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LARGE TIME BEHAVIOR OF ENTROPY SOLUTIONS TO A UNIPOLAR HYDRODYNAMIC MODEL OF SEMICONDUCTORS*

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Abstract. In this paper, we study the large time behavior of entropy solutions to the one-dimensional unipolar hydrodynamic model for semiconductors in the form of Euler–Poisson equations. First of all, a large time behavior framework for the time-increasing entropy solutions is given. In this framework, the global entropy solutions (which increase slowly with time) are proved to decay exponentially fast to the corresponding stationary solutions. Then, for an application purpose, the existence and time-increasing-rate of the global entropy solutions with large initial data is considered by using a modified fractional step Lax–Friedrichs scheme and the theory of compensated compactness. By using the large time behavior framework, the global entropy solutions are proved to decay exponentially fast to the stationary solutions when the adiabatic index $\gamma > 3$, without any assumption on smallness or regularity for the initial data.

Key words. Compressible Euler equation, entropy solution, large time behavior.

AMS subject classifications. 35M20, 35Q35, 76W05.

1. Introduction

In this paper, we study isentropic Euler–Poisson equations for the unipolar hydrodynamical model of semiconductor device

$$\begin{cases} n_t + J_x = 0, \\ J_t + (\frac{J^2}{n} + p(n))_x = nE - \frac{J}{\tau}, \\ E_x = n - b(x), \end{cases} \tag{1.1}$$

in the region $\Pi_T = \mathbf{R} \times [0,T)$ for any fixed $T > 0$. Here $n \geq 0$, J, and E represent the electron density, electron current density and the electric field, respectively. The pressure-density relation, $p(n)$, satisfies $p(n) = n^\gamma$ $(\gamma > 1)$. The function $b(x) > 0$, called the doping profile, stands for the density of fixed, positively charged background ions. The coefficient τ denotes the relaxation time. Since our interest here is the large time behavior of the solutions rather than the limit of relaxation times, so without loss of generality, we assume throughout this paper

$$\tau = 1. \tag{1.2}$$

The textbook [27] is a good reference for the derivation of the hydrodynamic model of semiconductors.

For the system (1.1), the initial conditions are

$$\begin{cases} n(x,0) = n_0(x) \geq 0, \\ J(x,0) = J_0(x), \end{cases} \tag{1.3}$$

with

$$\lim_{x\to\pm\infty} n_0(x) = n_\pm > 0, \qquad \lim_{x\to\pm\infty} J_0(x) = J_\pm. \tag{1.4}$$

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And the "boundary" condition at far field $x=-\infty$ is

$$\lim_{x\to-\infty} E(x,t)=E_-, \quad \text{for } a.e.\ t\geq 0, \tag{1.5}$$

which describes an outside electric field due to an applied bias. The letters $n_\pm$, $J_\pm$ and E_- are given state constants and we assume $J_+=J_-=E_-=0$. Moreover, we assume the doping profile $b(x)$ satisfies

$$\begin{aligned} &b(x)\in C^2(\mathbf{R}),\ b'(x)\in L^1(\mathbf{R})\cap H^1(\mathbf{R}),\\ &\lim_{x\to\pm\infty} b(x)=n_\pm, \quad b^*=\sup_x b(x)\geq \inf_x b(x)=b_*>0. \end{aligned} \tag{1.6}$$

As which was stated in the author's other paper [17, 18], the boundary condition (1.5), or replaced by

$$\lim_{x\to+\infty} E(x,t)=E^+, \quad \text{for } a.e.\ t\geq 0,$$

is necessary and natural. Otherwise, the far field state functions $J(\pm\infty,t)$ and $E(\pm\infty,t)$ will be underdetermined.

Recently, many efforts were made for the system (1.1) and its steady-state model

$$\begin{cases} \tilde{J}_x=0,\\ (\dfrac{\tilde{J}^2}{\tilde{n}}+p(\tilde{n}))_x=\tilde{n}\tilde{E}-\dfrac{\tilde{J}}{\tau},\\ \tilde{E}_x=\tilde{n}-b(x). \end{cases} \tag{1.7}$$

The mathematical study was initiated by Degond and Markowich in [4] for the steady-state hydrodynamic model (1.7) with bounded domain. In [4], the authors discussed the uniqueness of subsonic stationary solutions for sufficiently large τ. Then, some other kinds of subsonic, transonic, and supersonic solutions are obtained, cf. [1, 5, 8, 25, 28, 31] *et al*. On the other hand, for the time-dependent hydrodynamic model (1.1), there are many results on the existence and asymptotic behavior of smooth solutions (see [10, 11, 12, 17, 18, 19, 20, 24, 29] for example). Yet, Wang and Chen in [32] proved that for smooth initial data with large C^1 norm, the corresponding solution of Euler–Poisson system will develop singularities in finite time. Therefore, it is important to consider weak solutions of the system (1.1). As far as weak solutions are concerned, Zhang [34], Marcati, and Natalini [26] investigated the global existence of entropy solutions to the initial-boundary value problem and the Cauchy problem, respectively. Yet, Marcati and Natalini's result [26], which need the initial data to be finite mass, cannot be used to problem (1.1)–(1.5). As to the large time behavior of entropy solutions, Huang, Pan, and Yu [16] proved the uniformly bounded entropy solutions, which satisfy

$$0\leq n(x,t)\leq C_0, \quad \text{for } a.e.\ (x,t)\in\Pi_T, \tag{1.8}$$

decay exponentially to the stationary solutions (the subsonic solution of (2.4), (2.5)) with the smallness assumption on the amplitude of $\bar{J}$. In my other paper [33], I generalized the result of [16] to the isothermal Euler–Poisson equations with spherical symmetry. However, condition (1.8) is stiff and still be an open problem for L^∞ entropy solutions, although it seems natural from physical point of view. For example, the bounds obtained in [34] and [26] grow with time. In this paper, instead of proving the difficult result (1.8), we will first give a large time behavior framework for time-increasing entropy solutions. In this framework, by using energy method and entropy dissipation estimate, any global entropy solutions of (1.1)–(1.5) satisfying

$$0 \le n(x,t) \le Mt^{\alpha}, \quad \text{for constant } M>0 \text{ and } 0\le \alpha<2 \tag{1.9}$$

are proved to decay exponentially fast to the corresponding stationary solutions, too. For an application purpose, we consider the existence and time-increasing-rate of the global entropy solutions by using a modified fractional step Lax–Friedrichs scheme [13] and the theory of compensated compactness[6, 9, 22, 23]. And then, from the large time behavior framework, we conclude the obtained global entropy solutions decay exponentially fast to the stationary solutions when the adiabatic index $\gamma>3$, without any assumption on smallness or regularity for the initial data.

Now, we give the definition of entropy solution. Consider the locally bounded measurable functions $n(x,t)$, $J(x,t)$, $E(x,t)$, where $E(x,t)$ is absolutely continuous in x, a.e. in t.

Definition 1.1. *The vector function (n,J,E) is said to be a weak solution of problem (1.1)–(1.5), if it satisfies the system* (1.1) *(with $\tau=1$) in the distributional sense, verifies the initial and limiting restrictions (1.3)–(1.5). Furthermore, a weak solution of system (1.1)–(1.5) is called an entropy solution if it satisfies the entropy inequality*

$$\partial_t \eta + \partial_x q - \partial_J \eta (nE-J) \le 0, \quad \text{in} \quad \mathcal{D}', \tag{1.10}$$

where (η,q) are mechanical entropy-entropy flux pair satisfying

$$\eta(n,J) = \frac{1}{2}\frac{J^2}{n} + \frac{n^\gamma}{\gamma-1}, \ q(n,J) = \frac{1}{2}\frac{J^3}{n^2} + \frac{\gamma}{\gamma-1} n^{\gamma-1} J. \tag{1.11}$$

Notation. *Throughout this paper, C_0, C_i, et al always denote some specific positive constants, and C, $\tilde{C}$ denotes the generic positive constant.*

The organization of this paper is as follows. In Section 2, some preliminaries, including the existence of stationary solutions, are given. In Section 3, the large time behavior framework for time-increasing entropy solutions is considered. As an application of this framework, the existence, time-increasing-rate and large time behavior of entropy solutions to problem (1.1)–(1.5) are considered in Section 4.

2. Preliminaries

In this section, we will give some preliminaries which will be used later. We first give an inequality, which is important in the proof of our main results:

Lemma 2.1. *For $n\ge 0, \tilde{n}>0, \gamma\ge 1$, we have*

$$n^\gamma - \tilde{n}^\gamma - \gamma \tilde{n}^{\gamma-1}(n-\tilde{n}) \le \frac{\gamma}{\tilde{n}}(n^\gamma - \tilde{n}^\gamma)(n-\tilde{n}). \tag{2.1}$$

Proof. If $n=\tilde{n}$ or $\gamma=1$, the result is obvious. Otherwise, noticing

$$n^\gamma - \tilde{n}^\gamma - \gamma \tilde{n}^{\gamma-1}(n-\tilde{n}) = \gamma(\gamma-1)(n-\tilde{n})^2 \int_0^1 \Big(\tilde{n} + \theta(n-\tilde{n})\Big)^{\gamma-2}(1-\theta)d\theta, \tag{2.2}$$

and

$$\int_0^1 \Big(\tilde{n} + \theta(n-\tilde{n})\Big)^{\gamma-2}(1-\theta)d\theta \le \int_0^1 \Big(\tilde{n} + \theta(n-\tilde{n})\Big)^{\gamma-2} d\theta$$

$$
\begin{aligned}
&= \frac{1}{\gamma-1}\Big(\frac{n^{\gamma-1}-\tilde{n}^{\gamma-1}}{n-\tilde{n}}\Big) = \frac{1}{\gamma-1}\Big(\frac{1}{\tilde{n}}\frac{n^\gamma-\tilde{n}^\gamma}{n-\tilde{n}} - \frac{n^{\gamma-1}}{\tilde{n}}\Big)\\
&\le \frac{1}{\gamma-1}\frac{1}{\tilde{n}}\frac{n^\gamma-\tilde{n}^\gamma}{n-\tilde{n}},
\end{aligned}
\tag{2.3}
$$

we get our result. □

REMARK 2.1. Compared with the result of Lemma 5.2 in [30], this one can be used to time-increasing density $n(x,t)$.

Next, we consider the steady solution of Equation (1.1)

$$
\begin{cases}
\tilde{J}_x = 0,\\
(\frac{\tilde{J}^2}{\tilde{n}} + p(\tilde{n}))_x = \tilde{n}\tilde{E} - \tilde{J},\\
\tilde{E}_x = \tilde{n} - b(x),
\end{cases}
\tag{2.4}
$$

under the conditions

$$
\tilde{n}(x) - b(x) \in H^1(\mathbf{R}), \quad \tilde{J} = \bar{J} = 0, \quad \tilde{E}(-\infty) = E_- = 0. \tag{2.5}
$$

Here, we concern the classical solutions in the region where the subsonic condition

$$
\inf_x (p'(\tilde{n}) - \frac{\bar{J}^2}{\tilde{n}}) > 0
$$

and the positivity of the density $\inf_x \tilde{n}(x) > 0$ holds, then (2.4), (2.5) becomes

$$
\begin{cases}
\tilde{n}\tilde{E} - p(\tilde{n})_x = 0\\
\tilde{E}_x = \tilde{n} - b(x)
\end{cases}
\tag{2.6}
$$

$$
\tilde{n}(x) - b(x) \in H^1(\mathbf{R}), \quad \tilde{E}(-\infty) = E_- = 0. \tag{2.7}
$$

The correlative system

$$
\begin{cases}
(\tilde{n}\tilde{E} - p(\tilde{n})_x)_x = 0\\
\tilde{E}_x = \tilde{n} - b(x)
\end{cases}
\tag{2.8}
$$

with constrain (2.7) was considered in [24]. Using the similar method, we have the following theorem:

THEOREM 2.1. *Assume that (1.6) holds, then problem (2.6), (2.7) has a unique solution $(\tilde{n},\tilde{E})$, such that*

$$
b_* \le \tilde{n}(x) \le b^*, \quad x \in \mathbf{R}.
$$

REMARK 2.2. There are some other properties of the stationary solution, see [24] for details. Here we only give the one related to this paper.

To prove the existence of entropy solutions, we introduce some basic facts about the homogeneous compressible Euler equation

$$
\begin{cases}
\partial_t n + \partial_x J = 0,\\
\partial_t J + \partial_x\big(\frac{J^2}{n} + p(n)\big) = 0,
\end{cases}
\tag{2.9}
$$

which can be rewritten in the following form:

$$v_t + f(v)_x = 0, \tag{2.10}$$

where $v = (n,J)^T$, $f(v) = (J, \frac{J^2}{n} + p(n))^T$. The eigenvalues of (2.9) are

$$\lambda_1 = \frac{J}{n} - n^\theta, \quad \lambda_2 = \frac{J}{n} + n^\theta, \tag{2.11}$$

and the Riemann invariants are

$$w = \frac{J}{n} + \frac{n^\theta}{\theta}, \quad z = \frac{J}{n} - \frac{n^\theta}{\theta}, \tag{2.12}$$

where $\theta = \frac{\gamma - 1}{2}$. The classical Riemann problem for (2.9) is

$$\begin{cases} (2.9), & t > 0, \quad x \in \mathbf{R}, \\ (n,J)|_{t=0} = \begin{cases} (n_l, J_l), & x < 0, \\ (n_r, J_r), & x > 0, \end{cases} \end{cases} \tag{2.13}$$

where n_l, J_l, n_r, and J_r are constants satisfying

$$0 \le n_l, \ n_r, \ \left|\frac{J_l}{n_l}\right|, \ \left|\frac{J_r}{n_r}\right| < \infty.$$

There are two distinct types of rarefaction waves and shock waves, which are labeled 1-rarefaction or 2-rarefaction waves and 1-shock or 2-shock waves, respectively. The following two Lemmas are elementary, we only state the results. For more details, see [3].

Lemma 2.2. *There exists a global weak solution of Riemann problem (2.13) which is piecewise smooth function satisfying*

$$\begin{aligned} & w(x,t) \equiv w(n(x,t), J(x,t)) \le \max\{w(n_l, J_l), w(n_r, J_r)\}, \\ & z(x,t) \equiv z(n(x,t), J(x,t)) \ge \min\{z(n_l, J_l), z(n_r, J_r)\}, \\ & w(x,t) \ge z(x,t). \end{aligned} \tag{2.14}$$

This means if the Riemann data lies in $\bigwedge = \{(n,J) : w \le w_0, z \ge z_0, w \ge z\}$, then the Riemann solution of (2.13) lies in $\bigwedge$, as well.

Lemma 2.3. *If* $\{(n,J) : a \le x \le b\} \subset \bigwedge$, *then*

$$\left(\frac{1}{b-a} \int_a^b n dx, \frac{1}{b-a} \int_a^b J dx \right) \in \bigwedge. \tag{2.15}$$

The solutions of problem (2.13) are the building blocks of our approximate solutions, which are constructed by the Lax–Friedrichs scheme with operator splitting.

3. Large time behavior framework of time-increasing entropy solutions

In this part, we will give the large time behavior framework for any entropy solutions satisfying

$$0 \le n(x,t) \le M t^\alpha, \quad \text{for constant } M > 0 \text{ and } 0 \le \alpha < 2. \tag{3.1}$$

As expected, the entropy inequality will play an important role in our analysis. For this purpose, we introduce the new variables:

$$\begin{aligned}\eta^* &= \frac{J^2}{2n} + \frac{n^\gamma}{\gamma-1} - \frac{\tilde{n}^\gamma}{\gamma-1} - \frac{\gamma}{\gamma-1}\tilde{n}^{\gamma-1}(n-\tilde{n}) \\ &= \eta - \frac{\tilde{n}^\gamma}{\gamma-1} - \frac{\gamma}{\gamma-1}\tilde{n}^{\gamma-1}(n-\tilde{n}),\end{aligned} \tag{3.2}$$

$$\begin{aligned}q^* &= \frac{J^3}{2n^2} + \frac{\gamma}{\gamma-1}n^{\gamma-1}J - \frac{\gamma}{\gamma-1}\tilde{n}^{\gamma-1}J \\ &= q - \frac{\gamma}{\gamma-1}\tilde{n}^{\gamma-1}J,\end{aligned} \tag{3.3}$$

where η and q are the entropy-entropy flux pair defined in (1.11). From this definition, η^* is non-negative for any $(x,t)\in\mathbf{R}\times\mathbf{R}^+$. The following theorem is our main result in this part.

THEOREM 3.1 (Large time behavior framework). *Suppose (n,J,E) is any entropy solution of problem (1.1)–(1.5) satisfying (3.1), $(\tilde{n},\tilde{E})$ is the corresponding stationary solution obtained in Theorem 2.1 If*

$$(E-\tilde{E})(x,0)\in L^2(\mathbf{R}), \quad \eta^*(x,0)\in L^1(\mathbf{R}), \tag{3.4}$$

and

$$\int_{-\infty}^{+\infty}\Big(n_0(s)-\tilde{n}(s)\Big)ds = 0 \ for\ any\ t>0, \tag{3.5}$$

then there exist positive constants $T(\alpha)$, C, and $\tilde{C}$ such that

$$\begin{aligned}&\int_{-\infty}^{+\infty}\Big((E-\tilde{E})^2(x,t)+\eta^*(x,t)\Big)dx \\ &\le Ce^{-\tilde{C}t^{\frac{2-\alpha}{2}}}\int_{-\infty}^{+\infty}\Big((E-\tilde{E})^2(x,0)+\eta^*(x,0)\Big)dx\end{aligned} \tag{3.6}$$

satisfies for any $t>T(\alpha)$.

Proof. Introduce the new variable

$$y(x,t) = -\int_{-\infty}^{x}\Big(n(s,t)-\tilde{n}(s)\Big)ds = -(E-\tilde{E}), \quad x\in\mathbf{R}. \tag{3.7}$$

Obviously, y is absolutely continuous in x for a.e. $t>0$. Then, noticing the conservation of mass, we have

$$y_x = -(n-\tilde{n}), \quad y_t = J. \tag{3.8}$$

Of course, $y(-\infty)=y(+\infty)=0$ follows from (3.5). From (1.1), (2.4), and (3.8), we get y admits the equation

$$y_{tt} + (\frac{J^2}{n})_x + (n^\gamma - \tilde{n}^\gamma)_x + y_t = -\tilde{n}y - \tilde{E}y_x + yy_x. \tag{3.9}$$

Multiplying y with (3.9) and integrating over $(-\infty,+\infty)$, we get

$$\begin{aligned}&\frac{d}{dt}\int_{-\infty}^{+\infty}(yy_t+\frac{1}{2}y^2)dx+\int_{-\infty}^{+\infty}\left((n^\gamma-\tilde{n}^\gamma)(n-\tilde{n})+(\tilde{n}-\frac{\tilde{E}_x}{2})y^2\right)dx\\&\le\int_{-\infty}^{+\infty}y_t^2dx+\int_{-\infty}^{+\infty}\frac{y_t^2}{n}y_x\,dx,\end{aligned}\tag{3.10}$$

that is,

$$\frac{d}{dt}\int_{-\infty}^{+\infty}(yy_t+\frac{1}{2}y^2)dx+\int_{-\infty}^{+\infty}\Big((n^\gamma-\tilde{n}^\gamma)(n-\tilde{n})\Big)dx+b_*\int_{-\infty}^{+\infty}y^2dx\le\int_{-\infty}^{+\infty}\frac{\tilde{n}}{n}y_t^2dx.\tag{3.11}$$

On the other hand, noticing the definition of η^* and q^* in (3.2) and (3.3) the following entropy inequality holds in the sense of distribution:

$$\begin{aligned}\eta_t^*+q_x^*&=\eta_t+q_x-\frac{\gamma}{\gamma-1}\tilde{n}^{\gamma-1}(n-\tilde{n})_t-\frac{\gamma}{\gamma-1}(\tilde{n}^{\gamma-1}J)_x\\&\le JE-\frac{J^2}{n}-\frac{\gamma}{\gamma-1}\tilde{n}^{\gamma-1}(n-\tilde{n})_t-\frac{\gamma}{\gamma-1}(\tilde{n}^{\gamma-1}J)_x\\&=JE-\frac{J^2}{n}-\gamma\tilde{n}^{\gamma-2}y_t\tilde{n}_x.\end{aligned}\tag{3.12}$$

Noticing

$$\begin{aligned}JE&=J\tilde{E}+J(E-\tilde{E})=y_t\tilde{E}-yy_t\\&=\frac{P(\tilde{n})_x}{\tilde{n}}y_t-yy_t=\gamma\tilde{n}^{\gamma-2}y_t\tilde{n}_x-yy_t,\end{aligned}\tag{3.13}$$

and using the theory of divergence-measure fields [2], we have

$$\frac{d}{dt}\int_{-\infty}^{+\infty}(\eta^*+\frac{1}{2}y^2)dx+\int_{-\infty}^{+\infty}\frac{y_t^2}{n}dx\le 0.\tag{3.14}$$

Let $\lambda(t)=\sqrt{M}t^{\frac{\alpha}{2}}+b^*+2=\tilde{M}t^{\frac{\alpha}{2}}+b^*+2$, where the constant $M=\tilde{M}^2$ are the same as which in (3.1). It is easy to see that $\lambda(t)\ge\sup_x\sqrt{n(x,t)}+b^*+2$. Multiplying (3.14) by $\lambda(t)$ and adding the result to (3.11) to obtain

$$\begin{aligned}&\frac{d}{dt}\int_{-\infty}^{+\infty}\left(\lambda\eta^*+\frac{\lambda}{2}y^2+\frac{1}{2}y^2+yy_t\right)dx-\frac{\tilde{M}\alpha}{2}t^{\frac{\alpha}{2}-1}\int_{-\infty}^{+\infty}(\eta^*+\frac{y^2}{2})dx\\&+\int_{-\infty}^{+\infty}\Big((n^\gamma-\tilde{n}^\gamma)(n-\tilde{n})\Big)dx+b_*\int_{-\infty}^{+\infty}y^2dx\\&+\int_{-\infty}^{+\infty}(\lambda-\tilde{n})\frac{y_t^2}{n}dx\le 0.\end{aligned}\tag{3.15}$$

In view of (3.2), $\alpha<2$, and Lemma 2.1, we have

$$\begin{aligned}&\frac{1}{2}\left[\int_{-\infty}^{+\infty}\Big((n^\gamma-\tilde{n}^\gamma)(n-\tilde{n})\Big)dx+b_*\int_{-\infty}^{+\infty}y^2dx+\int_{-\infty}^{+\infty}(\lambda-\tilde{n})\frac{y_t^2}{n}dx\right]\\&\ge\frac{\tilde{M}\alpha}{2}t^{\frac{\alpha}{2}-1}\int_{-\infty}^{+\infty}\left(\eta^*+\frac{y^2}{2}\right)dx\end{aligned}\tag{3.16}$$

for $t > t_* = \max\{1, (\frac{2b_*}{\tilde{M}\alpha})^{\frac{2}{\alpha-2}}, (\frac{b_*}{\gamma\tilde{M}\alpha})^{\frac{2}{\alpha-2}}\}$. Since we consider the large time behavior, without loss of generality, we assume $t > t_*$. Then (3.15) becomes

$$\frac{d}{dt}\int_{-\infty}^{+\infty}\Big(\lambda\eta^* + \frac{\lambda}{2}y^2 + \frac{1}{2}y^2 + yy_t\Big)dx + \frac{1}{2}\int_{-\infty}^{+\infty}\Big((n^\gamma - \tilde{n}^\gamma)(n-\tilde{n})\Big)dx$$
$$+\frac{b_*}{2}\int_{-\infty}^{+\infty} y^2 dx + \frac{1}{2}\int_{-\infty}^{+\infty}(\lambda - \tilde{n})\frac{y_t^2}{n}dx \le 0. \tag{3.17}$$

Again, from Lemma 2.1, we have

$$\begin{aligned}
&\lambda\eta^* + \frac{1+\lambda}{2}y^2 + yy_t\\
\le& \frac{\sqrt{n}y^2}{2} + \frac{y_t^2}{2\sqrt{n}} + \frac{\lambda y_t^2}{2n} + \frac{y^2}{2} + \frac{\lambda}{\gamma-1}\Big(n^\gamma - \tilde{n}^\gamma - \gamma\tilde{n}^{\gamma-1}(n-\tilde{n})\Big) + \frac{\lambda}{2}y^2\\
\le& \Big(\frac{2\lambda+1}{2b_*} + \frac{\lambda+1}{2} + \frac{\gamma\lambda}{b_*(\gamma-1)}\Big)\Big(b_* y^2 + (\lambda-\tilde{n})\frac{y_t^2}{n} + (n^\gamma - \tilde{n}^\gamma)(n-\tilde{n})\Big).
\end{aligned} \tag{3.18}$$

Then (3.17) and (3.18) imply that there exist two positive constant C_1 and C_2 depending only on b_* and γ such that

$$\frac{d}{dt}\int_{-\infty}^{+\infty}\Big(\lambda\eta^* + \frac{1+\lambda}{2}y^2 + yy_t\Big)dx + \frac{1}{C_1+\lambda C_2}\int_{-\infty}^{+\infty}\Big(\lambda\eta^* + \frac{1+\lambda}{2}y^2 + yy_t\Big)dx \le 0, \tag{3.19}$$

that is, there exists a positive constant C_3 such that

$$\frac{d}{dt}\int_{-\infty}^{+\infty}\Big(\lambda\eta^* + \frac{1+\lambda}{2}y^2 + yy_t\Big)dx + \frac{1}{C_3}t^{-\frac{\alpha}{2}}\int_{-\infty}^{+\infty}\Big(\lambda\eta^* + \frac{1+\lambda}{2}y^2 + yy_t\Big)dx \le 0 \tag{3.20}$$

satisfies for $t > t_*$. Noticing the relationship

$$\begin{aligned}
&\lambda\eta^* + \frac{1+\lambda}{2}y^2 + yy_t\\
\ge& -\frac{\sqrt{n}y^2}{2} - \frac{y_t^2}{2\sqrt{n}} + \frac{(1+\lambda)y^2}{2} + \frac{\lambda y_t^2}{2n} + \frac{\lambda}{\gamma-1}\Big(n^\gamma - \tilde{n}^\gamma - \gamma\tilde{n}^{\gamma-1}(n-\tilde{n})\Big)\\
\ge& \frac{y^2}{2} + \eta^* \ge 0,
\end{aligned} \tag{3.21}$$

let

$$F(x,t) = \lambda\eta^* + \frac{1+\lambda}{2}y^2 + yy_t,$$

then (3.20) becomes

$$\frac{d}{dt}\Big(\ln\int_{-\infty}^{+\infty}F(x,t)dx\Big) + \frac{1}{C_3}t^{-\frac{\alpha}{2}} \le 0. \tag{3.22}$$

Integrating (3.22) from 0 to t, we have

$$\int_{-\infty}^{+\infty}F(x,t)dx \le \int_{-\infty}^{+\infty}F(x,0)dxe^{-C_4 t^{\frac{2-\alpha}{2}}}, \tag{3.23}$$

where $C_4=\frac{2}{C_3(2-\alpha)}$. Since (3.21) and

$$F(x,t)\le(2\lambda+1)(y^2+\eta^*),$$

there exist two positive constants C_5 and C_6 such that

$$\int_{-\infty}^{+\infty}(y^2+\eta^*)dx\le C_6\int_{-\infty}^{+\infty}[y^2(x,0)+\eta^*(x,0)]dxe^{-C_5t^{\frac{2-\alpha}{2}}}. \tag{3.24}$$

Then, we finish the proof of Theorem 3.1. □

4. The existence and time-increasing-rate of entropy solutions

To utilize the large time behavior framework obtained in Section 3, we consider the existence and time-increasing-rate of entropy solutions to problem (1.1)–(1.5). In [26], Marcati and Natalini studied the existence of entropy solution for the Cauchy problem (1.1) with finite initial mass, i.e. $n_0(x)\in L^1(\mathbf{R}),b(x)\in L^1(\mathbf{R})$. However, the initial data of problem (1.1)–(1.5) do not satisfy the finite mass condition. Huang, Li, and Yu considered this problem at the end of paper [13]. In order to get the growth estimates of entropy solution and the completeness result of this paper, we briefly introduce the process.

Step 1: *Construction of approximate solutions in the region* $\Pi_T,T>0$.

Let the space mesh length be $l=2^{-s}$, where s is a positive integer, and the time mesh length be h such that the ratio $\frac{h}{l}$ is equal to some fixed constant depending on T. This will make the CFL condition satisfy and guarantee the stability of the numerical schemes. Given a partition of the strip Π_T, for any $T>0$

$$S_k=\{(x,t)|kh\le t<(k+1)h\},\quad k\ge 0, \tag{4.1}$$

set

$$I_k=\{j|k+j \text{ is even}, -\frac{1}{l^2}-k\le j\le\frac{1}{l^2}+k\}, \tag{4.2}$$

and

$$Q_{kj}:=S_k\cap\{(x,t)|(j-1)l\le x<(j+1)l;j+k \text{ is even}\}. \tag{4.3}$$

Let

$$n_0^l(x)=\begin{cases}0, & x<-\frac{1}{l}\quad\text{or}\quad x\ge\frac{1}{l},\\ n_0, & -\frac{1}{l}\le x<\frac{1}{l},\end{cases}\qquad J_0^l(x)=\begin{cases}0, & x<-\frac{1}{l}\quad\text{or}\quad x\ge\frac{1}{l},\\ J_0, & -\frac{1}{l}\le x<\frac{1}{l},\end{cases} \tag{4.4}$$

$$b^l(x)=\begin{cases}b(x), & -\frac{1}{l}\le x<\frac{1}{l},\\ 0, & \text{for others } x.\end{cases} \tag{4.5}$$

Set

$$v^l(x,0)=\frac{1}{2l}\int_{(j-2)l}^{jl}v_0^l(x)dx,\qquad j \text{ is even},x\in[(j-2)l,jl). \tag{4.6}$$

For $(x,t)\in Q_{0j}$, define

$$v^l(x,t)=v_R^l(x,t)+H\Big(v_R^l(x,t),E_R^l(x,t)\Big)t, \tag{4.7}$$

where

$$H\Big(v_R^l(x,t),E_R^l(x,t)\Big)=(0,n_R^lE_R^l-J_R^l)^T(x,t),$$

and $v_R^l(x,t)=(n_R^l,J_R^l)(x,t)^T$ are the solutions of (2.9) with the initial data

$$v_0^l(x)=\begin{cases} v^l((j-1)l,0), & x<jl,\\ v^l((j+1)l,0), & x>jl,\end{cases} \tag{4.8}$$

and

$$E_R^l(x,t)=\begin{cases} E_-, \quad x<-\frac{1}{l}-l,\\ E_+=\int_{-\infty}^{+\infty}\Big(n_R^l(y,t)-b^l(y)\Big)dy+E_-, \quad x\geq\frac{1}{l}+l,\\ \int_{-\frac{1}{l}-l}^{x}\Big(n_R^l(y,t)-b^l(y)\Big)dy+E_-, \quad -\frac{1}{l}-l\leq x<\frac{1}{l}+l.\end{cases} \tag{4.9}$$

Suppose that $v^l(x,t)=(n^l,J^l)^T(x,t)$ and $E_R^l(x,t)$ have been defined for $t<kh$, set

$$v^l(x,kh)=\frac{1}{2l}\int_{(j-1)l}^{(j+1)l}v^l(x,kh-0)dx, \qquad x\in[(j-1)l,(j+1)l),j+k=odd, \tag{4.10}$$

$$E_R^l(x,kh)=E_R^l(x,kh-0), \tag{4.11}$$

and define

$$v^l(x,t)=v_R^l(x,t)+H\Big(v_R^l(x,t),E_R^l(x,t)\Big)(t-kh), \qquad (x,t)\in Q_{k,j}, \tag{4.12}$$

where $v_R^l(x,t)=(n_R^l,J_R^l)^T(x,t)$ are the solutions of (2.9) with the initial data

$$v_k^l(x)=\begin{cases} v^l((j-1)l,kh), & x<jl,\\ v^l((j+1)l,kh), & x>jl,\end{cases} \tag{4.13}$$

and $E_R^l(x,t)$

$$=\begin{cases} E_-, & x<-\frac{1}{l}-(k+1)l,\\ \int_{-\frac{1}{l}-(k+1)l}^{x}\Big(n_R^l(y,t)-b^l(y)\Big)dy+E_-, & -\frac{1}{l}-(k+1)l\leq x<\frac{1}{l}+(k+1)l,\\ E_+, & x\geq\frac{1}{l}+(k+1)l.\end{cases} \tag{4.14}$$

REMARK 4.1. From the construction procedure above, there are two parts with simple approximate solutions in the strip Π_T:

(1) $\bar{n}^l=0,\ \bar{J}^l=0,\ E_R^l=E_-,$ when $(x,t)\in Q_{kj},\ j<-\frac{1}{l^2}-k;$
(2) $\bar{n}^l=0,\ \bar{J}^l=0,\ E_R^l=E_+,$ when $(x,t)\in Q_{kj},\ j>\frac{1}{l^2}+k.$

Step 2: *L^∞ boundedness of the approximation solutions.*

In this step, a key point is to prove the uniform boundedness of E_R^l with respect to l. From the construction procedure, for any $(k-1)h\le t<kh$, $-\frac{1}{l}-kl\le x<\frac{1}{l}+kl$, we have

$$E_R^l(x,t)=E_-+\int_{-\frac{1}{l}-kl}^{x}\Big(n_R^l(x,t)-b^l(x)\Big)dx. \tag{4.15}$$

It is easy to see that there exists $j\in Z$ such that $k+j$ is even, $(j-2)l\le x<jl$. Then there exist two constants M,N such that

$$\begin{aligned}E_R^l(x,t)&\le E_-+\int_{-\frac{1}{l}-kl}^{jl}n_R^l(x,t)dx-\int_{-\frac{1}{l}-kl}^{x}b^l(x)dx\\&\le E_-+\int_{-\frac{1}{l}-(k+1)l}^{(j+1)l}n_R^l(x,(k-1)h+0)dx-\int_{-\frac{1}{l}}^{x}b^l(x)dx\\&\le E_-+\int_{-\frac{1}{l}}^{(j+k)l}n_R(x,0)dx-\int_{-\frac{1}{l}}^{x}b^l(x)dx\\&\le E_-+\int_{-\frac{1}{l}}^{x}\Big(n_0-b(x)\Big)dx+\int_{(j-2)l}^{(j+k)l}n_0dx\\&\le C+\|n_0\|_{L^\infty}Mt\le Nt,\end{aligned} \tag{4.16}$$

and

$$\begin{aligned}E_R^l(x,t)&\ge E^-+\int_{-\frac{1}{l}-kl}^{(j-2)l}n_R^l(x,t)dx-\int_{-\frac{1}{l}-kl}^{x}b^l(x)dx\\&\ge E^-+\int_{-\frac{1}{l}-(k-1)l}^{(j-3)l}n_R^l(x,(k-1)h+0)dx-\int_{-\frac{1}{l}}^{x}b^l(x)dx\\&\ge E^-+\int_{-\frac{1}{l}}^{(j-2-k)l}n_R^l(x,0)dx-\int_{-\frac{1}{l}}^{x}b^l(x)dx\\&\ge E^-+\int_{-\frac{1}{l}}^{(j-2-k)l}\Big(n_0-b(x)\Big)dx-\int_{(j-2-k)l}^{jl}b^l(x)dx\\&\ge -C-\|b\|_{L^\infty}Mt\ge -Nt,\end{aligned} \tag{4.17}$$

where we used the fact

$$\int_{il}^{jl}n_R^l(x,kh-0)dx\le\int_{\frac{l}{h}(t-(k-1)h)+(i-1)l}^{-\frac{l}{h}(t-(k-1)h)+(j+1)l}n_R^l(x,t)dx\le\int_{il-l}^{jl+l}n_R^l(x,(k-1)h+0)dx, \tag{4.18}$$

and

$$\int_{il}^{jl}n_R^l(x,kh-0)dx\ge\int_{-\frac{l}{h}(t-(k-1)h)+(i+1)l}^{\frac{l}{h}(t-(k-1)h)+(j-1)l}n_R^l(x,t)dx\ge\int_{il+l}^{jl-l}n_R^l(x,(k-1)h+0)dx. \tag{4.19}$$

To obtain a uniform bound for the approximate solutions, we estimate the Riemann invariants $w^l(x,t)$ and $z^l(x,t)$. For $kh\le t<(k+1)h$, we have

$$w^l(x,t)=w_R^l(x,t)+\Big(E_R^l-\frac{w_R^l+z_R^l}{2}\Big)(t-kh)$$

$$
\begin{aligned}
&= \Big(1-\frac{t-kh}{2}\Big) w_R^l(x,t) - \frac{t-kh}{2} z_R^l(x,t) + E_R^l(t-kh) \\
&\le \Big(1-\frac{t-kh}{2}\Big) \sup_x \{w_R^l(x,kh+0)\} - \frac{t-kh}{2} \inf_x \{z_R^l(x,kh+0)\} + Nth, \quad (4.20)
\end{aligned}
$$

and

$$
\begin{aligned}
z^l(x,t) &= z_R^l(x,t) + \Big(E_R^l - \frac{w_R^l + z_R^l}{2}\Big)(t-kh) \\
&\ge \Big(1-\frac{t-kh}{2}\Big) \inf_x \{z_R^l(x,kh+0)\} - \frac{t-kh}{2} \sup_x \{w_R^l(x,kh+0)\} - Nth, \quad (4.21)
\end{aligned}
$$

where

$$
\begin{aligned}
&(w^l, z^l)(x,t) = \Big(w(n^l,m^l), z(n^l,m^l)\Big)(x,t), \\
&(w_R^l, z_R^l)(x,t) = \Big(w(n_R^l,m_R^l), z(n_R^l,m_R^l)\Big)(x,t).
\end{aligned} \tag{4.22}
$$

From the construction of the approximate solutions, Lemma 2.2, and Lemma 2.3, it is easy to see that

$$
\begin{cases}
\sup_x w^l(x,t) \le \alpha_0 + Nt^2, \\
\inf_x z^l(x,t) \ge -\alpha_0 - Nt^2.
\end{cases}
$$

Recalling the definition of Riemann invariants, we have a positive constant $\Lambda > 0$ such that

$$
\begin{cases}
n^l(x,t) \le \Lambda t^{\frac{4}{\gamma-1}}, \\
|J(x,t)| \le \Lambda n(x,t) t^2.
\end{cases}
$$

Step 3: *Compactness of entropy dissipation measure and the existence of entropy solutions.*

That is, we first prove the H_{loc}^{-1} compactness of entropy dissipation measures

$$
\eta(v^l)_t + q(v^l)_x
$$

associated with any entropy pair (η, q) and approximate solutions v^l of the Lax–Friedrichs scheme, and then use the compensated compactness framework for Euler equation with $\gamma > 1$ ([6, 7, 22, 23])to get the global existence result. This step is classical and we omit it. Therefore, we have the following existence theorem:

THEOREM 4.1 (Existence). *Suppose the initial data $n_0(x), J_0(x)$ satisfy*

$$
\begin{cases}
0 \le n_0(x) \le C_0, \quad |J_0(x)| \le C_0 n_0(x), \quad a.e. \quad x \in \mathbf{R}, \\
n_0 - b(x) \in L^1(\mathbf{R}), \ b(x) \in L^1_{loc}(\mathbf{R}),
\end{cases}
$$

for a certain constant $C_0 > 0$. Then there exists a global entropy solution $\Big(n(x,t), J(x,t), E(x,t)\Big)$ of problem (1.1)–(1.5) in the region Π_T, satisfying

$$
0 \le n(x,t) \le C_1 t^{\frac{4}{\gamma-1}}, \ |J(x,t)| \le C_1 n(x,t) t^2, \ |E(x,t)| \le C_2 t, \quad a.e. \ (x,t) \in \Pi_T,
$$

where C_1 and C_2 are two positive constants.

From the large time behavior framework and Theorem 4.1, we need $\frac{4}{\gamma-1}<2$, i.e. $\gamma>3$ to get the following result for Euler–Poisson Equation (1.1)–(1.5).

THEOREM 4.2 (Large time behavior). *Suppose (n,J,E) be the entropy solution obtained in Theorem 4.1, $(\tilde{n},\tilde{E})$ be the corresponding stationary solution. If*

$$(E-\tilde{E})(x,0)\in L^2(\mathbf{R}), \qquad \eta^*(x,0)\in L^1(\mathbf{R}), \tag{4.23}$$

then for any $\gamma>3$, there exist constants $T(\gamma),C$ and $\tilde{C}$ such that

$$\int_{-\infty}^{+\infty}\Big((E-\tilde{E})^2(x,t)+\eta^*(x,t)\Big)dx\le C\int_{-\infty}^{+\infty}\Big((E-\tilde{E})^2(x,0)+\eta^*(x,0)\Big)dxe^{-\tilde{C}t^{\frac{\gamma-3}{\gamma-1}}} \tag{4.24}$$

satisfies for any $t>T(\gamma)$, where η^* is the relative mechanical energy which is defined in (3.2).

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