_4^2))+3\delta\|\nabla U\|_4^2+\delta\|\varepsilon^{\frac12}\Delta P_\alpha\|^2. \qquad (4.14)
\end{aligned}
$$

In order to control the term with δ, we multiply (4.8) by $\varepsilon\rho(p_0+\varepsilon\tilde p^\varepsilon)\nabla P_\alpha$ and integrate the resultant equality by parts over Ω to obtain

$$\frac{d}{dt}\int_\Omega\varepsilon\tilde\rho^\varepsilon U_\alpha\nabla P_\alpha dx+\int_\Omega|\nabla P_\alpha|^2dx$$

$$
\begin{aligned}
&= \Big(\varepsilon\int_\Omega \tilde\rho^\varepsilon U_\alpha \nabla P_{\alpha t}dx + \varepsilon\int_\Omega \tilde\rho^\varepsilon_t U_\alpha \nabla P_\alpha dx\Big) + \varepsilon\int_\Omega (\mu'\Delta U_\alpha + (\mu'+\nu')\nabla \mathrm{div}\, U_\alpha)\nabla P_\alpha dx \\
&\quad -\varepsilon\int_\Omega \tilde\rho^\varepsilon \nabla P_\alpha\Big(\tilde u^\varepsilon\cdot\nabla U_\alpha + \partial_x^\alpha(U\cdot\nabla u_\varepsilon) + [\partial_x^\alpha, \tilde u^\varepsilon]\nabla U + \varepsilon^{-1}[\partial_x^\alpha, \rho^{-1}(p_0+\varepsilon\tilde p^\varepsilon)]\nabla P \\
&\quad -[\partial_x^\alpha, \rho^{-1}(p_0+\varepsilon\tilde p^\varepsilon)](\mu'\Delta U + (\mu'+\nu')\nabla \mathrm{div}\, U)\Big)dx + \kappa'\int_\Omega \tilde\rho^\varepsilon \nabla\Delta\tilde\rho^\varepsilon_\alpha \nabla P_\alpha dx \\
&\quad +\varepsilon\int_\Omega \tilde\rho^\varepsilon \partial_x^\alpha f_2 \nabla P_\alpha dx =: H_1+H_2+H_3+H_4+H_5. \qquad (4.15)
\end{aligned}
$$

We estimate the H_i's as follows. By using integration by parts, it follows from (4.7), $(2.2)_1$, Lemmas 4.1–4.3 that

$$
\begin{aligned}
H_1 &= \varepsilon\int_\Omega (\rho(p_0+\varepsilon\tilde p^\varepsilon)\mathrm{div}\, U_\alpha + \varepsilon\rho'(p_0+\varepsilon\tilde p^\varepsilon)\nabla\tilde p^\varepsilon U_\alpha)(\tilde u^\varepsilon\cdot\nabla P_\alpha + [\partial_x^\alpha, \tilde u^\varepsilon]\nabla P \\
&\quad + \partial_x^\alpha(U\cdot\nabla p_\varepsilon) + \varepsilon^{-1}[\partial_x^\alpha, q^{-1}(p_0+\varepsilon\tilde p^\varepsilon)]\mathrm{div}\, U - \partial_x^\alpha f_1)dx \\
&\quad -\varepsilon\int_\Omega \rho'(p_0+\varepsilon\tilde p^\varepsilon)(\varepsilon\tilde u^\varepsilon\nabla\tilde p^\varepsilon + q^{-1}(p_0+\varepsilon\tilde p^\varepsilon)\mathrm{div}\,\tilde u^\varepsilon)U_\alpha\nabla P_\alpha dx \\
&\le C\varepsilon(\|\mathrm{div}\, U_\alpha\| + \varepsilon\|\nabla\tilde p^\varepsilon U_\alpha\|)(\|\tilde u^\varepsilon \nabla P_\alpha\| + \|[\partial_x^\alpha, \tilde u^\varepsilon]\nabla P\| + \|\partial_x^\alpha(U\cdot\nabla p_\varepsilon)\| \\
&\quad + \varepsilon^{-1}\|[\partial_x^\alpha, q^{-1}(p_0+\varepsilon\tilde p^\varepsilon)]\mathrm{div}\, U\| + \|\partial_x^\alpha f_1\|) \\
&\le C\varepsilon(\|\mathrm{div}\, U_\alpha\| + \varepsilon\|\nabla\tilde p^\varepsilon\|_{L^\infty}\|U_\alpha\|)(\|\tilde u^\varepsilon\|_{L^\infty}\|\nabla P_\alpha\| + \|\nabla\tilde u^\varepsilon\|_3\|\nabla P\|_3 + \|U\|_4\|\nabla p_\varepsilon\|_4 \\
&\quad + \varepsilon^{-1}\|\nabla q^{-1}(p_0+\varepsilon\tilde p^\varepsilon)\|_3\|\mathrm{div}\, U\|_3 + \|\partial_x^\alpha f_1\|) \\
&\le C(M_0)\|\mathrm{div}\, U_\alpha\|^2 + C(M_0)\varepsilon^2 + C(M_0)(1+D^4)(\|P\|_4^2 + \|\varepsilon^{\frac12}\nabla P_\alpha\|^2 + \|U\|_4^2),
\end{aligned}
$$

and

$$
\begin{aligned}
H_2 &= -\varepsilon\int_\Omega \big(\mu'\nabla U_\alpha + (\mu'+\nu')\mathrm{div}\, U_\alpha\big)\Delta P_\alpha dx \\
&\le \frac{\delta}{2}\|\varepsilon^{\frac12}\Delta P_\alpha\|^2 + C(\|\mathrm{div}\, U_\alpha\|^2 + \|\nabla U_\alpha\|^2).
\end{aligned}
$$

Next, same as I_2, I_3, I_4, and I_6, it is easy to compute

$$
H_3 \le C\|\nabla U\|_4^2 + C(M_0)(1+D^8)(\|P\|_4^2 + \|\varepsilon^{\frac12}\nabla P_\alpha\|^2 + \|U\|_4^2),
$$

and

$$
H_5 \le C(M_0)\varepsilon^2 + C(M_0)\|\varepsilon^{\frac12}\nabla P_\alpha\|^2.
$$

Finally, noting (4.12) and (4.13), we have

$$
\begin{aligned}
H_4 &= -\kappa'\int_\Omega \Big(\rho(p_0+\varepsilon\tilde p^\varepsilon)\Delta\tilde\rho^\varepsilon_\alpha\Delta P_\alpha + \varepsilon\rho'(p_0+\varepsilon\tilde p^\varepsilon)\Delta\tilde\rho^\varepsilon_\alpha\nabla\tilde p^\varepsilon\nabla P_\alpha\Big)dx \\
&= -\kappa'\varepsilon\int_\Omega \rho(p_0+\varepsilon\tilde p^\varepsilon)\rho'(p_0+\varepsilon\tilde p^\varepsilon)(\Delta P_\alpha)^2dx - \kappa'\varepsilon\int_\Omega \rho(p_0+\varepsilon\tilde p^\varepsilon)\Delta P_\alpha\Big([\partial_x^\alpha, \rho'(p_0+\varepsilon\tilde p^\varepsilon)]\Delta P \\
&\qquad + \partial_x^\alpha(\rho'(p_0+\varepsilon\tilde p^\varepsilon)\Delta p_\varepsilon) + \varepsilon\partial_x^\alpha(\rho''(p_0+\varepsilon\tilde p^\varepsilon)(\nabla\tilde p^\varepsilon)^2)\Big)dx \\
&\quad -\kappa'\varepsilon\int_\Omega \rho'(p_0+\varepsilon\tilde p^\varepsilon)\nabla\tilde p^\varepsilon\nabla P_\alpha\Big(\rho'(p_0+\varepsilon\tilde p^\varepsilon)\Delta P_\alpha + [\partial_x^\alpha, \rho'(p_0+\varepsilon\tilde p^\varepsilon)]\Delta P \\
&\qquad + \varepsilon\partial_x^\alpha(\rho'(p_0+\varepsilon\tilde p^\varepsilon)\Delta p_\varepsilon) + \varepsilon\partial_x^\alpha(\rho''(p_0+\varepsilon\tilde p^\varepsilon)(\nabla\tilde p^\varepsilon)^2)\Big)dx
\end{aligned}
$$

$$\leq -\kappa'\varepsilon\int_\Omega \rho(p_0+\varepsilon\tilde{p}^\varepsilon)\rho'(p_0+\varepsilon\tilde{p}^\varepsilon)(\Delta P_\alpha)^2dx+\frac{\delta}{2}\|\varepsilon^{\frac{1}{2}}\Delta P_\alpha\|^2$$
$$+C(M_0)(1+D^{10})(\|P\|_4^2+\|\varepsilon^{\frac{1}{2}}\nabla P\|_4^2)+C(M_0)\varepsilon^2.$$

Hence, inserting the above estimates on the H_i's into (4.15) yields

$$\frac{d}{dt}\int_\Omega \varepsilon\rho(p_0+\varepsilon\tilde{p}^\varepsilon)U_\alpha\nabla P_\alpha dx+\int_\Omega(|\nabla P_\alpha|^2+\kappa'\varepsilon\rho(p_0+\varepsilon\tilde{p}^\varepsilon)\rho'(p_0+\varepsilon\tilde{p}^\varepsilon)(\Delta P_\alpha)^2)dx$$
$$\leq C_1\|\nabla U_\alpha\|^2+\delta\|\varepsilon^{\frac{1}{2}}\Delta P_\alpha\|^2$$
$$+C(M_0)(1+D^{10})(\|P\|_4^2+\|\varepsilon^{\frac{1}{2}}\nabla P\|_4^2+\|U\|_4^2)+C(M_0)\varepsilon^2. \tag{4.16}$$

Finally, choose proper constant $\lambda>0$, which satisfies $\lambda C_1<\frac{1}{4}\mu'$ and

$$\frac{1}{2}(\tilde{\rho}^\varepsilon|U_\alpha|^2+\kappa'\varepsilon|\rho'(p_0+\varepsilon\tilde{p}^\varepsilon)\nabla P_\alpha|^2)+\lambda\varepsilon\rho(p_0+\varepsilon\tilde{p}^\varepsilon)U_\alpha\nabla P_\alpha\geq\frac{1}{4}(|U_\alpha|^2+\kappa'\varepsilon|\nabla P_\alpha|^2).$$

Further, let us take $\delta>0$ with $3\delta<\frac{1}{4}\mu'$ and $\delta(\lambda+1)<\frac{1}{2}\kappa' c_1$. Therefore, combining (4.14) and $\lambda(>0)$(4.16), we obtain

$$\frac{d}{dt}(\|P\|_4^2+\|U\|_4^2+\sum_{|\beta|=5}\|\varepsilon^{\frac{1}{2}}\partial_x^\beta P\|^2)+(\|\nabla U\|_4^2+\|\nabla P\|_4^2+\sum_{|\gamma|=4}\|\varepsilon^{\frac{1}{2}}\Delta\partial_x^\gamma P\|^2)$$
$$\leq C(M_0)(1+D^{10})(\|P\|_4^2+\|U\|_4^2+\sum_{|\beta|=5}\|\varepsilon^{\frac{1}{2}}\partial_x^\beta P\|^2)+C(M_0)\varepsilon^2. \tag{4.17}$$

Then we integrate (4.17) from 0 to T with $[0,T]\subset[0,\min\{T_\varepsilon,T_0\})$ to obtain

$$\|P\|_4^2+\|U\|_4^2+\sum_{|\beta|=5}\|\varepsilon^{\frac{1}{2}}\partial_x^\beta P\|^2+\int_0^T(\|\nabla U\|_4^2+\|\nabla P\|_4^2+\sum_{|\gamma|=4}\|\varepsilon^{\frac{1}{2}}\Delta\partial_x^\gamma P\|^2)dt$$
$$\leq C(M_0)T\varepsilon^2+C(M_0)\int_0^T(1+D^{10})(\|P\|_4^2+\|U\|_4^2+\sum_{|\beta|=5}\|\varepsilon^{\frac{1}{2}}\partial_x^\beta P\|^2)dt. \tag{4.18}$$

Here we have used the fact the initial data are in equilibrium. Furthermore, we apply Gronwall's lemma to (4.18) to get

$$\|P\|_4^2+\|U\|_4^2+\sum_{|\beta|=5}\|\varepsilon^{\frac{1}{2}}\partial_x^\beta P\|^2\leq C(M_0)T_0\varepsilon^2\exp\Big[C(M_0)\int_0^T(1+D^{10})dt\Big]. \tag{4.19}$$

Since $\|P\|_4^2+\|U\|_4^2+\sum_{|\beta|=5}\|\varepsilon^{\frac{1}{2}}\partial_x^\beta P\|^2=D^2$, it follows that

$$D(T)^2\leq C(M_0)T_0\varepsilon^2\exp\Big[C(M_0)\int_0^T(1+D^{10})dt\Big]\equiv Q(T). \tag{4.20}$$

Thus, it holds that

$$Q'(t)=C(M_0)(1+D^{10})Q(t)\leq C(M_0)Q(t)+C(M_0)Q^6(t).$$

Applying the nonlinear Gronwall-type inequality in [25] to the last inequality yields

$$Q(t)\leq e^{C(M_0)T_0}$$

for $t\in[0,\min\{T_0,T_\varepsilon\})$ if we choose ε small enough that

$$Q(0)=C(M_0)T_0\varepsilon^2\le e^{-C(M_0)T_0}.$$

Because of (4.20), there exists a constant c, independent of ε, such that

$$D(T)\le c\varepsilon. \tag{4.21}$$

for $T\in[0,\min\{T_0,T_\varepsilon\})$. Finally, Theorem 3.2 is concluded from (4.21). This completes the proof. ∎

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