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REPRESENTATION OF DISSIPATIVE SOLUTIONS TO A NONLINEAR VARIATIONAL WAVE EQUATION*

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Abstract. The paper introduces a new way to construct dissipative solutions to a second order variational wave equation. By a variable transformation, from the nonlinear PDE one obtains a semilinear hyperbolic system with sources. In contrast with the conservative case, here the source terms are discontinuous and the discontinuities are not always crossed transversally. Solutions to the semilinear system are obtained by an approximation argument, relying on Kolmogorov's compactness theorem. Reverting to the original variables, one recovers a solution to the nonlinear wave equation where the total energy is a monotone decreasing function of time.

Key words. Nonlinear variational wave equation, global existence, dissipative solutions.

AMS subject classifications. 35Q35.

1. Introduction

We consider the Cauchy problem for a nonlinear wave equation in one space dimension:

$$u_{tt} - c(u)\big(c(u)u_x\big)_x = 0, \tag{1.1}$$

with initial data

$$u(0,x) = u_0(x), \qquad\qquad u_t(0,x) = u_1(x). \tag{1.2}$$

The function $c:\mathbb{R}\mapsto\mathbb{R}_+$, determining the wave speeds, is assumed to be smooth and uniformly positive. As long as the solution remains smooth, it is well known that its energy

$$\mathcal{E}(t) \doteq \frac{1}{2}\int \Big[u_t^2(t,x) + c^2\big(u(t,x)\big)\,u_x^2(t,x)\Big]\,dx \tag{1.3}$$

remains constant. It is thus natural to seek global solutions within the set of functions having bounded energy, i.e. with $u\in H^1(\mathbb{R})$ and $u_t, u_x \in \mathbf{L}^2(\mathbb{R})$ at a.e. time t. In this functional space, the existence of globally defined weak solutions was first proved in [11, 12] (see also [10]). For smooth initial data, formation of singularities was studied in [8].

A different approach was introduced in [5], relying on a transformation of both independent and dependent variables. In the new variables, the equation, (1.1), is replaced by a semilinear system, which always admits global smooth solutions (for smooth initial data). Going back to the original variables, this method yields a family of solutions to the original equation, (1.1), continuously depending on the initial data (in appropriate norms). A careful construction of a semigroup of solutions, with R^2, S^2 possibly replaced by positive measures, can be found in [10]. We recall here the main properties of these global solutions, defined both forward and backward in time:

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(P) The solution $t \mapsto u(t,\cdot)$ takes values in $H^1(\mathbb{R})$ for every $t \in \mathbb{R}$. In the t-x plane, the function $u = u(t,x)$ is Hölder continuous with exponent $1/2$. The map $t \mapsto u(t,\cdot)$ is continuously differentiable as a map with values in $\mathbf{L}^p_{\text{loc}}$, for all $1 \le p < 2$. Moreover, it is Lipschitz continuous w.r.t. the $\mathbf{L}^2$ distance, i.e.

$$\left\| u(t,\cdot) - u(s,\cdot) \right\|_{\mathbf{L}^2} \le L\,|t-s| \tag{1.4}$$

for some constant L and all $t,s \in \mathbb{R}$. The equation, (1.1), is satisfied in the integral sense:

$$\iint \Big[\phi_t\, u_t - \big(c(u)\phi\big)_x c(u)\, u_x \Big]\, dxdt = 0 \tag{1.5}$$

for all test functions $\phi \in \mathcal{C}^1_c$, continuously differentiable with compact support in the t-x plane. Concerning the initial conditions at $t=0$, the first equality in (1.2) is satisfied pointwise, while the second holds in $\mathbf{L}^p_{\text{loc}}$ for $p \in [1,2[$.

The solutions constructed in [11, 12] are **dissipative**, with energy $t \mapsto \mathcal{E}(t)$ which is non-increasing. On the other hand, the solutions obtained in [5] and [10] are **conservative** in the sense that the energy $\mathcal{E}(t) = \mathcal{E}_0$ equals a fixed constant for almost all times t. At an exceptional set of times of measure zero, one can still define a conserved energy in terms of a positive Radon measure. However, this will not be absolutely continuous w.r.t. Lebesgue measure (for example, it may contain Dirac masses). At these particular times, the integral in (1.3) accounts only for the absolutely continuous part of the energy, and is thus strictly smaller than $\mathcal{E}_0$.

By putting the equation into a semilinear form, this variable change provides a transparent way to understand singularity formation, and construct a semigroup of conservative solutions continuously depending on the initial data. A natural question, which motivated the present paper, is whether the same variable transformation can be used to generate a semigroup of dissipative solutions. We recall that, in connection with the Camassa–Holm equation, semigroups of conservative and dissipative solutions have been constructed respectively in [2] and [3], based on a similar approach.

To implement such a program, the main difficulty can be explained as follows. Using characteristic variables, one obtains a semilinear system whose right hand side is Lipschitz continuous in the case of conservative solutions, but discontinuous in the case dissipative solutions. Because of these discontinuities, the existence of solutions does not follow from general theory and must be studied with care. A guiding principle is that, if all discontinuities are crossed transversally, then the Cauchy problem is still well posed. This is indeed what happens for the Camassa–Holm equation [3]. However, in connection with (1.1) we now encounter a "borderline" situation, illustrated by the system with discontinuous right hand side

$$w_Y = \begin{cases} \cos z - \cos w & \text{if } \max\{w,z\} < \pi, \\ 0 & \text{if } \max\{w,z\} \ge \pi, \end{cases}$$

$$z_X = \begin{cases} \cos w - \cos z & \text{if } \max\{w,z\} < \pi, \\ 0 & \text{if } \max\{w,z\} \ge \pi. \end{cases}$$

As w approaches the discontinuity we have $w \approx \pi$, and hence $w_Y \approx \cos z + 1$. This is strictly positive, except when $z \approx \pm\pi$. This lack of transversality renders the system much harder to study.

Our analysis shows that, assuming $c'(u) > 0$ for all u, global dissipative solutions of (1.1) can indeed be constructed by solving a semilinear system with discontinuous right hand side. However, in contrast with [2, 3, 5], solutions are not obtained as the unique fixed points of a contractive transformation. Instead, the existence proof relies here on a compactness argument, based on the Kolmogorov-Riesz theorem [9]. The drawback of this approach is that it does not guarantee the uniqueness of solutions. We speculate that the issue of uniqueness might be resolved by the analysis of characteristics, as in [1, 6].

The paper is organized as follows. In Section 2 we review the variable transformation introduced in [5] and describe the new semilinear system with discontinuous right hand side, which corresponds to dissipative solutions. At the end of this section we can state our main results, on the global existence of solutions to the semilinear system and to the original wave equation, (1.1). Section 3 is the core of the paper. The discontinuous semilinear system is here approximated by a family of Lipschitz continuous systems, admitting unique solutions. As the approximation parameter $\varepsilon \to 0$, a compactness argument yields a subsequence strongly converging to an exact solution. In Section 4, returning to the original variables $u(t,x)$, we obtain the global existence of a weak solution to the nonlinear wave equations, (1.1)–(1.2), forward in time. The proof that this solution satisfies all properties (P) is very similar to the one in [5], and we thus omit most of the details.

2. An equivalent semilinear system

We briefly review the variable transformations introduced in [BZ]. These reduce the quasilinear wave equation, (1.1), to a semilinear system, in characteristic variables. Throughout this section, all equations are derived assuming that the solution is smooth. At a later stage we will prove that the same equations remain meaningful and provide a solution to the original equation, (1.1), also for general initial data $(u_0,u_1)\in H^1\times \mathbf{L}^2$. Define

$$\begin{cases} R \doteq u_t + c(u)u_x, \\ S \doteq u_t - c(u)u_x, \end{cases} \tag{2.1}$$

so that

$$u_t = \frac{R+S}{2}, \qquad u_x = \frac{R-S}{2c}. \tag{2.2}$$

If u is a smooth solution of (1.1), these variables satisfy

$$\begin{cases} R_t - cR_x = \frac{c'}{4c}(R^2 - S^2), \\ S_t + cS_x = \frac{c'}{4c}(S^2 - R^2). \end{cases} \tag{2.3}$$

We can thus regard R,S as the densities of backward and forward moving waves, respectively. Multiplying the first equation in (2.3) by R and the second one by S, we obtain two balance laws for R^2 and S^2, namely

$$\begin{cases} (R^2)_t - (cR^2)_x = \frac{c'}{2c}(R^2S - RS^2), \\ (S^2)_t + (cS^2)_x = -\frac{c'}{2c}(R^2S - RS^2). \end{cases} \tag{2.4}$$

As a consequence, for a smooth solution $u = u(t,x)$ the following quantities are conserved:

$$E \doteq \frac{1}{2}\left(u_t^2 + c^2u_x^2\right) = \frac{R^2+S^2}{4}, \qquad M \doteq -u_t u_x = \frac{S^2 - R^2}{4c}. \tag{2.5}$$

Indeed

$$\begin{cases} E_t + (c^2 M)_x = 0, \\ M_t + E_x = 0. \end{cases} \tag{2.6}$$

One can think of $R^2/4$ and $S^2/4$ as the energy densities associated with backward and forward moving waves, respectively. Because of the sources on the right hand sides of (2.4), some energy is transferred from forward to backward waves, or viceversa. However, the total amount of energy remains constant. To deal with possibly unbounded values of R,S, it is convenient to introduce a new set of dependent variables:

$$w \doteq 2\arctan R, \qquad z \doteq 2\arctan S, \tag{2.7}$$

so that

$$R = \tan\frac{w}{2}, \qquad S = \tan\frac{z}{2}.$$

From (2.3) one derives the equations

$$\begin{cases} w_t - c w_x = \dfrac{2}{1+R^2}(R_t - cR_x) = \dfrac{c'}{2c}\dfrac{R^2 - S^2}{1+R^2}, \\ z_t + c z_x = \dfrac{2}{1+S^2}(S_t + cS_x) = \dfrac{c'}{2c}\dfrac{S^2 - R^2}{1+S^2}. \end{cases} \tag{2.8}$$

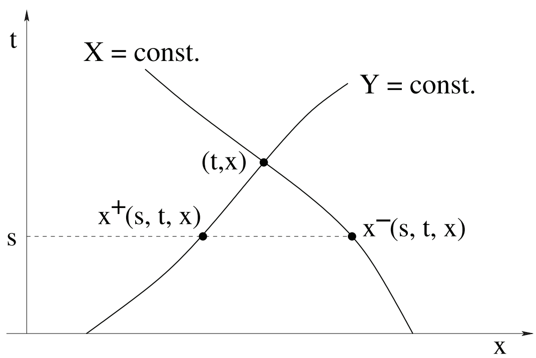

FIG. 2.1. *Forward and backward characteristics through the point* (t,x).

To reduce the system to a semilinear one, we perform a further change of independent variables. Consider the equations for the forward and backward characteristics (Figure 2.1):

$$\dot{x}^+ = c(u), \qquad \dot{x}^- = -c(u). \tag{2.9}$$

The characteristics passing through a given point (t,x) will be denoted by

$$s \mapsto x^+(s,t,x), \qquad s \mapsto x^-(s,t,x), \tag{2.10}$$

respectively. Let initial data $(u_0,u_1) \in H^1 \times \mathbf{L}^2$ be given, as in (1.2). In turn, this determines the functions R,S in (2.1) at time $t=0$:

$$\begin{cases} R(0,\cdot) = u_1 + c(u_0)u_{0,x} \in \mathbf{L}^2(\mathbb{R}), \\ S(0,\cdot) = u_1 - c(u_0)u_{0,x} \in \mathbf{L}^2(\mathbb{R}). \end{cases} \tag{2.11}$$

As coordinates (X,Y) of a point (t,x) we shall use the quantities

$$X \doteq \int_0^{x^-(0,t,x)} \big(1+R^2(0,x)\big)\,dx, \qquad Y \doteq \int_{x^+(0,t,x)}^0 \big(1+S^2(0,x)\big)\,dx. \tag{2.12}$$

Notice that this implies

$$X_t - c(u)X_x = 0, \qquad Y_t + c(u)Y_x = 0, \tag{2.13}$$

$$(X_x)_t - (cX_x)_x = 0, \qquad (Y_x)_t + (cY_x)_x = 0. \tag{2.14}$$

We also observe that

$$X_x(t,x) = \lim_{h\to 0} \frac{1}{h}\int_{x^-(0,t,x)}^{x^-(0,t,x+h)} \big(1+R^2(0,x)\big)\,dx,$$

$$Y_x(t,x) = \lim_{h\to 0} \frac{1}{h}\int_{x^+(0,t,x)}^{x^+(0,t,x+h)} \big(1+S^2(0,x)\big)\,dx.$$

For any smooth function f, using (2.13)–(2.14) one finds

$$\begin{cases} f_t + cf_x = f_X X_t + f_Y Y_t + cf_X X_x + cf_Y Y_x = (X_t + cX_x) f_X = 2cX_x f_X, \\ f_t - cf_x = f_X X_t + f_Y Y_t - cf_X X_x - cf_Y Y_x = (Y_t - cY_x) f_Y = -2cY_x f_Y. \end{cases} \tag{2.15}$$

We now introduce the further variables

$$p \doteq \frac{1+R^2}{X_x}, \qquad q \doteq \frac{1+S^2}{-Y_x}. \tag{2.16}$$

These quantities are related to the partial derivatives X_x, Y_x by the identities

$$(X_x)^{-1} = \frac{p}{1+R^2} = p\cos^2\frac{w}{2}, \qquad (-Y_x)^{-1} = \frac{q}{1+S^2} = q\cos^2\frac{z}{2}. \tag{2.17}$$

Notice that, if the quantity $1+R^2$ were exactly conserved along backward characteristics, we would have

$$(1+R^2)_t - \big[c(u)(1+R^2)\big]_x = 0,$$

and hence $p \equiv 1$. In general, the variable p describes by how much the quantity $1+R^2$ fails to be conserved along backward characteristics. Similarly, q describes by how much the quantity $1+S^2$ is not conserved along forward characteristics.

Starting with the nonlinear wave equation, (1.1), using X,Y as independent variables we thus obtain a semilinear hyperbolic system with smooth coefficients for the variables u,w,z,p,q. Following [5], we consider the set of equations

$$\begin{cases} w_Y = \theta \cdot \frac{c'(u)}{8c^2(u)}(\cos z - \cos w)\, q, \\ z_X = \theta \cdot \frac{c'(u)}{8c^2(u)}(\cos w - \cos z)\, p, \end{cases} \tag{2.18}$$

$$\begin{cases} p_Y = \theta \cdot \frac{c'(u)}{8c^2(u)}\big[\sin z - \sin w\big]\, pq, \\ q_X = \theta \cdot \frac{c'(u)}{8c^2(u)}\big[\sin w - \sin z\big]\, pq. \end{cases} \tag{2.19}$$

To obtain conservative solutions, the above equations should hold everywhere, with $\theta\equiv 1$. On the other hand, to construct dissipative solutions, we here choose

$$\theta=\begin{cases} 1 & \text{if} \quad \max\{w,z\}<\pi, \\ 0 & \text{if} \quad \max\{w,z\}\geq\pi. \end{cases} \tag{2.20}$$

Finally, the function $u=u(X,Y)$ can be recovered by integrating any of the two equations

$$\begin{cases} u_Y = \frac{\sin z}{4c}\, q, \\ u_X = \frac{\sin w}{4c}\, p. \end{cases} \tag{2.21}$$

Given initial data $(u_0,u_1)\in H^1\times\mathbf{L}^2$ as in (1.2), the corresponding boundary data for the system (2.18)–(2.21) is constructed as follows. We first observe that the line $t=0$ corresponds to a curve γ in the X-Y plane, say

$$Y=\varphi(X), \qquad X\in\mathbb{R},$$

where $Y\doteq\varphi(X)$ if and only if

$$X=\int_0^x \big(1+R^2(0,x)\big)\,dx, \quad Y=-\int_0^x \big(1+S^2(0,x)\big)\,dx \qquad \text{for some } \ x\in\mathbb{R}.$$

We can use the variable x as a parameter along the curve γ. The assumptions on u_0 and u_1 imply that the corresponding functions $R(0,\cdot)$ and $S(0,\cdot)$ defined at (2.11) are both in $\mathbf{L}^2$. The initial energy is computed by

$$\mathcal{E}_0\doteq\frac{1}{4}\int\big[R^2(0,x)+S^2(0,x)\big]\,dx<\infty. \tag{2.22}$$

The two functions

$$X(x)\doteq\int_0^x \big(1+R^2(0,x)\big)\,dx, \qquad Y(x)\doteq\int_x^0 \big(1+S^2(0,x)\big)\,dx \tag{2.23}$$

are well defined and absolutely continuous. Clearly, X is strictly increasing while Y is strictly decreasing. Therefore, the map $X\mapsto\varphi(X)$ is continuous and strictly decreasing. From (2.22) it follows

$$-X-4\mathcal{E}_0\leq\varphi(X)\leq -X+4\mathcal{E}_0. \tag{2.24}$$

As (t,x) ranges over the domain $]0,\infty[\times\mathbb{R}$, the corresponding variables (X,Y) range over the domain

$$\Omega^+\doteq\big\{(X,Y);\ \ Y\geq\varphi(X)\big\}. \tag{2.25}$$

Along the non-characteristic curve

$$\gamma\doteq\big\{(X,Y);\ \ Y=\varphi(X)\big\}\subset\mathbb{R}^2$$

parameterized by $x\mapsto\big(X(x),Y(x)\big)$, we can now assign the boundary data $(\bar w,\bar z,\bar p,\bar q,\bar u)\in\mathbf{L}^\infty$ defined by

$$\begin{cases} w(X,\varphi(X)) = \bar w(X) = 2\arctan R(0,x), \\ z(\varphi^{-1}(Y),Y) = \bar z(Y) = 2\arctan S(0,x), \\ u(X,\varphi(X)) = \bar u(X) = u_0(x), \end{cases} \qquad \begin{cases} \bar p\equiv 1, \\ \bar q\equiv 1. \end{cases} \tag{2.26}$$

Our first main result provides the global existence of solutions to the discontinuous semilinear system.

THEOREM 2.1 (existence of solutions to the semilinear system). *Let $c = c(u)$ be a smooth function satisfying*

$$c(u) \geq c_0 > 0, \qquad c'(u) > 0 \qquad \text{for all } u \in \mathbb{R}. \tag{2.27}$$

Then, for any $(u_0, u_1) \in H^1(\mathbb{R}) \times \mathbf{L}^2(\mathbb{R})$, the semilinear system (2.18)–(2.21), with boundary data given by (2.26), (2.11), has a solution defined for all $(X,Y) \in \Omega^+$.

In order to transform this solution back into the original variables, we set $f = x$ and then $f = t$ in (2.15), and obtain

$$\begin{cases} x_X = \dfrac{(1+\cos w)p}{4}, \\ x_Y = -\dfrac{(1+\cos z)q}{4}, \end{cases} \qquad \begin{cases} t_X = \dfrac{(1+\cos w)p}{4c}, \\ t_Y = \dfrac{(1+\cos z)q}{4c}. \end{cases} \tag{2.28}$$

By a direct calculation one finds $x_{XY} = x_{YX}$ and $t_{XY} = t_{YX}$. We can thus integrate the above equations and recover (t,x) as functions of (X,Y). In turn, this yields a function $\tilde{u}(t,x)$ implicitly defined by

$$\tilde{u}\big(t(X,Y), x(X,Y)\big) \doteq u(X,Y). \tag{2.29}$$

THEOREM 2.2 (existence of dissipative solutions to the wave equation). *Let $c = c(u)$ be a smooth function satisfying (2.27) and consider initial data $(u_0, u_1) \in H^1(\mathbb{R}) \times \mathbf{L}^2(\mathbb{R})$. Let (w,z,u,p,q) be a solution to the discontinuous semilinear system (2.18)–(2.21) with boundary data (2.26). Then the function $\tilde{u}(t,x)$ in (2.29) is well defined, and provides a dissipative solution to the Cauchy problem (1.1)–(1.2).*

A proof of Theorem 2.1 will be given in Section 3, while Theorem 2.2 is proved in Section 4.

3. Global solutions of the discontinuous semilinear system

The proof of Theorem 2.1 will be given in several steps.

1. To construct solutions (w,z,u,p,q) to the system (2.18)–(2.21) we use an approximation technique. For any $\varepsilon > 0$, consider the system of PDEs:

$$\begin{cases} w_Y(X,Y) = \theta_\varepsilon \cdot \frac{c'(u)}{8c^2(u)} (\cos z - \cos w) q + \varepsilon, \\ z_X(X,Y) = \theta_\varepsilon \cdot \frac{c'(u)}{8c^2(u)} (\cos w - \cos z) p + \varepsilon, \end{cases} \tag{3.1}$$

$$\begin{cases} p_Y = \theta_\varepsilon \cdot \frac{c'(u)}{8c^2(u)} \big[\sin z - \sin w\big] pq, \\ q_X = \theta_\varepsilon \cdot \frac{c'(u)}{8c^2(u)} \big[\sin w - \sin z\big] pq, \end{cases} \tag{3.2}$$

$$u_Y = \frac{\sin z}{4c(u)} q. \tag{3.3}$$

Here the coefficient $\theta_\varepsilon = \theta_\varepsilon(\max\{w,z\})$ is defined by setting

$$\theta_\varepsilon \doteq \begin{cases} 1 & \text{if} \quad \max\{z,w\} \leq \pi, \\ 0 & \text{if} \quad \max\{z,w\} \geq \pi + \varepsilon^3, \end{cases} \tag{3.4}$$

and by requiring θ_ε to be an affine function of $\max\{z,w\}$ on the interval $[\pi,\pi+\varepsilon^3]$. Let initial data (2.26) be given along the curve $\gamma=\big\{(X,Y);\ Y=\varphi(X)\big\}\subset\mathbb{R}^2$. For convenience, we extend it to the outer region $\{(X,Y);\ Y<\varphi(X)\}$ by letting u,w,p be constant along vertical lines (where X is constant) and z,q be constant along horizontal lines (where Y is constant). We observe that, for any $\varepsilon>0$, the right hand sides of (3.1)–(3.2) and (3.3) are Lipschitz continuous. Hence, given the initial data

$$\begin{cases} w(X,\varphi(X))=\bar{w}(X),\\ z(Y,\varphi^{-1}(Y))=\bar{z}(Y),\end{cases}\qquad \begin{cases} p(X,\varphi(X))=1,\\ q(Y,\varphi^{-1}(Y))=1,\end{cases}\qquad u(X,\varphi(X))=\bar{u}(X)), \tag{3.5}$$

this semilinear hyperbolic system admits a unique local solution, say $(w_\varepsilon,z_\varepsilon,p_\varepsilon,q_\varepsilon,u_\varepsilon)$. Indeed, this solution can be obtained as the unique fixed point of an integral transformation.

We claim that this solution is globally defined on the entire domain $\Omega^+=\{(X,Y);\ Y\geq\varphi(X)\}$. This is not immediately obvious because the equations (3.2) are quadratic w.r.t. p,q. For a given $M>0$, consider the domain

$$\Omega_M\doteq\{(X,Y);\ Y\geq\varphi(X),\quad X\leq M,\quad Y\leq M\}\ \subset\ \Omega^+. \tag{3.6}$$

To achieve the global existence on Ω_M, it suffices to prove the uniform a priori bounds

$$\begin{cases} 0<C^{-1}\leq p(X,Y)\leq C,\\ 0<C^{-1}\leq q(X,Y)\leq C,\end{cases}\qquad \text{for all } (X,Y)\in\Omega_M, \tag{3.7}$$

for some constant C depending on M.

Observe that (3.2) implies $p_Y+q_X=0$. Hence the differential form $pdX-qdY$ has zero integral along any closed curve in Ω_M. For any $(X,Y)\in\Omega_M$, consider the closed curve $\Gamma=\Gamma_1\cup\Gamma_2\cup\Gamma_3$, where Γ_1 is the portion of boundary γ between $(\varphi^{-1}(Y),Y)$ and $(X,\varphi(X))$, Γ_2 is the vertical segment from $(X,\varphi(X))$ to (X,Y) and Γ_3 is the horizontal segment from (X,Y) to $(\varphi^{-1}(Y),Y)$. Integrating $pdX-qdY$ on Γ and using the initial data (3.5), one obtains

$$\begin{aligned}\int_{\varphi^{-1}(Y)}^{X} p(X',Y)\,dX'+\int_{\varphi(X)}^{Y} q(X,Y')\,dY' &= X-\varphi^{-1}(Y)+Y-\varphi(X)\\ &\leq 2\,(|X|+|Y|+4\mathcal{E}_0)\,,\end{aligned} \tag{3.8}$$

because of (2.24). To complete the proof of (3.7), assume

$$C_0\doteq\sup_u\frac{c'(u)}{8c^2(u)}<+\infty. \tag{3.9}$$

Integrating the first equation of (3.2) along line segment from $(X,\varphi(X))$ to (X,Y) and using (2.24), (3.8), since $p,q>0$ we obtain

$$\begin{aligned} p(X,Y) &= \exp\left(\int_{\varphi(X)}^{Y}\theta\frac{c'}{8c^2}(\sin z-\sin w)q(X,Y')\,dY'\right)\\ &\leq \exp\left(2C_0\int_{\varphi(X)}^{Y}q(X,Y')\,dY'\right)\end{aligned}$$

$$\leq \exp(8C_0(M+2\mathcal{E}_0)). \tag{3.10}$$

In the same way, we also obtain

$$p(X,Y) = \exp\left(-2C_0\int_{\varphi(X)}^{Y} q(X,Y')\,dY'\right)$$
$$\geq \exp(-8C_0(M+2\mathcal{E}_0)). \tag{3.11}$$

Together (3.10) and (3.11) yield the first estimate in (3.7). The second estimate is obtained in a nearly identical manner.
In turn, this implies that the maps

$$Y \mapsto w(X,Y), \qquad X \mapsto z(X,Y)$$

are uniformly Lipschitz continuous on the domain Ω_M, say with Lipschitz constant L.

2. Our next goal is to show that, as $\varepsilon \to 0$, this sequence of approximations is compact in $\mathbf{L}^1_{loc}(\mathbb{R}^2)$. For this purpose, some a priori estimates are needed. Fix $\varepsilon > 0$ and consider the corresponding solution (w,z,p,q,u) of (3.1)–(3.4). To shorten notation, in the following we drop the the subscript ε. Define the maps

$$X \mapsto Y^\pi(X) \doteq \inf\{Y \in [0,M];\ \ w(X,Y) = \pi\},$$
$$Y \mapsto X^\pi(Y) \doteq \inf\{X \in [0,M];\ \ z(X,Y) = \pi\},$$

Observe that, for $\varepsilon > 0$ small enough, if $Y^\pi(X) < M$, then

$$w(X,Y) \geq \pi + \varepsilon^3 \qquad \text{for all } Y \geq Y^\pi(X) + \varepsilon. \tag{3.12}$$

Indeed, for $Y \in [Y^\pi(X), Y^\pi(X)+\varepsilon]$ we have

$$w_Y(X,Y) \geq \theta_\varepsilon \cdot \frac{c'(u)}{8c^2(u)}(1-\cos w)q + \varepsilon \geq \frac{\varepsilon}{2}. \tag{3.13}$$

As soon as w becomes greater than $\pi + \varepsilon^3$ we have $\theta_\varepsilon = 0$ and $w_Y = \varepsilon$. This implies (3.12).

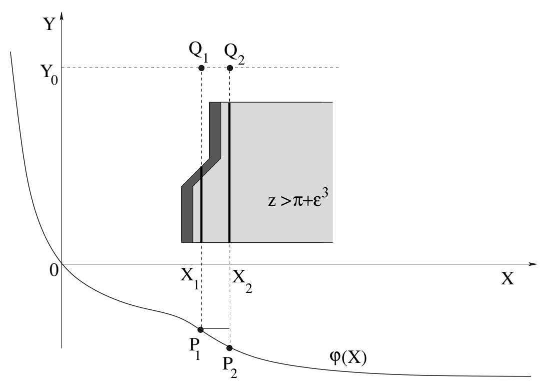

FIG. 3.1. *Estimating the difference* $w(X_2,Y_0) - w(X_1,Y_0)$.

3. The heart of the proof is provided by the next lemma.

LEMMA 3.1. *Given $M>0$ there exists a constant C such that, for any $\varepsilon\in]0,1]$, the solution of (3.1)–(3.5) satisfies the following estimates. For any $X_1<X_2\leq M$ and any $Y_0\in[\varphi(X_1)\ M]$, one has*

$$\begin{aligned}&|w(X_1,Y_0)-w(X_2,Y_0)|+|p(X_1,Y_0)-p(X_2,Y_0)|+|u(X_1,Y_0)-u(X_2,Y_0)|\\ &\leq C\Big\{|\bar w(X_1)-\bar w(X_2)|+|\bar u(X_1)-\bar u(X_2)|+|X_1-X_2|+|\varphi(X_1)-\varphi(X_2)|\\ &\quad+meas\Big(\{Y\leq M;[X_1,X_2]\cap[X^\pi(Y),X^\pi(Y)+\varepsilon]\neq\emptyset\}\Big)\Big\}^{1/2}+C\varepsilon. \qquad (3.14)\end{aligned}$$

Moreover, for $Y_1<Y_2$ and any $X_0\in[\varphi^{-1}(Y_1),M]$, one has

$$\begin{aligned}&|z(X_0,Y_1)-z(X_0,Y_2)|+|q(X_0,Y_1)-q(X_0,Y_2)|\\ &\leq C\Big\{|\bar z(Y_1)-\bar z(Y_2)|+|\bar u(Y_1)-\bar u(Y_2)|+|Y_1-Y_2|+|\varphi^{-1}(Y_1)-\varphi^{-1}(Y_2)|\\ &\quad+meas\Big(\{X\leq M;\ \ [Y_1,Y_2]\cap[Y^\pi(X),Y^\pi(X)+\varepsilon]\ \neq\emptyset\}\Big)\Big\}^{1/2}+C\varepsilon. \qquad (3.15)\end{aligned}$$

The above estimates can be explained with the aid of Figure 3.1. For $i=1,2$, consider the points $P_i=(X_i,\varphi(X_i))$, $Q_i=(X_i,Y_0)$. Then $w(Q_i)$ can be computed by solving the ODE in (3.1) on the interval $Y\in[\varphi(X_i),Y_0]$, with initial data $w=\bar w$ at $Y=\varphi(X_i)$. For $i=1,2$, the right hand sides of these ODEs are almost the same, except in the case where $z(X_1,Y)<\pi$ but $z(X_2,Y)>\pi$. This motivates the presence of the last term on the right hand side of (3.14) and (3.15).

Proof. For notational convenience, we lump together different variables and write

$$\alpha(X,Y)\doteq(w,p,u)(X,Y), \qquad\qquad \beta(X,Y)\doteq(z,q)(X,Y).$$

We first consider the easier case where

$$w(X_i,Y)<\pi \qquad\qquad \text{for all } Y\leq Y_0,\quad i=1,2. \qquad (3.16)$$

Since the maps $X\mapsto\beta(X,Y)$ are uniformly bounded and Lipschitz continuous on bounded sets, we can find a constant κ such that

(i) If $[X_1,X_2]\cap[X^\pi(Y),X^\pi(Y)+\varepsilon]=\emptyset$, then

$$\frac{\partial}{\partial Y}\Big|\alpha(X_1,Y)-\alpha(X_2,Y)\Big|\leq\kappa\Big(|X_1-X_2|+|\alpha(X_1,Y)-\alpha(X_2,Y)|\Big). \qquad (3.17)$$

(ii) If $[X_1,X_2]\cap[X^\pi(Y),X^\pi(Y)+\varepsilon]\ \neq\emptyset$, then

$$\frac{\partial}{\partial Y}\Big|\alpha(X_1,Y)-\alpha(X_2,Y)\Big|\leq\kappa. \qquad (3.18)$$

The differential inequalities (3.17)–(3.18) are complemented by the estimate on the initial data

$$
\begin{aligned}
&\Big|\alpha(X_1,\varphi(X_1))-\alpha(X_2,\varphi(X_1))\Big| \\
&\leq \Big|\bar w(X_1)-\bar w(X_2)\Big|+\Big|\bar p(X_1)-\bar p(X_2)\Big|+\Big|\bar u(X_1)-\bar u(X_2)\Big|+\kappa|\varphi(X_1)-\varphi(X_2)|,
\end{aligned}
\tag{3.19}
$$

for a suitable constant κ.

Using the differential inequalities (3.17)–(3.18) on the interval $Y\in[\varphi(X_1),Y_0]$ together with (3.19), by a Gronwall-type estimate we obtain

$$
\begin{aligned}
&|w(X_1,Y_0)-w(X_2,Y_0)|+|p(X_1,Y_0)-p(X_2,Y_0)|+|u(X_1,Y_0)-u(X_2,Y_0)| \\
&\leq C_1\Big\{|\bar w(X_1)-\bar w(X_2)|+|\bar u(X_1)-\bar u(X_2)|+|X_1-X_2|+|\varphi(X_1)-\varphi(X_2)| \\
&\quad+\text{meas}\Big(\{Y\leq M;[X_1,X_2]\cap[X^\pi(Y),X^\pi(Y)+\varepsilon]\ \neq\emptyset\}\Big)\Big\},
\end{aligned}
\tag{3.20}
$$

for a suitable constant C_1. In the case where

$$
z(X,Y_i)\leq\pi \qquad\qquad \text{for all } X\leq X_0,\quad i=1,2, \tag{3.21}
$$

a nearly identical argument yields

$$
\begin{aligned}
&|z(X_0,Y_1)-z(X_0,Y_2)|+|q(X_0,Y_1)-q(X_0,Y_2)| \\
&\leq C\Big\{|\bar z(Y_1)-\bar z(Y_2)|+|\bar u(Y_1)-\bar u(Y_2)|+|Y_1-Y_2|+|\varphi^{-1}(Y_1)-\varphi^{-1}(Y_2)| \\
&\quad+\text{meas}\Big(\{X\leq M;[Y_1,Y_2]\cap[Y^\pi(X),Y^\pi(X)+\varepsilon]\ \neq\emptyset\}\Big)\Big\}.
\end{aligned}
\tag{3.22}
$$

We now study the more difficult case where (3.16) does not hold. To fix the ideas, assume that, for some $Y_*\in[\varphi(X_1),Y_0]$, we have

$$
w(X_i,Y)<\pi \qquad \text{for all } Y<Y_*,\quad i=1,2, \qquad w(X_1,Y_*)=\pi. \tag{3.23}
$$

For $Y_0\geq Y_*$, using the triangle inequality we can write

$$
\begin{aligned}
&|\alpha(X_1,Y_0)-\alpha(X_2,Y_0)| \\
&\leq |\alpha(X_1,Y_0)-\alpha(X_1,Y_*)|+|\alpha(X_1,Y_*)-\alpha(X_2,Y_*)|+|\alpha(X_2,Y_*)-\alpha(X_2,Y_0)| \\
&= A_1+A_2+A_3.
\end{aligned}
$$

Recalling (3.12) we have the bound

$$
A_1\leq\kappa\varepsilon
$$

for some constant κ. Moreover, by the previous arguments we already know that the estimate (3.20) holds when Y_0 is replaced by Y_*. This yields a bound on A_2.

We now work toward an estimate of A_3. Call

$$
Y^*\doteq\sup\{Y\in[Y_*,Y_0];\quad w(X_2,Y)<\pi\}.
$$

Since

$$
|\alpha(X_2,Y_0)-\alpha(X_2,Y^*)|\leq\kappa\varepsilon,
$$

it suffices to estimate the magnitude of the difference $|\alpha(X_2,Y^*)-\alpha(X_2,Y_*)|$.

The relevant equations in (3.1)–(3.3) are

$$\begin{cases} w_Y = \dfrac{c'(u)}{8c^2(u)}(\cos z - \cos w)q + \varepsilon, \\[2mm] p_Y = \dfrac{c'(u)}{8c^2(u)}(\sin z - \sin w)pq, \\[2mm] u_Y = \dfrac{\sin z}{4c(u)}q, \end{cases} \tag{3.24}$$

with initial data $w(Y_*, X_2) \approx \pi$. Since in these computations $X = X_2$ is fixed, we shall omit this variable and write $w(Y) = w(X_2, Y)$, $z(Y) = z(X_2, Y)$, etc. Roughly speaking, two cases can occur:

(i) $\sin z(Y) \approx 0$. In this case $w_Y, p_Y, u_Y \approx 0$. Hence all these functions remain almost constant.

(ii) $\sin z(Y)$ is not close to zero. In this case $\cos z(Y)$ is much bigger than -1, hence $w_Y(Y)$ is strictly positive. Therefore, $w(\cdot)$ will increase, reaching $\pi + \varepsilon^3$ within a short time. After this happens, $|\alpha_Y(\cdot)| \leq 2\varepsilon$, hence α remains almost constant.

In both cases, the difference $|\alpha(Y) - \alpha(Y_*)|$ remains small.

Relying on the previous ideas, we now work out a rigorous proof. Set

$$\delta \doteq \pi - w(Y_*).$$

Notice that δ is bounded by the right hand side of (3.20). Choose constants $0 < c_0 < C_0$ such that

$$c_0 \leq \frac{c'(u)}{8c(u)} \leq C_0. \tag{3.25}$$

From the first equation in (3.24) we deduce the lower bound

$$w(Y) \geq w(Y_*) - C_0(1 - \cos(\pi - 2\delta)) \qquad Y \in [Y_*, Y^*]. \tag{3.26}$$

In addition, we have

$$\begin{cases} |p_Y| \leq C_0\Big(|\sin z(Y)| + |\sin 2\delta|\Big) + \varepsilon, \\[2mm] |u_Y| \leq C_0 |\sin z(Y)|. \end{cases} \tag{3.27}$$

$$\delta \geq w(Y^*) - w(Y_*) \geq \int_{Y_*}^{Y^*} \frac{c'(u)}{8c^2(u)}(\cos z(Y) - \cos(\pi - 2\delta))\, dY. \tag{3.28}$$

$$\int_{Y_*}^{Y^*} \frac{c'(u)}{8c^2(u)}\Big(\cos z(Y) + 1 - \delta^2\Big)\, dY \leq \delta. \tag{3.29}$$

Using (3.25), from (3.29) we deduce

$$\begin{aligned} \int_{Y_*}^{Y^*} |\sin z(Y)|\, dY &= \int_{Y_*}^{Y^*} 2\Big|\sin\frac{z(Y)}{2}\Big|\Big|\cos\frac{z(Y)}{2}\Big|\, dY \\ &\leq 2|Y^* - Y_*|^{1/2}\left(\int_{Y_*}^{Y^*} \cos^2\frac{z(Y)}{2}\, dY\right)^{1/2} \end{aligned}$$

$$\leq C' \left(\int_{Y_*}^{Y^*} 1 + \cos z(Y)\, dY \right)^{1/2} \leq C\delta^{1/2} \tag{3.30}$$

for some constants C', C. By (3.27) we thus have

$$|p(Y^*) - p(Y_*)| \leq \int_{Y_*}^{Y^*} |p_Y|\, dY \leq C(\delta^{1/2} + \varepsilon) \tag{3.31}$$

and

$$|u(Y^*) - u(Y_*)| \leq \int_{Y_*}^{Y^*} |u_Y|\, dY \leq C\delta^{1/2}, \tag{3.32}$$

possibly with a larger constant C. Since δ is bounded by the right hand side of (3.20), by a suitable choice of C we obtain (3.14).
Using (3.22), a similar argument yields (3.15). □

4. Recalling the definition of the domain Ω_M at (3.6), consider any rectangle $[a,b] \times [c,d] \subset \Omega_M$, and let (ξ, ζ) be any vector such that $[a+\xi, b+\xi] \times [c+\zeta, d+\zeta] \subset \Omega_M$.
Consider any solution of (3.1)–(3.5), for some $\varepsilon \in]0,1]$. Since the components w, p, u are uniformly Lipschitz continuous w.r.t. Y, we have the easy estimate

$$\int_a^b \int_c^d |\alpha(X,Y) - \alpha(X, Y+\zeta)|\, dY\, dX \leq \int_a^b C|\zeta|\, dX \leq C(b-a)|\zeta|. \tag{3.33}$$

for some constant C.
Next, using (3.14) with $X_1 \doteq X$, $X_2 \doteq X + \xi$, we obtain

$$\begin{aligned}
&\int_a^b \int_c^d |\alpha(X,Y) - \alpha(X+\xi, Y)|\, dY\, dX \\
\leq\, & C(d-c) \int_a^b \Big\{ \big|\bar{w}(X) - \bar{w}(X+\xi)\big| + \big|\bar{u}(X) - \bar{u}(X+\xi)\big| + |\xi| + \big|\varphi(X) - \varphi(X+\xi)\big| \\
&\quad + \text{meas}\Big(\{Y \leq M;\ \ [X, X+\xi] \cap [X^\pi(Y), X^\pi(Y)+\varepsilon] \neq \emptyset\} \Big) \Big\}^{1/2} dX + C\varepsilon \\
\leq\, & C' \int_a^b \Big(A(X)^{1/2} + B(X)^{1/2} \Big)\, dX + C\varepsilon,
\end{aligned} \tag{3.34}$$

where

$$A(X) \doteq \big|\bar{w}(X) - \bar{w}(X+\xi)\big| + \big|\bar{u}(X) - \bar{u}(X+\xi)\big| + |\xi| + \big|\varphi(X) - \varphi(X+\xi)\big|,$$

$$B(X) \doteq \text{meas}\Big(\{Y \leq M; [X, X+\xi] \cap [X^\pi(Y), X^\pi(Y)+\varepsilon] \neq \emptyset\} \Big).$$

Since the initial data $\bar{w}$ is bounded and measurable, while $\bar{p} \equiv 1$ and $\bar{u}, \varphi$ are continuous, there exists some modulus of continuity ψ such that

$$\begin{aligned}
&\int_a^b \Big(\big|\bar{w}(X) - \bar{w}(X+\xi)\big| + \big|\bar{u}(X) - \bar{u}(X+\xi)\big| + |\xi| + \big|\varphi(X) - \varphi(X+\xi)\big| \Big)\, dX \\
&\leq \psi(|\xi|).
\end{aligned} \tag{3.35}$$

Calling $A_0 \doteq \frac{\psi(|\xi|)}{b-a}$, we have

$$\begin{aligned}\int_a^b A(X)^{1/2}\,dX &= \int_{[a,b]\cap\{A\le A_0\}} A(X)^{1/2}\,dX + \int_{[a,b]\cap\{A>A_0\}} A(X)^{1/2}\,dX \\ &\le \sqrt{\psi(|\xi|)\cdot(b-a)} + \frac{1}{\sqrt{A_0}}\int_a^b A(X)\,dX \le 2\sqrt{\psi(|\xi|)\cdot(b-a)}.\end{aligned} \tag{3.36}$$

To estimate the integral of $B^{1/2}$ we observe that

$$\int_a^b B(X)\,dX \le (d-c)(|\xi|+\varepsilon). \tag{3.37}$$

The same arguments as in (3.36), with $\psi(|\xi|)$ replaced by the right hand side of (3.37), now yield

$$\int_a^b B(X)^{1/2}\,dX \le 2\sqrt{(d-c)(|\xi|+\varepsilon)\cdot(b-a)}. \tag{3.38}$$

Together, the estimates (3.33) and (3.36)–(3.38) yield a bound of the form

$$\int_a^b\int_c^d |\alpha(X,Y)-\alpha(X+\xi,Y+\zeta)|\,dY\,dX \le \Phi(|\xi|+|\zeta|+\varepsilon) \tag{3.39}$$

for some continuous function Φ, with $\Phi(0)=0$. Nearly identical estimates hold for the functions $\beta=(z,q)$.

5. Given any sequence $\varepsilon_n\to 0$, consider the corresponding approximate solutions $U^\varepsilon \doteq (u^\varepsilon,w^\varepsilon,z^\varepsilon,p^\varepsilon,q^\varepsilon)$. In order to use the Kolmogorov-Riesz compactness theorem [9] and prove that a subsequence U^{ε_n} admits a subsequence converging in $\mathbf{L}^1_{loc}(\Omega^+)$, the following property must be proved:

   (P) *For any $\epsilon>0$ there exists $\rho>0$ such that, on any rectangle $Q\subset\Omega^+$, one has*

   $$\int_Q |U^{\varepsilon_n}(X+\xi,Y+\zeta)-U^{\varepsilon_n}(X,Y)|\,dX\,dY \le \epsilon \tag{3.40}$$

   whenever $|\xi|+|\zeta|\le\rho$, $n\ge 1$.
   By the previous step, we have an estimate of the form

   $$\int_Q \big|U^{\varepsilon_n}(X+\xi,Y+\zeta)-U^{\varepsilon_n}(X,Y)\big|\,dX\,dY \le \Phi(|\xi|+|\zeta|+\varepsilon_n). \tag{3.41}$$

   Choose $\rho'>0$ small enough so that $\Phi(2\rho')<\epsilon$. If $|\xi|+|\zeta|<\rho'$ then (3.40) holds for all $\varepsilon_n<\rho'$. Since there are only finitely many functions U^{ε_n} with $\varepsilon_n\ge\rho'$, by choosing $\rho\in]0,\rho']$ small enough, we can guarantee that (3.40) holds whenever $|\xi|+|\zeta|<\rho$ and U^ε is one of these finitely many functions with $\varepsilon_n>\rho'$.

6. Using the Kolmogorov–Riesz compactness theorem, we obtain a sequence $\varepsilon\to 0$ such that

   $$(u^\varepsilon,w^\varepsilon,z^\varepsilon,p^\varepsilon,q^\varepsilon)(X,Y) \to (u,w,z,p,q)(X,Y) \qquad \text{for a.e. } (X,Y)\in\Omega^+.$$

   By Lipschitz continuity, for any given $M>0$ this implies:

- For a.e. $X\in\mathbb{R}$ one has the uniform convergence: $(u^\varepsilon,w^\varepsilon,p^\varepsilon)(X,Y) \to (u,w,p)(X,Y)$, for all $Y\in[\varphi(X),M]$.
- For a.e. $Y\in\mathbb{R}$ one has the uniform convergence: $(z^\varepsilon,q^\varepsilon)(X,Y) \to (z,q)(X,Y)$, for all $X\in[\varphi^{-1}(Y),M]$.

7. It remains to show that the limit functions (u,w,z,p,q) provide a solution to the set of equations (2.18)–(2.21). This is nontrivial, because the right hand side of the equations (2.18)–(2.21) is discontinuous at $w=\pi$ or $z=\pi$. Comparing the functions $\theta,\theta_\varepsilon$ in (2.20) and (3.4), one should be aware that general it is not true that $\theta_\varepsilon\to\theta$ as $\varepsilon\to 0$. For example, this convergence fails if $w^\varepsilon=z^\varepsilon=\pi-\varepsilon$. Our proof is based on the following a priori estimate. For any given $\eta,\varepsilon>0$, consider the set

$$A_{\eta,\varepsilon}\doteq\Big\{(X,Y)\in\Omega_M;\quad \max\{w^\varepsilon(X,Y),z^\varepsilon(X,Y)\}\geq\pi-\eta\Big\}.$$

We claim that

$$\iint_{A_{\eta,\varepsilon}}\Big(|w_Y^\varepsilon|+|p_Y^\varepsilon|+|z_X^\varepsilon|+|q_X^\varepsilon|\Big)\,dXdY\leq C\eta^{1/3}, \tag{3.42}$$

for some constant C, uniformly valid on the region where $0<\varepsilon\leq\eta$. The idea of the proof is quite simple: on the set where $|w-z|$ is small the derivatives w_Y,p_Y,z_X,q_X are all close to zero. On the other hand, the set where $|w-z|$ is large has small measure. To simplify notation, we here omit the superscript $^\varepsilon$. More precisely, for any $\delta>0$ consider the sets

$$S_\delta\doteq\Big\{(X,Y)\in A_{\eta,\varepsilon}\,;w\leq z-\delta\leq z\leq\pi\Big\},$$
$$S'_\delta\doteq\Big\{(X,Y)\in A_{\eta,\varepsilon}\,;z\leq w-\delta\leq w\leq\pi\Big\}.$$

Observe that, for suitable constants $0<c<C$, we have

(i) $z_X(X,Y)\geq\ c\cdot\delta^2$ for all $(X,Y)\in S_\delta$.

(ii) If $z(X,Y)\geq\pi-\eta$, then

$$\begin{cases} z_X(X',Y)\geq -C\cdot\eta,\\ z(X',Y)\geq\pi-C\eta,\end{cases}\qquad\text{for all}\quad (X',Y)\in\Omega_M,\quad X'\geq X.$$

Nearly identical estimates hold for w,w_Y. From (i)–(ii) we deduce

$$\text{meas}(S_\delta\cup S'_\delta)=\ \mathcal{O}(1)\cdot\eta\delta^{-2}. \tag{3.43}$$

Choosing $\delta=\eta^{1/3}$ we obtain

$$\begin{aligned}&\left(\iint_{S_\delta\cup S'_\delta}+\iint_{A_{\eta,\varepsilon}\setminus(S_\delta\cup S'_\delta)}\right)\Big(|w_Y^\varepsilon|+|p_Y^\varepsilon|+|z_X^\varepsilon|+|q_X^\varepsilon|\Big)\,dXdY\\ &=\mathcal{O}(1)\cdot\eta\delta^{-2}+\mathcal{O}(1)\cdot\delta=\mathcal{O}(1)\cdot\eta^{1/3}.\end{aligned} \tag{3.44}$$

8. Thanks to (3.42), by choosing a further subsequence $\varepsilon_n\downarrow 0$ and setting $\eta_n=\varepsilon_n^{1/3}$, we can assume that

$$\iint_{A_{\eta_n,\varepsilon_m}}\Big(|w_Y^{\varepsilon_m}|+|p_Y^{\varepsilon_m}|+|z_X^{\varepsilon_m}|+|q_X^{\varepsilon_m}|\Big)\,dXdY\leq C\varepsilon_n^{1/3}\leq 2^{-n} \tag{3.45}$$

for every $m\geq n\geq 1$.
Calling

$$A_0\doteq\{(X,Y)\in\Omega_M;\quad \max\{w(X,Y),z(X,Y)\}\geq\pi\},$$

for every $n\geq 1$ we have the estimate

$$\begin{aligned}&\iint_{A_0}\Big(|w_Y|+|p_Y|+|z_X|+|q_X|\Big)\,dXdY\\ &\leq\iint_{A_{\eta_n,\varepsilon_m}}\Big(|w_Y|+|p_Y|+|z_X|+|q_X|\Big)\,dXdY\\ &\leq\limsup_{m\to\infty}\iint_{A_{\eta_{n-1},\varepsilon_m}}\Big(|w_Y^{\varepsilon_m}|+|p_Y^{\varepsilon_m}|+|z_X^{\varepsilon_m}|+|q_X^{\varepsilon_m}|\Big)\,dXdY\leq 2^{-n+1}.\end{aligned}\tag{3.46}$$

Hence the left hand side of (3.46) is zero.

9. To complete the proof, setting

$$B_0\doteq\{(X,Y)\in\Omega_M;\quad \max\{w(X,Y),z(X,Y)\}<\pi\},$$

we need to show that

$$\iint_{B_0}\Lambda(X,Y)\,dXdY=0,\tag{3.47}$$

where Λ accounts for the differences between the right and left hand sides of (2.18)–(2.21). More precisely:

$$\begin{aligned}\Lambda\doteq&\left|w_Y-\frac{c'(u)}{8c^2(u)}(\cos z-\cos w)q\right|+\left|z_X-\frac{c'(u)}{8c^2(u)}(\cos w-\cos z)p\right|\\ &+\left|p_Y-\frac{c'(u)}{8c^2(u)}(\sin z-\sin w)pq\right|+\left|q_X-\frac{c'(u)}{8c^2(u)}(\sin w-\sin z)pq\right|+\left|u_Y-\frac{\sin z}{4c}\right|.\end{aligned}\tag{3.48}$$

Toward this goal, for any $\nu\geq 1$ call

$$B_\nu\doteq\{(X,Y)\in\Omega_M;\quad \max\{w(X,Y),z(X,Y)\}\leq\pi-2^{-\nu}\}.$$

We can choose a sequence $\varepsilon_n\downarrow 0$ such that

$$(u^{\varepsilon_n},w^{\varepsilon_n},z^{\varepsilon_n},p^{\varepsilon_n},q^{\varepsilon_n})(X,Y)\to(u,w,z,p,q)(X,Y)\qquad\text{for a.e. }(X,Y)\in B_\nu.$$

By Egoroff's theorem, for any $\delta>0$, there exists a subset $F\subseteq B_\nu$ with meas$(F)<\delta$ and

$$(u^{\varepsilon_n},w^{\varepsilon_n},z^{\varepsilon_n},p^{\varepsilon_n},q^{\varepsilon_n})\to(u,w,z,p,q)\qquad\text{uniformly for any }(X,Y)\in B_\nu\backslash F.$$

Then for any $(X,Y)\in B_\nu\backslash F$, there exists some $\nu_1>\nu\geq 1$ such that

$$\max\{w^{\varepsilon_n}(X,Y),z^{\varepsilon_n}(X,Y)\}\leq\pi-2^{-\nu_1}.$$

It implies that

$$B_\nu\backslash F\subseteq B_{\nu_1,\varepsilon_n}\doteq\{(X,Y)\in\Omega_M;\quad \max\{w^{\varepsilon_n}(X,Y),z^{\varepsilon_n}(X,Y)\}\leq\pi-2^{-\nu_1}\}.$$

Let Λ^{ε_n} be the same form of Λ in (3.48) with (u,w,z,p,q) replaced by $(u^{\varepsilon_n},w^{\varepsilon_n},z^{\varepsilon_n},p^{\varepsilon_n},q^{\varepsilon_n})$. Equations (3.1)–(3.3) implies that $\Lambda^{\varepsilon_n}=2\varepsilon_n$ on B_{ν_1,ε_n}. Thus we obtain

$$\begin{aligned}\iint_{B_\nu}\Lambda(X,Y)\,dXdY &= \iint_{B_\nu\setminus F}\Lambda(X,Y)\,dXdY+\iint_F\Lambda(X,Y)\,dXdY\\ &=\lim_{\varepsilon_n\downarrow 0}\iint_{B_\nu\setminus F}\Lambda^{\varepsilon_n}(X,Y)\,dXdY+\iint_F\Lambda(X,Y)\,dXdY\\ &\le \lim_{\varepsilon_n\downarrow 0}\iint_{B_{\nu_1,\varepsilon_n}}\Lambda^{\varepsilon_n}(X,Y)\,dXdY+C\delta=C\delta. \end{aligned}\tag{3.49}$$

Hence

$$\iint_{B_\nu}\Lambda(X,Y)\,dXdY=0,\tag{3.50}$$

for every $\nu\ge 1$. Letting $\nu\to\infty$ in (3.50) and using Lebesgue's monotone convergence theorem, we conclude (3.47). □

4. Global existence of dissipative solutions

Going back to the original variables, we now prove that the limit function u provides a dissipative solution to the original wave equation, (1.1). The proof of Theorem 2.2 will be given in several steps.

1. As in [5], by setting $f=x$ and then $f=t$ in (2.15), we obtain

$$\begin{cases} x_X=\dfrac{(1+\cos w)p}{4},\\ x_Y=-\dfrac{(1+\cos z)q}{4}, \end{cases}\qquad \begin{cases} t_X=\dfrac{(1+\cos w)p}{4c},\\ t_Y=\dfrac{(1+\cos z)q}{4c}. \end{cases}\tag{4.1}$$

Conversely, whenever $\cos w\ne -1$ and $\cos z\ne -1$, one has

$$\begin{cases} X_x=\dfrac{2}{(1+\cos w)p},\\ Y_x=-\dfrac{2}{(1+\cos z)q}, \end{cases}\qquad \begin{cases} X_t=\dfrac{2c}{(1+\cos w)p},\\ Y_t=\dfrac{2c}{(1+\cos z)q}. \end{cases}\tag{4.2}$$

As in [5], we can recover (t,x) by integrating either one of the equations for x and for t in (4.1). Indeed, from the equations (2.18)–(2.21) it follows that $x_{XY}=x_{YX}$ and $t_{XY}=t_{YX}$ for a.e. X,Y.

2. For any $(\bar t,\bar x)$, we now define $u(\bar t,\bar x)\doteq u(X,Y)$ where (X,Y) is any point such that $x(X,Y)=\bar x$ and $t(X,Y)=\bar t$. We claim that the above definition of $u(t,x)$ is independent of the choice of (X,Y). Indeed (see Figure 4.1), suppose that there are two different points (X_1,Y_1) and (X_2,Y_2) such that

$$x(X_1,Y_1)= x(X_2,Y_2)=\bar x,\qquad t(X_1,Y_1)=t(X_2,Y_2)=\bar t.\tag{4.3}$$

Two cases must be considered.

CASE 1: $X_1\le X_2$, $Y_1\le Y_2$. We then consider the set

$$\Gamma_{\bar x}\doteq\{(X,Y);\ x(X,Y)\le \bar x\}$$

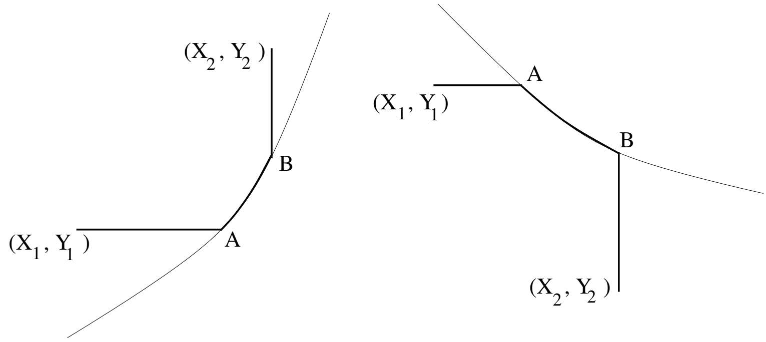

FIG. 4.1. *Proving that the map $(t,x)\mapsto u(t,x)$ is well defined.*

with boundary $\partial\Gamma_{\bar x}$. By (4.1), $x(X,Y)$ is increasing w.r.t. X and decreasing w.r.t. Y. This boundary can thus be represented as the graph of a Lipschitz continuous function, namely

$$X-Y=\phi(X+Y).$$

We now construct the Lipschitz continuous curve γ as in Figure 4.1, left, consisting of

- a horizontal segment joining (X_1,Y_1) with a point $A=(X_A,Y_A)$ on $\partial\Gamma_{\bar x}$, with $Y_A=Y_1$,
- a portion of the boundary $\partial\Gamma_{\bar x}$,
- a vertical segment joining (X_2,Y_2) to a point $B=(X_B,Y_B)$ on $\partial\Gamma_{\bar x}$, with $X_B=X_2$.

We can parameterize this curve in a Lipschitz continuous way, say $\gamma:[\xi_1,\xi_2]\mapsto\mathbb{R}^2$, using the parameter $\xi=X+Y$. Observe that the map $(X,Y)\mapsto(t,x)$ is constant along γ. By (4.1) this implies $(1+\cos w)X_\xi=(1+\cos z)Y_\xi=0$, hence $\sin w\cdot X_\xi=\sin z\cdot Y_\xi=0$. We now compute

$$u(X_2,Y_2)-u(X_1,Y_1)=\int_\gamma\big(u_X\,dX+u_Y\,dY\big)=\int_{\xi_1}^{\xi_2}\left(\frac{p\sin w}{4c}X_\xi-\frac{q\sin z}{4c}Y_\xi\right)d\xi=0,$$

proving our claim.

CASE 2: $X_1\le X_2$, $Y_1\ge Y_2$. In this case, we consider the set

$$\Gamma_{\bar t}\doteq\Big\{(X,Y);\ \ t(X,Y)\le\bar t\Big\},$$

and construct a curve γ connecting (X_1,Y_1) with (X_2,Y_2) as in Figure 4.1, right. Details are nearly identical to those of Case 1.

3. In this step we prove that the function u provides a weak solution to the original nonlinear wave equation, (1.1). According to (1.5), we need to show that

$$\iint\phi_t\big[(u_t+cu_x)+(u_t-cu_x)\big]-\big(c(u)\phi\big)_x\big[(u_t+cu_x)-(u_t-cu_x)\big]\,dxdt=0 \tag{4.4}$$

for every test function $\phi\in\mathcal{C}_c^\infty([0,\infty[\times\mathbb{R})$. We now express the double integral in terms of the variables X,Y, using the change of variable formula

$$dxdt=\frac{pq}{2c(1+R^2)(1+S^2)}\,dXdY=\frac{pq}{2c}\cos^2\frac{w}{2}\cos^2\frac{z}{2}\,dXdY, \tag{4.5}$$

see [5] for details. Since only the absolutely continuous part of the measure $c'(u)u_x^2$ is accounted in the double integral (4.4), using (2.15) we obtain that (4.4) is equivalent to

$$\begin{aligned} 0 &= \iint R\big[\phi_t - (c\phi)_x)\big] + S\big[\phi_t + (c\phi)_x\big]\,dxdt \\ &= \iint -2cY_x\phi_Y R + 2cX_x\phi_X S + \theta c'\phi(u_X X_x + u_Y Y_x)(S-R)\,dxdt. \end{aligned} \tag{4.6}$$

REMARK 4.1. The integrand in (1.5), or equivalently (4.6), contains the term $c'(u)u_x^2$ which multiplies the test function ϕ. Computing the same integral in terms of X,Y, a straightforward use of the change of variable formula would lead to a Radon measure. However, it is only the absolutely continuous part of this measure that actually contributes to the integral (1.5), i.e. the part with $\max\{w,z\}<\pi$. For this reason, in (4.6) we need to insert the additional factor θ. We observe that, in the conservative case [5], this factor was not needed, because in that case the corresponding Radon measure is already absolutely continuous with density $c'(u)u_x^2$, for a.e. time t.

Using (4.5) to change variables and the identities

$$\begin{cases} \dfrac{1}{1+R^2} = \cos^2\dfrac{w}{2} = \dfrac{1+\cos w}{2}, \\ \dfrac{1}{1+S^2} = \cos^2\dfrac{z}{2} = \dfrac{1+\cos z}{2}, \end{cases} \qquad \begin{cases} \dfrac{R}{1+R^2} = \dfrac{\sin w}{2}, \\ \dfrac{S}{1+S^2} = \dfrac{\sin z}{2}, \end{cases} \tag{4.7}$$

the double integral in (4.6) can be written as

$$\begin{aligned} &\iint \left\{ \frac{R}{1+R^2}p\phi_Y + \frac{S}{1+S^2}q\phi_X + \theta\frac{c'pq}{8c^2}\left(\frac{\sin w}{1+S^2} - \frac{\sin z}{1+R^2}\right)(S-R)\phi \right\} dXdY \\ &= \iint \left\{ \frac{p\sin w}{2}\phi_Y + \frac{q\sin z}{2}\phi_X \right. \\ &\quad \left. + \theta\frac{c'pq}{8c^2}\left(\sin w \sin z - \sin w\cos^2\frac{z}{2}\tan\frac{w}{2} - \sin z\cos^2\frac{w}{2}\tan\frac{z}{2}\right)\phi \right\} dXdY \\ &= \iint \left\{ \frac{p\sin w}{2}\phi_Y + \frac{q\sin z}{2}\phi_X + \theta\frac{c'pq}{8c^2}\big[\cos(w-z)-1\big]\phi \right\} dXdY. \end{aligned} \tag{4.8}$$

Since u,w,p are Lipschitz continuous functions of Y, while u,z,q are Lipschitz continuous functions of X, after integrating by parts, it suffices to check that the identity

$$\left(\frac{p\sin w}{2}\right)_Y + \left(\frac{q\sin z}{2}\right)_X = \theta\frac{c'pq}{8c^2}\big[\cos(w-z)-1\big] \tag{4.9}$$

holds at a.e. point (X,Y). By (2.18) and (2.19) we have

$$\begin{aligned} \left(\frac{p\sin w}{2}\right)_Y + \left(\frac{q\sin z}{2}\right)_X &= p_Y\frac{\sin w}{2} + p\left(\frac{\sin w}{2}\right)_Y + q_X\frac{\sin z}{2} + q\left(\frac{\sin z}{2}\right)_X \\ &= \theta\frac{c'pq}{16c^2}\big[(\sin z - \sin w)\sin w + \cos w(\cos z - \cos w)\big] \end{aligned}$$

$$
\begin{aligned}
&+\theta\frac{c'pq}{16c^2}\big[(\sin w-\sin z)\sin z+\cos z(\cos w-\cos z)\big]\\
=&\theta\frac{c'pq}{8c^2}\big[\cos(w-z)-1\big] \qquad (4.10)
\end{aligned}
$$

which implies (4.9) and hence (4.4). Therefore, the function u provides a weak solution to (1.1).

4. It remains to prove that the weak solution u is dissipative. Toward this goal, consider any $0\le t_1<t_2$ and a large radius $r>0$, and define

$$\Omega^r\doteq\big\{(X,Y);\ \ X\le r,\ \ Y\le r,\ \ t_1\le t(X,Y)\le t_2\big\}.$$

We can represent the above set as

$$\Omega^r\doteq\big\{(X,Y);\ \ X\le r,\ \ Y\le r,\ \ \phi_1(X+Y)\le X-Y\le\phi_2(X+Y)\big\},$$

for some functions $\phi_1<\phi_2$, Lipschitz continuous with constant 1. With reference to Figure 4.2, assume that

$$x(A)=a,\qquad x(B)=b,\qquad x(C)=c,\qquad x(D)=d,$$

for some $a<b$ and $c<d$. Moreover, let γ_1, γ_2 be the lower and upper portions of the boundary of Ω^r. From the equations (2.18)–(2.19) it follows that the 1-form

$$E\,dx-(cM^2)\,dt=\frac{(1-\cos w)p}{8}\,dX-\frac{(1-\cos z)q}{8}\,dY \qquad (4.11)$$

is closed. Therefore, the integral of the 1-form (4.11) along the boundary of Ω^r is zero. In particular, this yields

$$
\begin{aligned}
&\int_{\gamma_1}\left\{\frac{(1-\cos w)p}{8}\,dX-\frac{(1-\cos z)q}{8}\,dY\right\}-\int_{\gamma_2}\left\{\frac{(1-\cos w)p}{8}\,dX-\frac{(1-\cos z)q}{8}\,dY\right\}\\
=&\int_{AC}\frac{(1-\cos w)p}{8}\,dX+\int_{BD}\frac{(1-\cos z)q}{8}\,dY\ge 0. \qquad (4.12)
\end{aligned}
$$

We now observe that, at time t_1, the total energy inside the interval $[a,b]$ is computed by

$$
\begin{aligned}
&\int_a^b\frac{1}{2}\big[u_t^2(t_1,x)+c^2(u(t_1,x))u_x^2(t_1,x)\big]\,dx\\
=&\int_{\gamma_1\cap\{w(X,Y)<\pi\}}\frac{(1-\cos w)p}{8}\,dX-\int_{\gamma_1\cap\{z(X,Y)<\pi\}}\frac{(1-\cos z)q}{8}\,dY. \qquad (4.13)
\end{aligned}
$$

A nearly identical formula yields the energy at time t_2 inside the interval $[c,d]$. From the basic equations (2.18)–(2.20) one obtains the implications

$$
\begin{aligned}
&w(X,Y)=\pi\Longrightarrow w(X,Y')=\pi\quad\text{and}\quad q(X,Y')=q(X,Y)\quad\text{for all}\ \ Y'\ge Y,\\
&z(X,Y)=\pi\Longrightarrow z(X',Y)=\pi\quad\text{and}\quad p(X',Y)=p(X,Y)\quad\text{for all}\ \ X'\ge X.
\end{aligned}
\qquad (4.14)
$$

Combining (4.13) with (4.12) and using (4.14) we now obtain

$$\int_a^b\frac{1}{2}\big(u_t^2(t_2,x)+c^2(u(t_2,x))u_x^2(t_2,x)\big)\,dx-\int_c^d\frac{1}{2}\big(u_t^2(t_1,x)+c^2(u(t_1,x))u_x^2(t_1,x)\big)\,dx$$

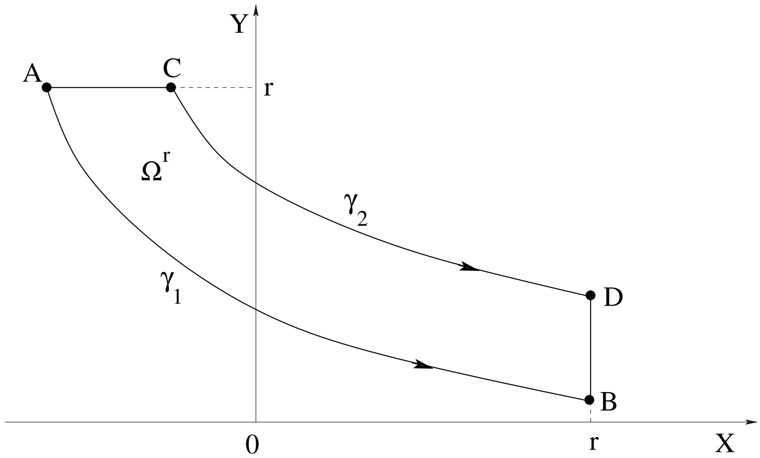

FIG. 4.2. *The set* Ω^r *considered in step* **4** *of the proof. In the contour integrations at (4.12), the curve* γ_1 *is oriented from* A *to* B*, while* γ_2 *is oriented from* C *to* D*.*

$$\begin{aligned}
&= \int_{\gamma_1 \setminus \{w(X,Y)=\pi\}} \frac{(1-\cos w)p}{8}\,dX - \int_{\gamma_1 \setminus \{z(X,Y)=\pi\}} \frac{(1-\cos z)q}{8}\,dY \\
&\quad - \int_{\gamma_2 \setminus \{w(X,Y)=\pi\}} \frac{(1-\cos w)p}{8}\,dX + \int_{\gamma_2 \setminus \{z(X,Y)=\pi\}} \frac{(1-\cos z)q}{8}\,dY \\
&= \int_{\gamma_1} \left\{ \frac{(1-\cos w)p}{8}\,dX - \frac{(1-\cos z)q}{8}\,dY \right\} - \int_{\gamma_2} \left\{ \frac{(1-\cos w)p}{8}\,dX - \frac{(1-\cos z)q}{8}\,dY \right\} \\
&\quad - \int_{\gamma_1 \cap \{w(X,Y)=\pi\}} \frac{p}{4}\,dX + \int_{\gamma_1 \cap \{z(X,Y)=\pi\}} \frac{q}{4}\,dY + \int_{\gamma_2 \cap \{w(X,Y)=\pi\}} \frac{p}{4}\,dX \\
&\quad - \int_{\gamma_2 \cap \{z(X,Y)=\pi\}} \frac{q}{4}\,dY \geq 0. \qquad (4.15)
\end{aligned}$$

Indeed, as shown in Figure 4.3, by (4.14) it follows

$$\begin{aligned}
0 &\leq \int_{\gamma_1 \cap \{w(X,Y)=\pi\}} \frac{p}{4}\,dX \leq \int_{\gamma_2 \cap \{w(X,Y)=\pi\}} \frac{p}{4}\,dX, \\
0 &\leq -\int_{\gamma_1 \cap \{z(X,Y)=\pi\}} \frac{q}{4}\,dY \leq -\int_{\gamma_2 \cap \{z(X,Y)=\pi\}} \frac{q}{4}\,dY.
\end{aligned} \qquad (4.16)$$

Letting $r \to +\infty$, we have $a,c \to -\infty$ while $b,d \to +\infty$. Hence (4.15) yields the desired inequality on the total energy:

$$\begin{aligned}
\mathcal{E}(t_2) &= \int_{-\infty}^{\infty} \frac{1}{2} \left(u_t^2(t_2,x) + c^2(u(t_2,x)) u_x^2(t_2,x) \right) dx \\
&\leq \int_{-\infty}^{\infty} \frac{1}{2} \left(u_t^2(t_1,x) + c^2(u(t_1,x)) u_x^2(t_1,x) \right) dx = \mathcal{E}(t_1),
\end{aligned}$$

showing that the weak solution $u = u(t,x)$ is dissipative. □

REMARK 4.2. Within the proof, we checked that the differential form (4.11) is closed. This might suggest that, as in [5], our solution is still conservative. The key difference can be explained as follows (see Figure 4.4). The contour

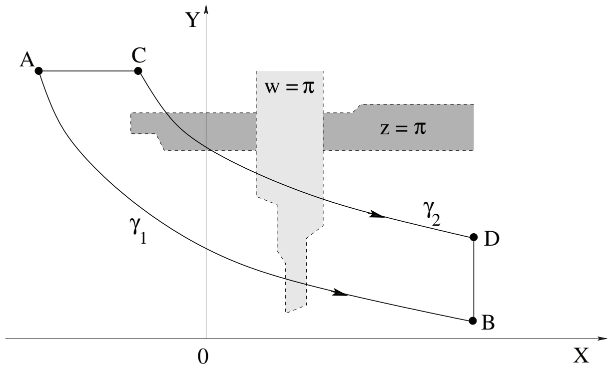

FIG. 4.3. *The set where $w=\pi$ grows as Y increases. Similarly, the set where $z=\pi$ grows as X increases. By (4.14), this yields the inequalities in (4.16).*

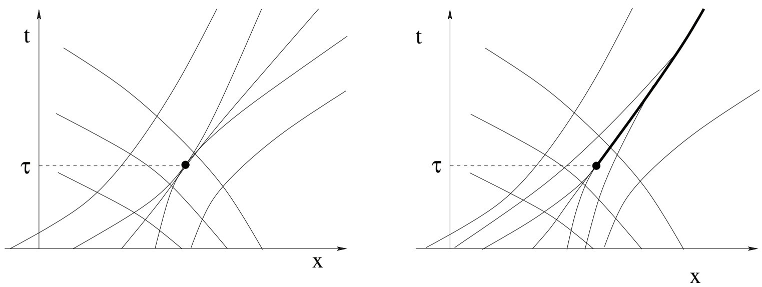

FIG. 4.4. *Left: characteristic curves in a conservative solution. At time τ a positive amount of energy is concentrated at a single point. However, for $t>\tau$ the energy measure is again absolutely continuous. Right: the characteristics curves in a dissipative solution with the same initial data. When some of the energy concentrates at one point, it remains inside the singular part of the energy measure $\mu^{(t)}$ for all subsequent times. The equations (2.18)–(2.21) imply that this singular part of the energy is formally transported along characteristics. However it does not affect the solution u at any time $t>\tau$.*

integral

$$\int_{\gamma_1} \left\{ \frac{(1-\cos w)p}{8}\, dX - \frac{(1-\cos z)q}{8}\, dY \right\}$$

yields the total energy at time t_1 inside the interval $[a,b]$. In general, this energy is a positive Radon measure $\mu^{(t_1)}$ on the real line. On the other hand the integral

$$\int_a^b \frac{1}{2}\left[u_t^2(t_1,x)+c^2(u(t_1,x))u_x^2(t_1,x)\right] dx$$

accounts only for the absolutely continuous part of this measure. In the solution constructed in [5], the measure $\mu^{(t)}$ is absolutely continuous for a.e. time t. Hence $\mathcal{E}(t)=\mathcal{E}(0)$ for a.e. t. On the other hand, in our dissipative solution the singular part of the energy measure (corresponding to $w=\pi$ or $z=\pi$) is always increasing with time. Hence the absolutely continuous part can only decrease.

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