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GLOBAL CLASSICAL SOLUTIONS OF THE "ONE AND ONE-HALF" DIMENSIONAL VLASOV–MAXWELL–FOKKER–PLANCK SYSTEM*

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Abstract. We study the "one and one-half" dimensional Vlasov–Maxwell–Fokker–Planck system and obtain the first results concerning well-posedness of solutions. Specifically, we prove the global-in-time existence and uniqueness in the large of classical solutions to the Cauchy problem and a gain in regularity of the distribution function in its momentum argument.

Key words. Kinetic Theory, Vlasov, Fokker–Planck equation, global existence.

AMS subject classifications. 35L60, 35Q83, 82C22, 82D10.

1. Introduction

From a mathematical perspective, the fundamental non-relativistic equations which describe the time evolution of a collisionless plasma are given by the Vlasov–Maxwell system:

$$\text{(VM)} \qquad \begin{cases} \partial_t f + v \cdot \nabla_x f + (E + v \times B) \cdot \nabla_v f = 0 \\ \rho(t,x) = \int f(t,x,v)\,dv, \quad j(t,x) = \int v f(t,x,v)\,dv \\ \partial_t E = \nabla \times B - j, \quad \nabla \cdot E = \rho \\ \partial_t B = -\nabla \times E, \quad \nabla \cdot B = 0. \end{cases}$$

Here, f represents the distribution of (positively-charged) ions in the plasma, while ρ and j are the charge and current density, and E and B represent electric and magnetic fields generated by the charge and current. The independent variables, $t \geq 0$ and $x, v \in \mathbb{R}^3$ represent time, position, and momentum, respectively, and physical constants, such as the charge and mass of particles, as well as the speed of light, have been normalized to one.

In order to include collisions of particles with a background medium in the physical formulation, a diffusive term is added to the Vlasov equation in (VM). With this, the equations are referred to as the Vlasov–Maxwell–Fokker–Planck system. Since basic questions of well-posedness remain unknown even in lower dimensions, we study a dimensionally-reduced version of this model for which $x \in \mathbb{R}$ and $v \in \mathbb{R}^2$, the so-called "one and one-half dimensional" analogue, given by

$$\text{(VMFP)} \qquad \begin{cases} \partial_t f + v_1 \partial_x f + K \cdot \nabla_v f = \Delta_v f \\ K_1 = E_1 + v_2 B, \quad K_2 = E_2 - v_1 B \\ \rho(t,x) = \int f(t,x,v)\,dv - \phi(x), \quad j(t,x) = \int v f(t,x,v)\,dv \\ \partial_t E_2 = -\partial_x B - j_2, \ \ \partial_t B = -\partial_x E_2, \ \ \partial_x E_1 = \rho, \ \ \partial_t E_1 = -j_1. \end{cases}$$

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This system is the lowest-dimensional analogue that one may study and include electromagnetic effects, as imposing $v \in \mathbb{R}$ changes the model into the one-dimensional Vlasov–Poisson system. In (VMFP) we assume a single species of particles described by $f(t,x,v)$ in the presence of a given, fixed background $\phi \in C^1(\mathbb{R}) \cap H^1(\mathbb{R}) \cap L^1(\mathbb{R})$ that is neutralizing in the sense that

$$\int \phi(x)\, dx = \iint f(0,x,v)\, dv\, dx.$$

The electric and magnetic fields are given by $E(t,x) = \langle E_1(t,x), E_2(t,x)\rangle$ and $B(t,x)$, respectively. For initial data we take a nonnegative particle density f^0 with bounded moments $v_0^b \partial_x^k f^0 \in L^2(\mathbb{R}^3)$, along with fields $E_2^0, B^0 \in H^1(\mathbb{R})$. Additionally, we specify data for E_1, namely

$$(E_1 DAT) \qquad E_1(0,x) = \int_{-\infty}^{x} \left(\int f^0(y,w)\, dw - \phi(y) \right) dy.$$

In fact, this particular choice of data for E_1 is the only one which leads to a solution possessing finite energy (see [5] and [12]). The inclusion of the neutralizing density ϕ is also necessary in order to arrive at finite energy solutions for (VMFP) with a single species of ion.

The analysis of (VM) has seen some progress in recent decades. For instance, the global existence of weak solutions, which also holds for the relativistic system (RVM), was shown in [3]. Unlike its relativistic analogue, however, no results currently exist that ensure global existence of classical solutions. Hence, the current work is focused in this direction. Alternatively, a wide array of results have been obtained for the electrostatic simplification of (VM) – the Vlasov–Poisson system, obtained by taking $B \equiv 0$ within the model. The Vlasov–Poisson system does not include magnetic effects, and the electric field is given by an elliptic equation rather than a system of hyperbolic PDEs. This simplification has led to a great deal of progress concerning the electrostatic system, including theorems regarding the well-posedness of solutions [10, 11, 14, 15]. The book [6] can provide a general reference to information concerning kinetic equations of plasma dynamics, including (VM) and (VMFP).

Independent of these advances, many of the most basic existence and regularity questions remain unsolved for (VMFP). For much of the existence theory for collisionless models, one is mainly focused on bounding the velocity support of the distribution function f, assuming that f^0 possesses compact momentum support, as this condition has been shown to imply global existence [7]. Hence, one of the main difficulties which arises for (VMFP) is the introduction of particles that are propagated with arbitrarily large momenta, stemming from the inclusion of the diffusive Fokker–Planck operator. Thus, the momentum support is necessarily unbounded and many known tools are unavailable. Though the v-support of the distribution function is not bounded, we are able to overcome this issue by controlling large enough moments of the distribution to guarantee sufficient decay of f in its momentum argument. This also allows us to control nonlinear terms that arise within derivative estimates. As an additional difference arising from the Fokker–Planck operator, we note that when studying collisionless systems, in which $\Delta_v f$ is omitted, L^∞ is typically the proper space in which to estimate both the particle distribution and the fields. With the addition of the diffusion operator, though, the natural space in which to estimate f is now L^2. Thus, to take advantage of the gain in regularity that should result from the Fokker–Planck term, we iterate in a weighted L^2 setting. Other crucial features which appear include conservation of mass, and the symmetry of the diffusive operator. The main advantage of the diffusion operator is that it allows one to estimate spatial derivatives of the density in $L^2(\mathbb{R}^3)$ independent of the momentum derivatives. This is not true for the Vlasov–Maxwell system, which is conservative rather than dissipative. Additionally, the appearance of the Laplacian allows the particle distribution to gain regularity in its momentum argument in comparison to its initial data. Finally, we note that our methods utilize an extra conservation law arising from the structure of the one-and-one-half dimensional system in order to bound the electric and magnetic fields. Hence, they do not immediately apply to higher-dimensional analogues of (VMFP), though many of the other ideas presented below will likely be useful in the two, two-and-one-half, and three dimensional settings.

Though this is the first investigation of the well-posedness of (VMFP) in the large, others have studied Vlasov–Maxwell models incorporating a Fokker–Planck term for small initial data. Both Yu and Yang [17] and Chae [1] constructed global classical solutions to the three-dimensional Vlasov–Maxwell–Fokker–Planck system for initial data sufficiently close to Maxwellian using Kawashima estimates and the well-known energy method. Additionally, Lai [8, 9] arrived at a similar result for a one and one-half dimensional "relativistic" Vlasov–Maxwell–Fokker–Planck system using classical estimates. The system in this work features a relativistic transport term, but still utilizes the Laplacian Δ_v as the Fokker–Planck term. We note that the relativistic transport operator yields an extremely beneficial result, known as the cone estimate (see [5]), whereas the non-relativistic transport within (VMFP) does not. Thus, one essential component of the current paper is to overcome the lack of bounds on energy inside the light cone. Finally, we mention [12], which arrived at similar results to our own but studied the relativistic Vlasov–Maxwell system with a Lorentz-invariant diffusion operator. While we utilize some of the tools introduced within [12], and related articles [4, 13], we also introduce a number of new methods to overcome the loss of the cone estimate, finite speed of propagation, and *a priori* field bounds in order to arrive at the first large data global classical solutions to (VMFP) set in any dimension, see Theorem 1.2 below. First we state a local existence theorem:

THEOREM 1.1. *Let $a > 8$ and $\varepsilon > 0$ and denote*

$$v_0 = \sqrt{1 + |v|^2}.$$

Assume that $\phi \in C^1(\mathbb{R}) \cap H^1(\mathbb{R}) \cap L^1(\mathbb{R})$. Assume that f^0 is continuous, nonnegative, and bounded and possesses a partial derivative with respect to x such that

$$\iint v_0^{a+2+\varepsilon}(f^0)^2\, dv\, dx + \iint v_0^{a-2+\varepsilon}(\partial_x f^0)^2\, dv\, dx$$

is finite. Assume that $E_2^0, B^0 \in C^1(\mathbb{R}) \cap H^1(\mathbb{R})$. Then there is $T > 0$ depending only on

$$\iint \left[v_0^{a+2+\varepsilon}(f^0)^2 + v_0^{a-2+\varepsilon}(\partial_x f^0)^2\right]\, dv\, dx + \|E_2^0\|_{H^1}^2 + \|B^0\|_{H_1}^2,$$

$f \in C([0,T] \times \mathbb{R}^3) \cap C^1((0,T] \times \mathbb{R}^3)$ with second order partial derivatives with respect to v_1, v_2 that are continuous on $(0,T] \times \mathbb{R}^3$, and $(E,B) \in C^1([0,T] \times \mathbb{R})$ for which (VMFP) holds, (E_1DAT) holds, and

$$(f, E_2, B)\big|_{t=0} = (f^0, E_2^0, B^0).$$

Moreover, f is nonnegative and bounded, and

$$\iint \left[v_0^{a+2+\varepsilon} f^2 + v_0^{a-2+\varepsilon}(\partial_x f)^2\right]\, dv\, dx + \|E(t)\|_{H^1} + \|B(t)\|_{H^1}$$

is bounded on $[0,T]$. *Lastly, the above solution is unique.*

Note that f^0 is not assumed to be smooth in v. Now we may state the main result:

THEOREM 1.2. *In addition to the hypotheses of Theorem 1.1, assume that* $E_2^0, B^0 \in L^1(\mathbb{R})$ *and* $v_0^\delta f^0 \in L^\infty(\mathbb{R}^3)$ *for some* $\delta > a+2+\varepsilon$, *and* $v_0^2 f^0 \in L^1(\mathbb{R}^3)$. *Then, the local solution of Theorem 1.1 may be extended to* $[0,\infty)\times\mathbb{R}^3$.

We note that Theorems 1.1 and 1.2 can be altered to accommodate a friction term. In the model with friction, the Vlasov equation is changed to

$$\partial_t f + v_1\partial_x + k\cdot\nabla_v f = \nabla_v\cdot(\nabla_v f + vf).$$

The new term is lower order and does not change either of the results.

As additional evidence of the gain in regularity in v we also state:

PROPOSITION 1.3. *Assume the hypotheses of Theorem 1.2 hold. Then for all* $t>0$

$$\iint \left(f^2 + t\left|\nabla_v f\right|^2 + \frac{1}{2}t^2\left|\nabla_v^2 f\right|^2\right) dv\,dx \le C_t.$$

This paper proceeds as follows. The proof of Theorem 1.1 is postponed to Section 4 and Sections 2 and 3 assume the result of this theorem. In Section 2 we state six lemmas and show how Theorem 1.2 follows from them. The proofs of these lemmas and Proposition 1.3 are contained within Section 3.

Throughout the paper C denotes a positive generic constant that may change from line to line. When necessary, we will specifically identify the quantities upon which C may depend. Regarding norms, we will abuse notation and allow the reader to differentiate certain norms via context. For instance $\|f(t)\|_\infty = \sup_{x\in\mathbb{R},v\in\mathbb{R}^2}|f(t,x,v)|$ whereas $\|B(t)\|_\infty = \sup_{x\in\mathbb{R}}|B(t,x)|$, with analogous statements for $\|\cdot\|_2$ and $<\cdot,\cdot>$ which denote the L^2 norm and inner product, respectively.

2. Global existence

Throughout this section we assume the hypotheses of Theorem 1.1 hold. Let T be the maximal time of existence and, in order to prove Theorem 1.2 by contradiction, assume T is finite.

To begin, we will first prove a result that will allow us to estimate the particle density and its moments. When studying collisionless kinetic equations, one often wishes to integrate along the Vlasov characteristics in order to derive estimates. However, the appearance of the Fokker–Planck term changes the structure of the operator in (VMFP), and the values of the distribution function are not conserved along such curves. Hence, the following lemma (similar to that of [2]) will be utilized to estimate the particle distribution in such situations.

LEMMA 2.1. *Let* $g\in L^1((0,T),L^\infty(\mathbb{R}^3))$ *and* $h_0\in L^\infty(\mathbb{R}^3)\cap L^2(\mathbb{R}^3)$ *be given. Let* $F(t,x,v) = \mathcal{F}(t,x,v) + \mathcal{B}(t,x)\langle v_2,-v_1\rangle$ *be given with* $\mathcal{F}\in W^{1,\infty}((0,T)\times\mathbb{R}^3;\mathbb{R}^2)$ *and* $\mathcal{B}\in W^{1,\infty}((0,T)\times\mathbb{R};\mathbb{R})$. *Assume* $h(t,x,v)$ *is a weak solution of*

$$\begin{cases} \mathcal{L}h \;=\; \partial_t h + v_1\partial_x h + F(t,x,v)\cdot\nabla_v h - \Delta_v h = g(t,x,v) \\ \qquad\qquad h(0,x,v) = h_0(x,v) \end{cases} \tag{2.1}$$

so that $h \in L^2((0,T) \times \mathbb{R}; H^1(\mathbb{R}^2))$ *satisfies*

$$\int_0^T \iint \Big[h\,(-\partial_t \phi - v_1 \partial_x \phi) + \nabla_v h \cdot (F\phi + \nabla_v \phi) - g\phi \Big] dvdxdt \\ - \iint h_0(x,v)\phi(0,x,v)dvdx = 0$$

for every $\phi \in \mathcal{D}([0,T) \times \mathbb{R}^3)$. *Then, for every* $t \in [0,T]$

$$\|h(t)\|_\infty \leq \|h_0\|_\infty + \int_0^t \|g(s)\|_\infty \, ds.$$

Another useful tool will be the conservation of mass and energy growth identities, which we establish in the next result.

Lemma 2.2 (Conservation Laws). *Assume* $v_0^2 f^0 \in L^1(\mathbb{R}^3)$. *Then, for every* $t \in [0,T)$,

$$\|f(t)\|_1 = \|f^0\|_1$$

and

$$\iint |v|^2 f(t,x,v)\, dv\, dx + \int (|E|^2 + B^2) dx \leq C(1+t).$$

Next, we state a lemma that will allow us to control v_2 moments of the particle distribution.

Lemma 2.3 (Propagation of v_2-moments). *Let* $p \in [0,\infty)$ *be given and assume the hypotheses of Lemma 2.2 with* $E_2^0, B^0 \in L^1(\mathbb{R})$. *Let* $R(s) = \sqrt{1+s^2}$. *If* $\|R(v_2)^p f^0\|_\infty < \infty$, *then for any* $t \in [0,T)$

$$\|R(v_2)^p f(t)\|_\infty < C_T.$$

With control of velocities in the v_2 direction, we are able to control the induced electric and magnetic fields. Bounds on moments of the particle density then follow from this result.

Lemma 2.4 (Control of fields and moments). *Assume there is* $\delta > 4$ *such that* $v_0^\delta f^0 \in L^\infty(\mathbb{R}^3)$, $v_0^2 f^0 \in L^1(\mathbb{R}^3)$, *and* $B^0 \in L^1(\mathbb{R})$. *Then, for any* $t \in [0,T)$

$$\|v_0^\delta f(t)\|_\infty \leq C_T, \tag{2.2}$$

$$\|E(t)\|_\infty + \|B(t)\|_\infty \leq C_T, \tag{2.3}$$

and

$$\left\| \int v_0^{\beta-2} f(t)\, dv \right\|_\infty \leq C_T \tag{2.4}$$

for any $\beta \in [0,\delta)$.

Thus, once control of the fields is obtained, any higher moment of the particle distribution function can be controlled as well, assuming that the initial distribution possesses the same property. Next, we utilize energy estimates to bound the density and its derivatives in $L^2(\mathbb{R}^3)$.

LEMMA 2.5. *Assume the hypotheses of Lemma 2.4 hold, then for every* $t \in (0,T]$

$$\frac{d}{dt}\|f(t)\|_2^2 = -2\|\nabla_v f(t)\|_2^2$$

and thus

$$\|f(t)\|_2 \leq \|f^0\|_2.$$

If additionally, $v_0^\gamma f^0 \in L^2(\mathbb{R}^3)$ *for some* $\gamma > 0$, *then*

$$\frac{d}{dt}\|v_0^\gamma f(t)\|_2^2 \leq C_T\|v_0^\gamma f(t)\|_2^2 - 2\|v_0^\gamma \nabla_v f(t)\|_2^2$$

and thus

$$\|v_0^\gamma f(t)\|_2 \leq C_T$$

for every $t \in [0,T)$.

LEMMA 2.6. *Assume the hypotheses of Lemma 2.4 hold with* $\delta > 8$. *Then for every* $\gamma \in \left(2, \dfrac{\delta-4}{2}\right) \cap \left(2, \dfrac{a-2+\varepsilon}{2}\right]$ *and* $t \in [0,T)$ *we have*

$$\|v_0^\gamma \partial_x f(t)\|_{L^2} + \|\partial_x E(t)\|_{L^2} + \|\partial_x B(t)\|_{L^2} \leq C_T.$$

Proof. Now we may prove Theorem 1.2. Applying Lemma 2.5 with $\gamma = \dfrac{a+2+\varepsilon}{2}$ yields

$$\iint v_0^{a+2+\varepsilon} f^2 \, dv \, dx \leq C_T.$$

Applying Lemma 2.6 with $\gamma = \dfrac{a-2+\varepsilon}{2}$ yields

$$\iint v_0^{a-2+\varepsilon}(\partial_x f)^2 \, dv \, dx + \int (|\partial_x E|^2 + (\partial_x B)^2) \, dx \leq C_T.$$

Also by Lemma 2.2

$$\int (|E|^2 + B^2) \, dx \leq C_T.$$

Taking $(f(t), E_2(t), B(t))$ as an initial condition and applying Theorem 1.1 we find the solution may be extended to $[0, t+\tau]$ with $\tau > C_T$. This contradicts the maximality of T and completes the proof. □

3. Proofs of lemmas and estimates

The first result (Lemma 2.1) is very close to a previous lemma [12], in which this property was shown for the relativistic Fokker–Planck operator. One alteration necessary in the proof of [12, Lemma 1] is to change the relativistic velocity $\hat{v}_1$ to v_1, which does not affect the conclusion. Also, here F is not in L^∞, but $\nabla_v \cdot F = \nabla_v \cdot \mathcal{F}$ and the proof of [12, Lemma 1] still applies. Hence, we omit any additional details.

Proof. (Lemma 2.2.) We begin with conservation of mass. Integrating the Vlasov equation over all (x, v) we find

$$\frac{d}{dt} \iint f(t, x, v)\, dv\, dx = 0.$$

Thus, using the decay of f^0 we find for every $t \in [0, T)$

$$\iint f(t, x, v)\, dvdx = \iint f^0(x, v)\, dv\, dx < \infty. \tag{3.1}$$

To arrive at the estimate of the total energy, we multiply the Vlasov equation by $|v|^2$ and integrate in v. The Fokker–Planck term becomes

$$\int |v|^2 \Delta_v f\, dv = -\int 2v \cdot \nabla_v f\, dv = 4 \int f\, dv$$

after two integrations by parts. Hence, using the divergence structure of the Vlasov equation, we arrive at the local energy identity

$$\partial_t e + \partial_x m = 4 \int f(t, x, v)\, dv \tag{3.2}$$

where

$$e(t, x) = \int |v|^2 f(t, x, v)\, dv + (|E(t, x)|^2 + |B(t, x)|^2)$$

and

$$m(t, x) = \int v_1 |v|^2 f(t, x, v) dv + 2E_2(t, x) B(t, x).$$

We integrate (3.2) over all space to deduce the global energy identity

$$\frac{d}{dt} \int e(t, x) dx = 4 \iint f^0(x, v)\, dx\, dv$$

whence we find

$$\int e(t, x)\, dx \leq C(1 + t)$$

for all $t \in [0, T)$. □

Now we utilize the conservation laws to prove Lemma 2.3.

Proof. (Lemma 2.3) We begin by bounding the potential associated to the electric and magnetic fields. By Lemma 2.2 we have

$$\int |j_2(t, x)|\, dx \leq C(1 + t)$$

and hence

$$\left| \int \int_0^t j_2(\tau, y \pm (t - \tau))\, d\tau dy \right| \leq \int_0^t \int |j_2(\tau, y \pm (t - \tau))|\, dy d\tau \leq C(1 + t)^2.$$

Since $B^0, E_2^0 \in L^1(\mathbb{R})$, it follows that

$$\int |B(t,x)|\, dx \le C(1+t)^2$$

and we may define

$$A(t,x) = \int_{-\infty}^{x} B(t,y)\, dy.$$

Note that $\partial_x A = B$ and $\partial_t A = -E_2$. Moreover, using Maxwell's equations, we find

$$(\partial_t^2 - \partial_x^2)A = j_2$$

and thus

$$A(t,x) = \frac{1}{2}(A(0,x-t) + A(0,x+t)) + \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} \int v_2 f(s,y,v)\, dvdyds. \tag{3.3}$$

The (x,v)-integral can be bounded using Cauchy–Schwarz and Lemma 2.2 as

$$\begin{aligned}\int_{x-t+s}^{x+t-s} \int v_2 f(s,y,v)\, dvdy \le & \left(\iint f(x,y,v)\, dvdy\right)^{1/2} \left(\iint v_2^2 f(x,y,v)\, dvdy\right)^{1/2} \\ \le & \|f^0\|_1^{1/2} \left(\iint |v|^2 f(s,y,v)\, dvdy\right)^{1/2} \\ \le & C(1+s)^{1/2}.\end{aligned}$$

Hence, using the assumptions on initial data and integrating, we find

$$\|A(t)\|_\infty \le C(1+t)^{3/2} \le C_T.$$

Next, we utilize the identity

$$\partial_t A + v_1 \partial_x A = -E_2 + v_1 B = -K_2$$

within the Vlasov–Fokker–Planck equation. In particular, let $\psi \in C^2(\mathbb{R})$ be given and multiply this equation by $\psi(v_2 + A(t,x))$. Denoting the VFP operator by

$$\mathcal{V}h := \partial_t h + v_1 \partial_x h + K \cdot \nabla_v h - \Delta_v h,$$

we find

$$\mathcal{V}(\psi(v_2 + A)f) = -2\psi'(v_2 + A)\partial_{v_2} f - \psi''(v_2 + A)f. \tag{3.4}$$

Next, define the function $R(x) = \sqrt{1+x^2}$. To prove the first assertion, we use $\psi(x) = R^p(x)$ within (3.4) and derive the equation

$$\mathcal{V}(R^p(v_2+A)f) = -2p(v_2+A)R^{p-2}(v_2+A)\partial_{v_2} f - pR^{p-4}(v_2+A)[1+(p-1)|v_2+A|^2]f.$$

Using the identity

$$\partial_{v_2} f = R^{-p}(v_2+A)\partial_{v_2}(R^p(v_2+A)f) - p(v_2+A)R^{-2}(v_2+A)f$$

the right side becomes

$$-2p(v_2+A)R^{-2}(v_2+A)\partial_{v_2}(R^p(v_2+A)f) + pR^{p-4}(v_2+A)[-1+(p+1)|v_2+A|^2]f.$$

Hence, if this first term is included within the VFP operator by defining

$$\overline{K} = K + \left\langle 0, 2p\frac{v_2+A}{1+|v_2+A|^2}\right\rangle$$

to form the new operator $\overline{\mathcal{V}}$, we find

$$\overline{\mathcal{V}}(R^p(v_2+A)f) = pR^{p-4}(v_2+A)[-1+(p+1)|v_2+A|^2]f.$$

We note that the term on the right side satisfies

$$|pR^{p-4}(v_2+A)[-1+(p+1)|v_2+A|^2]f| \le CR^{p-2}(v_2+A)f \le CR^p(v_2+A)f.$$

We invoke Lemma 2.1 with $h = R^p(v_2+A)f$ and $\mathcal{L} = \overline{\mathcal{V}}$ so that

$$\|R^p(v_2+A(t))f(t)\|_\infty \le \|R^p(v_2+A(0))f^0\|_\infty + C\int_0^t \|CR^p(v_2+A(s))f(s)\|_\infty\,ds.$$

By Gronwall's inequality we find

$$\|R^p(v_2+A(t))f(t)\|_\infty \le C_T$$

for $t \in [0,T)$. Finally, the previously established control of $\|A(t)\|_\infty$ yields the first result as for $p \ge 0$

$$\begin{aligned} R^p(v_2)f(t,x,v) &= (1+|v_2+A(t,x)-A(t,x)|^2)^{p/2}f(t,x,v)\\ &\le C(R^p(v_2+A(t,x)) + |A(t,x)|^p)f(t,x,v)\\ &\le C\|R^p(v_2+A(t))f(t)\|_\infty + \|A(t)\|_\infty^p\|f(t)\|_\infty\\ &\le C_T. \end{aligned}$$

Hence, taking supremums we find

$$\|R^p(v_2)f(t)\|_\infty \le C_T.$$

□

Using this result, we may bound the fields and moments of the distribution function.

Proof. (Lemma 2.4.) We first bound E_1 using conservation of mass so that

$$\|E_1(t)\|_\infty = \sup_{x\in\mathbb{R}}\left|\int_{-\infty}^x \left(\int f(t,x,v)\,dv - \phi(x)\right)\,dx\right| \le \iint f(t,x,v)\,dvdx + \|\phi\|_1 \le C.$$

Next, we estimate the other field components. Using the transported field equations, we find

$$(E_2 \pm B)(t,x) = (E_2 \pm B)(0, x\mp t) - \int_0^t \int v_2 f(s, x\mp(t-s), v)\,dvds. \tag{3.5}$$

Note that $E_2^0, B^0 \in L^\infty(\mathbb{R})$ by the Sobolev embedding theorem. Thus, for any $\varepsilon_1, \varepsilon_2 > 0$ we have

$$\begin{aligned}
|(E_2 \pm B)(t,x)| \leq & C\left(1+\int_0^t\int R^{-(1+\varepsilon_1)}(v_1)\left[R^{1+\varepsilon_1}(v_1)f^{\frac{1+\varepsilon_1}{\gamma}}(s,x\mp(t-s),v)\right]\right. \\
& \left. |v_2|R^{-(2+\varepsilon_2)}(v_2)\left[R^{2+\varepsilon_2}(v_2)f^{1-\frac{1+\varepsilon_1}{\gamma}}(s,x\mp(t-s),v)\right]dvds\right) \\
\leq & C\left(1+\int_0^t \|R^\gamma(v_1)f(s)\|_\infty^{\frac{1+\varepsilon_1}{\gamma}}\|R^q(v_2)f(s)\|_\infty^{\frac{\gamma-(1+\varepsilon_1)}{\gamma}}ds\right)
\end{aligned}$$

where $q = \dfrac{(2+\varepsilon_2)\gamma}{\gamma-(1+\varepsilon_1)}$. We choose $\gamma > 1+\varepsilon_1$ and note that $\delta > 4$ ensures that we may also choose $q \leq \gamma \leq \delta$. Define the function

$$F(t) := \sup_{s\in[0,t]} \|v_0^\gamma f(s)\|_\infty .$$

Invoking Lemma 2.3 with $p = q$ we find

$$\begin{aligned}
\|(E_2 \pm B)(t)\|_\infty &\leq C_T\left(1+\left[\sup_{s\in[0,t]}\|R^\gamma(v_1)f(s)\|_\infty\right]^{\frac{1+\varepsilon_1}{\gamma}}\right) \\
&\leq C_T\left(1+F(t)^{\frac{1+\varepsilon_1}{\gamma}}\right).
\end{aligned} \tag{3.6}$$

Using the identity

$$E_2(t,x) = \frac{1}{2}([E_2(t,x)+B(t,x)]+[E_2(t,x)-B(t,x)])$$

we see that the same bound holds for $\|E_2(t)\|_\infty$.

Next, we multiply VFP by v_0^γ and use the same method as in the proof of Lemma 2.3 to derive the equation

$$\mathcal{V}(v_0^\gamma f) = \gamma v_0^{\gamma-2}(v\cdot E)f - 2\gamma v_0^{-2}v\cdot\nabla_v(v_0^\gamma f)+\gamma(\gamma|v|^2-2)v_0^{\gamma-4}f. \tag{3.7}$$

If the second term on the right side is included within the VFP operator, we define

$$\overline{K} = K + 2\gamma\frac{v}{1+|v|^2}$$

to form a new operator, $\overline{\mathcal{V}}$. We find

$$\begin{aligned}
\overline{\mathcal{V}}(v_0^\gamma f) &= \gamma v_0^{\gamma-2}(v\cdot E)f+\gamma(\gamma|v|^2-2)v_0^{\gamma-4}f \\
&=: \mathrm{I}+\mathrm{II}.
\end{aligned}$$

Clearly,

$$\mathrm{II} \leq C\|v_0^{\gamma-2}f(t)\|_\infty \leq C\|v_0^\gamma f(t)\|_\infty \leq CF(t).$$

Estimating I requires the field estimates, which yield

$$\begin{aligned}
I &\leq Cv_0^{\gamma-2}(|v_1|+|v_2|\cdot\|E_2(t)\|_\infty)f \\
&\leq C\left(\|v_0^{\gamma-1}f(t)\|_\infty + C_T\||v_2|^{\frac{\gamma}{2}}f(t)\|_\infty^{\frac{2}{\gamma}}\|v_0^\gamma f(t)\|_\infty^{1-\frac{2}{\gamma}}\left(1+F(t)^{\frac{1+\varepsilon_1}{\gamma}}\right)\right) \\
&\leq C_T\left(F(t)+F(t)^{\frac{\gamma-2}{\gamma}}+F(t)^{\frac{\gamma-1+\varepsilon_1}{\gamma}}\right)
\end{aligned}$$

since $\gamma/2 \le \delta$. We combine these estimates and invoke Lemma 2.1 with $h = v_0^\gamma f$ and $\mathcal{L} = \overline{\mathcal{V}}$ so that

$$\|v_0^\gamma f(t)\|_\infty \le \|v_0^\gamma f^0\|_\infty + C_T \int_0^t \left(F(s) + F(s)^{\frac{\gamma-2}{\gamma}} + F(s)^{\frac{\gamma-1+\varepsilon_1}{\gamma}} \right) ds.$$

Taking the supremum in t and choosing $\varepsilon_1 \le 1$

$$F(t) \le F(0) + C_T \int_0^t \left(F(s) + F(s)^{\frac{\gamma-2}{\gamma}} + F(s)^{\frac{\gamma-1+\varepsilon_1}{\gamma}} \right) ds \le F(0) + C_T \int_0^t (1 + F(s))\, ds.$$

Gronwall's inequality then yields the bound $F(t) \le C_T$ for any $t \in [0, T)$ and $1 \le \gamma \le \delta$. The bound on moments of the distribution function follows immediately and the field bound

$$\|E_2(t)\|_\infty + \|B(t)\|_\infty \le C_T$$

then follows from (3). Finally, using the bound on moments of the density, control of the v-integral follows since we have

$$\int v_0^{\gamma-2} f(t,x,v)\, dv \le \|v_0^\delta f(t)\|_\infty \cdot \int v_0^{-2-(\delta-\gamma)}\, dv \le C_T$$

for $\gamma < \delta$ and taking the supremum in x yields (2.4). ∎

Proof. (Lemma 2.5) We proceed by using dissipative estimates. First, we compute:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|f(t)\|_2^2 &= \langle -v_1 \partial_x f - K \cdot \nabla_v f + \Delta_v f, f \rangle \\ &= -\langle v_1 \partial_x f, f\rangle - \langle K \cdot \nabla_v f, f\rangle + \langle \Delta_v f, f\rangle.\end{aligned}$$

Notice that the first two terms are pure derivatives in x and v, respectively. Thus,

$$\langle v_1 \partial_x f, f\rangle = \frac{1}{2} \iint \partial_x (v_1 f^2)\, dv dx = 0$$

and

$$\langle K \cdot \nabla_v f, f\rangle = \frac{1}{2} \iint \nabla_v \cdot (K f^2)\, dv dx = 0.$$

Finally, $\langle \Delta_v f, f\rangle = -\|\nabla_v f(t)\|_2^2$. Hence,

$$\frac{d}{dt}\|f(t)\|_2^2 = -2\|\nabla_v f(t)\|_2^2 \le 0$$

and the first conclusion follows.

Similarly, we may multiply by $v_0^{2\gamma}$ and proceed in the same manner. Within this estimate we will use $v_0 \ge 1$ in order to increase moments of the estimates where necessary so as to match the results of the lemma. Computing the time derivative

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v_0^\gamma f(t)\|_2^2 &= \iint v_0^{2\gamma} f[-v_1 \partial_x f - K \cdot \nabla_v f + \Delta_v f]\, dv dx \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}.\end{aligned}$$

The first term vanishes as it is a pure x-derivative. For II, we integrate by parts and use the field bounds of Lemma 2.4 so that

$$\begin{aligned}\mathrm{II} &= -\frac{1}{2}\iint v_0^{2\gamma}\nabla_v\cdot(Kf^2)\,dvdx\\ &= \gamma\iint v_0^{2\gamma-2}v\cdot Ef^2\,dvdx\\ &\le C_T\|v_0^\gamma f(t)\|_2^2.\end{aligned}$$

To estimate III, we integrate by parts twice in the first term and once in the second term to find

$$\begin{aligned}\mathrm{III} &= -\iint \nabla_v(v_0^{2\gamma}f)\cdot\nabla_v f\,dvdx\\ &= -\iint(2\gamma v_0^{2\gamma-2}vf + v_0^{2\gamma}\nabla_v f)\cdot\nabla_v f\,dvdx\\ &\le C\|v_0^\gamma f(t)\|_2^2 - \|v_0^\gamma\nabla_v f(t)\|_2^2.\end{aligned}$$

Combining the estimates, we find

$$\frac{d}{dt}\|v_0^\gamma f(t)\|_2^2 \le C_T\|v_0^\gamma f(t)\|_2^2 - 2\|v_0^\gamma\nabla_v f(t)\|_2^2$$

as in the statement of the lemma. Additionally, because the second term on the right side of the inequality is nonpositive, we invoke Gronwall's inequality and find

$$\|v_0^\gamma f(t)\|_2^2 \le C_T\|v_0^\gamma f^0\|_2^2 \le C_T,$$

for every $\gamma \ge 0$ for which the norm of the initial data $\|v_0^\gamma f^0\|_2$ is finite. ∎

Proof. (Lemma 2.6.) If f were C^3 we could compute

$$\begin{aligned}&\frac{d}{dt}\iint v_0^{2\gamma}(\partial_x f)^2\,dvdx\\ &= \iint 2v_0^{2\gamma}\partial_x f(\Delta_v\partial_x f - v_1\partial_x^2 f - K\cdot\nabla_v\partial_x f - \partial_x K\cdot\nabla_v f)\,dvdx\\ &= -2\iint v_0^{2\gamma}|\nabla_v\partial_x f|^2\,dvdx + \iint(\partial_x f)^2(\Delta_v v_0^{2\gamma} + K\cdot\nabla_v v_0^{2\gamma})\,dvdx\\ &\quad + 2\iint f\partial_x K\cdot(v_0^{2\gamma}\nabla_v\partial_x f + \partial_x f\nabla_v v_0^{2\gamma})\,dvdx\\ &\le -2\iint v_0^{2\gamma}|\nabla_v\partial_x f|^2\,dvdx + C\iint(\partial_x f)^2 v_0^{2\gamma}(1+|E|)\,dvdx\\ &\quad + C\sqrt{\iint f^2|\partial_x K|^2 v_0^{2\gamma}\,dvdx}\left(\sqrt{\iint v_0^{2\gamma}|\nabla_v\partial_x f|^2\,dvdx} + \sqrt{\iint v_0^{2\gamma}(\partial_x f)^2\,dvdx}\right).\end{aligned}$$

Using the inequalities $-x^2 + Ax \le \frac{1}{4}A^2$ and $2xy \le x^2 + y^2$ yields

$$\begin{aligned}\frac{d}{dt}\iint v_0^{2\gamma}(\partial_x f)^2\,dvdx \le\,& C\iint(\partial_x f)^2 v_0^{2\gamma}(1+|E|)\,dvdx\\ &+ C\iint f^2|\partial_x K|^2 v_0^{2\gamma}\,dvdx,\end{aligned}$$

and hence

$$\begin{aligned}\iint v_0^{2\gamma}(\partial_x f)^2\,dvdx \leq & C + C\int_0^t \iint v_0^{2\gamma}(\partial_x f)^2(1+|E|)\,dvdxd\tau \\ & + C\int_0^t \iint v_0^{2\gamma} f^2|\partial_x K|^2\,dvdxd\tau. \qquad (3.8)\end{aligned}$$

By a standard regularization argument, it follows that (3.8) holds for the solution (f, E, B) with the regularity stated in Theorem 1.1. Applying (2.3) and (2.4) with $\beta = 2\gamma + 4$ yields

$$\begin{aligned}\iint v_0^{2\gamma}(\partial_x f)^2\,dvdx \leq & C + C_T\int_0^t \iint v_0^{2\gamma}(\partial_x f)^2\,dvdxd\tau \\ & + C\int_0^t \int (|\partial_x E|^2 + (\partial_x B)^2)\|f(t)\|_{L^\infty}\int f v_0^{2\gamma+2}\,dvdxd\tau \\ \leq & C + C_T\int_0^t \iint v_0^{2\gamma}(\partial_x f)^2\,dvdxd\tau \\ & + C_T\int_0^t \int (|\partial_x E|^2 + (\partial_x B)^2)\,dxd\tau. \qquad (3.9)\end{aligned}$$

Similarly, if E and B were C^2 we could compute

$$\begin{aligned}\frac{d}{dt}\int [|\partial_x E|^2 + (\partial_x B)^2]\,dx = & -2\int \partial_x E \cdot \partial_x j\,dx \\ \leq & 2\sqrt{\int |\partial_x E|^2\,dx}\sqrt{\int |\partial_x j|^2\,dx}\end{aligned}$$

so

$$\begin{aligned}\int [|\partial_x E|^2 + (\partial_x B)^2]\,dx \leq & C + 2\int_0^t \sqrt{\int |\partial_x E|^2\,dx}\sqrt{\int |\partial_x j|^2\,dx}\,d\tau \\ \leq & C + \int_0^t \int (|\partial_x E|^2 + |\partial_x j|^2)\,dxd\tau. \qquad (3.10)\end{aligned}$$

By a standard regularization argument it follows that (3.10) holds for E and $B \in C^1$. Since $\gamma > 2$, we have

$$\begin{aligned}|\partial_x j|^2 \leq & \left(\int |\partial_x f| v_0 dv\right)^2 \\ \leq & \int |\partial_x f|^2 v_0^{2\gamma} dv \int v_0^{2-2\gamma} dv \leq C\int |\partial_x f|^2 v_0^{2\gamma}\,dv.\end{aligned}$$

Thus adding (3.9) and (3.10) yields

$$\begin{aligned}& \iint v_0^{2\gamma}(\partial_x f)^2\,dvdx + \int (|\partial_x E|^2 + (\partial_x B)^2)\,dx \\ \leq & C + C_T\int_0^t \left(\int |\partial_x E|^2\,dxd\tau + \iint v_0^{2\gamma}(\partial_x f)^2\,dvdx\right)\,d\tau.\end{aligned}$$

An application of Gronwall's inequality completes the proof. ∎

Proof. (Proposition 1.3.) If f were C^4 we could compute the following:

$$\begin{aligned}
\frac{d}{dt}\iint f^2\,dvdx &= -2\iint |\nabla_v f|^2\,dvdx,\\
\frac{d}{dt}\iint |\nabla_v f|^2\,dvdx &= -2\iint (|\nabla_v^2 f|^2 + \partial_{v_1} f\partial_x f)\,dvdx,\\
\frac{d}{dt}\iint |\nabla_v^2 f|^2\,dvdx &= -2\iint |\nabla_v^3 f|^2\,dvdx - 4\iint \nabla_v\partial_{v_1} f\cdot\nabla_v\partial_x f\,dvdx\\
&= -2\iint |\nabla_v^3 f|^2\,dvdx + 4\iint \partial_x f\Delta_v\partial_{v_1} f\,dvdx,
\end{aligned}$$

so

$$\begin{aligned}
&\frac{d}{dt}\iint (f^2 + t|\nabla_v f|^2 + \frac{1}{2}t^2|\nabla_v^2 f|^2)\,dvdx\\
&= -\iint |\nabla_v f|^2\,dvdx - t\iint |\nabla_v^2 f|^2\,dvdx - 2t\iint \partial_{v_1} f\partial_x f\,dvdx\\
&\quad + \frac{1}{2}t^2\iint (-2|\nabla_v^3 f|^2 + 4\partial_x f\Delta_v\partial_{v_1} f)\,dvdx\\
&\le -\iint |\nabla_v f|^2\,dvdx + 2t\sqrt{\iint (\partial_x f)^2\,dvdx}\,\sqrt{\iint (\partial_{v_1} f)^2\,dvdx}\\
&\quad + t^2\left(-\iint |\nabla_v^3 f|^2\,dvdx + 2\sqrt{\iint (\partial_x f)^2\,dvdx}\,\sqrt{\iint (\Delta_v\partial_{v_1} f)^2\,dvdx}\right).
\end{aligned}$$

Using the inequality $-x^2 + Ax \le \frac{1}{4}A^2$ twice yields

$$\frac{d}{dt}\iint (f^2 + t|\nabla_v f|^2 + \frac{1}{2}t^2|\nabla_v^2 f|^2)\,dvdx \le Ct^2\iint (\partial_x f)^2\,dvdx$$

and

$$\begin{aligned}
&\iint (f^2 + t|\nabla_v f|^2 + \frac{1}{2}t^2|\nabla_v^2 f|^2)\,dvdx\\
&\le \iint (f^0)^2\,dvdx + C\int_0^t \tau^2\iint (\partial_x f)^2\,dvdxd\tau. \qquad (3.11)
\end{aligned}$$

Again by a standard regularization argument, it follows that (3.11) holds for the solution constructed in Theorem 1.1. By Lemma 2.6,

$$\iint (\partial_x f)^2\,dvdx \le C_t$$

so the proposition follows from (3.11). ∎

4. Local existence

Define $b = a - 4$,

$$\mathcal{F} = \left\{ f : \mathbb{R}^3 \to [0,\infty) \,\middle|\, v_0^{\frac{a}{2}} f,\ v_0^{\frac{b}{2}}\partial_x f \in L^2(\mathbb{R}^3) \right\}$$

and

$$\|f\|_{\mathcal{F}} = \|v_0^{\frac{a}{2}} f\|_{L^2} + \|v_0^{\frac{b}{2}} \partial_x f\|_{L^2}.$$

For the time being we consider smooth initial data (f^0, E_2^0, B^0).

Let $T \in (0,1), R > 1, \psi : \mathbb{R} \to [0,1]$ be smooth with $s \leq -1 \Rightarrow \psi(s) = 1$ and $s \geq 0 \Rightarrow \psi(s) = 0$. Define $\psi^R(v) = \psi(|v| - R)$. For $(E, B) \in C([0,T];\ H^1(\mathbb{R}))$ smooth define

$$\mathcal{L}^R(E, B) = (\tilde{f}, \tilde{E}, \tilde{B})$$

by

$$K = E + \psi^R(v) B(v_2, -v_1), \tag{4.1}$$

$$\partial_t \tilde{f} + v_1 \partial_x \tilde{f} + K \cdot \nabla_v \tilde{f} = \Delta_v \tilde{f}, \qquad \tilde{f}(0,\cdot,\cdot) = f^0, \tag{4.2}$$

$$\tilde{\rho} = \int \tilde{f} dv - \phi,\ \tilde{j} = \int v \tilde{f}\, dv, \tag{4.3}$$

$$\tilde{E}_1 = \int_{-\infty}^{x} \tilde{\rho}\, dy, \tag{4.4}$$

$$\partial_t \tilde{E}_2 + \partial_x \tilde{B} = -\tilde{j}_2, \qquad \partial_t \tilde{B} + \partial_x \tilde{E}_2 = 0, \tag{4.5}$$

$$(\tilde{E}_2, \tilde{B})(0,\cdot) = (E_2^0, B^0). \tag{4.6}$$

Note that K is bounded and hence by Proposition A.1 of [2], (4.2) has a solution $\tilde{f} \in L^2([0,T] \times \mathbb{R}; H^1(\mathbb{R}^2))$. The reference, [16], may also be used for this.

Let $\alpha = a + 2 + \varepsilon,\ \beta = b + 2 + \varepsilon$, and

$$C_0 > \|E_2^0\|_{H^1} + \|B^0\|_{H^1} + \|v_0^{\frac{\alpha}{2}} f^0\|_{L^2} + \|v_0^{\frac{\beta}{2}} \partial_x f^0\|_{L^2}.$$

We assume that

$$\|(E, B)(t)\|_{H^1} \leq 10 C_0 \tag{4.7}$$

on $[0, T]$. Within the remainder of this section constants may depend on α, T and C_0 but not on R or $\nabla_v f^0$.

First, using (4.7) and the Sobolev embedding theorem,

$$\begin{aligned} \frac{d}{dt} \iint v_0^\alpha \tilde{f}^2\, dv dx =& -2 \iint v_0^\alpha |\nabla_v \tilde{f}|^2\, dv dx \\ &+ \iint \tilde{f}^2 (\Delta_v v_0^\alpha + E \cdot \nabla_v v_0^\alpha)\, dv dx \\ \leq& C \iint \tilde{f}^2 v_0^\alpha\, dv dx. \end{aligned} \tag{4.8}$$

Hence, by Gronwall's inequality

$$\iint v_0^\alpha \tilde{f}^2\, dv dx \leq C. \tag{4.9}$$

Similarly, and by using the Cauchy–Schwarz inequality,

$$\begin{aligned}
&\frac{d}{dt}\iint v_0^\beta(\partial_x\tilde f)^2\,dvdx\\
=&-2\iint v_0^\beta|\nabla_v\partial_x\tilde f|^2\,dvdx+\iint(\partial_x\tilde f)^2(\Delta_v v_0^\beta+E\cdot\nabla_v v_0^\beta)\,dvdx\\
&+2\iint \tilde f\partial_x K\cdot(v_0^\beta\nabla_v\partial_x\tilde f+\partial_x\tilde f\nabla_v v_0^\beta)\,dvdx\\
\le&-2\iint v_0^\beta|\nabla_v\partial_x\tilde f|^2\,dvdx+C\iint(\partial_x\tilde f)^2 v_0^\beta\,dvdx\\
&+C\iint \tilde f(|\partial_x E|+|\partial_x B|)(v_0^{\beta+1}|\nabla_v\partial_x\tilde f|+|\partial_x\tilde f|v_0^\beta)\,dvdx\\
\le&-2\iint v_0^\beta|\nabla_v\partial_x\tilde f|^2\,dvdx+C\iint(\partial_x\tilde f)^2 v_0^\beta\,dvdx\\
&+C\sqrt{\iint \tilde f^2 v_0^{\beta+2}(|\partial_x E|^2+(\partial_x B)^2)\,dvdx}\left[\sqrt{\iint v_0^\beta|\nabla_v\partial_x\tilde f|^2\,dvdx}\right.\\
&\left.+\sqrt{\iint v_0^\beta(\partial_x\tilde f)^2\,dvdx}\right].
\end{aligned}\tag{4.10}$$

Using $-x^2+Ax\le\frac{1}{4}A^2$ and $2xy\le x^2+y^2$ yields

$$\begin{aligned}
\frac{d}{dt}\iint v_0^\beta(\partial_x\tilde f)^2\,dvdx\le&C\iint v_0^\beta(\partial_x\tilde f)^2\,dvdx\\
&+C\iint \tilde f^2 v_0^{\beta+2}dv(|\partial_x E|^2+(\partial_x B)^2)\,dx.
\end{aligned}\tag{4.11}$$

Also by (4.9)

$$\begin{aligned}
\int \tilde f^2 v_0^{\beta+2}dv=\int \tilde f^2 v_0^{\frac{\alpha+\beta}{2}}\,dv=&\int_{-\infty}^x\int 2\tilde f\partial_x\tilde f v_0^{\frac{\alpha+\beta}{2}}\,dvdy\\
\le&2\|\tilde f(t)v_0^{\frac{\alpha}{2}}\|_{L^2}\|\partial_x\tilde f(t)v^{\frac{\beta}{2}}\|_{L^2}\\
\le&C+\iint v_0^\beta(\partial_x\tilde f)^2\,dvdx
\end{aligned}\tag{4.12}$$

so using (4.7) and (4.11) yields

$$\frac{d}{dt}\iint v_0^\beta(\partial_x\tilde f)^2\,dvdx\le C+C\iint v_0^\beta(\partial_x\tilde f)^2\,dvdx.$$

Hence

$$\iint v_0^\beta(\partial_x\tilde f)^2\,dvdx\le C.\tag{4.13}$$

Next consider $(\mathcal{E},\mathcal{B})\in C([0,T];H^1(\mathbb{R}))$ smooth for which (4.7) holds and define $(\tilde F,\tilde{\mathcal{E}},\tilde{\mathcal{B}})=\mathcal{L}^{\mathcal{R}}(\mathcal{E},\mathcal{B})$ where $R\le\mathcal{R}$ and $\mathcal{K}$, $\tilde F$, $\tilde P$, $\tilde J$, $\tilde{\mathcal{E}}$, and $\tilde{\mathcal{B}}$ are defined as in equations (4.1)–(4.6).

Let $G=(E,B)-(\mathcal{E},\mathcal{B})$ and note that

$$|K-\mathcal{K}|\le v_0|G|+v_0|\mathcal{B}|\;|\psi^R(v)-\psi^{\mathcal{R}}(v)|$$

$$\leq v_0|G| + v^{1+\frac{\varepsilon}{2}}|\mathcal{B}|R^{-\frac{\varepsilon}{2}} \tag{4.14}$$

and similarly

$$|\partial_x K - \partial_x \mathcal{K}| \leq v_0|\partial_x G| + v_0^{1+\frac{\varepsilon}{2}}|\partial_x \mathcal{B}|R^{-\frac{\varepsilon}{2}}. \tag{4.15}$$

Let $\tilde{g} = \tilde{f} - \tilde{F}$. Proceeding as before in (4.10) and (4.11) we have

$$\begin{aligned}
\frac{d}{dt}\iint v_0^a \tilde{g}^2\, dvdx = & -2\iint v_0^a |\nabla_v \tilde{g}|^2\, dvdx \\
& + \iint \tilde{g}^2(\Delta_v v_0^a + E\cdot\nabla_v v_0^a)\, dvdx \\
& + 2\iint \tilde{F}(K-\mathcal{K})\cdot(v_0^a \nabla_v \tilde{g} + \tilde{g}\nabla_v v_0^a)\, dvdx \\
\leq & C\iint \tilde{g}^2 v_0^a\, dvdx + C\iint \tilde{F}^2|K-\mathcal{K}|^2 v_0^a\, dvdx. \qquad (4.16)
\end{aligned}$$

By (4.14), the Sobolev embedding theorem, and (4.9) we have

$$\begin{aligned}
&\iint \tilde{F}^2|K-\mathcal{K}|^2 v_0^a\, dvdx \leq \iint \tilde{F}^2(G^2 + \mathcal{B}^2R^{-\varepsilon})v_0^\alpha\, dvdx \\
\leq & C(\|G(t)\|^2_{L^\infty} + \|\mathcal{B}(t)\|^2_{L^\infty}R^{-\varepsilon}) \leq C(\|G(t)\|^2_{H^1} + R^{-\varepsilon}).
\end{aligned}$$

Substitution into (4.16) and using Gronwall's inequality yields

$$\iint v_0^a\, \tilde{g}^2\, dvdx \leq C\int_0^t \|G(\tau)\|^2_{H^1} d\tau + CR^{-\varepsilon}t. \tag{4.17}$$

Again proceeding as in (4.10) and (4.11) we have

$$\begin{aligned}
&\frac{d}{dt}\iint v_0^b(\partial_x\tilde{g})^2\, dvdx \\
= & -2\iint v_0^b|\nabla_v\partial_x\tilde{g}|\, dvdx + \iint(\partial_x\tilde{g})^2(\Delta_v v_0^b + K\cdot\nabla_v v_0^b)\, dvdx \\
& + 2\iint(\partial_x\tilde{F}(K-\mathcal{K}) + \tilde{g}\,\partial_x K + \tilde{F}\partial_x(K-\mathcal{K}))\cdot(v_0^b\nabla_v\partial_x\,\tilde{g} + \partial_x\tilde{g}\nabla_v v_0^b)\, dvdx \\
\leq & C\iint v_0^b(\partial_x\tilde{g})^2\, dvdx + C\iint(\partial_x\tilde{F})^2|K-\mathcal{K}|^2 v_0^b\, dvdx \\
& + C\iint \tilde{g}^2|\partial_x K|^2 v_0^b\, dvdx + C\iint \tilde{F}^2|\partial_x(K-\mathcal{K})|^2 v_0^b\, dvdx. \qquad (4.18)
\end{aligned}$$

By (4.14), (4.13), and the Sobolev embedding theorem we have

$$\begin{aligned}
&\iint(\partial_x\tilde{F})^2|K-\mathcal{K}|^2 v_0^b\, dvdx \\
\leq & C\iint(\partial_x\tilde{F})^2(|G|^2 + |\mathcal{B}|^2R^{-\varepsilon})v_0^\beta\, dvdx \\
\leq & C(\|G(t)\|^2_{H^1} + \|\mathcal{B}(t)\|^2_{H^1}R^{-\varepsilon}) \leq C(\|G(t)\|^2_{H^1} + R^{-\varepsilon}). \qquad (4.19)
\end{aligned}$$

Note that (using (4.17))

$$\int \tilde{g}^2 v_0^{b+2} dv \leq 2\iint |\tilde{g}||\partial_x\tilde{g}|v_0^{\frac{a+b}{2}}\, dvdx$$

$$\begin{aligned}
&\leq 2\left(\iint \tilde{g}^2 v_0^a \, dvdx\right)^{\frac{1}{2}} \left(\iint (\partial_x \tilde{g})^2 v_0^b \, dvdx\right)^{\frac{1}{2}} \\
&\leq C\int_0^t \|G(\tau)\|_{H^1}^2 d\tau + CR^{-\varepsilon}t + \iint (\partial_x \tilde{g})^2 v_0^b \, dvdx \qquad (4.20)
\end{aligned}$$

so by (4.20), (4.7), and the Sobolev embedding theorem

$$\begin{aligned}
&\iint \tilde{g}^2 |\partial_x K|^2 v_0^b \, dvdx \leq \iint \tilde{g}^2 (|\partial_x E|^2 + (\partial_x B)^2) v_0^{b+2} \, dvdx \\
\leq &C\int_0^t \|G(\tau)\|_{H^1}^2 d\tau + CR^{-\varepsilon}t + C\iint (\partial_x \tilde{g})^2 v_0^b \, dvdx. \qquad (4.21)
\end{aligned}$$

Using (4.15), (4.12), (4.13), and (4.7) we have

$$\begin{aligned}
&\iint \tilde{F}^2 |\partial_x (K - \mathcal{K})|^2 v_0^b \, dvdx \\
\leq &\iint \tilde{F}^2 (|\partial_x G|^2 + (\partial_x \mathcal{B})^2 R^{-\varepsilon}) v_0^\beta \, dvdx \\
\leq &C\int (|\partial_x G|^2 + (\partial_x \mathcal{B})^2 R^{-\varepsilon}) \, dx \\
\leq &C\|G(t)\|_{H^1}^2 + CR^{-\varepsilon}. \qquad (4.22)
\end{aligned}$$

Substitution of (4.19), (4.21), and (4.22) into (4.18) yields

$$\begin{aligned}
&\frac{d}{dt}\iint v_0^b (\partial_x \tilde{g})^2 \, dvdx \\
\leq \; &C\iint v_0^b (\partial_x \tilde{g})^2 \, dvdx + C\|G(t)\|_{H^1}^2 + CR^{-\varepsilon} + C\int_0^t \|G(\tau)\|_{H^1}^2 \, d\tau.
\end{aligned}$$

By Gronwall's inequality we have

$$\iint v_0^b (\partial_x \tilde{g})^2 \, dvdx \leq C\int_0^t \|G(\tau)\|_{H^1}^2 \, d\tau + CR^{-\varepsilon}t. \qquad (4.23)$$

Next we consider the fields. We have

$$\frac{d}{dt}\int (|\tilde{E}|^2 + \tilde{B}^2) dx = -2\int \tilde{E} \cdot \tilde{j} dx.$$

By (4.9)

$$\int |\tilde{j}|^2 dx \leq \int \left(\int \tilde{f} v_0^\alpha \, dv\right) \left(\int v_0^{-\alpha} |v|^2 \, dv\right) dx \leq C$$

so

$$\frac{d}{dt}\int \left(|\tilde{E}|^2 + \tilde{B}^2\right) dx \leq C\left(\int |\tilde{E}|^2 dx\right)^{\frac{1}{2}}$$

and

$$\int \left(|\tilde{E}|^2 + \tilde{B}^2\right) dx \leq C_0 + Ct \qquad (4.24)$$

follows. Similarly, by (4.13)

$$\int |\partial_x \tilde{j}|^2 \, dx \le \int \left(\int (\partial_x \tilde{f})^2 v_0^\beta dv \right) \left(\int v_0^{2-\beta} dv \right) dx \le C$$

so

$$\begin{aligned} \frac{d}{dt} \int \left(|\partial_x \tilde{E}|^2 + \left(\partial_x \tilde{B}\right)^2 \right) dx \le & 2\|\partial_x \tilde{E}(t)\|_{L^2} \|\partial_x \tilde{j}(t)\|_{L^2} \\ \le & C\|\partial_x \tilde{E}(t)\|_{L^2} \end{aligned}$$

and

$$\int \left(|\partial_x \tilde{E}|^2 + (\partial_x \tilde{B})^2 \right) dx \le C_0 + Ct \tag{4.25}$$

follows. In the same manner (4.17) yields

$$\begin{aligned} \int |\tilde{j} - \tilde{J}|^2 dx \le & \int \left(\int \tilde{g}^2 v_0^a \, dv \right) \left(\int v_0^{-a} |v|^2 \, dv \right) dx \\ \le & C \int_0^t \|G(\tau)\|_{H^1}^2 d\tau + CR^{-\varepsilon} t \end{aligned}$$

and, letting $\tilde{G} = (\tilde{E}, \tilde{B}) - (\tilde{\mathcal{E}}, \tilde{\mathcal{B}})$,

$$\int |\tilde{G}|^2 dx \le C \int_0^t \|G(\tau)\|_{H^1}^2 + CR^{-\varepsilon} t$$

follows. Lastly (4.23) yields

$$\begin{aligned} \int |\partial_x (\tilde{j} - \tilde{J})|^2 \, dx \le & C \int \left(\int (\partial_x \tilde{g})^2 v_0^b \, dv \right) \left(\int v_0^{2-b} dv \right) dx \\ \le & C \int_0^t \|G(\tau)\|_{H^1}^2 \, d\tau + CR^{-\varepsilon} t \end{aligned}$$

and

$$\int |\partial_x \tilde{G}|^2 \, dx \le C \int_0^t \|G(\tau)\|_{H^1}^2 d\tau + CR^{-\varepsilon} t$$

follows.

From (4.24) and (4.25) we have

$$\|(\tilde{E}, \tilde{B})(t)\|_{H^1}^2 \le 2C_0 + CT \le 10 C_0$$

for T suitably restricted. Also,

$$\begin{aligned} & \|(\tilde{f} - \tilde{F})(t)\|_{\mathcal{F}}^2 + \|(\tilde{E}, \tilde{B})(t) - (\tilde{\mathcal{E}}, \tilde{\mathcal{B}})(t)\|_{H^1}^2 \\ \le & C \int_0^t \|(E, B)(\tau) - (\mathcal{E}, \mathcal{B})(\tau)\|_{H^1}^2 d\tau + CR^{-\varepsilon} t. \end{aligned} \tag{4.26}$$

Define $(f^{n+1}, E^{n+1}, B^{n+1}) = \mathcal{L}^{2^n}(E^n, B^n)$ for $n \ge 0$ where f^0, E^0, B^0 are determined by the initial conditions. By (4.26) we have

$$\|(f^{n+1} - f^n)(t)\|_{\mathcal{F}}^2 + \|(E^{n+1}, B^{n+1})(t) - (E^n, B^n)(t)\|_{H^1}^2$$

$$\leq C \int_0^t \|(E^n, B^n)(\tau) - (E^{n-1}, B^{n-1})(\tau)\|_{H^1}^2 \, d\tau + C 2^{-n\varepsilon} t. \tag{4.27}$$

Suppose $A > 1$ and

$$0 \leq x^{n+1}(t) \leq C \left(\int_0^t x^n(\tau) d\tau + A^{-n} t \right)$$

for $n \geq 0$. Then by induction

$$\begin{aligned} x^n(t) &\leq \left(\|x^0\|_{L^\infty} + 1 \right) \frac{(Ct)^n}{n!} + A^{-n} \sum_{\ell=1}^{n-1} \frac{(CAt)^\ell}{\ell!} \\ &\leq \left(\|x^0\|_{L^\infty} + 1 \right) \frac{(CT)^n}{n!} + A^{-n} e^{CAT}. \end{aligned}$$

Since this bound is summable, it follows from (4.27) that (E^n, B^n) is Cauchy in $C([0,T]; H^1)$ and f^n is Cauchy in $C([0,T]; \mathcal{F})$. Let $(f, E, B) = \lim_{n\to\infty} (f^n, E^n, B^n)$.

We will now use the explicitly known fundamental solution for the linear equation

$$\partial_t f + v_1 \partial_x f = \Delta_v f,$$

namely, for $0 \leq \tau < t,\ x, y \in \mathbb{R}, v, w \in \mathbb{R}^2$

$$\begin{aligned} &\mathcal{G}(t, x, v, \tau, y, w) \\ =&[4\pi(t-\tau)]^{-1} e^{\frac{-|v-w|^2}{4(t-\tau)}} \left[\frac{\pi}{3} (t-\tau)^3 \right]^{-\frac{1}{2}} \exp\left(-3 \frac{(x - y - \frac{1}{2}(t-\tau)(v_1 + w_1))^2}{(t-\tau)^3} \right). \end{aligned}$$

This may be derived from line (2.5) of [16] by letting $\beta \to 0^+$ (with $N = 2$) then integrating in y_2. It may also be derived directly by Fourier transform. Since

$$\begin{cases} \partial_t f^{n+1} + v_1 \partial_x f^{n+1} = \Delta_v f^{n+1} - K^n \cdot \nabla_v f^{n+1} \\ f^{n+1}(0, \cdot, \cdot) = f^0 \end{cases}$$

it follows by theorems II.2 and II.3 of [16] that

$$f^{n+1} = H + \int_0^t \iint \mathcal{G}(t, x, v, \tau, y, w) (-K^n \cdot \nabla_w f^{n+1}) \Big|_{(\tau, y, w)} dw dy d\tau$$

where

$$H(t, x, v) = \iint \mathcal{G}(t, x, v, 0, y, w) f^0(y, w) \, dw dy.$$

It is easy to check that

$$\iint |\nabla_w \mathcal{G}(t, x, v, \tau, y, w)| \, dw dy \leq C (t-\tau)^{-\frac{1}{2}}$$

and it follows that

$$f^{n+1} = H + \int_0^t \iint \nabla_w \mathcal{G}(t, x, v, \tau, y, w) \cdot (K^n f^{n+1}) \Big|_{(\tau, y, w)} dw dy d\tau.$$

Now by (4.14) and the Sobolev embedding theorem

$$\begin{aligned}
&\iint |Kf - K^n f^{n+1}|^2 \, dvdx \le C \iint \left(|K - K^n|^2 f^2 + |K^n|^2 |f - f^{n+1}|^2 \right) dvdx \\
\le& C \int \left(|(E,B) - (E^0, B^n)|^2 + B^2 (2^n)^{-\frac{\varepsilon}{2}} \right) \int f^2 v_0^{2+\varepsilon} \, dvdx \\
&+ C \int |(E^n, B^n)|^2 \int |f - f^{n+1}|^2 v_0^2 \, dvdx \\
\le& C \left(\|(E,B)(t) - (E^n, B^n)(t)\|_{H^1}^2 + 2^{-\frac{n\varepsilon}{2}} \right) \iint f^2 v_0^{2+\varepsilon} \, dvdx \\
&+ C \iint (f - f^{n+1})^2 v_0^2 \, dvdx \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Also,

$$\left(\iint |\nabla_w \mathcal{G}(t,x,v,\tau,y,w)|^2 \, dwdy \right)^{\frac{1}{2}} \le C(t-\tau)^{-\frac{1}{2}}$$

so it follows that

$$f = H + \int_0^t \iint \nabla_w \mathcal{G}(t,x,v,\tau,y,w) \cdot (Kf) \Big|_{(\tau,y,w)} dwdyd\tau. \tag{4.28}$$

By Lemma 2.1

$$0 \le f^n \le \sup f^0$$

so

$$0 \le f \le \sup f^0 \tag{4.29}$$

follows.

Thus far we have assumed (f^0, E_2^0, B^0) to be smooth. Now consider (f^0, E_2^0, B^0) as in Theorem 1.1. Consider a sequence $(f^{0k}, E_2^{0k}, B^{0k})$ of smooth initial conditions with

$$\|v_0^{\frac{\alpha}{2}} (f^{0k} - f^0)\|_{L^2} + \|v_0^{\frac{\beta}{2}} \partial_x (f^{0k} - f^0)\|_{L^2} + \|(E_0^{0k}, B^{0k}) - (E_2^0, B^0)\|_{H^1} \to 0.$$

By a limiting procedure like the above we conclude that (4.28) and (4.29) hold for (f^0, E_2^0, B^0) as in Theorem 1.1.

By Theorem II.3 of [16] $H \in C([0,\infty) \times \mathbb{R}^3) \cap C^1((0,\infty) \times \mathbb{R}^3)$ with $\partial_{v_i} \partial_{v_j} H$ continuous on $t > 0$ and

$$\begin{cases} \partial_t H + v_1 \partial_x H = \Delta_v H \\ H(0, \cdot, \cdot) = f^0. \end{cases}$$

We claim that f is differentiable in v. First we show that f is Hölder continuous in v. For example, if $\overline{v}_1 < \overline{\overline{v}}_1$ then

$$\begin{aligned}
&\left| \nabla_w \mathcal{G}(t,x,\overline{\overline{v}}_1, v_2, \tau, y, w) - \nabla_w \mathcal{G}(t,x,\overline{v}_1, v_2, \tau, y, w) \right|^2 \\
=& \left| \int_{\overline{v}_1}^{\overline{\overline{v}}_1} \nabla_w \partial_{v_1} \mathcal{G}(t,x,v,\tau,y,w) \, dv_1 \right|^2
\end{aligned}$$

$$\leq \left|\overline{\overline{v}}_1 - \overline{v}_1\right| \int_{\overline{v}_1}^{\overline{\overline{v}}_1} \left|\nabla_w \partial_{v_1} \mathcal{G}\right|^2 dv_1$$

and

$$\begin{aligned}
&\iint \left|\nabla_w \mathcal{G}(t,x,\overline{\overline{v}}_1,v_2,\tau,y,w) - \nabla_w \mathcal{G}(t,x,\overline{v}_1,v_2,\tau,y,w)\right|^2 dwdy \\
\leq& \left|\overline{\overline{v}}_1 - \overline{v}_1\right| \int_{\overline{v}_1}^{\overline{\overline{v}}_1} \iint \left|\nabla_w \partial_{v_1} \mathcal{G}\right|^2 dwdydv_1 \\
\leq& C\left(\overline{\overline{v}}_1 - \overline{v}_1\right)^2 (t-\tau)^{-2}
\end{aligned}$$

so

$$\begin{aligned}
&\iint \left|\left(\nabla_w \mathcal{G}|_{\overline{v}_1}^{\overline{\overline{v}}_1}\right)(fK)\Big|_{(\tau,y,w)}\right| dwdy \\
\leq& \sqrt{C(\overline{\overline{v}}_1 - \overline{v}_1)^2 (t-\tau)^{-2}} \sqrt{\iint |fK|^2\, dwdy} \leq C\left|\overline{\overline{v}}_1 - \overline{v}_1\right| (t-\tau)^{-1}.
\end{aligned}$$

But we also have

$$\begin{aligned}
&\iint \left|\nabla_w \mathcal{G}(t,x,v,\tau,y,w) \cdot (fK)(\tau,y,w)\right| dwdy \\
\leq& \sqrt{\iint \left|\nabla_w \mathcal{G}\right|^2 dwdy}\ \sqrt{\iint |fK|^2\, dwdy} \leq C(t-\tau)^{-\frac{1}{2}}
\end{aligned}$$

so by (4.28)

$$\left|(f-H)|_{\overline{v}_1}^{\overline{\overline{v}}_1}\right| \leq C\int_0^t \min\left(\left|\overline{\overline{v}}_1 - \overline{v}_1\right| (t-\tau)^{-1}, (t-\tau)^{-\frac{1}{2}}\right) d\tau.$$

It follows that f is Hölder continuous in v_1 with exponent θ for all $\theta \in (0,1)$. The Hölder continuity in v_2 may be shown similarly. By (4.28) we may write

$$f = H + \int_0^t \iint \nabla_w \mathcal{G}(t,x,v,\tau,y,w) \cdot \left(fK\,|_{(\tau,y,w)} - fK\big|_{(\tau,x,v)}\right) dwdyd\tau.$$

Now it follows that f is differentiable in v and

$$\partial_{v_i} f = \partial_{v_i} H + \int_0^t \iint \nabla_w \partial_{v_i} \mathcal{G} \cdot \left(fK\,|_{(\tau,y,w)} - fK\big|_{(\tau,x,v)}\right) dwdyd\tau.$$

We now may integrate by parts in (4.28) and obtain

$$f = H - \int_0^t \iint \mathcal{G}(t,x,v,\tau,y,w)(K\cdot\nabla_w f)(\tau,y,w)\, dwdyd\tau.$$

Note that by (4.12), (4.13), and the Sobolev embedding theorem.

$$\begin{aligned}
&\iint |\partial_x (fK)|^2\, dvdx \\
\leq& C\iint \left(f^2|\partial_x K|^2 + |K|^2(\partial_x f)^2\right) dvdx
\end{aligned}$$

$$\leq C \int \left(|\partial_x E|^2 + (\partial_x B)^2\right) \int f^2 v_0^2 \, dv dx + C \int (|E|^2 + |B|^2) \int (\partial_x f)^2 v_0^2 \, dv dx$$
$$\leq C \int \left(|\partial_x E|^2 + (\partial_x B)^2\right) dx + C \iint (\partial_x f)^2 v_0^2 \, dv dx \leq C.$$

Now it follows from (4.28) that

$$\partial_x f = \partial_x H + \int_0^t \iint \nabla_w \mathcal{G}(t,x,v,\tau,y,w) \cdot \partial_y (fK)\Big|_{(\tau,y,x)} dw dy d\tau.$$

In particular, f is Hölder continuous in x with exponent θ for each $\theta \in (0,1)$. Hence, Theorem II.1 of [16] applies and shows that f has the regularity stated in Theorem 1.1. The regularity of E and B follows from this.

Finally, suppose that $(F, \mathcal{E}, \mathcal{B})$ is another solution with the same initial value as (f, E, B). Then, by (4.12), (4.13), and (4.16),

$$\begin{aligned} &\frac{d}{dt} \iint v_0^b (f-F)^2 \, dv dx \\ \leq& C \iint v_0^b (f-F)^2 \, dv dx + C \int \left(|E - \mathcal{E}|^2 + (B - \mathcal{B})^2\right) \int F^2 v_0^{b+2} \, dv dx \\ \leq& C \iint v_0^b (f-F)^2 dv dx + C \int \left(|E - \mathcal{E}|^2 + (B - \mathcal{B})^2\right) dx. \end{aligned} \tag{4.30}$$

Also

$$\frac{d}{dt} \int \left(|E - \mathcal{E}|^2 + (B - \mathcal{B})^2\right) dx = -2 \int (E - \mathcal{E}) \cdot \int (f - F) \, v \, dv dx.$$

Since

$$\begin{aligned} \left(\int |f - F| v_0 \, dv\right)^2 \leq& \int (f-F)^2 v_0^b dv \int v_0^{2-b} dv \\ \leq& C \int (f-F)^2 v_0^b dv, \end{aligned}$$

we have

$$\begin{aligned} \frac{d}{dt} \int \left(|E - \mathcal{E}|^2 - |B - \mathcal{B}|^2\right) dx \leq& C \sqrt{\int |E - \mathcal{E}|^2 \, dx} \sqrt{\iint (f-F)^2 v_0^b \, dv dx} \\ \leq& C \int |E - \mathcal{E}|^2 dx + C \iint (f-F)^2 v_0^b \, dv dx. \end{aligned} \tag{4.31}$$

Uniqueness follows from (4.30) and (4.31) and the proof is complete.

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