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ERROR ANALYSIS OF A DYNAMIC MODEL ADAPTATION PROCEDURE FOR NONLINEAR HYPERBOLIC EQUATIONS*

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Abstract. In order to theoretically validate a dynamic model adaptation method, we consider a simple case where the *model error* can be thoroughly analyzed. The dynamic model adaptation consists of detecting at each time step the region where a given *fine model* can be replaced by a corresponding *coarse model* in an automatic way, without deteriorating the accuracy of the result, and to couple the two models, each being computed on its respective region. Our fine model is a 2×2 system which involves a small time scale; setting this time scale to 0 leads to a classical conservation law, the *coarse model*, with a flux which depends on the unknown and on space and time. The adaptation method provides an intermediate *adapted* solution which results from the coupling of both models at each time step. In order to obtain sharp and rigorous error estimates for the model adaptation procedure, a simple fine model is investigated, and smooth transitions between fine and coarse models are considered. We refine existing stability results for conservation laws with respect to the flux function which enables us to know how to balance the time step, the threshold for the domain decomposition, and the size of the transition zone. Numerical results are presented at the end and show that our estimate is optimal.

Key words. Conservation laws, error estimate, model adaptation, thick coupling interface.

AMS subject classifications. 35L65, 35B45, 35B30, 35A35.

1. Introduction

In the context of the simulation of complex fluids, the inner heterogeneities of a flow may lead to use several models with different degrees of accuracy. In an ideal situation, these models can be related one another through asymptotic analysis: singular limits, homogenization, space reduction, etc. A model may be preferred to the others according to the local features of the flow: one aims at using the simplest one without deteriorating the accuracy of the results in comparison with the results which would be obtained with the finest model. It naturally gives rise to problems of coupling if different models are used in different zones of the computational domain at the same time. Moreover, in the case of transient flows, these zones may evolve in agreement with the structures of the flow. Therefore, we have to tackle the problem of the automatic detection of the best model to use among a given hierarchy of models; this is what we call *dynamic model adaptation*. The difficulty is twofold:

- to estimate the decrease of accuracy due to the local use of a coarser model instead of a reference model,

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- to handle the use of different models in different regions of the computational domain.

A byproduct of these two issues is the necessity of estimating the error due to the coupling of the different models. Such estimates are usually called *modeling error estimates* and go back to the early works of Oden et al. (see [34, 31, 30, 33]) for heterogeneous materials, [32] for Solid Mechanics, and [1] for laminated plates and shells. *A posteriori* error estimators were developed in a general setting in [5] in order to equilibrate the modeling error with the numerical error for a global adaptive method. Since these pioneer works, model adaptation has seen a large increase of interest, see for instance [29, 2, 28, 17, 18, 15].

The applications we have in mind here prevents us from following the methodologies proposed in the literature. Indeed, we are interested in compressible fluids with stiff effects such as phase transition, thermal exchanges, and drag force in two-phase flows, and more generally in nonlinear hyperbolic systems of PDEs. Model adaptation for several of these applications have already been addressed in [27] by the authors, in a more heuristic way due to the complexity of the investigated models. Given a system of balance laws with a stiff source term, we proposed in [27] an error indicator to dynamically replace this model by its equilibrium approximation. To do so, a Chapman–Enskog expansion is performed at the numerical level and we use the first-order corrector as the modeling error indicator. Even if it is impossible to theoretically assess the relevance of this approach, the numerical results are very good. We then obtain a fully dynamic model adaptation method for nonlinear systems of hyperbolic equations (very similar methods are proposed in [17, 18] for kinetic equations and their hydrodynamic limits).

In the present paper, we aim at providing theoretical arguments in favor of our approach. Due to the actual knowledge in this field of Mathematics (see for instance [16]), we have no other choice than to restrict ourselves to much simpler models than those investigated in [27], so as to be allowed to use error estimates for nonlinear hyperbolic equations. The hierarchy of models we focus on is based on the classical relaxation framework, and the coupling is performed between a 2×2 relaxation (fine) system with its (coarse) equilibrium scalar conservation law. The *fine model* is

$$\begin{cases}\partial_t u_f(x,t)+\partial_x F(u_f(x,t),v_f(x,t))=0,\\ \partial_t v_f(x,t)+\partial_x G(u_f(x,t),v_f(x,t))=\frac{1}{\tau}(v_{\mathrm{eq}}(x,t)-v_f(x,t)),\end{cases}\tag{1.1}$$

where $u_f,v_f:\mathbb{R}\times\mathbb{R}_+\to\mathbb{R}$ are the unknowns. Functions F, G, and v_{eq} are assumed to be smooth and the constant $\tau>0$ denotes the characteristic time and may be considered to be small. As a result, if this time is neglected in the sense $\tau\to 0$, one obtains the *coarse model*

$$\begin{cases}\partial_t u_c(x,t)+\partial_x F(u_c(x,t),v_{\mathrm{eq}}(x,t))=0,\\ v_c(x,t)=v_{\mathrm{eq}}(x,t).\end{cases}\tag{1.2}$$

The dynamic model adaptation consists of automatically replacing (1.1) by (1.2) in time and space with a sharp control of the error induced by this replacement and by the coupling at the transition zones between the two models.

In this relaxation setting, we may mention some convergence results for the interface coupling of the Jin–Xin relaxation system with its limit conservation law initiated in [6], the subject has received some recent interest [21, 14]. However, the interface remains fixed, there is no error estimate nor adaptation procedure, which is the aim of our present work.

In practice, the algorithm to define our *adapted* solution (u_a,v_a) relies on a discrete-in-time procedure. To evolve from time t_n to time $t_{n+1}=t_n+\Delta t_a$, where Δt_a is a given time step, we perform the following computations (we only present in this introduction a simplified and slightly false version, see Section 2.3 for the full description of the method):

1. Assume that $(u_a,v_a)(\cdot,t_n)$ is known and let $\Sigma>0$ be a given threshold.
2. Solve the Cauchy problem for all $x\in\mathbb{R}$ (we assume here that it can be done easily)

$$\begin{cases}\partial_t v_i(x,t)+\partial_x G(u_a(x,t_n),v_i(x,t))=\frac{1}{\tau}(v_{\mathrm{eq}}(x,t)-v_i(x,t)), & t\in[t_n,t_{n+1}],\\ v_i(x,t_n)=v_a(x,t_n).\end{cases} \tag{1.3}$$

3. Define the *fine domain* by $\Omega_f^{(n)}=\{x\in\mathbb{R}\mid \|v_i(.,t)-v_{\mathrm{eq}}(.,t)\|_{1,\infty}\geq\Delta t_a\Sigma$, for all $t\in[t_n,t_{n+1}]\}$.
4. Computation of v_a:
   (a) Define a regularized characteristic function $\chi_\delta(.,t)\in\mathcal{C}(\mathbb{R};[0,1])$ such that $\chi_\delta(x,t)=1$ if $x\in\Omega_f^{(n)}$, $\chi_\delta(x,t)=0$ if $d(x,\Omega_f^{(n)})\geq\delta$, $t\in[t_n,t_{n+1}]$.
   (b) Define

$$v_a=\chi_\delta v_i+(1-\chi_\delta)v_{\mathrm{eq}} \text{ in } \mathbb{R}\times[t_n,t_{n+1}]. \tag{1.4}$$

5. Computation of u_a:
   solve the scalar conservation law

$$\partial_t u_a+\partial_x F(u_a,v_a)=0 \tag{1.5}$$

   for $x\in\mathbb{R}$ and $t\in[t_n,t_{n+1}]$, with initial data $u_a(\cdot,t_n)$.

The goal is to perform an error analysis between the solution (u_a,v_a) of the model adaptation algorithm and the solution (u_f,v_f) of the fine model (1.1) in order to fix the different parameters of the model adaptation algorithm (the time step Δt_a, the threshold Σ, and the buffer size δ) and to obtain an error estimate between (u_a,v_a) and (u_f,v_f).

Because of step 2 of the previous algorithm, where we have to solve the equation of the fine problem (with a perturbed initial data $v_a(\cdot,t_n)$ instead of $v_f(\cdot,t_n)$), there is no hope to save some computational time with this method. Nevertheless, it appears to be important to quantify rigorously the error $\|u_f(.,t)-u_a(.,t)\|$ made by such an adaptation procedure in order to justify the numerical behavior of coupling algorithms in more complex cases, as for example those presented in [27], where carrying out such an analysis is out of reach.

In [27], we have described the dynamic adaptation procedure directly at the numerical level. We had at our disposal a numerical indicator, derived from a Chapman–Enskog expansion, which depends only on the solution u (precisely on an approximation u_j^n of $u_a(x,t_n)$), for evaluating the distance between v and v_{eq}. Unfortunately neither at the discrete nor at the continuous level do we have any theoretical result for evaluating this distance. Hence, since we want here to quantify precisely the model error $\|u_f(.,t)-u_a(.,t)\|$, we make in step 2 the assumption that it is possible to compute v_i (which means that it is possible to compute v_f). Going a step further in this direction, in order to lead the analysis on a precise background, we now restrict ourselves to an even simpler case where $G=0$ in the model (1.1).

Hence, the hierarchy of models we now focus on is based on the coupling of a scalar conservation law with a stiff ordinary differential equation.

$$\begin{cases} \partial_t u_f(x,t) + \partial_x F(u_f(x,t), v_f(x,t)) = 0, \\ \partial_t v_f(x,t) = \dfrac{1}{\tau}(v_{\mathrm{eq}}(x,t) - v_f(x,t)). \end{cases} \tag{1.6}$$

Due to the possibility to exactly compute v_f in the fine model (1.6), the different models rely on scalar conservation laws of the form

$$\partial_t u + \partial_x f(u,x,t) = 0. \tag{1.7}$$

From the mathematical point of view, it is well-known that the classical theory of well-posedness for such equations requires that f be a smooth function of its three variables (see [23]). It is obviously the same when one aims at deriving error estimates (see [24, 11, 26, 3, 19, 9, 22, 10, 13, 25] and Appendix A.2). The main tool we use is an estimate obtained when considering two conservation laws with different flux functions. This has been studied by [26, 3] and recently by [25] (see also [13]). We adapt these results in Appendix A.2 to our setting. We then have to pay a special attention to the smoothness with respect to (w.r.t.) x of the underlying problems to solve. Coupling problems with infinitely thin coupling interfaces can be found for instance in [20, 12, 21, 27] but developing error estimates for such problems seems out of reach: to our knowledge, the only example of (numerical) error estimates for a discontinuous flux f w.r.t. x is done in [8] and this result does not apply here. We thus have to consider a regularizing buffer zone for the model coupling, as done by [17, 18] and [4]. This is the reason why we use the regularized characteristic function χ_δ in (1.4) instead of the classical characteristic function $\mathbf{1}_{\Omega_f^{(n)}}$. Of course, the error estimate between the solution (u_a, v_a) of the model adaptation and the solution (u_f, v_f) of the fine model (1.6) blows up when δ tends to 0 so that we have to carefully calibrate δ in order to control the error due to the model adaptation. Thanks to the estimate of Appendix A.2, we are able to balance the threshold parameter Σ and the buffer size δ, see (3.35) and Theorem 3.9. Even if our analysis is restricted to simple models which may be considered far from realistic problems, we believe that the rates we obtain between parameters of the adaptive method can be used for more complex models.

Let us now present the outline of the paper. In Section 2, the fine model and the associated coarse model are described in details, with a special care to the smoothness of the different functions. Then, a discrete-in-time model adaptation algorithm is given. Section 3 first collects the error estimates due to the adaptation, w.r.t. the different parameters of the adaptive algorithm. Using the error analysis developed in Appendix A, we are able to prove Theorem 3.9 where an error estimate between the solution of the fine model (1.6) and the solution provided by the dynamic model adaptation procedure is given. In the last section, we numerically illustrate the dynamic model adaptation. The fine model corresponds to a transport equation of a chemical component u_f with a speed v_f which depends on the external medium. With sufficiently small numerical parameters in order to avoid any interaction between the modeling error and the numerical error, we verify that the error estimate is optimal. The final section is devoted to some related ongoing works and possible extensions.

2. The models and the dynamic model adaptation

2.1. The fine model. We first consider the solution $(u_f,v_f):\mathbb{R}\times\mathbb{R}_+\to\mathbb{R}^2$ of the so-called *fine model*, that is

$$\begin{cases}\partial_t u_f(x,t)+\partial_x F(u_f(x,t),v_f(x,t))=0,\\ \partial_t v_f(x,t)=\dfrac{1}{\tau}(v_{\mathrm{eq}}(x,t)-v_f(x,t)),\\ u_f(x,0)=u_0(x),\\ v_f(x,0)=v_0(x).\end{cases}\tag{2.1}$$

In the above system, the flux function F is supposed to belong to $\mathcal{C}^2(\mathbb{R}^2;\mathbb{R})$, with uniformly bounded first order and second order derivatives. Moreover, we assume that $F(0,v)=0$ for all $v\in\mathbb{R}$. The relaxation time τ is assumed to be constant and strictly positive. Concerning the equilibrium state v_{eq}, it is supposed to belong to $\mathcal{C}^2(\mathbb{R}_+\times\mathbb{R}_+)$, and to be constant outside the cylinder $(-R,R)\times\mathbb{R}_+$ for some $R>0$. The initial data v_0 is supposed to belong to $\mathcal{C}^2(\mathbb{R})$, and to be equal to v_{eq} for $|x|\geq R$. As a consequence, $v_f\equiv v_{\mathrm{eq}}$ outside $(-R,R)\times\mathbb{R}_+$.

In the system (2.1), the second equation is linear and decoupled from the first one, so that it can be solved apart. Therefore, v_f is explicitly given by: $\forall(x,t)\in\mathbb{R}\times\mathbb{R}_+$,

$$v_f(x,t)=v_0(x)e^{-\frac{t}{\tau}}+\frac{1}{\tau}\int_0^t v_{\mathrm{eq}}(x,a)e^{\frac{a-t}{\tau}}\,da.\tag{2.2}$$

It is easy to check that $v_f\in\mathcal{C}^2(\mathbb{R}\times\mathbb{R}_+)$ is uniformly bounded as well as its first and second order derivatives, so that the function $s\mapsto F(s,v_f(x,t))$ is regular enough to ensure the existence and the uniqueness of the Kruzhkov entropy solution u_f (see [23]) to the problem

$$\begin{cases}\partial_t u_f+\partial_x F(u_f,v_f)=0 \quad \text{in } \mathbb{R}\times\mathbb{R}_+,\\ u_f(\cdot,0)=u_0.\end{cases}$$

In the sequel, we will assume that $u_0\in L^\infty\cap BV(\mathbb{R})$, so that, thanks to [23, 13, 25], the solution u_f itself belongs to $L^\infty_{\mathrm{loc}}\cap BV_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}_+)$. The stability result is recalled in Appendix A (see Theorem A.1).

2.2. The coarse model. Roughly speaking, in the case where τ is small, v_f should be close to v_{eq}. Therefore, a natural *coarse model* for approximating the solution of (2.1) consists of

$$\begin{cases}\partial_t u_c(x,t)+\partial_x F(u_c(x,t),v_{\mathrm{eq}}(x,t))=0, \quad (x,t)\in\mathbb{R}\times(0,\infty),\\ u_c(x,0)=u_0(x), \quad x\in\mathbb{R},\\ v_c(x,t)=v_{\mathrm{eq}}(x,t), \quad (x,t)\in\mathbb{R}\times\mathbb{R}_+.\end{cases}\tag{2.3}$$

Here again, due to the regularity of v_{eq}, the problem (2.3) admits a unique Kruzhkov entropy solution u_c belonging to $L^\infty_{\mathrm{loc}}\cap BV_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}_+)$ for $u_0\in L^\infty\cap BV(\mathbb{R})$.

2.3. Adaptive modeling. We aim now at solving the coarse model (2.3) where u_c is close to u_f, and to turn back to the resolution of the system (2.1) in the zones where u_c is not a satisfactory approximation of u_f. To do so, we introduce a time-dependent partition of $\mathbb{R}$, i.e:

$$\mathbb{R}=\overline{\Omega}_f(t)\cup\overline{\Omega}_c(t), \qquad \Omega_f(t)\cap\Omega_c(t)=\emptyset, \qquad \forall t\geq 0.\tag{2.4}$$

In order to define the sets Ω_f and Ω_c, we introduce three threshold values $\Sigma>0$, Σ' and $\Sigma''>0$, and a time step $\Delta t_a>0$ for the adaptation procedure. For $n\in\mathbb{N}$, we denote by I_n the interval $[t_n,t_{n+1})$, where $t_k=k\Delta t_a$. We also introduce the size of the regularized buffer zones, denoted δ in the following.

Let us now describe the model adaptation procedure which defines the functions $(u_a,v_a):\mathbb{R}\times\mathbb{R}_+\to\mathbb{R}^2$ intended to approximate the solution (u_f,v_f) of the fine model (2.1):

INITIALIZATION:
Define $\left(u_a^{(-1)},v_a^{(-1)}\right)(\cdot,0)=(u_0,v_0)(\cdot)$.

FROM t_n TO t_{n+1}:

1. Define the *indicator* function $v_i^{(n)}:\mathbb{R}\times I_n\to\mathbb{R}$ as the solution of the Cauchy problem

$$\begin{cases}\partial_t v_i^{(n)}=\dfrac{1}{\tau}\left(v_{\mathrm{eq}}-v_i^{(n)}\right),\\ v_i^{(n)}(\cdot,t_n)=v_a^{(n-1)}(\cdot,t_n).\end{cases}\tag{2.5}$$

2. Define the open subset $\Omega_f^{(n)}$ of $\mathbb{R}$ by

$$\Omega_f^{(n)}=\Big\{x\in\mathbb{R}\;\Big|\;|v_{\mathrm{eq}}(x,t)-v_i^{(n)}(x,t)|>\Delta t_a\Sigma,\\ |\partial_x v_{\mathrm{eq}}(x,t)-\partial_x v_i^{(n)}(x,t)|>\Delta t_a\Sigma',\text{ or }|\partial_{xx}^2 v_i^{(n)}|>\Sigma'',\forall t\in I_n\Big\},\tag{2.6}$$

and $\Omega_c^{(n)}=\overbrace{\mathbb{R}\setminus\Omega_f^{(n)}}^{\circ}$.
Define $\Omega_f(t)=\Omega_f^{(n)}$ and $\Omega_c(t)=\Omega_c^{(n)}$ for all $t\in I_n$.

3. Define a *regularized characteristic function* $\chi_\delta\in\mathcal{C}^2(\mathbb{R};[0,1])$ such that

$$\chi_\delta(x,t)=\begin{cases}1 & \text{if } x\in\Omega_f^{(n)},\\ 0 & \text{if } d\left(x,\Omega_f^{(n)}\right)\ge\delta,\end{cases}\tag{2.7}$$

for all $t\in I_n$ and $x\in\mathbb{R}$, and such that there exist α_1 and α_2 depending neither on Ω_f nor on δ such that

$$\|\partial_x\chi_\delta(\cdot,t)\|_\infty\le\frac{\alpha_1}{\delta},\qquad \|\partial_{xx}^2\chi_\delta(\cdot,t)\|_\infty\le\frac{\alpha_2}{\delta^2}.\tag{2.8}$$

(Notice that such a function χ_δ always exists.)

4. Solution to the *model coupling problem*:
Define $v_a^{(n)}\in\mathcal{C}^2(\mathbb{R}\times I_n;\mathbb{R})$ by

$$v_a^{(n)}=\chi_\delta v_i^{(n)}+(1-\chi_\delta)v_{\mathrm{eq}}.\tag{2.9}$$

Define $u_a^{(n)}\in\mathcal{C}([t_n,t_{n+1}];L^1_{loc}(\mathbb{R}))$ as the unique Kruzhkov entropy solution of the scalar conservation law

$$\begin{cases}\partial_t u_a^{(n)}+\partial_x F\left(u_a^{(n)},v_a^{(n)}\right)=0,\\ u_a^{(n)}(\cdot,t_n)=u_a^{(n-1)}(\cdot,t_n).\end{cases}\tag{2.10}$$

Thanks to this algorithm, we are now able to define the *solution of the dynamic model coupling procedure* by: for all $t \in I_n$, for almost all $x \in \mathbb{R}$,

$$(u_a, v_a)(x,t) = \sum_{n=0}^{\infty} (u_a^{(n)}, v_a^{(n)})(x,t) \; \mathbf{1}_{I_n}(t).$$

We also introduce for the following the global indicator function

$$v_i(\cdot,t) = \sum_{n=0}^{\infty} v_i^{(n)}(\cdot,t) \; \mathbf{1}_{I_n}.$$

Let us note that for all $t \in I_0$, we have $v_i = v_f$, which yields

$$v_a^{(0)} = \chi_\delta v_f + (1-\chi_\delta) v_{\mathrm{eq}}. \tag{2.11}$$

Remark 2.1. It is worth noting that neither v_a nor v_i are continuous w.r.t. time due to the adaptation procedure. Nevertheless, the functions are piecewise smooth w.r.t. time, discontinuities appearing only at the ends $(t_k)_{k \geq 1}$ of the time steps. Therefore, the function u_a is uniquely defined. Moreover, in view of Theorem A.1 presented in Appendix A (see also [25]), the total variation of $u_a^{(n)}(\cdot,t)$ is controlled for all $t \in I_n$, so that the application $t \mapsto TV(u_a(\cdot,t))$ is locally bounded on $\mathbb{R}_+$.

Remark 2.2. Notice that, thanks to the assumption $v_f \equiv v_{\mathrm{eq}}$ on $\{|x| > R\} \times \mathbb{R}_+$, one has $\Omega_f(t) \subset \{|x| < R\}$ for all $t \geq 0$.

3. Quantifying the modeling error linked to adaptation

We restrict our study to an arbitrary finite time horizon $T > 0$, and, for the sake of simplicity, we assume that there exists a positive integer N_a such that $T = N_a \Delta t_a$.

3.1. Quantification of the error $\|v_f - v_a\|$.

Lemma 3.1. *There exists c depending only on τ and T (but not on Σ, Σ', δ, and Δt_a) such that, for all $n \geq 0$,*

$$|v_a(x,t_n) - v_f(x,t_n)| \leq c \Sigma \mathbf{1}_{|x|<R}. \tag{3.1}$$

Proof. First, in view of Remark 2.2,

$$v_a(x,t_n) = v_f(x,t_n) \text{ if } |x| \geq R.$$

Now, fix $x \in (-R,R)$. Clearly, because of the definition of $v_a^{(0)}$, one has

$$|v_a(x,0) - v_f(x,0)| \leq \Delta t_a \Sigma \leq T \Sigma, \tag{3.2}$$

so that (3.1) holds for $n = 0$. Now, assume that (3.1) holds for $n \geq 0$, then

$$|v_a(x,t_{n+1}) - v_f(x,t_{n+1})| \leq \mathcal{A}_n + \mathcal{B}_n + \mathcal{C}_n, \tag{3.3}$$

where, due to the Definition (2.9) of $v_a^{(n)}$ and since $v_a(x,t_{n+1}) = v_a^{(n+1)}(x,t_{n+1})$, we have set

$$\mathcal{A}_n = |v_a^{(n+1)}(x,t_{n+1}) - v_i^{(n+1)}(x,t_{n+1})|,$$
$$\mathcal{B}_n = |v_a^{(n)}(x,t_{n+1}) - v_i^{(n)}(x,t_{n+1})|$$

$$\begin{aligned}&= (1-\chi_\delta(x,t_{n+1}))|v_{\mathrm{eq}}(x,t_{n+1}) - v_i^{(n)}(x,t_{n+1})|,\\ \mathcal{C}_n &= |v_i^{(n)}(x,t_{n+1}) - v_f(x,t_{n+1})|.\end{aligned}$$

Bearing in mind the definitions of $\Omega_f^{(n+1)}$ and $\Omega_f^{(n)}$, we have

$$\mathcal{A}_n \le \Delta t_a \Sigma \qquad \text{and} \qquad \mathcal{B}_n \le \Delta t_a \Sigma. \tag{3.4}$$

On the other hand, using the fact that $v_i^{(n)}(x,\cdot)$ and $v_f(x,\cdot)$ satisfy the same linear ODE on I_n for different initial data, $w^{(n)}(x,\cdot) := v_i^{(n)}(x,\cdot) - v_f(x,\cdot)$ is the solution on I_n of

$$\begin{cases}\partial_t w^{(n)} + \frac{1}{\tau} w^{(n)} = 0,\\ w^{(n)}(x,t_n) = v_a(x,t_n) - v_f(x,t_n).\end{cases}$$

Therefore, we can write

$$\mathcal{C}_n = e^{-\Delta t_a/\tau}|v_a(x,t_n) - v_f(x,t_n)|. \tag{3.5}$$

In view of (3.3), (3.4), and (3.5), we obtain that

$$|v_a(x,t_{n+1}) - v_f(x,t_{n+1})| \le 2\Delta t_a \Sigma + e^{-\Delta t_a/\tau}|v_a(x,t_n) - v_f(x,t_n)|,$$

yielding by induction that

$$|v_a(x,t_n) - v_f(x,t_n)| \le \frac{2\Delta t_a}{1-e^{-\Delta t_a/\tau}}\Sigma + e^{-n\Delta t_a/\tau}|v_a(x,0) - v_f(x,0)|.$$

It only remains to check that the function $t \mapsto \frac{2t}{1-e^{-t/\tau}}$ is increasing, so that, since $\Delta t_a \le T$,

$$|v_a(x,t_n) - v_f(x,t_n)| \le \frac{2T}{1-e^{-T/\tau}}\Sigma + |v_a(x,0) - v_f(x,0)|.$$

We conclude by using (3.2). ∎

Lemma 3.2. *There exists c depending only on τ and T (but not on Σ, Σ', δ, and Δt_a) such that*

$$|v_a(x,t) - v_f(x,t)| \le c\Sigma \mathbf{1}_{|x|<R}(x), \qquad \forall (x,t) \in \mathbb{R}\times[0,T). \tag{3.6}$$

Proof. In the case $t = T$, then (3.6) is nothing but (3.1) with $n = N_a$. Assume now that $t \in [0,T)$, then there exists a unique $n \in \{0,\dots,N-1\}$ such that $t \in I_n$. The triangle inequality yields

$$|v_a(x,t) - v_f(x,t)| \le |v_a(x,t) - v_i(x,t)| + |v_i(x,t) - v_f(x,t)|.$$

The first term of the right hand side is bounded by $\Delta t_a \Sigma$, while the second one is bounded by

$$|v_i(x,t) - v_f(x,t)| \le e^{-(t-t_n)/\tau}|v_a^{(n-1)}(x,t_n) - v_f(x,t_n)| \le e^{-(t-t_n)/\tau}(\mathcal{B}_{n-1} + \mathcal{C}_{n-1}),$$

where the quantities $\mathcal{B}_{n-1}$ and $\mathcal{C}_{n-1}$ have been introduced and controlled in the proof of Lemma 3.1. ∎

The proof of Lemmas 3.1 and 3.2 relies on two arguments, namely

1. v_f and v_i are solutions of the same contractive ODE with different initial data;
2. the error between v_a and v_i is controlled thanks to the adaptation procedure.

These two arguments still hold when one considers the space derivatives $\partial_x v_a$, $\partial_x v_f$, and $\partial_x v_i$ instead of the functions v_a, v_f, and v_i, but one has to take care of the perturbation introduced by the buffer of size δ. This leads to the following two lemmas.

LEMMA 3.3. *There exists c depending only on τ and T (but not on Σ, Σ', δ, and Δt_a) such that, for all $n \in \{0,\dots,N_a\}$,*

$$|\partial_x v_a(x,t_n) - \partial_x v_f(x,t_n)| \leq c\left(\Sigma' + \frac{\Sigma}{\delta}\right)\mathbf{1}_{|x|<R}(x), \qquad \forall x \in \mathbb{R}. \tag{3.7}$$

Proof. First, we use again Remark 2.2 to claim that, for all $t \geq 0$ and for all x such that $|x| \geq R$, one has

$$\partial_x v_a(x,t) = \partial_x v_f(x,t).$$

Consider now the case $x \in (-R,R)$. The Definition (2.11) of $v_a^{(0)}$ provides that

$$v_a(x,0) - v_f(x,0) = (1 - \chi_\delta(x,0))(v_{\rm eq}(x,0) - v_f(x,0)),$$

so that, in view of (2.7) and (2.8), one has

$$|\partial_x v_a(x,0) - \partial_x v_f(x,0)| \leq \left(|\partial_x v_{\rm eq}(x,0) - \partial_x v_f(x,0)| + \frac{\alpha_1}{\delta}|v_{\rm eq}(x,0) - v_f(x,0)|\right)\mathbf{1}_{\Omega_c^{(0)}(x)}.$$

It follows from the definition of $\Omega_f^{(0)}$ that

$$\left|\partial_x v_a^{(0)}(x,0) - \partial_x v_f(x,0)\right| \leq \Delta t_a\left(\Sigma' + \frac{\alpha_1 \Sigma}{\delta}\right) \leq c\left(\Sigma' + \frac{\Sigma}{\delta}\right). \tag{3.8}$$

Now, fix $n \in \{0,\dots,N_a - 1\}$, then, since $v_i^{(n+1)}(\cdot,t_{n+1}) \equiv v_a^{(n)}(\cdot,t_{n+1})$, we have

$$|\partial_x v_a(x,t_{n+1}) - \partial_x v_f(x,t_{n+1})| \leq \mathcal{D}_n + \mathcal{E}_n + \mathcal{F}_n, \tag{3.9}$$

where we have set

$$\begin{aligned}
\mathcal{D}_n &= \left|\partial_x\left(v_a^{(n+1)} - v_i^{(n+1)}\right)(x,t_{n+1})\right|,\\
\mathcal{E}_n &= \left|\partial_x\left(v_a^{(n)} - v_i^{(n)}\right)(x,t_{n+1})\right|,\\
\mathcal{F}_n &= \left|\partial_x\left(v_i^{(n)} - v_f\right)(x,t_{n+1})\right|.
\end{aligned}$$

In view of the Definition (2.9) of $v_a^{(n+1)}$, one has

$$v_a^{(n+1)} - v_i^{(n+1)} = (1 - \chi_\delta)\left(v_{\rm eq} - v_i^{(n+1)}\right) \qquad \text{on } \mathbb{R} \times I_{n+1},$$

so that, due to the definitions of $\Omega_f^{(n+1)}$ and χ_δ, we obtain that

$$\mathcal{D}_n \leq \Delta t_a\left(\Sigma' + \frac{\alpha_1 \Sigma}{\delta}\right)\mathbf{1}_{\Omega_c^{(n+1)}}(x). \tag{3.10}$$

Similarly,

$$\mathcal{E}_n \leq \Delta t_a \left(\Sigma' + \frac{\alpha_1 \Sigma}{\delta} \right) \mathbf{1}_{\Omega_c^{(n)}}(x). \tag{3.11}$$

In order to get a bound for $\mathcal{F}_n$, we notice that $z^{(n)} := \partial_x v_i^{(n)} - \partial_x v_f^{(n)}$ is the solution on I_n of

$$\begin{cases} \partial_t z^{(n)} + \frac{1}{\tau} z^{(n)} = 0, \\ z^{(n)}(x,t_n) = \partial_x v_a(x,t_n) - \partial_x v_f(x,t_n), \end{cases}$$

so that, as in the proof of Lemma 3.1, we obtain that

$$\mathcal{F}_n \leq e^{-\Delta t_a/\tau} |\partial_x v_a(x,t_n) - \partial_x v_f(x,t_n)|.$$

We conclude by mimicking the induction detailed in the proof of Lemma 3.1. ∎

We derive the following lemma, whose proof is left to the reader, from Lemma 3.3 by adapting the proof of Lemma 3.2.

Lemma 3.4. *There exists c depending only on τ and T (but not on Σ, Σ', δ, and Δt_a) such that*

$$|\partial_x v_a(x,t) - \partial_x v_f(x,t)| \leq c \left(\Sigma' + \frac{\Sigma}{\delta} \right) \mathbf{1}_{|x|<R}(x) \qquad \forall (x,t) \in \mathbb{R} \times [0,T). \tag{3.12}$$

The last estimate we need concerns the second order space derivative of v_a.

Lemma 3.5. *There exists c depending only on v_{eq}, v_f, and T such that*

$$|\partial_{xx}^2 v_a(x,t)| \leq c \left(1 + \Sigma'' + \frac{\Sigma}{\delta^2} + \frac{\Sigma'}{\delta} \right), \quad \forall (x,t) \in \mathbb{R} \times [0,T). \tag{3.13}$$

Proof. Here again, the proof is based on an induction argument. For $t \in I_0$, it follows from (2.11) that

$$\partial_{xx}^2 v_a^{(0)} = \chi_\delta \partial_{xx}^2 v_f + (1-\chi_\delta) \partial_{xx}^2 v_{\mathrm{eq}} + 2\partial_x (v_f - v_{\mathrm{eq}}) \partial_x \chi_\delta + (v_f - v_{\mathrm{eq}}) \partial_{xx}^2 \chi_\delta.$$

In view of the Definition (2.7) and of the properties (2.8) of the function χ_δ, we get that

$$\left| \partial_{xx}^2 v_a^{(0)} \right| \leq \max\left\{ \left|\partial_{xx}^2 v_f\right|, \left|\partial_{xx}^2 v_{\mathrm{eq}}\right| \right\} + \frac{2\alpha_1 \Sigma'}{\delta} + \frac{\alpha_2 \Sigma}{\delta^2} \quad \text{on } \mathbb{R} \times [0, \Delta t_a]. \tag{3.14}$$

In particular, due to the regularity of v_f and v_{eq}, the relation (3.13) holds for $(x,t) \in \mathbb{R} \times I_0$ for $c = c_0$ defined by

$$c_0 = \max\left\{ \|\partial_{xx}^2 v_f\|_{L^\infty(\mathbb{R}\times I_0)}, \|\partial_{xx}^2 v_{\mathrm{eq}}\|_{L^\infty(\mathbb{R}\times I_0)}, 2\alpha_1, \alpha_2 \right\}. \tag{3.15}$$

Now, fix $n \geq 1$, and assume that there exists c_{n-1} such that

$$|\partial_{xx}^2 v_a^{(n-1)}(x,t^n)| = |\partial_{xx}^2 v_i^{(n)}(x,t^n)| \leq c_{n-1} \left(1 + \Sigma'' + \frac{\Sigma}{\delta^2} + \frac{\Sigma'}{\delta} \right).$$

Then, for all $x\in\mathbb{R}$, the function $t\mapsto \partial^2_{xx} v_i^{(n)}(x,t)$ is the solution of the linear ODE

$$\begin{cases} \partial_t\partial^2_{xx} v_i^{(n)}(x,t) + \frac{1}{\tau}\partial^2_{xx} v_i^{(n)}(x,t) = \frac{1}{\tau}\partial^2_{xx} v_{\mathrm{eq}}(x,t), \\ \partial^2_{xx} v_i^{(n)}(x,t_n) = \partial^2_{xx} v_a^{(n-1)}(x,t_n). \end{cases}$$

In particular, for all $t\in\overline{I}_n$, one has

$$|\partial^2_{xx} v_i^{(n)}(x,t)| \le |\partial^2_{xx} v_i^{(n)}(x,t^n)| \le c_{n-1}\left(1+\Sigma''+\frac{\Sigma}{\delta^2}+\frac{\Sigma'}{\delta}\right). \tag{3.16}$$

Recall that the function $v_a^{(n)}$ is then defined by (2.9), so that, on $\mathbb{R}\times\overline{I}_n$, one has

$$\partial^2_{xx} v_a^{(n)} = \chi_\delta \partial^2_{xx} v_i^{(n)} + (1-\chi_\delta)\partial^2_{xx} v_{\mathrm{eq}} + 2\partial_x(v_i^{(n)} - v_{\mathrm{eq}})\partial_x\chi_\delta + (v_i^{(n)} - v_{\mathrm{eq}})\partial^2_{xx}\chi_\delta. \tag{3.17}$$

Since $v_a^{(n)} \equiv v_i^{(n)}$ in $\Omega_f^{(n)}\times\overline{I}_n$, one gets directly from (3.16) that

$$|\partial^2_{xx} v_a^{(n)}(x,t)| \le c_{n-1}\left(1+\Sigma''+\frac{\Sigma}{\delta^2}+\frac{\Sigma'}{\delta}\right), \quad \forall (x,t)\in\Omega_f^{(n)}\times\overline{I}_n. \tag{3.18}$$

Now, for $(x,t)\in\overline{\Omega_c^{(n)}\times I_n}$, one has

$$\left|\partial^2_{xx} v_i^{(n)}\right| \le \Sigma'', \quad \left|\partial_x(v_i^{(n)} - v_{\mathrm{eq}})\right| \le \Sigma', \quad \text{and} \quad \left|v_i^{(n)} - v_{\mathrm{eq}}\right| \le \Sigma.$$

Therefore, using again (2.8) in (3.17), we obtain

$$|\partial^2_{xx} v_a^{(n)}(x,t)| \le \|\partial^2_{xx} v_{\mathrm{eq}}\|_\infty + \Sigma'' + \frac{2\alpha_1\Sigma'}{\delta} + \frac{\alpha_2\Sigma}{\delta^2}, \quad \forall (x,t)\in\overline{\Omega_c^{(n)}\times I_n}. \tag{3.19}$$

In particular, it follows from (3.18) and (3.19) that

$$|\partial^2_{xx} v_a^{(n)}(x,t)| \le c_n\left(1+\Sigma''+\frac{\Sigma}{\delta^2}+\frac{\Sigma'}{\delta}\right),$$

where

$$c_n = \max\left\{c_{n-1},\, \|\partial^2_{xx} v_{\mathrm{eq}}\|_{L^\infty(\mathbb{R}\times(0,T))},\, 2\alpha_1,\, \alpha_2\right\}.$$

In view of the Definition (3.15) of c_0, we obtain by a straightforward induction that (3.13) holds with

$$c = \max\left\{\|\partial^2_{xx} v_f\|_{L^\infty(\mathbb{R}\times(0,T))},\, \|\partial^2_{xx} v_{\mathrm{eq}}\|_{L^\infty(\mathbb{R}\times(0,T))},\, 2\alpha_1,\, \alpha_2\right\}.$$

□

3.2. Quantification of the error $\|u_f - u_a\|$. In this section, our goal is to quantify the error produced by the adaptation procedure described in Section 2.3. To do so, we will overestimate

$$\|u_a - u_f\|_{C([0,T],L^1(\mathbb{R}))}, \tag{3.20}$$

where T is an arbitrary final time, thanks to quantities depending on Σ, Σ', and δ. Then, for a suitable choice of these quantities, we will guarantee that the modeling error (3.20) can be enforced to remain as small as desired.

Setting

$$f:(s,x,t)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}_+\mapsto F(s,v_a(x,t)) \tag{3.21}$$

and

$$g:(s,x,t)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}_+\mapsto F(s,v_f(x,t)), \tag{3.22}$$

the function u_a is then defined as the unique entropy solution to the problem

$$\begin{cases}\partial_t u_a+\partial_x f(u_a)=0 & \text{in } \mathbb{R}\times(0,\infty),\\ u_a(\cdot,0)=u_0 & \text{in } \mathbb{R},\end{cases}$$

while u_f can be seen as the unique entropy solution to the problem

$$\begin{cases}\partial_t u_f+\partial_x g(u_f)=0 & \text{in } \mathbb{R}\times(0,\infty),\\ u_f(\cdot,0)=u_0 & \text{in } \mathbb{R}.\end{cases}$$

Therefore, in order to quantify the difference between u_a and u_f, we will use a stability result w.r.t. to the flux function established in Appendix A.

Using Theorem A.3, we obtain that for all $t^\star>0$, one has

$$\|u_a(\cdot,t^\star)-u_f(\cdot,t^\star)\|_{L^1(\mathbb{R})}\le\inf_{\epsilon>0}\left(\epsilon C_1+\frac{C_2}{\epsilon}+C_3\right), \tag{3.23}$$

where, in view of the definition of the functions f and g, we have set

$$C_1=\alpha+\beta\|\partial_s\partial_x f\|_\infty+3\int_0^{t^\star}\sup_{s\in\mathbb{R}}\|\partial^2_{xx}f(s,\cdot,t)\|_\infty dt, \tag{3.24}$$

$$C_2=2\int_0^{t^\star}\int_{\mathbb{R}}\|(f-g)(\cdot,x,t)\|_\infty dxdt, \tag{3.25}$$

$$C_3=\int_0^{t^\star}\int_{\mathbb{R}}\|\partial_x(f-g)(\cdot,x,t)\|_\infty dxdt. \tag{3.26}$$

In the Definition (3.24) of C_1, the quantities α and β depend only on $t^\star$ (in an increasing way), u_0, F and on the fine solution v_f, but not on the adaptation procedure. They are made explicit in Appendix A.2.

Remark 3.1.

1. Theorem A.3 gives a localized in space error estimate. Nevertheless, since the flux functions $s\mapsto F(s,v_a(x,t))$ and $s\mapsto F(s,v_f(x,t))$ coincide on $\{|x|>R\}\times\mathbb{R}_+$, the contribution of the flux functions variation is located on a finite measure space. Then, the error produced by the adaptation procedure travels with a speed lower than or equal to $\|\partial_u F\|_\infty$, so that after a time t, the functions $u_f(\cdot,t)$ and $u_a(\cdot,t)$ may differ only on $\{|x|<R+t\|\partial_u F\|_\infty\}$. Therefore, considering the L^1-norm on the full space $\mathbb{R}$ is meaningful in (3.23).
2. Because of the previous point, the integrals w.r.t. the space variable appearing in (3.24)–(3.26) can be considered on the full space $\mathbb{R}$. As a consequence, the integration domains do not depend on ϵ as it is the case in the more general case presented in Appendix A.2. Therefore, the quantities defined as C_i^ϵ in Appendix A.2 do not depend on ϵ in the present case. This justifies the fact that we denote them by C_i, without ϵ.

LEMMA 3.6. *There exists $c>0$ depending only on the data F, u_0, v_f, v_{eq} and in an increasing way of $t^\star$ (but not on the adaptation procedure) such that C_1, defined by* (3.24), *satisfies*

$$C_1 \le c\left(1+\Sigma''+\Sigma'+\frac{\Sigma}{\delta}+(\Sigma')^2+\frac{\Sigma^2}{\delta^2}+\frac{\Sigma'}{\delta}+\frac{\Sigma}{\delta^2}\right). \tag{3.27}$$

Proof. Bearing in mind the Definition (3.21) of the flux function f, we obtain that

$$\partial_s\partial_x f(s,x,t) = \partial_s\partial_v F(s,v_a(x,t))\partial_x v_a(x,t),$$

so that

$$\|\partial_s\partial_x f\|_\infty \le \|\partial_s\partial_v F\|_\infty (\|\partial_x v_f\|_\infty + \|\partial_x(v_a-v_f)\|_\infty).$$

In view of Lemma 3.4, we deduce that

$$\|\partial_s\partial_x f\|_\infty \le \|\partial_u\partial_v F\|_\infty \left(\|\partial_x v_f\|_\infty + \Sigma' + \frac{2}{\delta}\Sigma\right). \tag{3.28}$$

Similarly, we have

$$\partial^2_{xx} f(u,x,t) = \partial^2_{vv}F(u,v_a(x,t))\,(\partial_x v_f(x,t))^2 + \partial_v F(u,v_a(x,t))\partial^2_{xx} v_a,$$

so that

$$\sup_{s\in\mathbb{R}} \left\|\partial^2_{xx} f(s,\cdot,t)\right\| \le \left\|\partial^2_{vv}F\right\|_\infty (\|\partial_x v_f\|_\infty + \|\partial_x(v_a-v_f)\|_\infty)^2 + \|\partial_v F\|_\infty \|\partial^2_{xx} v_a(\cdot,t)\|_\infty.$$

Using Lemmas 3.4 and 3.5, we deduce from the fact that $\left(\sum_{i=1}^k a_i\right)^2 \le k\sum_{i=1}^k (a_i)^2$ that

$$\begin{aligned}\sup_{s\in\mathbb{R}} \left\|\partial^2_{xx} f(s,\cdot,t)\right\| \le 3\|\partial^2_{vv}F\|_\infty &\left(\|\partial_x v_f\|^2_\infty + (\Sigma')^2 + \frac{1}{\delta^2}\Sigma^2\right)\\ &+ \|\partial_v F\|_\infty c\left(1+\Sigma''+\frac{\Sigma'}{\delta}+\frac{\Sigma}{\delta^2}\right).\end{aligned} \tag{3.29}$$

Hence, denoting by c a generic quantity depending only on the data F, u_0, v_f, v_{eq}, and, in an increasing way, on $t^\star$, we deduce from (3.24), (3.28), and (3.29) that (3.27) holds. ∎

LEMMA 3.7. *There exists c depending only on the data F, R, and $t^\star$ in an increasing way (but not on the adaptation procedure) such that C_2, defined by* (3.25), *satisfies*

$$C_2 \le c\Sigma. \tag{3.30}$$

Proof. In view of the regularity of F and of Lemma 3.2, we have

$$|f-g| = |F(s,v_a)-F(s,v_f)| \le \|\partial_v F\|_\infty |v_f - v_a| \le \|\partial_v F\|_\infty \Sigma \mathbf{1}_{(-R,R)}.$$

Estimate (3.30) then follows directly from the Definition (3.25) of C_2. ∎

LEMMA 3.8. *There exists c depending only on the data F, v_f, R, and, in an increasing way, on $t^\star$ (but not on the adaptation procedure) such that C_3, defined by* (3.26), *satisifes*

$$C_3 \le c\left(\Sigma+\Sigma'+\frac{\Sigma}{\delta}\right). \tag{3.31}$$

Proof. For all $s \in \mathbb{R}$, one has

$$\partial_x f(s,x,t) = \partial_x F(s,v_a(x,t)) = \partial_v F(s,v_a(x,t))\partial_x v_a(x,t).$$

Similarly,

$$\begin{aligned}\partial_x g(s,x,t) &= \partial_v F(s,v_f(x,t))\partial_x v_f(x,t)\\ &= \partial_v F(s,v_a(x,t))\partial_x v_f(x,t) + [\partial_v F(s,v_a(x,t)) - \partial_v F(s,v_a(x,t))]\,\partial_x v_f(x,t).\end{aligned}$$

Therefore,

$$|\partial_x(f-g)| \leq \|\partial_v F\|_\infty |\partial_x(v_f - v_a)| + \|\partial_x v_f\|_\infty \|\partial^2_{vv} F\|_\infty |v_f - v_a|.$$

The Lemmas 3.2 and 3.4 then yield

$$\sup_{s\in\mathbb{R}} |\partial_x(f-g)(s,x,t)| \leq \left(\|\partial_v F\|_\infty \left(\Sigma' + \frac{2}{\delta}\Sigma\right) + \|\partial_x v_f\|_\infty \|\partial^2_{vv} F\|_\infty \Sigma \right) \mathbf{1}_{(-R,R)}(x).$$

Estimate (3.31) follows from integrating on $\mathbb{R}\times(0,t^\star)$. ◻

We aim at letting δ, Σ, and Σ' go to 0, but it is expected that a good scaling can be selected so that the error contributions for C_1, C_2, and C_3 can be balanced. The first step consists of course in balancing the error contribution within each C_i. To do so, it appears from (3.27) and (3.31) that it is relevant to fix

$$\Sigma' = \frac{\Sigma}{\delta}. \tag{3.32}$$

We also propose to choose Σ'' as a constant, and for the sake of simplicity, we set

$$\Sigma'' = 1. \tag{3.33}$$

Because we investigate the limit $\delta,\Sigma,\Sigma' \to 0$, this implies that Σ is negligible w.r.t. $\Sigma' = \Sigma/\delta$. As a consequence of the choices (3.32) and (3.33), we obtain for Lemmas 3.6, 3.7, and 3.8 that

$$C_1 \leq c\left(1 + \frac{\Sigma}{\delta^2}\right), \qquad C_2 \leq c\Sigma, \quad \text{and} \quad C_3 \leq c\frac{\Sigma}{\delta}. \tag{3.34}$$

Now, balancing the contributions in C_1 suggest that we choose

$$\delta = \Sigma^{1/2}, \tag{3.35}$$

so that (3.34) turns to

$$C_1 \leq c, \qquad C_2 \leq c\Sigma, \quad \text{and} \quad C_3 \leq c\Sigma^{1/2}, \tag{3.36}$$

where c denotes a generic quantity depending on the data and on $t^\star$ in an increasing way.

THEOREM 3.9. *Fix $\Sigma > 0$, then with the choices* (3.32) *and* (3.35) *of the parameters δ and Σ', there exists c depending only on u_0, F, v_f, R, and T such that*

$$\|u_a - u_f\|_{\mathcal{C}([0,T];L^1(\mathbb{R}))} \leq c\Sigma^{1/2}. \tag{3.37}$$

Proof. Let $t^\star \in [0,T]$, then taking (3.36) into account in (3.23) provides that

$$\|u_a(\cdot,t^\star) - u_f(\cdot,t^\star)\|_{L^1(\mathbb{R})} \le c(t^\star) \inf_{\epsilon>0} \left(\epsilon + \frac{\Sigma}{\epsilon} + \Sigma^{1/2} \right),$$

where the dependence of c w.r.t. $t^\star$ has been stressed. Therefore, since c depends in an increasing way of $t^\star$, we deduce that

$$\|u_a(\cdot,t^\star) - u_f(\cdot,t^\star)\|_{L^1(\mathbb{R})} \le c(T) \inf_{\epsilon>0} \left(\epsilon + \frac{\Sigma}{\epsilon} + \Sigma^{1/2} \right).$$

The optimal choice of ϵ for minimizing the right hand side is clearly $\epsilon = \Sigma^{1/2}$, yielding

$$\|u_a(\cdot,t^\star) - u_f(\cdot,t^\star)\|_{L^1(\mathbb{R})} \le c(T)\Sigma^{1/2}.$$

Since both u_a and u_f belong to $\mathcal{C}([0,T];L^1_{\mathrm{loc}}(\mathbb{R}))$ (see [7]) and since $u_a - u_f$ is compactly supported in space with support in $[-R,R]$, we obtain that $u_a - u_f$ belongs to $\mathcal{C}([0,T];L^1(\mathbb{R}))$ and that (3.37) holds. ∎

4. Application to a model of transport with inertia

In this section, we illustrate the dynamic model adaptation procedure on the simple example of a transport equation. Let $v_{\mathrm{eq}} \in \mathcal{C}^2(\mathbb{R}\times\mathbb{R}_+)$ be the material speed of the flow, that is supposed to be given. Then we are interested in the concentration u of a chemical component convected by the flow. The coarse model consists of assuming that the speed of the chemical particles is exactly given by v_{eq}. This yields the equation

$$\begin{cases} \partial_t u_c + \partial_x(v_{\mathrm{eq}} u_c) = 0, \\ u_c(0) = u_0. \end{cases} \tag{4.1}$$

But taking into account the inertia of the particles leads to consider that their speed does not coincide with the material speed of the flow, but is given by v_f defined by the ODE

$$\begin{cases} \partial_t v_f = \frac{1}{\tau}(v_{\mathrm{eq}} - v_f), \\ v_f(\cdot,0) = v_0. \end{cases} \tag{4.2}$$

The resulting concentration u_f obeys the equation

$$\begin{cases} \partial_t u_f + \partial_x(v_f u_f) = 0, \\ u_f(0) = u_0, \end{cases} \tag{4.3}$$

where v_f is solution of (4.2).

As soon as the concentrations u_f and u_c remain uniformly bounded, the problem enters the frame exposed in Section 2. Therefore, we can apply the dynamic adaptation procedure described in Section 2.3 while controlling the error thanks to the analysis carried out in Section 3.

In order to illustrate our purpose, we have computed numerically the solutions u_c, u_f and u_a thanks to an explicit upwind finite volume scheme. The speed v_f, as well as the speed v_i necessary to build v_a are computed thanks to a fourth order Runge–Kutta scheme.

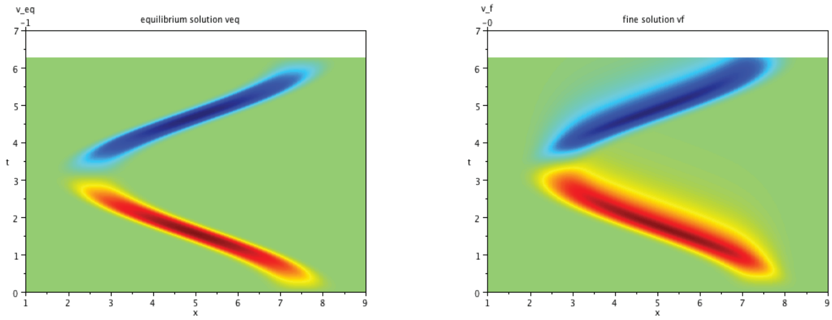

FIG. 4.1. *The equilibrium speed* v_{eq} *(left) and the effective speed* v_f *(right) plotted in the* (x,t)*-plane.*

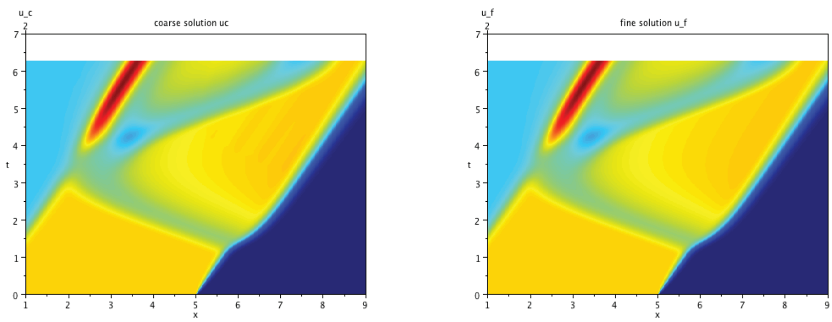

FIG. 4.2. *The concentration* u_c *computed with the coarse model* (4.1) *(left) and the concentration* u_f *computed with the fine model* (4.3) *(right) plotted in the* (x,t)*-plane.*

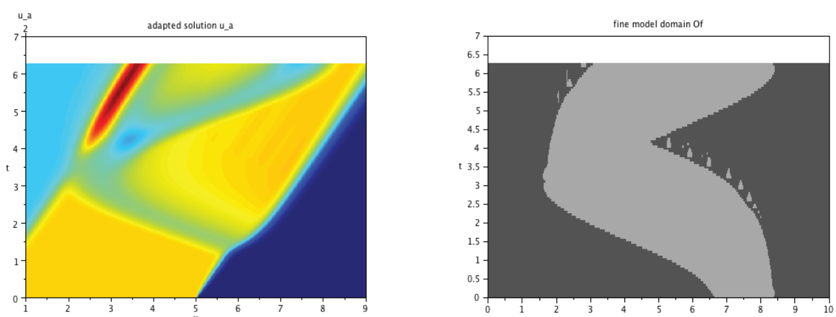

FIG. 4.3. *The concentration* u_a *computed* via *the adapted model described in Section 2.3 (left) and the evolution of the domains* Ω_f *(red) and* Ω_c *(blue) w.r.t. time (right) plotted in the* (x,t)*-plane.* $\Sigma=0.1$.

We define the function ν by

$$\nu(x)=\begin{cases}1 & \text{if } x\leq 0,\\ 1-8x^3(1-x) & \text{if } x\in[0,1/2],\\ 1-\nu(1-x) & \text{if } x\geq 1/2,\end{cases}$$

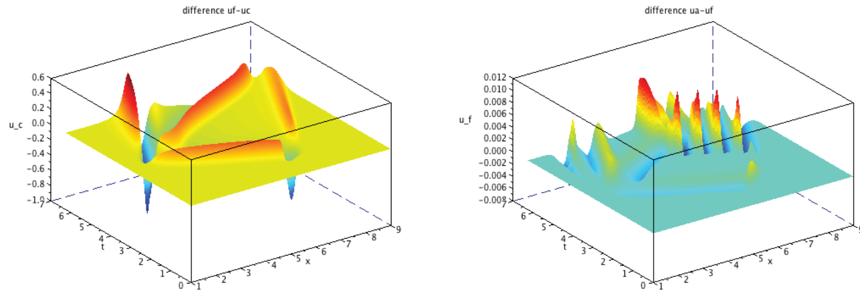

Fig. 4.4. *Plot in the* (x,t)*-plane of the difference between the fine and the coarse solutions* $u_c - u_f$ *(left), and the difference between the fine and the adapted solution* $u_a - u_f$ *(right,* $\Sigma = 0.1$*). Notice the difference in the scales along the vertical axes.*

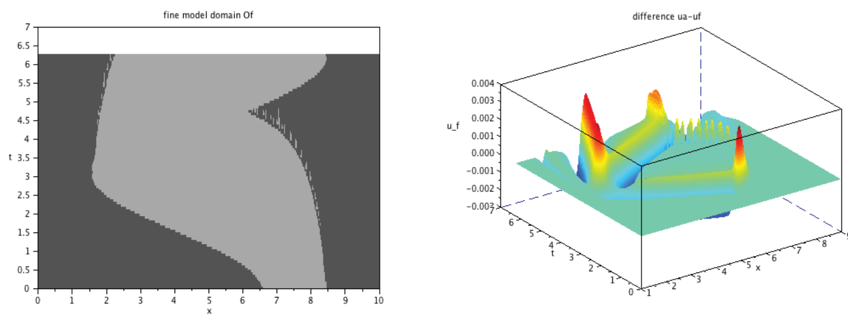

Fig. 4.5. *We plot in the* (x,t)*-plane the results of computations for* $\Sigma = 0.01$*, i.e., the fine domain* $\Omega_f(t)$ *(in light gray, left), and the difference* $u_a - u_f$ *(right). For the left figure, check that the size of the fine domain grows when* Σ *decreases. For the right figure, notice the difference in the scale along the vertical axes w.r.t. Figure 4.4.*

then the equilibrium speed, plotted in the (x,t)-plane on Figure 4.1, is given by

$$v_{\text{eq}}(x,t) = \nu(|x - 5 - 2.5\cos(t)|) + 0.3.$$

The relaxation parameter τ is set equal to 0.5.

For the adaptation time step, we have set $\Delta t_a = \pi/50$, and the numerical time step has been set to $\Delta t = \Delta t_a/10$. Following the analysis carried out in Section 3, we use Σ as a parameter, and Σ', Σ'', and δ are fixed by

$$\Sigma' = \delta = \Sigma^{1/2}, \qquad \Sigma'' = 1.$$

We define $u_0 = \mathbf{1}_{x \le 5}$, so that we can build the functions u_f, u_c and u_a for any choice of the parameter Σ.

We present now the results obtained for $\Sigma = 0.1$. On Figures 4.2 and 4.3, we can see that all the solutions look similar. Nevertheless, it appears on Figure 4.4 that, as expected by our study, the adapted solution u_a is much closer to the fine solution u_f than u_c.

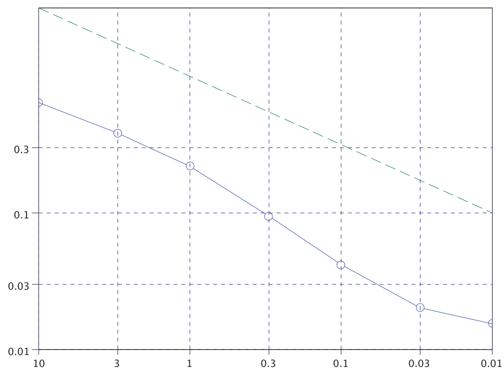

FIG. 4.6. *Numerical evaluation of* $\|u_a - u_f\|_{L^1((1,9)\times(0,2\pi))}$ *as a function of* Σ *(solid blue) plotted in* log*-scale, linear function with slope* $-1/2$ *(dashed green).*

We plot some results obtained for the lower value of $\Sigma = 0.01$ on Figure 4.5. As expected, the fine domain Ω_f in this case is larger than in the case where $\Sigma = 0.1$, and the error $|u_a - u_f|$ is smaller.

Finally, we present on Figure 4.6 the $L^1((1,9)\times(0,2\pi))$-error between the fine solution u_f and the adapted solution u_a. The saturation of the convergence if due to the numerical approximation of the solution. Indeed, refining the space and time discretization makes the saturation value decrease.

5. Conclusion

We derive in this work an error estimate for a simple algorithm of dynamic model adaptation applied to nonlinear hyperbolic equations. In order to perform this analysis, we have to consider thick interfaces of coupling. Using Theorem 3.9, we are able to define the size δ of the smooth buffer which connects the fine model to the coarse model w.r.t. the parameters of the model and of the procedure. Note that we were led to adapt the stability results obtained by [25] in our context. We also provide some numerical computations to illustrate the optimality of our result.

We only concentrate on a discrete-in-time procedure for the model adaptation. It would be interesting to include in the analysis the numerical error to obtain a full numerical procedure of model adaptation, as done for instance in [5]. The main tools can be found in the works of [9] and [22]. However, since our method is discrete w.r.t. time, the flux f of the underlying scalar conservation law

$$\partial_t u + \partial_x f(u,x,t) = 0$$

is discontinuous w.r.t. at each time step. As a consequence, we need to slightly refine the result of Chainais-Hillairet.

Appendix A. Stability results for scalar conservation laws.

In this appendix (precisely, in Theorem A.3), we state a new stability result for entropy solutions of scalar conservation laws w.r.t. their flux functions. Despite the fact that all this section is written in the one-dimensional space dimension, it can be extended to the multidimensional case without particular difficulty, except for the heavy notations involved in the study.

The stability result presented in Theorem A.3 is an extension to the case of time and space dependent flux functions of a result presented in [3]. It relies on stability estimates proved in [25] (see also [13]) that are recalled in Theorem A.1.

A.1. Total variation estimates for Kruzhkov entropy solutions. We consider two functions

$$f:\begin{cases}\mathbb{R}\times\mathbb{R}\times\mathbb{R}_+\to\mathbb{R}\\ (s,x,t)\mapsto f(s;x,t),\end{cases}\qquad\text{and}\qquad g:\begin{cases}\mathbb{R}\times\mathbb{R}\times\mathbb{R}_+\to\mathbb{R}\\ (s,x,t)\mapsto f(s;x,t),\end{cases}$$

being continuous on $\mathbb{R}\times\mathbb{R}\times\mathbb{R}_+$, and having continuous derivatives $\partial_s f$, $\partial_s\partial_x f$, and $\partial^2_{xx}f$ (resp. $\partial_s g$, $\partial_s\partial_x g$, and $\partial^2_{xx}g$). We assume that $f(0;x,t)=g(0;x,t)$ for all $(x,t)\in\mathbb{R}\times\mathbb{R}_+$, and that

$$L_f := \sup_{(x,t)\in\mathbb{R}\times\mathbb{R}_+}\sup_{s_1,s_2\in\mathbb{R}}\left|\frac{f(s_1;x,t)-f(s_2;x,t)}{s_1-s_2}\right|<\infty,$$
$$L_g := \sup_{(x,t)\in\mathbb{R}\times\mathbb{R}_+}\sup_{s_1,s_2\in\mathbb{R}}\left|\frac{g(s_1;x,t)-g(s_2;x,t)}{s_1-s_2}\right|<\infty,$$

that

$$\|\partial_s\partial_x f\|_\infty := \sup_{s\in\mathbb{R}}\sup_{(x,t)\in\mathbb{R}\times\mathbb{R}_+}|\partial_s\partial_x f(s;x,t)|<\infty,$$
$$\|\partial_s\partial_x g\|_\infty := \sup_{s\in\mathbb{R}}\sup_{(x,t)\in\mathbb{R}\times\mathbb{R}_+}|\partial_s\partial_x g(s;x,t)|<\infty,$$

and that, for all $T>0$, for $\phi\in\{f,g\}$,

$$(x,t)\mapsto\|\partial^2_{xx}\phi(\cdot;x,t)\|_\infty \text{ belongs to } L^1(\mathbb{R}\times(0,T))\cap L^\infty(\mathbb{R};L^1(0,T)). \tag{A.1}$$

Let $u\in L^\infty(\mathbb{R}\times\mathbb{R}_+)$ be the unique entropy solution of

$$\begin{cases}\partial_t u+\partial_x f(u;x,t)=0 \quad \text{in } \mathbb{R}\times\mathbb{R}_+,\\ u(\cdot,0)=u_0\in L^\infty(\mathbb{R}),\end{cases} \tag{A.2}$$

and let $v\in L^\infty(\mathbb{R}\times\mathbb{R}_+)$ be the unique entropy solution of

$$\begin{cases}\partial_t v+\partial_x g(v;x,t)=0 \quad \text{in } \mathbb{R}\times\mathbb{R}_+,\\ v(\cdot,0)=v_0\in L^\infty(\mathbb{R}).\end{cases} \tag{A.3}$$

Recall that u is defined by: $\forall\kappa\in\mathbb{R}$, $\forall\psi\in\mathcal{C}^\infty_c(\mathbb{R}\times\mathbb{R}_+)$,

$$\begin{aligned}&\iint_{\mathbb{R}\times\mathbb{R}_+}|u-\kappa|\partial_t\psi\,dxdt+\int_{\mathbb{R}}|u_0-\kappa|\psi(\cdot,0)dx\\ &+\iint_{\mathbb{R}\times\mathbb{R}_+}\operatorname{sign}(u-\kappa)(f(u;x,t)-f(\kappa;x,t))\partial_x\psi\,dxdt\\ &-\iint_{\mathbb{R}\times\mathbb{R}_+}\operatorname{sign}(u-\kappa)\partial_x f(\kappa;x,t)\psi\,dxdt\ge 0,\end{aligned} \tag{A.4}$$

while v is defined by: $\forall\kappa\in\mathbb{R}$, $\forall\psi\in\mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{R}_+)$,

$$\begin{aligned}&\iint_{\mathbb{R}\times\mathbb{R}_+}|v-\kappa|\partial_t\psi dxdt+\int_{\mathbb{R}}|v_0-\kappa|\psi(\cdot,0)dx\\&+\iint_{\mathbb{R}\times\mathbb{R}_+}\operatorname{sign}(v-\kappa)(g(v;x,t)-g(\kappa;x,t))\partial_x\psi dxdt\\&-\iint_{\mathbb{R}\times\mathbb{R}_+}\operatorname{sign}(v-\kappa)\partial_x g(\kappa;x,t)\psi dxdt\geq 0.\end{aligned}\tag{A.5}$$

THEOREM A.1 ([7, 25]). *The unique entropy solution u to the problem* (A.2) *belongs to $\mathcal{C}(\mathbb{R}_+,L^1_{loc}(\mathbb{R}))$ and*

$$\lim_{t\to 0}\|u(\cdot,t)-u_0\|_{L^1(\mathbb{R})}=0.\tag{A.6}$$

Moreover, if u_0 belongs to $BV(\mathbb{R})$, then, for all $T>0$, $u(\cdot,T)\in BV(\mathbb{R})$, and there exists c_f depending only on f such that

$$TV(u(T))\leq TV(u_0)e^{c_fT}+\int_0^T e^{c_f(T-t)}\int_{\mathbb{R}}\left\|\partial^2_{xx}f(\cdot;x,t)\right\|_\infty dxdt,$$

where

$$\left\|\partial^2_{xx}f(\cdot;x,t)\right\|_\infty:=\sup_{s\in\mathbb{R}}\left|\partial^2_{xx}f(s;x,t)\right|,$$

and

$$c_f=3\|\partial_x\partial_s f\|_{L^\infty(\mathbb{R}\times\mathbb{R}\times\mathbb{R}_+)}.$$

In what follows, we will often have to deal with the quantity

$$\int_0^T \mathrm{TV}(u(t))dt=TV(u_0)\frac{e^{c_fT}-1}{c_f}+\int_0^T\int_0^t e^{c_f(t-\tau)}\int_{\mathbb{R}}\|\partial^2_{xx}f(\cdot;x,t)\|_\infty dxd\tau dt.$$

We denote by

$$\begin{cases}C_{f,0}(T)=\frac{e^{c_fT}-1}{c_f},\\ C_{f,1}(T)=\int_0^T\int_0^t e^{c_f(t-\tau)}\int_{\mathbb{R}}\|\partial^2_{xx}f(\cdot;x,t)\|_\infty dxd\tau dt,\end{cases}$$

so that

$$\int_0^T \mathrm{TV}(u(t))dt=C_{f,0}(T)\mathrm{TV}(u_0)+C_{f,1}(T).\tag{A.7}$$

Similarly, we have

$$\int_0^T \mathrm{TV}(v(t))dt=C_{g,0}(T)\mathrm{TV}(v_0)+C_{g,1}(T),\tag{A.8}$$

where the quantities $C_{g,0}$ and $C_{g,1}$ are obtained from $C_{f,0}$ and $C_{f,1}$ by replacing the flux function f by the flux function g.

A.2. Stability w.r.t. the flux functions. We first state the following technical lemma.

LEMMA A.2. *For all $\kappa \in \mathbb{R}$, there exists $A_\kappa, B_\kappa \in L^\infty(\mathbb{R}\times\mathbb{R}_+)$ such that*

$$\begin{aligned}
&\iint_{\mathbb{R}\times\mathbb{R}_+} |v-\kappa|\partial_t\psi dxdt + \int_{\mathbb{R}} |v_0-\kappa|\psi(\cdot,0)dx \\
&+\iint_{\mathbb{R}\times\mathbb{R}_+} sign(v-\kappa)(f(v;x,t)-f(\kappa;x,t))\partial_x\psi dxdt \\
&-\iint_{\mathbb{R}\times\mathbb{R}_+} sign(v-\kappa)\partial_x f(\kappa;x,t)\psi dxdt \\
\geq &\iint_{\mathbb{R}\times\mathbb{R}_+} A_\kappa \partial_x\psi\, dxdt + \iint_{\mathbb{R}\times\mathbb{R}_+} B_\kappa\,\psi\, dxdt.
\end{aligned} \tag{A.9}$$

Moreover, for all $\kappa \in \mathbb{R}$,

$$|A_\kappa(x,t)| \leq 2\,\|(f-g)(\cdot;x,t)\|_\infty, \qquad \forall (x,t)\in\mathbb{R}\times\mathbb{R}_+, \tag{A.10}$$

while

$$|B_\kappa(x,t)| \leq \|\partial_x(f-g)(\cdot;x,t)\|_\infty, \qquad \forall (x,t)\in\mathbb{R}\times\mathbb{R}_+. \tag{A.11}$$

Proof. First of all, note that, thanks to (A.5), the function v satisfies: $\forall \kappa\in\mathbb{R}$, $\forall \psi \in \mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{R}_+)$,

$$\begin{aligned}
&\iint_{\mathbb{R}\times\mathbb{R}_+} |v-\kappa|\partial_t\psi dxdt + \int_{\mathbb{R}} |v_0-\kappa|\psi(\cdot,0)dx \\
&+\iint_{\mathbb{R}\times\mathbb{R}_+} \mathrm{sign}(v-\kappa)(f(v;x,t)-f(\kappa;x,t))\partial_x\psi dxdt \\
&-\iint_{\mathbb{R}\times\mathbb{R}_+} \mathrm{sign}(v-\kappa)\partial_x f(\kappa;x,t)\psi dxdt \\
\geq &\iint_{\mathbb{R}\times\mathbb{R}_+} A_\kappa \partial_x\psi\, dxdt + \iint_{\mathbb{R}\times\mathbb{R}_+} B_\kappa\,\psi\, dxdt,
\end{aligned}$$

where

$$\begin{aligned}
A_\kappa(x,t) &= \mathrm{sign}(v(x,t)-\kappa)\left[(f-g)(v(x,t);x,t)-(f-g)(\kappa;x,t)\right],\\
B_\kappa(x,t) &= \mathrm{sign}(v(x,t)-\kappa)\partial_x(f-g)(\kappa;x,t).
\end{aligned}$$

Clearly, the relations (A.10) and (A.11) hold for all $\kappa\in\mathbb{R}$. ∎

We now give a stability result which is an adaptation of Theorem 3.1 of [3] in the case of time-space depending flux functions. For $x_0\in\mathbb{R}$ and $R>0$, we denote

$$B(x_0,R) = \left\{x\in\mathbb{R} \;\middle|\; |x-x_0|<R\right\}.$$

THEOREM A.3. *Let $x_0\in\mathbb{R}$, and let $u_0\in BV(\mathbb{R})$, then, for all $T>0$ and for all $R>0$,*

$$\|u(\cdot,T)-v(\cdot,T)\|_{L^1(B(x_0,R))} \leq \|u_0-v_0\|_{L^1(B(x_0,R+L_fT))} + \inf_{\epsilon>0}\left(\epsilon C_1^\epsilon + \frac{C_2^\epsilon}{\epsilon} + C_3^\epsilon\right),$$

where

$$\begin{aligned}
C_1^\epsilon :=& TV(v_0)\left(1+e^{c_g T}+2\|\partial_s\partial_x f\|_\infty C_{g,0}(T)\right)\\
&+2\|\partial_s\partial_x f\|_\infty C_{g,1}(T)+3\int_0^T \sup_{s\in\mathbb{R}}\|\partial^2_{xx} f(s;\cdot,t)\|_{L^\infty(B_t^\epsilon)},\\
C_2^\epsilon :=& 2\int_0^T\int_{y\in B_t^\epsilon}\|(f-g)(\cdot;y,t)\|_\infty dydt,\\
C_3^\epsilon :=& \int_0^T\int_{y\in B_t^\epsilon}\|\partial_y(f-g)(\cdot;y,t)\|_\infty dydt.
\end{aligned}$$

Proof. For the sake of simplicity, we only perform the proof for $x_0=0$, but clearly, adapting it to any $x_0\in\mathbb{R}$ does not provide any additional difficulty. We follow the idea of [23, 24] and [3], and carry out a proof based on the doubling variable technique. Let $\xi:\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+\to\mathbb{R}_+$ be a smooth and compactly supported function. Then it follows from (A.4) and (A.12) that

$$\begin{aligned}
&\iint_{\mathbb{R}\times\mathbb{R}_+}\iint_{\mathbb{R}\times\mathbb{R}_+}|u(x,t)-\kappa|\partial_t\xi(x,t,y,s)dxdtdyds\\
&+\iint_{\mathbb{R}\times\mathbb{R}_+}\int_{\mathbb{R}}|u_0(x)-\kappa|\xi(x,0,y,s)dxdyds\\
&+\iint_{\mathbb{R}\times\mathbb{R}_+}\iint_{\mathbb{R}\times\mathbb{R}_+}\left\{\begin{matrix}\hfill\operatorname{sign}(u(x,t)-\kappa)\cdot\\ (f(u(x,t);x,t)-f(\kappa;x,t))\partial_x\xi(x,t,y,s)\end{matrix}\right\}dxdtdyds\\
&-\iint_{\mathbb{R}\times\mathbb{R}_+}\iint_{\mathbb{R}\times\mathbb{R}_+}\operatorname{sign}(u(x,t)-\kappa)\partial_x f(\kappa;x,t)\xi(x,t,y,s)dxdtdyds \qquad \geq 0,
\end{aligned}\tag{A.12}$$

while

$$\begin{aligned}
&\iint_{\mathbb{R}\times\mathbb{R}_+}\iint_{\mathbb{R}\times\mathbb{R}_+}|v(y,s)-\kappa|\partial_s\xi(x,t,y,s)dxdtdyds\\
&+\int_{\mathbb{R}}\iint_{\mathbb{R}\times\mathbb{R}_+}|v_0(y)-\kappa|\xi(x,t,y,0)dxdtdy\\
&+\iint_{\mathbb{R}\times\mathbb{R}_+}\iint_{\mathbb{R}\times\mathbb{R}_+}\left\{\begin{matrix}\hfill\operatorname{sign}(v(y,s)-\kappa)\cdot\\ (f(v(y,s);y,s)-f(\kappa;y,s))\partial_y\xi(x,t,y,s)\end{matrix}\right\}dxdtdyds\\
&-\iint_{\mathbb{R}\times\mathbb{R}_+}\iint_{\mathbb{R}\times\mathbb{R}_+}\operatorname{sign}(v(y,s)-\kappa)\partial_y f(\kappa;y,s)\xi(x,t,y,s)dxdtdyds\\
\geq&\iint_{\mathbb{R}\times\mathbb{R}_+}\iint_{\mathbb{R}\times\mathbb{R}_+}[B_\kappa(y,s)\xi(x,t,y,s)+A_\kappa(y,s)\partial_y\xi(x,t,y,s)]\,dxdtdyds.
\end{aligned}\tag{A.13}$$

Let $\rho,\tilde\rho\in\mathcal{C}_c^\infty(\mathbb{R};\mathbb{R}_+)$ be such that $\int_{\mathbb{R}}\rho(s)ds=\int_{\mathbb{R}}\tilde\rho(s)ds=1$ and $\operatorname{supp}\rho\subset[-1,1]$, $\operatorname{supp}\tilde\rho\subset[-1,0]$. We moreover assume that

$$\rho(0)=1, \text{ and } s\rho'(s)\leq 0,\quad \forall s\in\mathbb{R}.\tag{A.14}$$

Then, for $\epsilon,\delta>0$, we denote

$$\rho_\epsilon(s):=\frac{1}{\epsilon}\rho\left(\frac{s}{\epsilon}\right),\qquad \tilde\rho_\delta(s):=\frac{1}{\delta}\tilde\rho\left(\frac{s}{\delta}\right),$$

so that $\mathrm{supp}\rho_\epsilon \subset [-\epsilon,\epsilon]$, $\mathrm{supp}\tilde\rho_\delta \subset [-\delta,0]$ and $\int_{\mathbb{R}} \rho_\epsilon(s)ds = \int_{\mathbb{R}} \tilde\rho_\delta(s)ds = 1$. As a consequence of (A.14), one has also

$$-s\rho_\epsilon'(s) \geq 0 \quad \text{and} \quad -\int_{\mathbb{R}} s\rho_\epsilon'(s)ds = 1, \tag{A.15}$$

ensuring that $s \mapsto -s\rho_\epsilon'(s)$ is also an approximation of the unit.

Let $\psi \in \mathcal{C}_c^\infty(\mathbb{R}\times[0,T))$ (for some $T>0$), then, choosing

$$\xi(x,t,y,s) = \psi(x,t)\rho_\epsilon(x-y)\tilde\rho_\delta(t-s)$$

yields

$$\begin{aligned}
\partial_y \xi(x,t,y,s) &= -\psi(x,t)\rho_\epsilon'(x-y)\tilde\rho_\delta(t-s),\\
\partial_t \xi(x,t,y,s) + \partial_s \xi(x,t,y,s) &= \partial_t\psi(x,t)\rho_\epsilon(x-y)\tilde\rho_\delta(t-s),\\
\partial_x \xi(x,t,y,s) + \partial_y \xi(x,t,y,s) &= \partial_x\psi(x,t)\rho_\epsilon(x-y)\tilde\rho_\delta(t-s).
\end{aligned}$$

Choosing $\kappa = v(y,s)$ in (A.12) and $\kappa = u(x,t)$ in (A.13), then summing (recall that $\tilde\rho_\delta(s) = 0$ if $s \leq 0$) provides

$$T_1^{\epsilon,\delta} + T_2^{\epsilon,\delta} + T_3^{\epsilon,\delta} + T_4^{\epsilon,\delta} \geq T_5^{\epsilon,\delta} + T_6^{\epsilon,\delta}, \tag{A.16}$$

where

$$\begin{aligned}
T_1^{\epsilon,\delta} &= \iint_{\mathbb{R}\times\mathbb{R}_+}\iint_{\mathbb{R}\times\mathbb{R}_+} |u(x,t)-v(y,s)|\partial_t\psi(x,t)\rho_\epsilon(x-y)\tilde\rho_\delta(t-s)dxdtdyds,\\
T_2^{\epsilon,\delta} &= \iint_{\mathbb{R}\times\mathbb{R}_+}\int_{\mathbb{R}} |u_0(x)-v(y,s)|\psi(x,0)\rho_\epsilon(x-y)\tilde\rho_\delta(-s)dxdyds,\\
T_3^{\epsilon,\delta} &= \iint_{\mathbb{R}\times\mathbb{R}_+}\iint_{\mathbb{R}\times\mathbb{R}_+} \left\{\begin{array}{r} \mathrm{sign}(u(x,t)-v(y,s))\cdot \\ (f(u(x,t);x,t)-f(v(y,s);x,t)) \\ \partial_x\psi(x,t)\rho_\epsilon(x-y)\tilde\rho_\delta(t-s) \end{array}\right\} dxdtdyds,\\
T_4^{\epsilon,\delta} &= \iint_{\mathbb{R}\times\mathbb{R}_+}\iint_{\mathbb{R}\times\mathbb{R}_+} \left\{\begin{array}{r} \mathrm{sign}(u(x,t)-v(y,s))\cdot \\ \Big[(f(v(y,s);y,s)-f(v(y,s);x,t))\rho_\epsilon'(x-y) \\ -\partial_x f(v(y,s);x,t)\rho_\epsilon(x-y) \\ -(f(u(x,t);y,s)-f(u(x,t);x,t))\rho_\epsilon'(x-y) \\ +\partial_y f(u(x,t);y,s)\rho_\epsilon(x-y)\Big]\cdot \\ \tilde\rho_\delta(t-s)\psi(x,t) \end{array}\right\} dxdtdyds,\\
T_5^{\epsilon,\delta} &= -\iint_{\mathbb{R}\times\mathbb{R}_+}\iint_{\mathbb{R}\times\mathbb{R}_+} A_{u(x,t)}(y,s)\psi(x,t)\rho_\epsilon'(x-y)\tilde\rho_\delta(t-s)dxdtdyds,\\
T_6^{\epsilon,\delta} &= \iint_{\mathbb{R}\times\mathbb{R}_+}\iint_{\mathbb{R}\times\mathbb{R}_+} B_{u(x,t)}(y,s)\psi(x,t)\rho_\epsilon(x-y)\tilde\rho_\delta(t-s)dxdtdyds.
\end{aligned}$$

We can directly let δ tend to 0 in (A.16). Using the continuity in mean theorem and (A.6), this yields that

$$T_1^{\epsilon} + T_2^{\epsilon} + T_3^{\epsilon} + T_4^{\epsilon} \geq T_5^{\epsilon} + T_6^{\epsilon}, \tag{A.17}$$

where

$$T_1^\epsilon = \int_{\mathbb{R}}\iint_{\mathbb{R}\times\mathbb{R}_+} |u(x,t)-v(y,t)|\partial_t\psi(x,t)\rho_\epsilon(x-y)dxdtdy,$$

$$T_2^\epsilon = \int_{\mathbb{R}}\int_{\mathbb{R}} |u_0(x)-v_0(y)|\psi(x,0)\rho_\epsilon(x-y)dxdy,$$

$$T_3^\epsilon = \int_{\mathbb{R}}\iint_{\mathbb{R}\times\mathbb{R}_+} \left\{ \begin{array}{r} \mathrm{sign}(u(x,t)-v(y,t))\cdot \\ (f(u(x,t);x,t)-f(v(y,t);x,t)) \\ \partial_x\psi(x,t)\rho_\epsilon(x-y) \end{array} \right\} dxdtdy,$$

$$T_4^\epsilon = \int_{\mathbb{R}}\iint_{\mathbb{R}\times\mathbb{R}_+} \left\{ \begin{array}{r} \mathrm{sign}(u(x,t)-v(y,t))\cdot \\ \Big[(f(v(y,t);y,t)-f(v(y,t);x,t))\rho_\epsilon'(x-y) \\ -\partial_x f(v(y,t);x,t)\rho_\epsilon(x-y) \\ +f(u(x,t);x,t)-f(u(x,t);y,t)\rho_\epsilon'(x-y) \\ +\partial_y f(u(x,t);y,t)\rho_\epsilon(x-y)\Big]\psi(x,t) \end{array} \right\} dxdtdy,$$

$$T_5^\epsilon = -\int_{\mathbb{R}}\iint_{\mathbb{R}\times\mathbb{R}_+} A_{u(x,t)}(y,t)\psi(x,t)\rho_\epsilon'(x-y)dxdtdy,$$

$$T_6^\epsilon = \int_{\mathbb{R}}\iint_{\mathbb{R}\times\mathbb{R}_+} B_{u(x,t)}(y,t)\psi(x,t)\rho_\epsilon(x-y)dxdtdy.$$

Contrarily to the study performed by [23], we cannot let the parameter ϵ tend to 0, because of the presence of ρ_ϵ' in T_6^ϵ. The goal is now to derive ϵ-dependent estimates from (A.17), as it was already the case in [24].

Define the functions $Y_\theta : s \mapsto \min(1;\max(0;1-s/\theta))$ and

$$\phi_\theta(x,t) = Y_\theta(|x|-R-L_f(T-t)),$$

then, for a function $\chi \geq 0$ with compact support in $[0,T)$ to be specified latter, setting $\psi(x,t)=\phi_\theta(x,t)\chi(t)$ yields that

$$\begin{aligned} &|u(x,t)-v(y,t)|\phi_\theta(x,t)\chi'(t) \\ &\geq |u(x,t)-v(y,t)|\partial_t\psi(x,t) \\ &\quad +\mathrm{sign}(u(x,t)-v(y,t))(f(u(x,t);x,t)-f(v(y,t);x,t))\partial_x\psi(x,t). \end{aligned}$$

Therefore, we can write

$$T_1^\epsilon + T_3^\epsilon \leq \int_{\mathbb{R}}\iint_{\mathbb{R}\times\mathbb{R}_+} |u(x,t)-v(y,t)|\phi_\theta(x,t)\chi'(t)\rho_\epsilon(x-y)dxdydt. \tag{A.18}$$

For this choice of test function ψ, letting θ tend to 0 and denoting by

$$B_t = \big\{x\in\mathbb{R} \;\big|\; |x| < R+L_f(T-t)\big\}, \quad B_t^\epsilon = \big\{y\in\mathbb{R} \;\big|\; |y| < R+L_f(T-t)+\epsilon\big\},$$

taking (A.18) into account in the relation (A.17) provides

$$D_1^\epsilon + D_2^\epsilon + D_3^\epsilon \leq D_4^\epsilon + D_5^\epsilon, \tag{A.19}$$

where

$$D_1^\epsilon = \int_0^T \int_{y\in B_t^\epsilon}\int_{x\in B_t} |u(x,t)-v(y,t)|\chi'(t)\rho_\epsilon(x-y)dxdydt,$$

$$D_2^\epsilon = \int_{y\in B_0^\epsilon}\int_{x\in B_0} |u_0(x)-v_0(y)|\chi(0)\rho_\epsilon(x-y)dxdy,$$

$$D_3^\epsilon = \int_0^T \int_{y\in B_t^\epsilon}\int_{x\in B_t} \left\{ \begin{array}{r} \mathrm{sign}(u(x,t)-v(y,t))\cdot \\ \Big[(f(v(y,t);y,t)-f(v(y,t);x,t))\rho_\epsilon'(x-y) \\ -\partial_x f(v(y,t);x,t)\rho_\epsilon(x-y) \\ +f(u(x,t);x,t)-f(u(x,t);y,t)\rho_\epsilon'(x-y) \\ +\partial_y f(u(x,t);y,t)\rho_\epsilon(x-y)\Big]\chi(t) \end{array} \right\} dxdydt,$$

$$D_4^\epsilon = \int_0^T \int_{y\in B_t^\epsilon}\int_{x\in B_t} A_{u(x,t)}(y,t)\rho_\epsilon'(x-y)\chi(t)dxdydt,$$

$$D_5^\epsilon = \int_0^T \int_{y\in B_t^\epsilon}\int_{x\in B_t} B_{u(x,t)}(y,s)\rho_\epsilon(x-y)\chi(t)dxdydt.$$

From the triangular inequality, one has

$$D_2^\epsilon \le \int_{x\in B_0} |u_0(x)-v_0(x)|\chi(0)dx + \int_{B_0^\epsilon}\int_{B_0} |v_0(x)-v_0(y)|\chi(0)\rho_\epsilon(x-y)dxdy.$$

Since v_0 belongs to $BV(\mathbb{R})$, then

$$\int_{B_0^\epsilon}\int_{B_0} |v_0(x)-v_0(y)|\chi(0)\rho_\epsilon(x-y)dxdy \le \epsilon TV(v_0)\chi(0),$$

thus one has

$$D_2^\epsilon \le \int_{x\in B_0} |u_0(x)-v_0(x)|\chi(0)dx + \epsilon TV(v_0)\chi(0). \tag{A.20}$$

Similarly, the triangular inequality yields that

$$\begin{aligned} D_1^\epsilon \le & \int_0^T \int_{x\in B_t} |u(x,t)-v(x,t)|\chi'(t)dxdt \\ & + \int_0^T |\chi'(t)| \int_{y\in B_t^\epsilon}\int_{x\in B_t} |v(x,t)-v(y,t)|\rho_\epsilon(x-y)dxdydt. \end{aligned}$$

Since

$$\int_{y\in B_t^\epsilon}\int_{x\in B_t} |v(x,t)-v(y,t)|\rho_\epsilon(x-y)dxdy \le TV(v(\cdot,t))\epsilon,$$

then it follows from Theorem A.1 that for all $t\in[0,T]$,

$$TV(v(\cdot,t)) \le TV(v_0)e^{c_g T} + \int_0^T e^{c_g(T-\tau)}\int_{\mathbb{R}} \|\partial_{xx}^2 g(\cdot;x,t)\|_\infty dxdt := C_{BV},$$

where $c_g = 3\|\partial_x\partial_s g\|_\infty$. As a consequence, we obtain

$$D_1^\epsilon \le \int_0^T \int_{x\in B_t} |u(x,t)-v(x,t)|\chi'(t)dxdt + \epsilon C_{BV}\|\chi'\|_{L^1}. \tag{A.21}$$

Since $x\mapsto f(s;x,t)$ belongs to $\mathcal{C}^2(\mathbb{R},\mathbb{R})$, one has, for all $(x,y)\in B_t\times B_t^\epsilon$, that

$$f(v(y,t);y,t)-f(v(y,t);x,t) = \partial_x f(v(y,t);x,t)(y-x)+(y-x)^2\mu(x,y,t),$$

for some $\mu(x,y,t)\in\mathbb{R}$ with

$$|\mu(x,y,t)|\le\frac{1}{2}\sup_{s\in\mathbb{R}}\|\partial^2_{xx}f(s;\cdot,t)\|_{L^\infty(B^\epsilon_t)}. \tag{A.22}$$

Therefore,

$$\begin{aligned}&(f(v(y,t);y,t)-f(v(y,t);x,t))\rho'_\epsilon(x-y)-\partial_x f(v(y,t);x,t)\rho_\epsilon(x-y)\\ =&\,\partial_x f(v(y,t);x,t)\partial_x((y-x)\rho_\epsilon(x-y))+(x-y)^2\rho'_\epsilon(x-y)\mu(x,y,t),\end{aligned}$$

and similarly,

$$\begin{aligned}&(f(u(x,t);x,t)-f(u(x,t);y,t))\rho'_\epsilon(x-y)+\partial_y f(u(x,t);y,t)\rho_\epsilon(x-y)\\ =&\partial_y f(u(x,t);y,t)\partial_y((x-y)\rho_\epsilon(x-y))-(x-y)^2\rho'_\epsilon(x-y)\nu(x,y,t)\\ =&-\partial_y f(u(x,t);y,t)\partial_x((x-y)\rho_\epsilon(x-y))-(x-y)^2\rho'_\epsilon(x-y)\nu(x,y,t),\end{aligned} \tag{A.23}$$

for some $\nu(x,y,t)\in\mathbb{R}$ fulfilling

$$|\nu(x,y,t)|\le\frac{1}{2}\sup_{s\in\mathbb{R}}\|\partial^2_{xx}f(s;\cdot,t)\|_{L^\infty(B^\epsilon_t)}. \tag{A.24}$$

It follows from (A.1) that μ and ν belong to $L^1(\mathbb{R}\times\mathbb{R}\times(0,T))$. Denoting by

$$\begin{aligned}D^\epsilon_{31}=&\int_0^T\chi(t)\int_{B_t}\int_{B^\epsilon_t}\left\{\begin{array}{r}\operatorname{sign}(u(x,t)-v(y,t))\cdot\\ {[\partial_x f(v(y,t);x,t)-\partial_y f(u(x,t);y,t)]}\\ \cdot\partial_x((y-x)\rho_\epsilon(x-y))\end{array}\right\}dxdydt,\\ D^\epsilon_{32}=&\int_0^T\chi(t)\int_{B_t}\int_{B^\epsilon_t}\left\{\begin{array}{r}\operatorname{sign}(u(x,t)-v(y,t))\cdot\\ (\mu(x,y,t)+\nu(x,y,t))(x-y)^2\rho'_\epsilon(x-y)\end{array}\right\}dxdydt,\end{aligned}$$

we have

$$D^\epsilon_3=D^\epsilon_{31}+D^\epsilon_{32}. \tag{A.25}$$

Clearly, for all $x\in\mathbb{R}$, the function $y\mapsto(x-y)^2\rho'_\epsilon(x-y)$ is compactly supported in $[x-\epsilon,x+\epsilon]$ and

$$\left|(x-y)^2\rho'_\epsilon(x-y)\right|\le\epsilon(y-x)\rho'_\epsilon(x-y).$$

Using (A.14), (A.22) and (A.24), this ensures that

$$|D^\epsilon_{32}|\le\epsilon\int_0^T\chi(t)\sup_{s\in\mathbb{R}}\|\partial^2_{xx}f(s;\cdot,t)\|_{L^\infty(B^\epsilon_t)}\left(\int_{B_t}dx\right)dt. \tag{A.26}$$

Focus now on D^ϵ_{31}, that we rewrite under the form

$$D^\epsilon_{31}=\int_0^T\chi(t)\int_{x\in B_t}\int_{y\in B^\epsilon_t}[A(x,y,t)+B(x,y,t)]\partial_x[(y-x)\rho_\epsilon(x-y)]dxdydt, \tag{A.27}$$

where

$$A(x,y,t)=\operatorname{sign}(u(x,t)-v(y,t))\left(\tilde\partial_x f(v(y,t);x,t)-\tilde\partial_x f(u(x,t);x,t)\right),$$

$$B(x,y,t) = \mathrm{sign}(u(x,t)-v(y,t))\left(\tilde\partial_x f(u(x,t);x,t) - \partial_y f(u(x,t);y,t)\right),$$

where we have introduced the notation

$$\tilde\partial_x f(u(x,t);x,t) = \lim_{h\to 0}\frac{f(u(x,t);x+h,t)-f(u(x,t);x,t)}{h}.$$

Thanks to (A.14), one has

$$\int_{\mathbb{R}} \partial_x\left[(y-x)\rho_\epsilon(x-y)\right]dy = 0, \quad \forall x\in\mathbb{R}.$$

This implies that, for $\Upsilon\in\{A,B\}$,

$$\begin{aligned}
&\int_0^T \chi(t)\int_{x\in B_t}\int_{y\in B_t^\epsilon} \Upsilon(x,y,t)\partial_x[(y-x)\rho_\epsilon(x-y)]dydxdt\\
&=\int_0^T \chi(t)\int_{x\in B_t}\int_{y\in B_t^\epsilon} [\Upsilon(x,y,t)-\Upsilon(x,x,t)]\partial_x[(y-x)\rho_\epsilon(x-y)]dydxdt\\
&\le\int_0^T \chi(t)\int_{x\in B_t}\int_{y\in B_t^\epsilon} |\Upsilon(x,y,t)-\Upsilon(x,x,t)|(y-x)\rho_\epsilon'(x-y)dydxdt\\
&\quad+\int_0^T \chi(t)\int_{x\in B_t}\int_{y\in B_t^\epsilon} |\Upsilon(x,y,t)-\Upsilon(x,x,t)|\rho_\epsilon(x-y)dydxdt.
\end{aligned}\tag{A.28}$$

On the one hand, thanks to the regularity of f, the function

$$v\mapsto \mathrm{sign}(u(x,t)-v)\left(\tilde\partial_x f(v;x,t)-\tilde\partial_x f(u(x,t);x,t)\right)$$

is $\|\partial_s\partial_x f\|_\infty$-Lipschitz continuous. Therefore, it follows from the definition of $A(x,y,t)$ that

$$|A(x,y,t)-A(x,x,t)|\le \|\partial_s\partial_x f\|_\infty |v(y,t)-v(x,t)|.$$

Thus, using the fact the both ρ_ϵ and $s\mapsto -s\rho_\epsilon'(s)$ are approximations of the unit, the relation (A.28) ensures that

$$\begin{aligned}
&\int_0^T \chi(t)\int_{x\in B_t}\int_{y\in B_t^\epsilon} A(x,y,t)\partial_x[(y-x)\rho_\epsilon(x-y)]dydxdt\\
&\le 2\epsilon\|\partial_s\partial_x f\|_\infty\|\chi\|_\infty\int_0^T \mathrm{TV}(v(\cdot,t))dt.
\end{aligned}$$

Hence, using (A.8) provides that

$$\begin{aligned}
&\int_0^T \chi(t)\int_{x\in B_t}\int_{y\in B_t^\epsilon} A(x,y,t)\partial_x[(y-x)\rho_\epsilon(x-y)]dydxdt\\
&\le 2\epsilon\|\partial_s\partial_x f\|_\infty\|\chi\|_\infty\left(C_{g,0}(T)\mathrm{TV}(v_0)+C_{g,1}(T)\right).
\end{aligned}\tag{A.29}$$

On the other hand, the regularity of f ensures that

$$|B(x,y,t)-B(x,x,t)|\le \chi(t)\sup_{s\in\mathbb{R}}\left\|\partial^2_{xx}f(s;\cdot,t)\right\|_{L^\infty(B_t^\epsilon)}|x-y|.$$

As a consequence, it follows from (A.28) that

$$\int_0^T \chi(t) \int_{x\in B_t} \int_{y\in B_t^\epsilon} B(x,y,t)\partial_x[(y-x)\rho_\epsilon(x-y)]dydxdt$$
$$\leq 2\epsilon \int_0^T \chi(t)\sup_{s\in\mathbb{R}} \left\|\partial_{xx}^2 f(s;\cdot,t)\right\|_{L^\infty(B_t^\epsilon)} \left(\int_{B_t} dx\right) dt. \tag{A.30}$$

Taking (A.29) and (A.30) into account in (A.27) yields

$$D_{31}^\epsilon \leq 2\epsilon\|\chi\|_\infty \Bigg[\|\partial_s\partial_x f\|_\infty (C_{g,0}(T)\mathrm{TV}(v_0)+C_{g,1}(T))$$
$$+\int_0^T \sup_{s\in\mathbb{R}} \left\|\partial_{xx}^2 f(s;\cdot,t)\right\|_{L^\infty(B_t^\epsilon)} \left(\int_{B_t} dx\right) dt\Bigg]. \tag{A.31}$$

The relations (A.25), (A.26), and (A.31) thus provide that

$$D_3^\epsilon \leq \epsilon\|\chi\|_\infty \Bigg[3\int_0^T \sup_{s\in\mathbb{R}} \|\partial_{xx}^2 f(s,\cdot,t)\|_{L^\infty(B_t^\epsilon)} \left(\int_{B_t} dx\right) dt$$
$$+2\|\partial_s\partial_x f\|_\infty (C_{g,0}(T)\mathrm{TV}(v_0)+C_{g,1}(T))\Bigg] \tag{A.32}$$

Concerning D_4^ϵ, it follows from (A.10) and from $\int_{\mathbb{R}} |\rho_\epsilon'(x-y)|dx = 2/\epsilon$ that

$$D_4^\epsilon \geq -\frac{2}{\epsilon}\|\chi\|_\infty \int_0^T \int_{y\in B_t^\epsilon} \|(f-g)(\cdot;y,t)\|_\infty dydt. \tag{A.33}$$

Thanks to (A.11), one has

$$D_5^\epsilon \geq -\|\chi\|_\infty \int_0^T \int_{y\in B_t^\epsilon} \|\partial_y(f-g)(\cdot;y,t)\|_\infty dydt. \tag{A.34}$$

By choosing $\chi(t) = \min(1, \max(0,\lambda(T-t)))$ and letting λ tend to $+\infty$ in (A.19)–(A.21), (A.33) and (A.34), we obtain that

$$\int_{B_T} |u(x,T)-v(x,T)|dx \leq \int_{B_0} |u_0(x)-v_0(x)|dx + \epsilon C_1^\epsilon + \frac{C_2^\epsilon}{\epsilon} + C_3^\epsilon,$$

where the quantities $C_1^\epsilon, C_2^\epsilon$ and C_3^ϵ have been made explicit in Theorem A.3. Since the result above holds for all $\epsilon > 0$, it also holds for the optimal choice of ϵ, concluding the proof of Theorem A.3. ∎

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