

CONVERGENCE TO THE EQUILIBRIUM STATE FOR A BOSE–EINSTEIN 1D KAC GRAZING LIMIT MODEL*

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Abstract. The convergence to the equilibrium of the solution of a quantum Kac grazing limit model for Bose–Einstein identical particles is studied. Using the relative entropy method and a detailed analysis of the entropy production, the exponential decay rate is obtained under suitable assumptions on the mass and energy of the initial data. These theoretical results are further illustrated through numerical simulations.

Key words. Quantum Kac grazing limit model, relative entropy method, exponential decay.

AMS subject classifications. 35H10, 76P05, 84C40.

1. Introduction

In this paper we study a model equation governing the time evolution of a gas composed of Bose–Einstein identical particles which was introduced recently in [1].

Let $f(t, v)$ be the velocity distribution function of these particles at time $t > 0$ with the velocity $v \in \mathbb{R}$. According to quantum physics, the presence of a particle in the velocity range dv increases the probability that a particle will enter that range: the presence of $f(v)dv$ particles per unit volume increases this probability by the ratio $1 + f(v)$. Following Chapman and Cowling [6], this fundamental assumption yields the so-called Boltzmann–Bose–Einstein equation, that is the quantum Boltzmann equation for Bose–Einstein particles. This equation is extensively detailed in physics books together with numerical simulations (see [3, 16, 22] and references therein), with particular emphasis on blow-up phenomena, which is related to the famous Bose condensation effect.

However, there are not many rigorous mathematical results. We mention here for the spatially homogeneous isotropic case, a theory of weak solutions developed by Lu in [19, 21, 20], and another class of locally defined in time classical solutions by Escobedo et al. in [13, 14, 15]. We also refer to [2, 23] for more complete reviews on currently available mathematical results. On the other hand, Allemand and Toscani in [1] derived the following nonlinear Fokker–Planck equation (Kac grazing limit model, or Kac model)

$$\partial_t f = A_f(t) \partial_{vv} f + B_f(t) \partial_v (vf(1+f)) \quad (1.1)$$

with

$$A_f(t) = \int v^2 f(1+f) dv, \quad B_f(t) = \int f dv. \quad (1.2)$$

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Note here and below the integration is taken over \mathbb{R} without confusion. This model is obtained as the grazing collision limit of a one-dimensional Boltzmann equation for Bose–Einstein particles in the spirit of the Kac caricature of a Maxwell gas with a singular kernel [1]. However, the existence of good solutions for the Boltzmann–Bose–Kac model presented in [1] is still unknown, and is currently under investigation. We will therefore specialize to the above Kac model (1.1), and study the convergence of the solution to the Bose distribution by using the entropy method.

The rigorous study of the convergence to equilibrium is by now classical in kinetic theory. For example, using the classical logarithmic–Sobolev inequality of Gross [17], and the Csiszar–Kullback–Pinsker inequality [7, 18], the convergence to the equilibrium with exponential decay rate can be derived by using the relative entropy method for linear Fokker–Planck type equations. For the nonlinear Fokker–Planck–Landau equation, the trend to equilibrium has been obtained by Desvillettes and Villani in [9]. Toscani and Villani in [26] studied the convergence to the equilibrium for the Boltzmann equation. In [11], Desvillettes and Villani studied the trend to equilibrium for the spacial inhomogeneous linear Fokker–Planck equation. For more details about the trend to equilibrium for classical kinetic equations, we refer to [12, 10, 8] and the references therein.

In [5], Carrillo, Rosado and Salvarani studied the following model of a 1D quantum Fokker–Planck equation

$$\partial_t f = \partial_{vv} f + \partial_v(vf(1+f)). \quad (1.3)$$

Note the factor $1+f$ takes into account quantum effects. It is easy to see that the mass, $\int f(t,v)dv$, is conserved along the time evolution associated with (1.3). By using the relative entropy method, it was proved in [5] that the solutions of (1.3) converge to the Bose equilibrium function with an exponential decay rate. However, the above model (1.3), a simplified model of (1.1) with $A_f(t)$ and $B_f(t)$ given by (1.2) replaced by constant 1, does not conserve the kinetic energy, that is $\int v^2 f dv$. Moreover, the nonlinearity in the Kac model (1.1) is stronger.

For later use, let M and E be the mass and the energy defined by the initial data $f_0(v)$,

$$M = \int f_0(v)dv, \quad E = \int v^2 f_0(v)dv,$$

where we assume that $f_0 > 0$, $f_0 \in L^1(\mathbb{R})$, and $v^2 f_0 \in L^1(\mathbb{R})$. The entropy $H(f)$ is defined as

$$H(f)(t) = \int \gamma(f)(t,v)dv, \quad \text{with } \gamma(f) = f \log f - (1+f) \log(1+f).$$

Let us remark that the entropy used in [5] is the sum of the entropy defined above and the kinetic energy which is conserved for the Kac grazing limit model (1.1).

We shall work with smooth enough solutions: we show in the next section that one can get a priori weighted L^2 bounds on solutions, and similar estimates also hold true for higher derivatives as well.

The main result of this paper is given by Theorem 1.1 below. The constants Λ and C_N appearing in Assumption (1.4) are the constants appearing respectively in the Gagliardo–Nirenberg and Nash inequalities, see the next section for further details.

THEOREM 1.1. *Let $f(t, v)$ be the solution of the Kac grazing limit model (1.1) with initial data f_0 which is positive and satisfies*

$$\int f_0(v) dv = M, \quad \int v^2 f_0(v) dv = E$$

for some positive constants M and E . Assume moreover that there exists $\delta > 0$ such that

$$M \left\{ 4M \cdot \frac{M^3}{C_N^2 E} \cdot \left(1 + \frac{\Lambda^2 M^3}{9E} \right) - 1 \right\} < -\delta. \quad (1.4)$$

Then there exists a constant $C(f_0)$ depending on f_0 , such that

$$\|f(t) - f_\infty\|_{L^1} \leq C(f_0) \frac{e^{-\delta t/2}}{\sqrt{\delta}}, \quad \text{for all } t \geq \frac{\log 2}{2M(1 + \frac{\Lambda^2 M^3}{9E})}. \quad (1.5)$$

Here f_∞ is the Bose distribution with mass M and energy E defined by (3.5).

REMARK 1.2.

1. The constant $C(f_0)$ appearing in (1.5) is in fact explicit as can be seen from the proof below.
2. Since we are only interested in the large time behavior of the solution, the decay estimate (1.5) is only stated for time away from the initial time. However, a simple modification of the proof can be used to state a result valid for any time $t \geq 0$.
3. If the mass M is fixed, then assumption (1.4) is true if $\frac{M^3}{E}$ is small enough, that is if the energy E is large enough.

Comparing with the results obtained in [5] for the 1D quantum model (1.3), some additional assumptions on the initial data are needed in Theorem 1.1 for our model (1.1). In fact, a generalized logarithmic-Sobolev inequality developed in [4] for nonlinear diffusion equations was used directly in [5] to control the entropy production from below by the relative entropy. In the proof given in [5], an auxiliary nonlinear diffusion equation which has the same entropy and the equilibrium was introduced. On the other hand, for the Kac model (1.1), it is impossible to introduce such an auxiliary equation with the same relative entropy or the equilibrium state, and a compatible entropy production term. Instead of using the generalized logarithmic-Sobolev inequality, we follow some ideas used in [4] to get the decay rate of entropy production, and then to prove the convergence to equilibrium for the solution of the Kac grazing limit model. The constraint (1.4) on the initial data stated in Theorem 1.1 will be used to get the decay rate of the entropy production. Though it will be clear from the proofs below that this constraint is far from being sharp, we note that in the numerical simulations performed below, we don't take into account this constraint.

The rest of the paper is organized as follows: In Section 2, we will give some preliminary L^2 estimates. Then the entropy and the entropy equality will be introduced in Section 3. Based on a detailed study of the entropy production, we will get the exponential decay rate. Finally these theoretical results will be illustrated by numerical simulations in Section 4.

2. Preliminaries

In this section we will show some a priori weighted L^2 estimates on the solution, together with some control on the specific quantity $A_f(t)$ defined by (1.2). These estimates will be used in the next section in order to get the decay rate of the entropy production.

Before starting our estimates, we note first that it is not difficult to assert the positivity of the solution f using classical arguments, for example as in [5]: let ρ_ε be a Friedrich mollifier, and define the smoothed sign and absolute functions

$$\phi_\varepsilon = \rho_\varepsilon * \text{sign}, \quad \Phi_\varepsilon(x) = \int_0^x \phi_\varepsilon(y) dy.$$

Multiplying Kac equation by $\phi_\varepsilon(f)$, and integrating it over \mathbb{R} , we get

$$\begin{aligned} \frac{d}{dt} \int (\Phi_\varepsilon(f) - f) dv &= \frac{d}{dt} \int \Phi_\varepsilon(f) \\ &= -A_f(t) \int \phi'_\varepsilon(f) |\partial_v f|^2 dv - B_f(t) \int v f(1+f) \phi'_\varepsilon(f) \partial_v f dv. \end{aligned} \quad (2.1)$$

Note that

$$- \int v f \phi'_\varepsilon(f) \partial_v f dv = - \int v \partial_v (f \phi_\varepsilon(f) - \Phi_\varepsilon(f)) dv,$$

and

$$\begin{aligned} &- \int v f^2 \phi'_\varepsilon(f) \partial_v f dv \\ &= - \int v \partial_v (f^2 \phi_\varepsilon(f) - f \Phi_\varepsilon(f)) dv - \int v \partial_v \left(\int^f \xi^2 \phi'_\varepsilon(\xi) d\xi \right) dv. \end{aligned}$$

The first term on the right hand side of (2.1) is non positive since $\phi'_\varepsilon \geq 0$. And the second term on the right hand side of (2.1) vanishes as $\varepsilon \rightarrow 0$ from Lebesgue's dominated convergence theorem. Then, letting $\varepsilon \rightarrow 0$ on both side of (2.1), we have

$$\| |f(t)| - f(t) \|_{L^1} \leq \| |f_0| - f_0 \|_{L^1}.$$

If the initial data $f_0 \in L^1$ is non negative a.e. on \mathbb{R} , then the solution f (if it exists) belongs to L^1 and is always non negative a.e. on \mathbb{R} .

In conclusion, we have shown that if f , the smooth solution of Kac's model with initial data $f_0 \in L^1(\mathbb{R})$, is sufficiently decaying, there holds that the L^1 norm of f is non-increasing for $t > 0$. Furthermore if f_0 is non-negative a.e. in \mathbb{R} , the solution $f(t, v)$ also is non-negative in \mathbb{R} for any $t > 0$.

2.1. Weighted L^2 estimates of the solution. In this paragraph, we are going to show the following uniform in time L^2 estimates. The constants Λ and C_N which appear below are the constants appearing in the Gagliardo–Nirenberg and Nash inequality respectively, see the proof hereafter (in fact optimal constants are known).

PROPOSITION 2.1. *The L^2 norm of the solution f satisfies*

$$\|f(t)\|_{L^2}^2 \leq \frac{e^{2C_1 t}}{\frac{1}{\|f_0\|_{L^2}^2} + \frac{C_2}{C_1} (e^{2C_1 t} - 1)} \quad \text{for all } t \geq 0,$$

where

$$C_1(M, E) = M \left(1 + \frac{\Lambda^2 M^3}{9E} \right) \text{ and } C_2(M, E) = \frac{C_N^2 E}{M^4} = \frac{1}{M} \frac{C_N^2 E}{M^3}.$$

In particular, we have

$$\|f(t)\|_{L^2}^2 \leq 2 \frac{C_1}{C_2}, \quad \text{for } t \geq t_0 \equiv \frac{\log 2}{2C_1}. \quad (2.2)$$

Similarly, letting $\langle v \rangle = (1 + |v|^2)^{1/2}$, we have the weighted estimate

$$\|\langle v \rangle f(t)\|_{L^2}^2 \leq \frac{e^{2\tilde{C}_1 t}}{\frac{1}{\|\langle v \rangle f_0\|_{L^2}^2} + \frac{\tilde{C}_2}{\tilde{C}_1} \left(e^{2\tilde{C}_1 t} - 1 \right)} \quad \text{for } t \geq 0,$$

where

$$\tilde{C}_1(M, E) = 2E + M + \frac{M^2 \Lambda^2 (M + \frac{E}{2})^2}{9E} + 2 \frac{(M + \frac{E}{2})^4}{EC_N^2}, \quad \tilde{C}_2(M, E) = \frac{1}{2} \frac{EC_N^2}{(M + \frac{E}{2})^4}.$$

In particular, we have

$$\|\langle v \rangle f(t)\|_{L^2}^2 \leq 2 \frac{\tilde{C}_1}{\tilde{C}_2} \quad \text{for } t \geq t_1 \equiv \frac{\log 2}{2\tilde{C}_1}.$$

Proof. We multiply the equation by f and then integrate the resulting equality with respect to v to get

$$\frac{1}{2} \frac{d}{dt} \|f(t)\|_{L^2}^2 + A_f(t) \|\partial_v f(t)\|_{L^2}^2 = M \left(\frac{1}{2} \|f(t)\|_{L^2}^2 + \frac{1}{3} \|f(t)\|_{L^3}^3 \right). \quad (2.3)$$

Then, for any $\varepsilon > 0$, we use a Gagliardo–Nirenberg inequality (cf. (3.27) in [24]) to estimate the L^3 norm of f as

$$\begin{aligned} \|f(t)\|_{L^3}^3 &\leq \|f(t)\|_{L^1} \|f(t)\|_{L^\infty}^2 \leq \Lambda \|f(t)\|_{L^1} \|f(t)\|_{L^2} \|\partial_v f(t)\|_{L^2} \\ &\leq \varepsilon \|\partial_v f(t)\|_{L^2}^2 + \frac{\Lambda^2 M^2}{4\varepsilon} \|f(t)\|_{L^2}^2, \end{aligned}$$

where Λ denotes the constant arising in the Gagliardo–Nirenberg inequality. Recalling that $A_f(t) = E + \|vf\|_{L^2}^2$, we choose $\varepsilon = \frac{3E}{2M}$ and finally we have

$$\frac{d}{dt} \|f(t)\|_{L^2}^2 + E \|\partial_v f(t)\|_{L^2}^2 \leq \left(M + \frac{\Lambda^2 M^4}{9E} \right) \|f(t)\|_{L^2}^2. \quad (2.4)$$

The Nash inequality (cf. (6) in [25]) in one dimensional case reads

$$\|\partial_v f\|_{L^2} \geq C_N \frac{\|f\|_{L^2}^3}{\|f\|_{L^1}^2},$$

where C_N is a numerical constant. Using this Nash inequality in (2.4) gives

$$\frac{d}{dt} \|f(t)\|_{L^2}^2 \leq -C_2(M, E) \|f(t)\|_{L^2}^6 + C_1(M, E) \|f(t)\|_{L^2}^2,$$

with $C_1(M, E) = M + \Lambda^2 M^4 / (9E)$, $C_2(M, E) = C_N^2 E / M^4$. The above differential inequality can be solved explicitly in a standard way. For simplicity let us omit the dependence of the constants on the mass M and the energy E and denote $X(t) = \|f(t)\|_{L^2}^2$. Then we have

$$X'(t) \leq -C_2 X^3(t) + C_1 X(t),$$

which can be reduced to

$$\left(\frac{e^{2C_1 t}}{X(t)^2} \right)' \geq 2C_2 e^{2C_1 t}.$$

Integrating the above inequality over $[0, t]$ gives the upper bound of X^2 as

$$X^2(t) \leq \frac{e^{2C_1 t}}{\frac{1}{X(0)^2} + \frac{C_2}{C_1} (e^{2C_1 t} - 1)},$$

which is the uniform bound for the L^2 norm as stated in Proposition 2.1.

Now, let us choose t_0 such that $e^{2C_1 t_0} - 1 = 1$, that is

$$t_0 \equiv \frac{\log 2}{2C_1}.$$

It follows from the previous bound that

$$X^2(t) \leq \frac{e^{2C_1 t}}{\frac{C_2}{C_1} (e^{2C_1 t} - 1)} \leq \frac{C_1}{C_2} \left\{ 1 + \frac{1}{e^{2C_1 t} - 1} \right\}$$

and therefore

$$X^2(t) \leq 2 \frac{C_1}{C_2} \text{ for all } t \geq t_0.$$

The L^2 estimation of $\langle v \rangle f$ is similar. Set $g = \langle v \rangle f$. We multiply the Kac model (1.1) by $\langle v \rangle g$, then integrate it with respect to v over \mathbb{R} to get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|g(t)\|_{L^2}^2 &= A_f(t) \int \langle v \rangle g \partial_{vv} f \, dv + B_f(t) \int \langle v \rangle g \partial_v (vf(1+f)) \, dv \\ &= A_f(t) \left(\int \frac{v^2}{\langle v \rangle^4} g^2 \, dv - \|\partial_v g\|_{L^2}^2 \right) \\ &\quad + \frac{B_f(t)}{2} \int \frac{g^2}{\langle v \rangle^2} (1-v^2) \, dv + \frac{B_f(t)}{3} \int \frac{g^3}{\langle v \rangle^3} (1-3v^2) \, dv. \end{aligned}$$

Since $1/\langle v \rangle \leq 1$, $|v|/\langle v \rangle \leq 1$, we have

$$\frac{1}{2} \frac{d}{dt} \|g(t)\|_{L^2}^2 + A_f(t) \|\partial_v g(t)\|_{L^2}^2 \leq \left(A_f(t) + \frac{B_f(t)}{2} \right) \|g(t)\|_{L^2}^2 + \frac{B_f(t)}{3} \|g(t)\|_{L^3}^3.$$

We use the previous interpolation inequality and the Gagliardo–Nirenberg inequality to estimate the L^3 norm as

$$\begin{aligned} \|g(t)\|_{L^3}^3 &\leq \Lambda \|g(t)\|_{L^1} \|g(t)\|_{L^2} \|\partial_v g(t)\|_{L^2} \\ &\leq \varepsilon \|\partial_v g(t)\|_{L^2}^2 + \frac{\Lambda^2 \|g(t)\|_{L^1}^2}{4\varepsilon} \|g(t)\|_{L^2}. \end{aligned}$$

We take $\varepsilon = 3E/(2M)$ and use Nash inequality to get

$$\frac{d}{dt} \|g(t)\|_{L^2}^2 \leq \left(2A_f(t) + B_f(t) + \frac{M^2 \Lambda^2 \|g(t)\|_{L^1}^2}{9E} \right) \|g(t)\|_{L^2}^2 - \frac{EC_N^2}{\|g(t)\|_{L^1}^4} \|g(t)\|_{L^2}^6.$$

As $\langle v \rangle \leq 1 + v^2/2$, we have $\|g(t)\|_{L^1} \leq M + \frac{E}{2}$. Note that $A_f(t) \leq E + \|g(t)\|_{L^2}^2$. Then we have

$$\frac{d}{dt} \|g(t)\|_{L^2}^2 \leq \tilde{C}_1(M, E) \|g(t)\|_{L^2}^2 + 2\|g(t)\|_{L^2}^4 - \tilde{C}_2(M, E) \|g(t)\|_{L^2}^6,$$

with

$$\begin{aligned} \tilde{C}_1(M, E) &= 2E + M + \frac{M^2 \Lambda^2 (M + \frac{E}{2})^2}{9E}, \\ \tilde{C}_2(M, E) &= \frac{EC_N^2}{(M + \frac{E}{2})^4}. \end{aligned}$$

Letting $Y(t) = \|g(t)\|_{L^2}^2$, the differential inequality can be written as

$$Y'(t) \leq G(Y(t)) = \tilde{C}_1(M, E)Y(t) + 2Y(t)^2 - \tilde{C}_2(M, E)Y^3(t).$$

Note that $G(Y)$ has a unique positive zero point

$$Y_*(M, E) = \frac{1 + \sqrt{1 + \tilde{C}_1(M, E)\tilde{C}_2(M, E)}}{\tilde{C}_2(M, E)} > 0,$$

and $G(Y)$ is positive over $]0, Y_*[$ and negative on $]Y_*, +\infty[$. Then we get the global existence of $Y(t)$ which will take values between the initial value $Y(0)$ and the equilibrium point Y_* . In conclusion, we have

$$\|g(t)\|_{L^2}^2 \leq \max\{\|g_0\|_{L^2}^2, Y_*\},$$

with $g_0 = \langle v \rangle f_0$.

However, we can get similar estimations as for $X(t)$. To see this, having in mind the differential inequality satisfied by $Y(t)$, we use Young's inequality as follows:

$$2Y(t)^2 = 2\frac{1}{\sqrt{\varepsilon}}Y(t)^{1/2} \cdot \sqrt{\varepsilon}Y(t)^{3/2} \leq \frac{1}{\varepsilon}Y(t) + \varepsilon Y(t)^3$$

with $\varepsilon = \frac{1}{2}\tilde{C}_2(M, E)$, and then the differential inequality for $Y(t)$ is of the same form as for $X(t)$, that is we have

$$Y'(t) \leq \tilde{\tilde{C}}_1(M, E)Y(t) - \tilde{\tilde{C}}_2(M, E)Y^3(t), \quad t \geq 0,$$

where

$$\tilde{\tilde{C}}_1(M, E) = \tilde{C}_1(M, E) + \frac{2}{\tilde{C}_2(M, E)}, \quad \tilde{\tilde{C}}_2(M, E) = \frac{1}{2}\tilde{C}_2(M, E).$$

Then the same conclusions as for $X(t)$ also hold true.

REMARK 2.2.

1. We have only shown weighted L^2 estimations of f but similar estimates on higher derivatives also hold true. It is important to note that smoothness is not required for estimating convergence to equilibrium.
2. Note that we have the following interpolation inequality

$$\|f(t)\|_{L^2}^2 \geq \frac{M}{27^{1/2}} \left(\frac{M^3}{E} \right)^{1/2},$$

which follows classically by optimizing w.r.t. $R > 0$ the following inequality

$$M = \int_{|v| \leq R} f \, dv + \int_{|v| \geq R} f \, dv \leq \|f\|_{L^2} \sqrt{2R} + \frac{E}{R^2}.$$

Therefore an L^2 control on f implies automatically a control of $\frac{M^{5/2}}{E^{1/2}}$.

3. The point with the time values t_0 or t_1 is that for $t \geq \max\{t_0, t_1\}$, these L^2 norms are estimated only in terms of the mass M and the energy E : in particular, for sufficiently large time, these upper bounds do not involve the L^2 (weighted) norm of the initial data.
4. Note that for a fixed M , if we assume that M^3/E is small, then $C_1(M, E) \simeq M$. On the other hand, within the same range, C_2 will be very large. In particular, note that for $t \geq t_0$, it will follow that $\|f\|_{L^2}$ will be very small. The case of the other constants \tilde{C}_1 and \tilde{C}_2 is more complicated.
5. In order to deduce the differential inequality (2.4), we use the simple bound $A_f(t) \geq E$. However we can also make use of the other contribution of $A_f(t)$, namely $\|vf\|_{L^2}^2$. Indeed, for any $\alpha > 0$, we have, on one hand

$$\int_{|v| \geq \alpha} f \, dv = \int_{|v| \geq \alpha} |v|^{-1} |v| f \, dv \leq \frac{1}{\sqrt{\alpha}} \|vf\|_{L^2},$$

and on the other hand

$$M = \int_{|v| \leq \alpha} f \, dv + \int_{|v| \geq \alpha} f \, dv \leq \sqrt{\alpha} \|f\|_{L^2} + \int_{|v| \geq \alpha} f \, dv.$$

Therefore, we have, for any $\alpha > 0$

$$M \leq \sqrt{\alpha} \|f\|_{L^2} + \frac{1}{\sqrt{\alpha}} \|vf\|_{L^2}.$$

Optimizing w.r.t. α , we get

$$M \leq 2 \|f\|_{L^2}^{1/2} \|vf\|_{L^2}^{1/2},$$

that is

$$\|vf\|_{L^2}^2 \geq 2^{-4} M^4 \|f\|_{L^2}^{-2},$$

and finally

$$A_f(t) \geq E + 2^{-4} M^4 \|f\|_{L^2}^{-2}.$$

This lower bound can be used in the previous computations in order to slightly improve the constraint (1.4).

□

2.2. Estimate for the quantity $A_f(t)$. In this paragraph we study the time derivative of the quantity $A_f(t) = E + \|vf\|_{L^2}^2$.

PROPOSITION 2.3. *It holds that*

$$\frac{A'_f(t)}{A_f(t)} \leq 2\|f(t)\|_{L^2}^2.$$

Proof. We note that $g(t, v) = vf(t, v)$ satisfies

$$\partial_t g = A_f(t)\partial_{vv}g - 2A_f(t)\partial_v f + B_f(t)\partial_v(vg + g^2) - B_f(t)g(1 + f).$$

Then multiply this equality by g and integrate it with respect to v to get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|g(t)\|_{L^2}^2 + A_f(t)\|\partial_v g\|_{L^2}^2 + \frac{B_f(t)}{2}\|g(t)\|_{L^2}^2 \\ &= A_f(t)\|f(t)\|_{L^2}^2 - B_t(f) \int f(t, v)g^2(t, v) dv. \end{aligned}$$

As $A_f(t) = E + \|g(t)\|_{L^2}^2$, we have

$$A'_f(t) + A_f(t)(B_f(t) - 2\|f(t)\|_{L^2}^2) \leq B_f(t)E \leq B_f(t) \cdot A_f(t).$$

Using the definition of $A_f(t)$, the proof of Proposition 2.3 is completed. □

3. Relative entropy method and decay to equilibrium

In this section, we will prove Theorem 1.1 by using the relative entropy method. Firstly we will introduce the notions of entropy, entropy production and equilibrium functions related to the Kac model (1.1). Secondly we will show the decay rate of the entropy production. Finally the decay rate of the solution for the Kac model to the equilibrium can be derived.

3.1. Entropy and equilibrium. Let $\gamma(f) = f \log f - (1 + f) \log(1 + f)$. Note that $\gamma'(f) = \log \frac{f}{1+f}$. The entropy $H(f)$ is defined by

$$H(f) = \int \gamma(f) dv$$

and satisfies the entropy equality

$$\frac{d}{dt} H(f) = - \int \left(A_f(t) \frac{|\partial_v f|^2}{f(1+f)} + B_f(t)v\partial_v f \right) := -D(f). \tag{3.1}$$

The entropy production $D(f)$ can be given by different formulations. For example since $B_f(t) = \int_v f = - \int_v v\partial_v f$, $D(f)$ can be written as

$$\begin{aligned} D(f) &= \int \left(A_f(t) \frac{|\partial_v f|^2}{f(1+f)} + 2B_f(t)v\partial_v f \right) dv + B_f(t)^2 \\ &= \frac{1}{A_f(t)} \int f(1+f) \left(A_f(t)^2 |\partial_v \gamma'(f)|^2 + 2B_f(t)vA_f(t)\partial_v \gamma'(f) \right) dv \end{aligned} \tag{3.2}$$

$$+ \frac{1}{A_f(t)} \int f(1+f)v^2 B_f(t)^2 dv \tag{3.3}$$

$$= \frac{1}{A_f(t)} \int f(1+f) |A_f(t) \partial_v \gamma'(f) + B_f(t)v|^2 dv. \tag{3.4}$$

In particular, from (3.4) we get that $D(f)(t) \geq 0$.

Furthermore, we can use the expression of $A_f(t)$ into the entropy production $D(f)$ and write it in a more symmetric form as

$$\begin{aligned} D(f) &= \int_v \int_{v_*} \left(v_*^2 f_* (1+f_*) \frac{|\partial_v f|^2}{f(1+f)} - v \partial_v f v_* \partial_{v_*} f_* \right) dv dv_* \\ &= \frac{1}{2} \int_v \int_{v_*} f(1+f) f_* (1+f_*) |v \partial_{v_*} \gamma'(f_*) - v_* \partial_v \gamma'(f)|^2 dv dv_*, \end{aligned}$$

with $f_* = f(v_*)$. From the equality

$$v_* \partial_v \gamma'(f) = v \partial_{v_*} \gamma'(f_*),$$

the equilibrium f_∞ is given by

$$f_\infty(v) = \frac{1}{\exp(\lambda_1 v^2 - \lambda_2) - 1}, \tag{3.5}$$

where the constants $\lambda_1 > 0$ and λ_2 will be determined by the initial data. Note that the equilibrium f_∞ defined above is the so-called Bose distribution function.

REMARK 3.1. We can show that

$$\lambda_1 = \frac{B_{f_\infty}}{2A_{f_\infty}}, \tag{3.6}$$

keeping the same notations as in (1.2). In fact, we have

$$\begin{aligned} B_{f_\infty} &= \frac{1}{\sqrt{\lambda_1}} \int \frac{e^{\lambda_2}}{e^{v^2} - e^{\lambda_2}} dv \\ A_{f_\infty} &= \int \frac{v^2 e^{\lambda_1 v^2} e^{\lambda_2}}{(e^{\lambda_1 v^2} - e^{\lambda_2})^2} dv = -\partial_{\lambda_1} \int \frac{e^{\lambda_2}}{e^{\lambda_1 v^2} - e^{\lambda_2}} dv. \end{aligned}$$

Using the expressions of B_{f_∞} , we have

$$A_{f_\infty} = -\partial_{\lambda_1} B_{f_\infty} = \frac{1}{2\lambda_1^{3/2}} \int \frac{e^{\lambda_2}}{e^{v^2} - e^{\lambda_2}} dv = \frac{1}{2\lambda_1} B_{f_\infty}.$$

Note that (3.6) can also be obtained from the entropy production in the form (3.4).

The next natural step is to prove the following lemma

LEMMA 3.2. *The equilibrium f_∞ minimizes*

$$\left\{ H(f) : f(v) \text{ is positive, } \int f(v) dv = M, \int v^2 f(v) dv = E \right\}$$

with M and E fixed. As γ is convex, this minimizing function is unique. Moreover, given any solution $f(t, v)$ to the Kac model (1.1) with initial data f_0 of mass M and energy E , we have

$$H(f_\infty) \leq H(f)(t) \leq H(f_0), \quad t > 0,$$

and

$$\lim_{t \rightarrow \infty} H(f)(t) = H(f_\infty). \tag{3.7}$$

Let us immediately notice that the proof of the limit in (3.7) is nontrivial even if it is very natural in both math and physics. We will give only some rough ideas for proving (3.7) in the appendix¹.

Before ending this paragraph, we introduce the relative entropy $H(f|f_\infty)$ as

$$H(f|f_\infty) = H(f) - H(f_\infty) = \int [\gamma(f) - \gamma(f_\infty) - \gamma'(f_\infty)(f - f_\infty)] dv,$$

where we used the conservations of mass and energy to get the last equality.

3.2. Decay rate of the entropy production and the relative entropy.

To get the decay rate of the entropy production, we shall study the time derivative of $D(f)$. To simplify notations, we denote $\xi = A_f(t)\partial_v\gamma'(f) + B_f(t)v$. Hence the Kac equation and the entropy production $D(f)$ can be written as

$$\partial_t f = \partial_v[f(1+f)\xi], \quad D(f) = \frac{1}{A_f(t)} \int f(1+f)\xi^2.$$

Then we have

$$\begin{aligned} \frac{d}{dt}D(f) &= -\frac{A'_f(t)}{A_f^2(t)} \int f(1+f)\xi^2 + \frac{1}{A_f(t)} \int (1+2f)\xi^2 \partial_t f + \frac{2}{A_f(t)} \int f(1+f)\xi \partial_t \xi \\ &:= \text{I} + \text{II} + \text{III}. \end{aligned} \tag{3.8}$$

Next, we will calculate these three integrals. Firstly the integral I can be written as

$$\text{I} = -\frac{A'_f(t)}{A_f(t)} D(f).$$

Then the second integral, II, can be calculated as

$$\text{II} = \frac{1}{A_f(t)} \int (1+2f)\xi^2 \partial_v[f(1+f)\xi] = -\frac{2}{A_f(t)} \int \left(f^3 + \frac{3}{2}f^2 + f \right) \xi^2 \partial_v \xi dv.$$

We denote by $\phi(f) = f^3 + \frac{3}{2}f^2 + f$. Then using the expression of $\xi = A_f(t)\partial_v\gamma'(f) + B_f(t)v$, we can rewrite II as

$$\begin{aligned} \text{II} &= -\frac{B_f(t)}{A_f(t)} \int \phi(f)\xi^2 dv + 2 \int \left(\frac{\phi'(f)}{f(1+f)} - 2\frac{\phi(f)(1+2f)}{f^2(1+f)^2} \right) |\xi \partial_v f|^2 \\ &\quad + 4 \int \frac{\phi(f)}{f^2(1+f)^2} \xi \partial_v f \partial_v [f(1+f)\xi]. \end{aligned} \tag{3.9}$$

Finally, from the conservation of mass, $B_f(t) = M$, we get

$$\text{III} = 2\frac{A'_f(t)}{A_f(t)} \int \xi \partial_v f + 2 \int f(1+f)\xi \partial_{tv}^2 \gamma'(f)$$

¹We thank the referee for suggestions.

$$= 2 \frac{A'_f(t)}{A_f(t)} D(f) - 2 \int \frac{1}{f(1+f)} |\partial_v [f(1+f)\xi]|^2.$$

In conclusion, we get the derivative of the entropy dissipation under the form

$$\frac{d}{dt} D(f) = - \frac{B_f(t)}{A_f(t)} \int \phi(f) \xi^2 dv + \frac{A'_f(t)}{A_f(t)} D(f) - \{\dots\},$$

where $\{\dots\}$ denotes some positive term. As $\phi(f) > f(1+f)$, then we get the differential inequality

$$\frac{d}{dt} D(f) \leq \left(\frac{A'_f(t)}{A_f(t)} - B_f(t) \right) D(f). \quad (3.10)$$

Using Proposition 2.3 and then Proposition 2.1, we get

$$\frac{A'_f(t)}{A_f(t)} - B_f(t) \leq 2 \|f(t)\|_{L^2}^2 - B_f(t) \leq 4 \frac{C_1}{C_2} - M, \text{ for all } t \geq t_0 \equiv \frac{\log 2}{2C_1}.$$

Previous computations show that

$$4 \frac{C_1}{C_2} = 4M^2 \cdot \frac{M^3}{C_N^2 E} \cdot \left(1 + \frac{\Lambda^2 M^3}{9E}\right)$$

and

$$t_0 = \frac{\log 2}{2M \left(1 + \frac{\Lambda^2 M^3}{9E}\right)}.$$

Therefore, for all $t \geq t_0$, we get

$$\begin{aligned} \frac{A'_f(t)}{A_f(t)} - B_f(t) &\leq 4M^2 \cdot \frac{M^3}{C_N^2 E} \cdot \left(1 + \frac{\Lambda^2 M^3}{9E}\right) - M \\ &\leq M \left\{ 4M \cdot \frac{M^3}{C_N^2 E} \cdot \left(1 + \frac{\Lambda^2 M^3}{9E}\right) - 1 \right\}. \end{aligned}$$

At this point, we can now use our Assumption (1.4) about the existence of a suitable constant $\delta > 0$, in order to derive the following decay rate about the entropy production

$$D(f)(t) \leq D(f_{t_0}) e^{-\delta t} \text{ for all } t \geq t_0.$$

It remains to estimate $D(f_{t_0})$. This time we use Proposition 1 to get

$$\|f(t)\|_{L^2} \leq 2 \|f_0\|_{L^2}, \text{ for all } t \leq t_0,$$

and therefore by Proposition 2 again, we have

$$\frac{A'_f(t)}{A_f(t)} - B_f(t) \leq 4 \|f_0\|_{L^2} - M, \text{ for all } t \leq t_0.$$

Using the differential inequality (3.10) for the entropy production D , we get

$$D(f)(t) \leq D(f_0) e^{(4\|f_0\|_{L^2} - M)t}, \text{ for all } t \leq t_0,$$

and in particular, we have

$$D(f)(t) \leq D(f_0)e^{(4\|f_0\|_{L^2}-M)t_0}e^{-\delta t} \text{ for all } t \geq t_0.$$

We introduce the notation

$$\tilde{D}(f_0) \equiv D(f_0)e^{(4\|f_0\|_{L^2}-M)t_0}.$$

Then the above inequality can be written as

$$D(f)(t) \leq \tilde{D}(f_0)e^{-\delta t} \text{ for all } t \geq t_0.$$

We use the decay rate of the entropy production $D(f)$ in the entropy equality (3.1) to get

$$\frac{d}{dt}H(f|f_\infty) \geq -\tilde{D}(f_0)e^{-\delta t} \text{ for all } t \geq t_0.$$

Then integrating the above inequality over $]t_1, t_2[\subset]t_0, +\infty[$ gives

$$H(f|f_\infty)(t_2) - H(f|f_\infty)(t_1) \geq \tilde{D}(f_0) \frac{e^{-\delta t_2} - e^{-\delta t_1}}{\delta}.$$

Letting $t_2 \rightarrow +\infty$, and as $\lim_{t \rightarrow \infty} H(f|f_\infty)(t) = 0$, we get finally the decay rate of the relative entropy

$$H(f|f_\infty)(t) \leq \tilde{D}(f_0) \frac{e^{-\delta t}}{\delta} \text{ for all } t \geq t_0.$$

3.3. Decay rate of the L^1 norm and proof of Theorem 1.1. We can finally show the L^1 decay rate of the solution of (1.1) to the equilibrium. For this purpose, observe that there exists a function $y(t, v)$ which takes values between $f(t, v)$ and $f_\infty(v)$ and such that the relative entropy can be written as

$$\begin{aligned} H(f|f_\infty) &= H(f) - H(f_\infty) = \int (\gamma(f) - \gamma(f_\infty) - \gamma'(f_\infty)(f - f_\infty)) dv \\ &= \int \gamma''(y(t, v))(f - f_\infty)^2 dv, \end{aligned}$$

where we used the property of mass and energy conservations together with Taylor's formula in the last two equalities.

As in [5], using Cauchy-Schwartz inequality, we have

$$\begin{aligned} \|f(t) - f_\infty\|_{L^1(f < f_\infty)}^2 &\leq \int_{\{f < f_\infty\}} \frac{1}{\gamma''(y(t, v))} \int_{\{f < f_\infty\}} \gamma''(y(t, v))(f - f_\infty)^2 \\ &\leq \int f_\infty(1 + f_\infty) \int \gamma''(y(t, v))(f - f_\infty)^2 \\ &\leq C H(f|f_\infty) \leq C \tilde{D}(f_0) \frac{e^{-\delta t}}{\delta} \text{ for all } t \geq t_0. \end{aligned}$$

Hence using again the mass conservation, we get the following desired result

$$\|f(t) - f_\infty\|_{L^1(\mathbb{R})} = 2\|f(t) - f_\infty\|_{L^1(f < f_\infty)} \leq C(f_0) \frac{e^{-\delta t/2}}{\sqrt{\delta}} \text{ for all } t \geq t_0.$$

The proof of Theorem 1.1 is completed.

4. Convergence towards equilibria: numerical simulations

Under some conditions on the mass and energy, we have shown that the solution goes exponentially fast to the Bose equilibrium distribution. We will now perform some numerical simulations in this section, to show the equilibrium distributions for different initial states, and the exponential decay of the entropies, in particular extending the range of our main result.

Let us recall that the Bose distribution is given by

$$f_\infty = \frac{1}{\exp(\lambda_1 v^2 - \lambda_2) - 1}.$$

The numerical simulations are carried out for different initial conditions. The first example shows that if the initial data is concentrated near the center, then it will evolve to Bose distribution, with an entropy decaying exponentially fast to some final state.

Example 1. Consider the initial data

$$f_0 = \frac{0.1}{e^{(v-\pi/2)^2+0.1} - 1}. \quad (4.1)$$

The equilibrium distribution and evolution of entropy are shown in Figure 4.1.

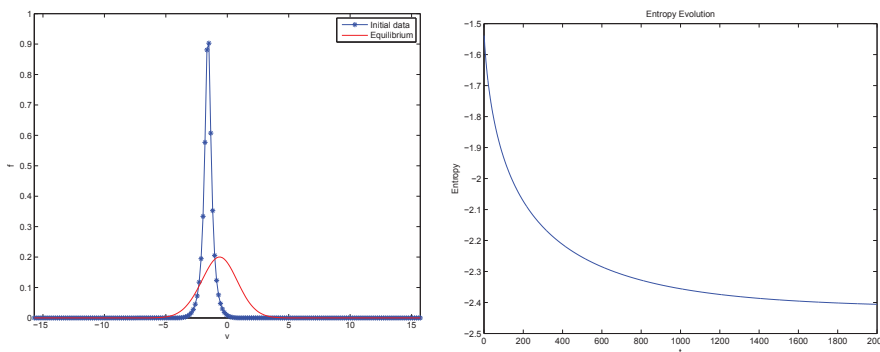


FIG. 4.1. Initial data f_0 in (4.1).

In Example 2, we take 10 times the initial data as in Example 1 and also observe the exponential decay of the entropy, with a different time scale used in the simulation.

Example 2. Consider the initial data

$$f_0 = \frac{1}{e^{(v-\pi/2)^2+0.1} - 1}. \quad (4.2)$$

The equilibrium distribution and evolution of entropy are shown in Figure 4.2.

The Bose distribution behaves clearly like a Gaussian function when $|v|$ is big. The next example shows the evolution of a Gaussian function to a Bose distribution.

Example 3. Consider the initial data

$$f_0 = 5 * e^{-v^2/2}. \quad (4.3)$$

The equilibrium distribution and evolution of entropy are shown in Figure 4.3. The

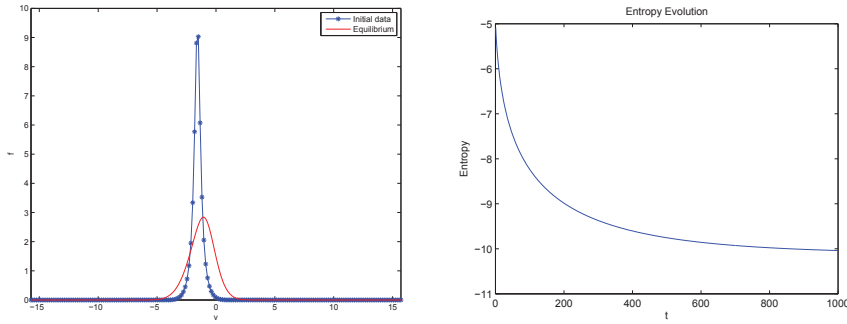


FIG. 4.2. Initial data f_0 in (4.2).

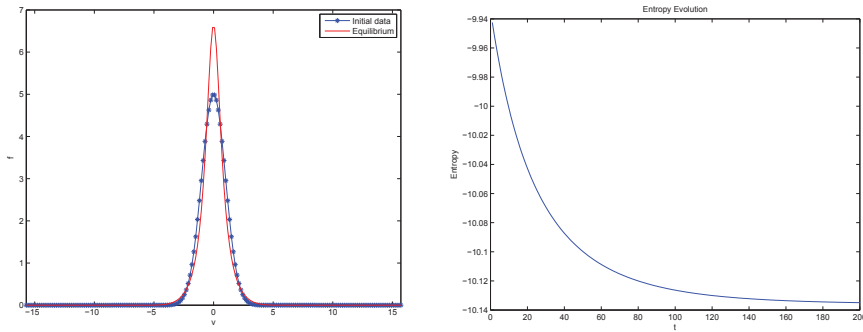


FIG. 4.3. Initial data f_0 in (4.3).

comparison shows that the Bose distribution is more singular near $|v|=0$, but behaves like a Gaussian function for $|v|$ big.

Example 4. Consider the initial data

$$f_0 = 8 * [e^{-(v+\pi/2)^2} + e^{-(v-\pi/2)^2}]. \tag{4.4}$$

The equilibrium distribution and evolution of entropy are shown in Figure 4.4.

The above example shows the evolution of the sum of two Gaussian functions. All the previous examples assumed smooth initial conditions. In the next example, we will consider more general cases with non smooth initial data, with a special focus on a continuous piecewise linear approximation of a characteristic function. Note that in order to prevent oscillation of numerical derivatives, the continuity of the initial data is required.

Example 5. Consider the initial data

$$f_0 = \begin{cases} \frac{5}{2} + \frac{2}{\pi}v & \text{for } v \in [-\frac{5\pi}{4}, 0], \\ \frac{5}{2} - \frac{2}{\pi}v & \text{for } v \in [0, \frac{5\pi}{4}], \\ 0 & \text{for others,} \end{cases} \tag{4.5}$$

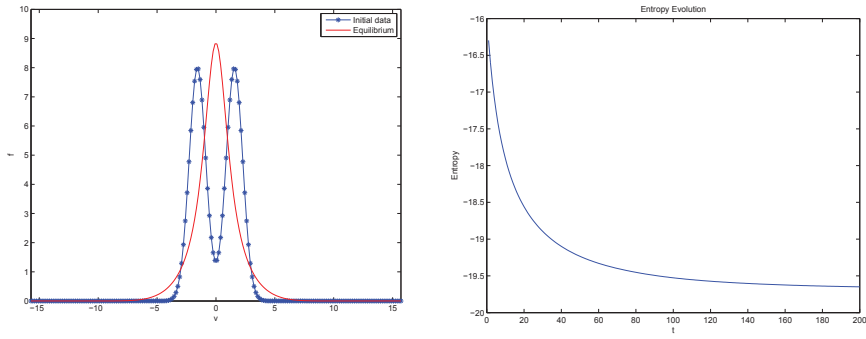


FIG. 4.4. Initial data f_0 in (4.4).

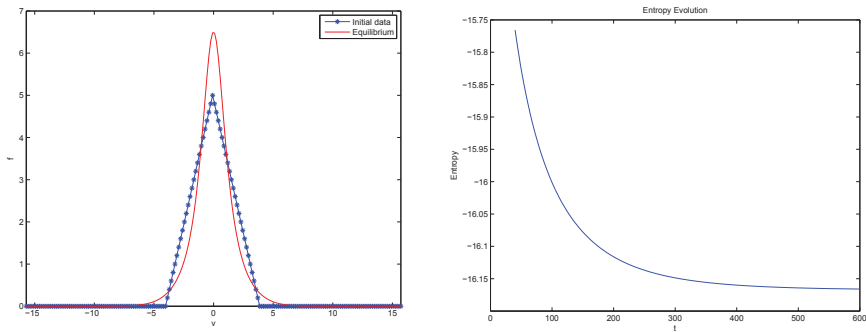


FIG. 4.5. Initial data f_0 in (4.5).

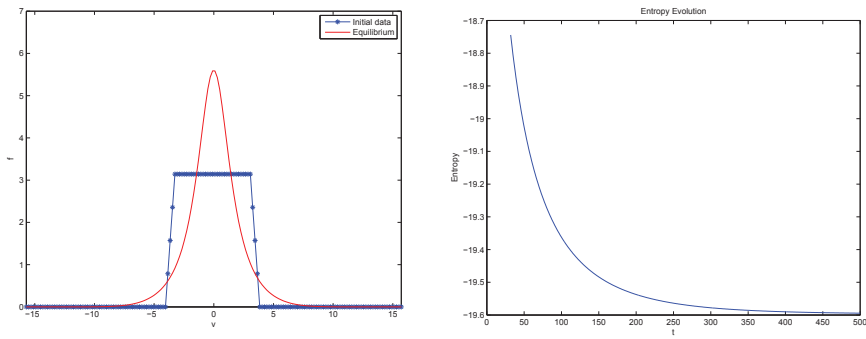


FIG. 4.6. Initial data f_0 in (4.6).

and

$$f_0 = \begin{cases} 4(v + \frac{5}{4}\pi) & \text{for } v \in [-\frac{5}{4}\pi, -\pi], \\ \pi & \text{for } v \in (-\pi, \pi), \\ -4(v - \frac{5}{4}\pi) & \text{for } v \in [\pi, \frac{5}{4}\pi], \\ 0 & \text{for others,} \end{cases} \quad (4.6)$$

respectively.

The equilibrium distributions and evolution of entropy with initial data (4.5) and (4.6) are shown in Figure 4.5 and Figure 4.6, respectively.

Note that this initial data does not belong to the space $L \log L$, since $f_0 = 0$ in some interval thus the entropy is ∞ at first several time steps. This more general case also shows the exponential decay of entropy.

All the above numerical results showed the quick convergence towards to equilibria, especially, we can see the exponential evolution of entropy. The numerical results further elaborate our main result stated in Theorem 1.1.

Appendix A. Sketch of the proof of (3.7) in Lemma 3.2. The proof of the limit (3.7) consists several steps.

Step 1. The limit function $g_\infty(v)$ of $f(t, v)$ as $t \rightarrow \infty$ has the same energy as the equilibrium f_∞ .

To prove this fact, it is enough to show the uniform decay of the energy tail, that is

$$\sup_{t \geq 0} \int_{|v| > R} v^2 f(t, v) dv \rightarrow 0 \quad \text{as } R \rightarrow \infty. \tag{A.1}$$

Recalling the Kac model

$$\partial_t f = A_f(t) \partial_{vv} f + B_f(t) \partial_v (vf(1+f)),$$

with

$$A_f(t) = \int v^2 f(1+f) dv, \quad B_f(t) = \int f dv,$$

then, for any test function $\varphi(v) \in C_c^2(\mathbb{R})$,

$$\begin{aligned} \frac{d}{dt} \int \varphi(v) f(t, v) dv &= A_f(t) \int \varphi(v) \partial_{vv} f + B_f(t) \int \varphi(v) \partial_v (vf(1+f)) \\ &= A_f(t) \int \varphi''(v) f - B_f(t) \int \varphi'(v) (vf(1+f)). \end{aligned}$$

Now we take

$$\phi(v) = e^{-1/v} \quad \text{for } x > 0, \quad \phi(v) = 0 \quad \text{for } x \leq 0,$$

it is easy to see that $\phi \in C^\infty(\mathbb{R})$ is positive and nondecreasing. Let

$$\varphi(v) = \frac{\phi(v^2 - 1)}{\phi(v^2 - 1) + \phi(4 - v^2)},$$

then $\varphi(v) \in C^\infty(\mathbb{R})$ has the following properties:

- (a). $\varphi(v)$ is even and $0 \leq \varphi(v) \leq 1$ for all $v \in \mathbb{R}$;
- (b). $\varphi(v) = 0$ for all $|v| \leq 1$, and $\varphi(v) = 1$ for all $|v| \geq 2$;
- (c). $\varphi'(v) = 0, \varphi''(v) = 0$ for all $|v| < 1$ or $|v| > 2$;
- (d). $v\varphi'(v) \geq 0$ for all $|v| \in \mathbb{R}$.

For any $R \geq 1$, let $\varphi_R(v) = \varphi(\frac{v}{R})$, we have $\varphi'_R(v) = \frac{1}{R}\varphi'(\frac{v}{R})$, $\varphi''_R(v) = \frac{1}{R^2}\varphi''(\frac{v}{R})$, and

$$\varphi'_R(v) = \varphi''_R(v) = 0 \quad \text{for } |v| < R \quad \text{or} \quad |v| > 2R,$$

then

$$\begin{aligned} \frac{d}{dt} \int v^2 \varphi_R(v) f(t, v) dv &= A_f(t) \int (v^2 \varphi_R(v))'' f dv - B_f(t) \int (v^2 \varphi_R(v))' (vf(1+f)) dv \\ &= A_f(t) \int \left(2\varphi\left(\frac{v}{R}\right)(v) + 4\frac{v}{R}\varphi'\left(\frac{v}{R}\right) + \frac{v^2}{R^2}\varphi''\left(\frac{v}{R}\right) \right) f dv \\ &\quad - B_f(t) \int \left(2v^2\varphi\left(\frac{v}{R}\right) + v^2\varphi'\left(\frac{v}{R}\right)\frac{v}{R} \right) f(1+f) dv \\ &=: I + II. \end{aligned}$$

For the first term,

$$\begin{aligned} I &= A_f(t) \int 2\varphi\left(\frac{v}{R}\right)(v) f dv + A_f(t) \int \left(4\frac{v}{R}\varphi'\left(\frac{v}{R}\right) + \frac{v^2}{R^2}\varphi''\left(\frac{v}{R}\right) \right) f dv \\ &= A_f(t) \int_{|v| \geq R} 2\varphi\left(\frac{v}{R}\right)(v) f dv + A_f(t) \int_{R \leq |v| \leq 2R} \left(4\frac{v}{R}\varphi'\left(\frac{v}{R}\right) + \frac{v^2}{R^2}\varphi''\left(\frac{v}{R}\right) \right) f dv \\ &\leq C_1 \int_{|v| \geq R} f(t, v) dv \leq \frac{C_1}{R^2} \int_{|v| \geq R} v^2 f(t, v) dv \leq \frac{C}{R^2}, \end{aligned}$$

where we used the conservation of energy and mass, and the fact that $A_f(t)$ is bounded. Next for the second term, by using property (d) above,

$$II \leq -2B_f(t) \int v^2 \varphi\left(\frac{v}{R}\right) f(1+f) dv \leq -2M \int v^2 \varphi_R(v) f(t, v) dv.$$

Then we have

$$\frac{d}{dt} \int v^2 \varphi_R(v) f(t, v) dv \leq \frac{C}{R^2} - 2M \int v^2 \varphi_R(v) f(t, v) dv.$$

Fixing $R \geq 1$ and setting

$$u(t) = \int v^2 \varphi_R(v) f(t, v) dv,$$

the above inequality can be rewritten as

$$u'(t) + 2Mu(t) \leq \frac{C}{R^2},$$

and so

$$(u(t)e^{2Mt})' = (u'(t) + 2Mu(t))e^{2Mt} \leq \frac{C}{R^2}e^{2Mt},$$

thus

$$u(t)e^{2Mt} = u(0) + \int_0^t (u(s)e^{2Ms})' ds \leq u(0) + \int_0^t \frac{C}{R^2} e^{2Ms} ds = u(0) + \frac{C}{R^2} \frac{e^{2Mt} - 1}{2M},$$

that is

$$u(t) \leq u(0)e^{-2Mt} + \frac{C}{2MR^2}(1 - e^{-2Mt}) \leq \max \left\{ u(0), \frac{C}{2MR^2} \right\}.$$

Recalling the definition of u , it follows that

$$\int v^2 \varphi_R(v) f(t, v) dv \leq \max \left\{ \int v^2 \varphi_R(v) f_0(v) dv, \frac{C}{2MR^2} \right\}.$$

Since $1_{\{|v| \geq 2R\}} \leq \varphi_R(v) \leq 1_{\{|v| \geq R\}}$, we have

$$\int_{\{|v| \geq 2R\}} v^2 f(t, v) dv \leq \max \left\{ \int_{\{|v| \geq R\}} v^2 f_0(v) dv, \frac{C}{2MR^2} \right\},$$

then

$$\sup_{t \geq 0} \int_{\{|v| \geq 2R\}} v^2 f(t, v) dv \leq \max \left\{ \int_{\{|v| \geq R\}} v^2 f_0(v) dv, \frac{C}{2MR^2} \right\} \rightarrow 0 \quad \text{as } R \rightarrow \infty,$$

which gives the proof of (A.1).

Step 2. Use the fact that the limit $\lim_{t \rightarrow \infty} H(f(t))$ exists and is finite and use the entropy dissipation $D(f(t))$ to find a sequence (e.g.) $t_n \in [n, n + 1], n = 1, 2, 3, \dots$ such that $D(f_n) \rightarrow 0$ as $n \rightarrow \infty$, where $f_n(v) = f(t_n, v)$. Then using weak compactness of $\{f_n\}_{n=1}^\infty$ there is subsequence $\{f_{n_k}\}_{k=1}^\infty$ and $0 \leq g_\infty \in L^1_2 \cap L^2$ such that $f_{n_k} \rightharpoonup g_\infty (k \rightarrow \infty)$ weakly in L^1 . Note from Step 1 that g_∞ also has the same energy E as that of f_∞ . Next it is easily proved that $H(f_{n_k}) \rightarrow H(g_\infty) (k \rightarrow \infty)$. Hence using monotone non-increase of $H(f(t))$ one obtains

$$\lim_{t \rightarrow \infty} H(f(t)) = H(g_\infty).$$

Then with the same argument as above one proves that

$$0 \leq H(f(t)) - H(g_\infty) \leq Ce^{-\delta t}, \quad \forall t \geq t_0$$

and

$$\|f(t) - g_\infty\|_{L^1} \leq Ce^{-\delta t/2}, \quad \forall t \geq t_0,$$

in particular, one has

$$\|f_{n_k} - g_\infty\|_{L^1} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Step 3. Choose a further subsequence, still denote it as $\{f_{n_k}\}_{k=1}^\infty$, such that $f_{n_k} \rightarrow g_\infty(v) (k \rightarrow \infty)$ a.e. $v \in \mathbb{R}$. Then using Fatou's Lemma one concludes that $g_\infty \in H^1(\mathbb{R})$:

$$\begin{aligned} \int (1 + \xi^2) |\hat{g}(\xi)|^2 d\xi &\leq \liminf_{k \rightarrow \infty} \int (1 + \xi^2) |\hat{f}_{n_k}(\xi)|^2 d\xi \\ &\leq C_0 \sup_{t \geq 0} [\|f(t)\|_{L^2}^2 + \|\partial_v f(t)\|_{L^2}^2] < \infty. \end{aligned}$$

Here we have supposed the $L^1 \cap L^2$ regularity for higher order derivatives of $f(t, v)$, see Remark 2.2. Then g_∞ is smooth. The higher regularity on g_∞ can be obtained if the

solution $f(t, v)$ has enough regularity and has the uniform bounds with the Sobolev norm of higher orders, then from the structure of $D(f)$ and that $\lim_{k \rightarrow \infty} D(f_{n_k}) = 0$ one can prove that $D(g_\infty) = 0$ which implies that g_∞ is a Bose–Einstein distribution. Since g_∞ also has the same mass and energy M, E as f_∞ , it follows that $g_\infty = f_\infty$. This proves (3.7).

We note here that the proof of the strict positivity of g_∞ is still open. Such strict positivity problems are even not easy for the classical Boltzmann equation for the Maxwellian molecules. For the present model, the main difficulty is that it involves derivatives. Another consideration is to assume that there is a function h on \mathbb{R} satisfying $h(v) > 0$ for all $v \in \mathbb{R}$, such that the solution $f(t, v)$ is bounded below by $h(v)$. This assumption, whatever it can be removed or not, is left as a future research issue.

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