COMMUN. MATH. SCI.
Vol. 13, No. 8, pp. 2223–2239

REAL OPTION MODEL OF DYNAMIC GROWTH PROCESSES WITH CONSUMPTION*

NENGSHENG FANG†, XINFENG RUAN‡, AND CAIXIU LIAO§

Abstract. A real option model is built upon a set of stochastic processes for some real investment decision making in incomplete markets. Typically, the optimal consumption level is obtained under a logarithmic utility constraint, and a partial integro-differential equation (PIDE) of the real option is deduced by martingale methods. Analytical formulation of the PIDE is solved by Fourier transformation. Two types of decision making strategies, i.e. option price and IRP (inner risk primium) comparisons, are provided. Finally, the Monte Carlo simulation and numerical computation are illustrated to verify the conclusions.

Key words. Real option, asset pricing, jump diffusion, optimal consumption strategy, risk premium.

AMS subject classifications. 35Q91, 60G20.

1. Introduction

Life circle theory [1, 2, 3] indicates that a representative entity[1] accumulates its endowment[2] continuously and stochastically. Meanwhile, the entity must consume part of the endowment to survive during its lifetime. Usually, the endowment can be traded[3] in an invisible market, and real investment decision making concerning an entity and its endowment deserves to be researched.

Plenty of daily phenomena and decision making satisfy the above description, for example, mate choice, corporate merger, employment, forest exploitation, and ocean development, etc. These investment decisions are often difficult to make since the underlying dynamics of growth and consumption are endogenous. Traditional discount cash flow (DCF) and net present value (NPV) methods are not flexible enough to deal with these problems [4, 5]. The real option approach, which is an extension of financial option theory, is highly recommended and widely applied to assess these project investments and natural resource explorations [6, 7, 8].

With the motivation above, we design a mathematical model to help make real investment decisions in this paper. We first use a set of stochastic differential equations with jumps to depict the dynamic process of a representative entity's endowment, and then find the optimal consumption level for the entity itself under a logarithmic utility

*Received: May 3, 2014; accepted (in revised form): October 5, 2014. Communicated by Shi Jin. This work is partly supported by the Ministry of Education Project of Humanities and Social Sciences (14YJCZH031), the Youth-Teacher Progress Funds of SWUFE (211QN10118), the Fundamental Research Funds for the Central Universities (JBK123456) and the National Natural Science Fund (11301194). Thanks for the research support from "Finance Seminar · SWUFE".

†School of Finance, Southwestern University of Finance and Economics, Chengdu, 611130; Collaborative Innovation Center of Financial Security, P.R. China.

‡School of Economic Mathematics, Southwestern University of Finance and Economics, Chengdu, 611130, P.R. China.

§School of Mathematical Sciences, South China Normal University, Guangzhou, 510631, P.R. China. Corresponding author: (liaocaixiu800919@163.com).

[1] The term "entity" is an abstract notion here which can be a person or a firm, etc.

[2] The term "endowment" contains all the wealth that can be valued. We just use the word "endowment" vaguely for expediency.

[3] Some special endowment may not exchange because of its naturally dependent attributes upon the entity, and it can only be shared with each other. In order to narrate, we just say "trade" in a standard way.

constraint. Under the optimal consumption level, we succeed in getting a partial integro-differential equation for the real option by martingale methods and finding its analytical solution by Fourier transform. Given a premium of the real option, one can find an implied option risk premium from the analytical solution. The implied risk premium is helpful for decision making in the field of real investment.

The dynamic of the endowment for the entity is ruled by a stochastic system which looks like the two types of equilibrium models in finance. One is the consumption-based asset pricing model in an exchange economy where a representative investor maximizes its expected utility by choosing an optimal level of consumption at each period [9, 10, 11, 12, 13, 14]. The other is the production-based asset pricing model in a production economy where a representative investor chooses an optimal level of consumption at each period and leaves the rest in the production to grow for the future consumption [15, 16, 17, 18, 19, 20, 21]. However, there are some essential differences in our model. First, no risk-free asset exists in the incomplete real investment market; besides, the entity's consumption is endogenous in the underlying process; moreover, the growth rate is a stochastic process as well as the volatility process. Thus, the dynamic process of the endowment is complicated but more close to economic reality. From the angle of mathematics, the process is no longer a Levy process as is usual. These differences are a great challenge since they imply that the real option model cannot be obtained and priced in a risk-neutral world as most of the classical financial option models are, e.g. the Black–Scholes model [22], Merton's jump-diffusion model [23], Heston's stochastic volatility option model [24], the alternative option with stochastic interested rate and jump-diffusion [25], and Kou's double exponential jump-diffusion model [26]. Our contribution is to build and solve a real option model in the physical world and to propose two investment criteria: the comparison of the real option price or the IRP (inner risk premium) with the market ones.

The remainder of this paper is organized as follows. In Section 2, we develop the underlying endowment process and find the optimal consumption strategy. In Section 3, we establish the pricing kernel and the real option model. In Section 4, we give the analytical expression of the option. In Section 5, we illustrate some numerical results and leave the code in the appendix for reference. Discussion and a conclusion are given last.

2. The endowment process and optimal consumption level

We develop in a continuous-time framework the following process of a representative entity's endowment with stochastic growth, fluctuation, and jumps from the realistic perspective. Moreover, we take the endogenous consumption into consideration.

Assume the endowments is valued[4] as $S(t)$[5] at time $t>0$, with growth rate $g(t)$ and variance $v(t)$. Let the accumulated consumption be $c(t)$, and let the Poisson jump process be $N(t)$ with constant jump density λ and log jump size x. The process is written as follows:

$$\begin{cases} dS(t)=S(t)g(t)dt-c(t)dt+\sqrt{v(t)}S(t)dW_S(t) \\ \qquad\qquad +S(t)(e^x-1)dN(t)-\lambda S(t)E(e^x-1)dt, S(0)=S_0>0, \\ dg(t)=\kappa_g(\theta_g-g(t))dt+\varepsilon_g\sqrt{v(t)}dW_g(t), g(0)=g_0, \\ dv(t)=\kappa_v(\theta_v-v(t))dt+\varepsilon_v\sqrt{v(t)}dW_v(t), v(0)=v_0>0. \end{cases} \tag{2.1}$$

[4]This is a fundamental assumption that the endowment can be measured in some way, but not necessarily in the form of money.

[5]Sometimes we use S_t, or omit the subscript $\cdot_t$ for simplicity, when there is no ambiguity, other variables are treated likewise in the paper.

Here $E[\cdot]$ is the expectation under some information filter $\{\mathcal{F}_t\}_{t\geq 0}$ in the probability space denoted by $(\Omega,\mathcal{F},P)$. The variables κ_g, θ_g, ε_g, κ_v, θ_v, ε_v, S_0, g_0, and v_0 are different constants. Here, $dW_S(t)$, $dW_g(t)$, $dW_v(t)$ are standard Brownian motions with constant correlation matrix Σ, but they are all independent of the Poisson jump. Denote

$$\Sigma := \begin{bmatrix} 1 & \rho_{Sg} & \rho_{Sv} \\ \rho_{Sg} & 1 & \rho_{gv} \\ \rho_{Sv} & \rho_{gv} & 1 \end{bmatrix}.$$

From Equation (2.1), the endowment market is incomplete since the process contains three Brownian motions and one Poisson jump process. Besides, there is not risk-free asset to use as a benchmark. Moreover, the spot growth rate, which can have a positive or negative value[6], is dependent upon the variance and, in the long run, has a mean-reverse level up to the external economic situation. The customized endowment process is different from some classical financial dynamics of assets, but to some extent is closer to a realistic situation in real investment and management fields.

The expected rate of $g(t)$ and $v(t)$ are directly calculated as

$$\bar{g} = e^{-\kappa_g T}(g_0 - \theta_g) + \theta_g, \quad \bar{v} = e^{-\kappa_v T} v_0 + \theta_v(1 - e^{-\kappa_v T}). \tag{2.2}$$

For tractability, we assume a logarithmic utility function for the representative entity

$$U(c) = \ln c. \tag{2.3}$$

Assume the entity's objective is to maximize the expected utility as follows: [7]

$$\max_c E\int_t^{\infty} e^{-\alpha(s-t)} U(c_s)ds, \tag{2.4}$$

where α is a constant discount rate.

THEOREM 2.1. *The representative entity's expected utility is positively related to the endowment $S(t)$ and the growth rate $g(t)$, and it is negatively related to the jump and the variance $v(t)$ for a given discount rate α. The optimal cumulative consumption level $c^*(t)$ is*

$$c^*(t) = \alpha S(t). \tag{2.5}$$

Proof. Denote the objective value function by

$$J(S_t, g_t, v_t) = \max_c E\int_t^{\infty} e^{-\alpha(s-t)} U(c_s)ds.$$

The condition of optimality is given by the following Bellman equation:

$$\alpha J - \max_c \{\mathcal{L}(J) + U\} = 0. \tag{2.6}$$

[6] We set $g(t)$ to be the form of the Visacek model, and it can be negative as well. However, we failed to get an analytical expression for the real option since the ODE for v is fractional. See footnote 9.

[7] The upper bound ∞ is represented formally in mathematics, which is usually a finite physical bound but sufficiently far from the time t.

Here,

$$\begin{aligned}\mathcal{L}(J)=&[Sg-c-\lambda SE(e^x-1)]\frac{\partial J}{\partial S}+\frac{1}{2}vS^2\frac{\partial^2 J}{\partial S^2}+\kappa_g(\theta_g-g)\frac{\partial J}{\partial \mathrm{g}}+\frac{1}{2}\varepsilon_g^2 v\frac{\partial^2 J}{\partial g^2}\\&+\kappa_v(\theta_v-v)\frac{\partial J}{\partial v}+\frac{1}{2}v\varepsilon_v^2\frac{\partial^2 J}{\partial v^2}+\varepsilon_g\rho_{Sg}vS\frac{\partial^2 J}{\partial S\partial g}\\&+\varepsilon_v\rho_{Sv}vS\frac{\partial^2 J}{\partial S\partial v}+\varepsilon_g\varepsilon_v\rho_{gv}v\frac{\partial^2 J}{\partial g\partial v}+\lambda E[J(Se^x,g,v)-J].\end{aligned}$$

Thus we get the following HJB equation which is an elliptic PIDE:

$$\begin{aligned}0=\max_c\{&[Sg-\lambda SE(e^x-1)]\frac{\partial J}{\partial S}+\frac{1}{2}vS^2\frac{\partial^2 J}{\partial S^2}+\kappa_g(\theta_g-g)\frac{\partial J}{\partial \mathrm{g}}+\frac{1}{2}\varepsilon_g^2 v\frac{\partial^2 J}{\partial g^2}\\&+\kappa_v(\theta_v-v)\frac{\partial J}{\partial v}+\frac{1}{2}v\varepsilon_v^2\frac{\partial^2 J}{\partial v^2}+\varepsilon_g\rho_{Sg}vS\frac{\partial^2 J}{\partial S\partial g}\\&+\varepsilon_v\rho_{Sv}vS\frac{\partial^2 J}{\partial S\partial v}+\varepsilon_g\varepsilon_v\rho_{gv}v\frac{\partial^2 J}{\partial g\partial v}\\&+\lambda E[J(Se^x,g,v)]-\lambda J-\alpha J+U-c\frac{\partial J}{\partial S}\}.\end{aligned}\tag{2.7}$$

In order to get the optimal consumption level, we take a partial derivative of the HJB Equation (2.7) with respect to c and obtain the first-order condition

$$0=\frac{\partial U}{\partial c}-\frac{\partial J}{\partial S}.\tag{2.8}$$

Enlightened by Cox, Ingersoll, and Ross [17], we guess that the HJB Equation (2.7) has a variables separated solution form of

$$J(S_t,g_t,v_t)=\frac{1}{\alpha}\ln S_t+Ag_t+Bv_t+D,\tag{2.9}$$

where A,B,D are constants independent of g_t and v_t. Then plugging equations (2.3) and (2.9) into (2.8), we get

$$c^*(t)=\alpha S_t,\tag{2.10}$$

and

$$\begin{aligned}0=&\frac{1}{\alpha}[g-\lambda E(e^x-1)]-\frac{1}{2\alpha}v+\kappa_g(\theta_g-g)A\\&+\kappa_v(\theta_v-v)B+\frac{\lambda}{\alpha}E[x]-\alpha[Ag+Bv+D]+\ln\alpha-1.\end{aligned}$$

By virtue of the arbitrariness of g_t and v_t, we have

$$\begin{cases}0=\frac{1}{\alpha}-\kappa_g A-\alpha A,\\0=-\frac{1}{2\alpha}-\kappa_v B-\alpha B,\\0=-\frac{\lambda}{\alpha}E(e^x-1)+\kappa_g\theta_g A+\kappa_v\theta_v B+\frac{\lambda}{\alpha}E[x]-\alpha D+\ln\alpha-1,\end{cases}$$

and get

$$\begin{cases}A=\frac{1}{(\kappa_g+\alpha)\alpha},\\B=-\frac{1}{2\alpha(\kappa_v+\alpha)},\\D=\frac{\lambda}{\alpha^2}E[x]-\frac{\lambda}{\alpha^2}E(e^x-1)+\frac{\kappa_g\theta_g}{(\kappa_g+\alpha)\alpha^2}-\frac{\kappa_v\theta_v}{2\alpha^2(\kappa_v+\alpha)}+\frac{\ln\alpha-1}{\alpha}.\end{cases}\tag{2.11}$$

The theorem results are immediately obtained from equations (2.9), (2.10), and (2.11). □

REMARK 2.2. In economics, $\alpha > 0$ means that one prefers consuming today with certainty to consuming the same quantity tomorrow with uncertainty. One can assume that the entity is constrained to consume $0 < c_t \leq S(t)$, i.e., $0 < \alpha < 1$ to preclude starvation or overdraft in its lifetime. However, advance by overdraft can be seen everywhere, thus we just assume $\alpha > 0$ here to satisfy the preferred hypothesis in economics. In fact, it is unnecessary in mathematics.

3. Pricing kernel and real option model

Assume the representative entity consumes its endowment according to the optimal strategy (2.5). Then process (2.1) is rewritten as

$$\begin{cases} \frac{dS(t)}{S(t)} = (g(t)-\alpha)dt + \sqrt{v(t)}dW_S(t) + (e^x-1)dN(t) - \lambda E(e^x-1)dt, S_0 > 0, \\ dg(t) = \kappa_g(\theta_g - g(t))dt + \varepsilon_g\sqrt{v(t)}dW_g(t), g(0) = g_0, \\ dv(t) = \kappa_v(\theta_v - v(t))dt + \varepsilon_v\sqrt{v(t)}dW_v(t), v_0 > 0. \end{cases} \tag{3.1}$$

Thus we integrate process (3.1) and get

$$\ln\frac{S(T)}{S(t)} = \int_t^T \left(g(s) - \alpha - \frac{1}{2}v(s) - \lambda E(e^x-1)\right)ds + \int_t^T \sqrt{v(s)}dW_S(s) + \sum_{i=1}^{N(T-t)} x_i. \tag{3.2}$$

THEOREM 3.1. *The pricing kernel for the endowment process is given by*

$$\frac{d\pi_t}{\pi_t} = -\mu(t)dt - \sqrt{v(t)}dW_S(t) + (e^y-1)dN(t) - \lambda E(e^y-1)dt. \tag{3.3}$$

Integrate to get

$$\frac{\pi_T}{\pi_t} = \exp\left\{\int_t^T \left(-\mu(s) - \frac{1}{2}v(s) - \lambda E(e^y-1)\right)ds - \int_t^T \sqrt{v(s)}dW_S(s) + \sum_{i=1}^{N(T-t)} y_i\right\}. \tag{3.4}$$

where

$$\mu(t) = g(t) - v(t) - \lambda E[(1-e^{-x})(e^x-1)], \tag{3.5}$$

and the random jump size y satisfies the following restriction:

$$E[(e^y - e^{-x})(e^x-1)] = 0. \tag{3.6}$$

Proof. Plugging equations (2.5), (3.2), and (3.4) into the martingale condition [27]

$$\pi_t S_t = E\left[\int_t^T \pi(s)c^*(s)ds + \pi_T S_T | \mathcal{F}_t\right], \quad \forall t, \tag{3.7}$$

we get

$$1 = E_t[\int_t^T \frac{\pi(\zeta)c^*(\zeta)}{\pi_t S_t}d\zeta + \frac{\pi_T S_T}{\pi_t S_t}]$$

$$=E_t[\,\alpha\int_t^T exp\{\int_t^\iota[-\mu(\zeta)-\frac{1}{2}v(\zeta)-\lambda E(e^y-1)]d\zeta-\int_t^\iota\sqrt{v(\zeta)}dW_S(\zeta)+\sum_{i=1}^{N(\iota-t)}y_i\} \tag{3.8}$$

$$*\,exp\{\int_t^\iota[g(\zeta)-\alpha-\frac{1}{2}v(\zeta)-\lambda E(e^x-1)]d\zeta+\int_t^\iota\sqrt{v(\zeta)}dW_S(\zeta)+\sum_{i=1}^{N(\iota-t)}x_i\}d\iota$$

$$+\,exp\{\int_t^T[-\mu(\zeta)-\frac{1}{2}v(\zeta)-\lambda E(e^y-1)]d\zeta-\int_t^T\sqrt{v(\zeta)}dW_S(\zeta)+\sum_{i=1}^{N(T-t)}y_i\}$$

$$*\,exp\{\int_t^T[g(\zeta)-\alpha-\frac{1}{2}v(\zeta)-\lambda E(e^x-1)]d\zeta+\int_t^T\sqrt{v(\zeta)}dW_S(\zeta)+\sum_{i=1}^{N(T-t)}x_i\}\,]$$

$$=E_t[\alpha\int_t^T exp\{\int_t^\iota[g(\zeta)-\mu(\zeta)-\alpha-v(\zeta)-\lambda E(e^y-1)-\lambda E(e^x-1)]d\zeta$$

$$+\sum_{i=1}^{N(\iota-t)}(x_i+y_i)\}d\iota+exp\{\int_t^T[g(\zeta)-\mu(\zeta)-\alpha-v(\zeta)-\lambda E(e^y-1)-\lambda E(e^x-1)]d\zeta$$

$$+\sum_{i=1}^{N(T-t)}(x_i+y_i)\}]$$

$$=E_t[\alpha\int_t^T exp\{\int_t^\iota[g(\zeta)-\mu(\zeta)-\alpha-v(\zeta)-\lambda E[(1-e^y)(e^x-1)]\}d\zeta d\iota$$

$$+exp\{\int_t^T[g(\zeta)-\mu(\zeta)-\alpha-v(\zeta)-\lambda E[(1-e^y)(e^x-1)]d\zeta\}]. \tag{3.9}$$

In the last equality, we utilized the identity property of the Poisson process

$$E(\prod_{i=1}^{N_\tau}e^{x_i+y_i})=\exp[\lambda E(e^{x+y}-1)\tau].$$

Define $\mu(t)=g(t)-v(t)-\lambda E[(1-e^{-x})(e^x-1)]$. One can verify that restriction (3.6) is sufficient to verify (3.8).

Remark 3.2. An extreme case of constraint (3.6) is that the jump y in the pricing kernel perfectly synchronizes with the jump x and $y=-x$. Since we should not have y in the growth rate $\mu(t)$ of the process $S(t)$, we put $y=-x$ into the definition of $\mu(t)$ which can be viewed as a pure discount rate which has subtracted the variance premium and jump premium formally. It is worth noting that the pricing kernel is not unique due to the incomplete market, because there is only one underlying process, but there are four risk sources-three Brownian motions and a jump.

Define a new probability measure Q by the Radon–Nikodym derivative

$$\frac{dQ}{dP}=exp\{\,-\lambda E(e^y-1)T+\sum_{i=1}^{N_T}y_i\}. \tag{3.10}$$

Then it holds true that (Lemma C.1 in [21])

$$E[f(x)e^y]=E^Q[f(x)]E(e^y). \tag{3.11}$$

□

4. The real option model and the pricing formula

Many real investment and management decisions are, in fact, some options of the underlying entity and can be considered in the frame of real options. To be exact, the decision relies upon the payment of the investor and the payoff gained from the entity. Take a European real investment option of endowment process (3.1), for example. Assume the strike price is K [8] and the payoff at the expiry date T is $[(S(T)-K)\varpi]^+$, where $\varpi=1$ for the call option and $\varpi=-1$ for the put option. Therefore, we can evaluate a price of the real option by virtue of the formal pricing kernel (3.4), and we get

$$V(S(t),K,g(t),v(t),t)=E\left[\frac{\pi_T}{\pi_t}[(S_T-K)\varpi]^+\big|\mathcal{F}_t\right], \tag{4.1}$$

which satisfies a PIDE given in Theorem 4.1.

THEOREM 4.1. *The European option prices $V(S(t),K,g(t),v(t),t)$ satisfy*

$$\begin{aligned}0=&-\mu V+\frac{\partial V}{\partial t}+[\mu-\alpha-\lambda^Q E^Q(e^x-1)]S\frac{\partial V}{\partial S}+\frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2}\\&+[\kappa_g(\theta_g-g)-\varepsilon_g\rho_{Sg}v]\frac{\partial V}{\partial g}+\frac{1}{2}\varepsilon_g^2 v\frac{\partial^2 V}{\partial g^2}\\&+[\kappa_v(\theta_v-v)-\varepsilon_v v\rho_{Sv}]\frac{\partial V}{\partial v}+\frac{1}{2}v\varepsilon_v^2\frac{\partial^2 V}{\partial v^2}+\varepsilon_g\rho_{Sg}vS\frac{\partial^2 V}{\partial S\partial g}\\&+\varepsilon_v\rho_{Sv}vS\frac{\partial^2 V}{\partial S\partial v}+\varepsilon_g\varepsilon_v\rho_{gv}v\frac{\partial^2 V}{\partial g\partial v}+\lambda^Q\{E^Q[V(Se^x,g,v,t)]-V\},\end{aligned} \tag{4.2}$$

with boundary condition

$$V(S(T),K,g(T),v(T),T)=[(S(T)-K)\varpi]^+, \tag{4.3}$$

where $\varpi=\pm1$ and $\lambda^Q=\lambda E[e^y]$ and $E^Q[\cdot]$ are defined in (3.11).

Proof. We apply the Feynman–Kac Theorem to the dynamics (3.1) to get the PIDE. Typically, applying the Itô formulation, we get

$$\begin{aligned}dV=&\frac{\partial V}{\partial t}dt+[g-\alpha-\lambda E(e^x-1)]S\frac{\partial V}{\partial S}dt+\frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2}dt\\&+\kappa_g(\theta_g-g)\frac{\partial V}{\partial g}dt+\frac{1}{2}\varepsilon_g^2 v\frac{\partial^2 V}{\partial g^2}dt+\kappa_v(\theta_v-v)\frac{\partial V}{\partial v}dt+\frac{1}{2}v\varepsilon_v^2\frac{\partial^2 V}{\partial v^2}dt\\&+\varepsilon_g\rho_{Sg}vS\frac{\partial^2 V}{\partial S\partial g}dt+\varepsilon_v\rho_{Sv}vS\frac{\partial^2 V}{\partial S\partial v}dt+\varepsilon_g\varepsilon_v\rho_{gv}v\frac{\partial^2 V}{\partial g\partial v}dt+S\sqrt{v}\frac{\partial V}{\partial S}dW_S(t)\\&+\varepsilon_g\sqrt{v}\frac{\partial V}{\partial g}dW_g(t)+\varepsilon_v\sqrt{v}\frac{\partial V}{\partial v}dW_v(t)+[V(Se^x,g,v,t)-V]dN(t),\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}d(\pi V)=&Vd\pi+\pi dV+d\pi dV\\=&V\pi\left\{-\mu dt-\sqrt{v}dW_S(t)-\lambda E(e^y-1)dt\right\}\end{aligned}$$

[8] Usually, K may not be money in the real investment. It might be some counterpart of the endowment, e.g. the investors' endowment or something like opportunity cost, etc.

$$
\begin{aligned}
&+\pi\left\{\frac{\partial V}{\partial t}dt+[g-\alpha-\lambda E(e^x-1)]S\frac{\partial V}{\partial S}dt+\frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2}dt\right.\\
&+\kappa_g(\theta_g-g)\frac{\partial V}{\partial g}dt+\frac{1}{2}\varepsilon_g^2 v\frac{\partial^2 V}{\partial g^2}dt+\kappa_v(\theta_v-v)\frac{\partial V}{\partial v}dt\\
&+\frac{1}{2}v\varepsilon_v^2\frac{\partial^2 V}{\partial v^2}dt+\varepsilon_g\rho_{Sg}vS\frac{\partial^2 V}{\partial S\partial g}dt+\varepsilon_v\rho_{vS}vS\frac{\partial^2 V}{\partial S\partial v}dt+\varepsilon_g\varepsilon_v\rho_{gv}v\frac{\partial^2 V}{\partial g\partial v}dt\\
&\left.+S\sqrt{v}\frac{\partial V}{\partial S}dW_S(t)+\varepsilon_g\sqrt{v}\frac{\partial V}{\partial g}dW_g(t)+\varepsilon_v\sqrt{v}\frac{\partial V}{\partial v}dW_v(t)\right\}\\
&-\pi\{vS\frac{\partial V}{\partial S}dt+\varepsilon_g\rho_{Sg}v\frac{\partial V}{\partial g}dt+v\varepsilon_v\rho_{Sv}\frac{\partial V}{\partial v}dt\}\\
&+[\pi e^y V(Se^x,g,v,t)-\pi V]\,dN(t).
\end{aligned}
$$

Due to the martingale condition $E[d(\pi(t)V(t))]=0$, we can obtain

$$
\begin{aligned}
0=&-[\mu+\lambda E(e^y-1)]V+\frac{\partial V}{\partial t}+[g-\alpha-\lambda E(e^x-1)-v]S\frac{\partial V}{\partial S}\\
&+\frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2}+[\kappa_g(\theta_g-g)-\varepsilon_g\rho_{Sg}v]\frac{\partial V}{\partial g}+\frac{1}{2}\varepsilon_g^2 v\frac{\partial^2 V}{\partial g^2}\\
&+[\kappa_v(\theta_v-v)-\varepsilon_v v\rho_{Sv}]\frac{\partial V}{\partial v}+\frac{1}{2}v\varepsilon_v^2\frac{\partial^2 V}{\partial v^2}+\varepsilon_g\rho_{Sg}vS\frac{\partial^2 V}{\partial S\partial g}\\
&+\varepsilon_v\rho_{Sv}vS\frac{\partial^2 V}{\partial S\partial v}+\varepsilon_g\varepsilon_v\rho_{gv}v\frac{\partial^2 V}{\partial g\partial v}+\lambda E[e^y V(Se^x,g,v,t)]-\lambda V.
\end{aligned}
$$

Making use of (3.5), (3.6), and (3.11), we can get (4.2) immediately. ∎

Although the PIDE (4.2) is a little complicated, it can be successfully solved by Fourier methods. Duffie, Pan, and Singleton [28] and Chacko and Das [29] present a transform analysis to price the valuation of options for affine jump-diffusions with stochastic volatility. Carr and Madan [30], Sepp [31], and Lewis [32, 33] summarize the Fourier transformation methods to obtain the transform-based solution of option prices. We employ the standard Fourier transformation methods to get an exact expression of the options as follows.

THEOREM 4.2. *Given an expected rate $\omega(t)$ of the real option, the analytical expression of the solution to the PIDE (4.2) is*

$$
V(S,K,g,v,t)=\frac{1+\varpi}{2}Se^{-\alpha\tau}+\frac{1-\varpi}{2}Ke^{-\int_t^T\mu(s)ds}-W, \tag{4.5}
$$

where

$$
W=\frac{e^{-\int_t^T\omega(s)ds}}{2\pi}\int_{iz_i-\infty}^{iz_i+\infty}e^{-iz\ln S}\frac{K^{iz+1}}{z^2-iz}e^{\Gamma(\tau)g+\Lambda(\tau)v+\Pi(\tau)}dz,
$$

$$
i=\sqrt{-1},\quad \tau=T-t,\quad z=z_r+iz_i,\quad 0<z_i<1,
$$

$$
\varpi=\pm 1\quad \textit{for the call/put option},\quad \Gamma(\tau)=-\frac{1+iz}{\kappa_g}(1-e^{-\kappa_g\tau}),
$$

$$
\Pi(\tau)=\delta\tau-(1+iz)\theta_g\big(\tau+\frac{e^{-\kappa_g\tau}-1}{\kappa_g}\big)+\kappa_v\theta_v\int_t^T\Lambda(s)ds,
$$

$$\delta = (1+iz)\lambda E[(1-e^{-x})(e^x-1)] + iz\alpha + iz\lambda^Q E^Q(e^x-1) + \lambda^Q E^Q[e^{-izx}-1],$$

and $\Lambda(\tau)$ *is determined by the ODE*

$$\begin{cases} \dot{\Lambda}(\tau) - \frac{1}{2}\varepsilon_v^2[\Lambda(\tau)]^2 + \eta\Lambda(\tau) + \vartheta = 0, \\ \Lambda(0) = 0, \end{cases}$$

where

$$\eta = \kappa_v + (1+iz)\varepsilon_v\rho_{Sv} + \frac{1+iz}{\kappa_g}\varepsilon_g\varepsilon_v\rho_{gv}(1-e^{-\kappa_g\tau}),$$

$$\vartheta = -1 - \frac{3}{2}iz + \frac{1}{2}z^2 - (1+iz)^2\varepsilon_g\rho_{Sg}\frac{1}{\kappa_g}(1-e^{-\kappa_g\tau}) - \frac{1}{2}\varepsilon_g^2\frac{(1+iz)^2}{\kappa_g^2}(1-e^{-\kappa_g\tau})^2.$$

REMARK 4.3. The difference of formulations (4.1) and (4.5) is that the former is an abstract backward pricing principle which needs the information of T, while the latter is an explicit forward pricing given at time t. The numerical computation speed of the latter is much faster than the former. The option price (4.5) is expressed in the physical world, thus the expected rate $\mu(t)$ of the underlying endowment and $\omega(t)$ of the real option are different. This is a major difference from classic financial options whose worlds exist risk-free with a rate equal to $\mu(t)$ and $\omega(t)$ in risk neutral pricing methods. For this reason, the real option prices are dependent upon $\omega(t)$ which is related to investors' aversion attitudes.

Proof. Notice that the call/put option payoff can be rewritten as

$$[(S_T-K)\varpi]^+ = \frac{1+\varpi}{2}S_T + \frac{1-\varpi}{2}K - \min\{S_T,K\}, \tag{4.6}$$

and by virtue of equations (4.1), (3.2), and (3.4), we get

$$E[\frac{\pi_T}{\pi_t}(\frac{1+\varpi}{2}S_T + \frac{1-\varpi}{2}K)] = \frac{1+\varpi}{2}S_t e^{-\alpha\tau} + \frac{1-\varpi}{2}e^{-\int_0^\tau \mu(s)ds}K.$$

Thus one may solve the call/put European options by the PIDE (4.2) with terminal payoff $V(S(T),K,g(T),v(T),T) = \min\{S_T,K\}$.

With some variable substitutions, $X = \ln S$, $\tau = T-t$, and $f(X,K;g,v,\tau) := V(S,K;g,v,t)$, the PIDE (4.2) can be rewritten as

$$\begin{cases} \frac{\partial f}{\partial \tau} = -\mu f + [\mu - \alpha - \lambda^Q E^Q(e^x-1) - \frac{1}{2}v]\frac{\partial f}{\partial X} + \frac{1}{2}v\frac{\partial^2 f}{\partial X^2} + [\kappa_g(\theta_g - g) - \varepsilon_g\rho_{Sg}v]\frac{\partial f}{\partial g} \\ \quad + \frac{1}{2}\varepsilon_g^2 v\frac{\partial^2 f}{\partial g^2} + [\kappa_v(\theta_v - v) - \varepsilon_v v\rho_{Sv}]\frac{\partial f}{\partial v} + \frac{1}{2}v\varepsilon_v^2\frac{\partial^2 f}{\partial v^2} + \varepsilon_g\rho_{Sg}v\frac{\partial^2 f}{\partial X\partial g} \\ \quad + \varepsilon_v\rho_{Sv}v\frac{\partial^2 f}{\partial X\partial v} + \varepsilon_g\varepsilon_v\rho_{gv}v\frac{\partial^2 f}{\partial g\partial v} + \lambda^Q E^Q[f(x+X,g,v,t)] - \lambda^Q f, \\ f(X,K;g,v,0) = \min\{e^X,K\}. \end{cases} \tag{4.7}$$

Let $F(z,K;g,v,\tau)$ be the Fourier transform of $f(X,K;g,v,\tau)$:

$$F(z,K;g,v,\tau) = \int_{-\infty}^{+\infty} e^{izX} f(X,K;g,v,\tau)dX.$$

The initial condition is a simple integral

$$F(z,K;g,v,0) = \int_{-\infty}^{+\infty} e^{izX}\min\{e^X,K\}dX$$

$$= \lim_{U\to\infty}\int_{-U}^{\ln K} e^{izX}e^{X}dX + \lim_{U\to\infty}\int_{lnK}^{U} Ke^{izX}dX$$
$$= \frac{K^{iz+1}}{z^2-iz}, 0<\Im Z<1. \tag{4.8}$$

Denote $z=z_r+iz_i$. The inverse Fourier transform is given by

$$f(X,K;g,v,\tau)=\frac{1}{2\pi}\int_{iz_i-\infty}^{iz_i+\infty} e^{-izX}F(z,K;g,v,\tau)dz.$$

The option price is an integral along a straight line in the complex z-plane parallel to the real axis. By introducing the transform (4.6), in the cases of call and put option expiration, this line can lie anywhere in the region $0<\Im z<1$. The integral is indeed independent of the choice of z_i, and we usually take $z_i=\frac{1}{2}$ as the value is halfway between the poles of the integrand which is continuous vs $z_r=\Re z$. The integral converges fastest along that contour if $z_{max}=\max z_r$ is chosen large enough [33].

Then, the PIDE (4.7) can be rewritten as

$$\begin{cases} \frac{\partial F}{\partial \tau}=-\mu F - iz[\mu-\alpha-\lambda^Q E^Q(e^x-1)-\frac{1}{2}v]F-\frac{1}{2}vz^2F \\ \qquad +[\kappa_g(\theta_g-g)-\varepsilon_g\rho_{Sg}v]\frac{\partial F}{\partial g}+\frac{1}{2}\varepsilon_g^2 v\frac{\partial^2 F}{\partial g^2}+[\kappa_v(\theta_v-v)-\varepsilon_v v\rho_{Sv}]\frac{\partial F}{\partial v}+\frac{1}{2}v\varepsilon_v^2\frac{\partial^2 F}{\partial v^2} \\ \qquad -iz\varepsilon_g\rho_{Sg}v\frac{\partial F}{\partial g}-iz\varepsilon_v\rho_{Sv}v\frac{\partial F}{\partial v}+\varepsilon_g\varepsilon_v\rho_{gv}v\frac{\partial^2 F}{\partial g\partial v}+\lambda^Q E^Q[e^{-izx}-1]F, \\ F(z,K;g,v,0)=\frac{K^{iz+1}}{z^2-iz}. \end{cases} \tag{4.9}$$

It is enough to solve the same PIDE with unitary initial value and then scale the solution. Hence,

$$V(S,K;g,v,t)=\frac{e^{-\int_0^\tau \omega(s)ds}}{2\pi}\int_{iz_i-\infty}^{iz_i+\infty} e^{-iz\ln S}\frac{K^{iz+1}}{z^2-iz}\hat{F}(z,K,g,v,\tau)dz, \tag{4.10}$$

and $\hat{F}(z,K;g,v,\tau)$ satisfies the following equation:

$$\begin{cases} \frac{\partial \hat{F}}{\partial \tau}=-\mu \hat{F} - iz[\mu-\alpha-\lambda^Q E^Q(e^x-1)-\frac{1}{2}v]\hat{F}-\frac{1}{2}vz^2\hat{F} \\ \qquad +[\kappa_g(\theta_g-g)-\varepsilon_g\rho_{Sg}v]\frac{\partial \hat{F}}{\partial g}+\frac{1}{2}\varepsilon_g^2 v\frac{\partial^2 \hat{F}}{\partial g^2}+[\kappa_v(\theta_v-v)-\varepsilon_v v\rho_{Sv}]\frac{\partial \hat{F}}{\partial v}+\frac{1}{2}v\varepsilon_v^2\frac{\partial^2 \hat{F}}{\partial v^2} \\ \qquad -iz\varepsilon_g\rho_{Sg}v\frac{\partial \hat{F}}{\partial g}-iz\varepsilon_v\rho_{Sv}v\frac{\partial \hat{F}}{\partial v}+\varepsilon_g\varepsilon_v\rho_{gv}v\frac{\partial^2 \hat{F}}{\partial g\partial v}+\lambda^Q E^Q[e^{-izx}-1]\hat{F}, \\ \hat{F}(z,K;g,v,0)=1. \end{cases} \tag{4.11}$$

According to [31, 33], we guess that Equation (4.11) has an exponential solution of the form

$$\hat{F}(z,K;g,v,\tau)=e^{\Gamma(\tau)g+\Lambda(\tau)v+\Pi(\tau)}, \tag{4.12}$$

with $\Gamma(0)=0$, $\Lambda(0)=0$, and $\Pi(0)=0$.

Substituting (4.12) into Equation (4.11), we get [9]

$$\dot{\Gamma}(\tau)g+\dot{\Lambda}(\tau)v+\dot{\Pi}(\tau)=-\mu-iz[\mu-\alpha-\lambda^Q E^Q(e^x-1)-\frac{1}{2}v]-\frac{1}{2}vz^2$$
$$+[\kappa_g(\theta_g-g)-\varepsilon_g\rho_{Sg}v]\Gamma(\tau)+\frac{1}{2}\varepsilon_g^2 v[\Gamma(\tau)]^2$$

[9] In order to get a homogeneous ODE in v and avoid the cross terms, we managed to design the process of $g(t)$ related to $\sqrt{v}$ shown in system (2.1).

$$
\begin{aligned}
&+[\kappa_v(\theta_v-v)-\varepsilon_v v\rho_{Sv}]\Lambda(\tau)\\
&+\frac{1}{2}v\varepsilon_v^2[\Lambda(\tau)]^2-iz\varepsilon_g\rho_{Sg}v\Gamma(\tau)-iz\varepsilon_v\rho_{Sv}v\Lambda(\tau)\\
&+\varepsilon_g\varepsilon_v\rho_{gv}v\Gamma(\tau)\Lambda(\tau)+\lambda^Q E^Q[e^{-izx}-1].
\end{aligned}
$$

Since g and v are arbitrary, taking (3.5) into consideration, we see that the ODEs are

$$
\begin{cases}\dot\Gamma(\tau)=-1-iz-\kappa_g\Gamma(\tau),\\ \Gamma(0)=0,\end{cases} \tag{4.13}
$$

$$
\begin{cases}\dot\Lambda(\tau)=1+\frac{3}{2}iz-\frac{1}{2}z^2-(1+iz)\varepsilon_g\rho_{Sg}\Gamma(\tau)+\frac{1}{2}\varepsilon_g^2\Gamma^2(\tau)-[\kappa_v+(1+iz)\varepsilon_v\rho_{Sv}]\Lambda(\tau)\\ \qquad\quad +\frac{1}{2}\varepsilon_v^2\Lambda^2(\tau)+\varepsilon_g\varepsilon_v\rho_{gv}\Lambda(\tau)\Gamma(\tau),\\ \Lambda(0)=0,\end{cases} \tag{4.14}
$$

$$
\begin{cases}\dot\Pi(\tau)=\delta+\kappa_g\theta_g\Gamma(\tau)+\kappa_v\theta_v\Lambda(\tau),\\ \Pi(0)=0,\end{cases} \tag{4.15}
$$

where

$$
\delta=(1+iz)\lambda E[(1-e^{-x})(e^x-1)]+iz\alpha+iz\lambda^Q E^Q(e^x-1)+\lambda^Q E^Q[e^{-izx}-1].
$$

The ODE (4.13) has a solution

$$
\Gamma(\tau)=-\frac{1+iz}{\kappa_g}(1-e^{-\kappa_g\tau}). \tag{4.16}
$$

Then, the ODE (4.14) can be written as

$$
\begin{cases}\dot\Lambda(\tau)-\frac{1}{2}\varepsilon_v^2[\Lambda(\tau)]^2+\eta\Lambda(\tau)+\vartheta=0,\\ \Lambda(0)=0,\end{cases} \tag{4.17}
$$

where

$$
\eta=\kappa_v+(1+iz)\varepsilon_v\rho_{Sv}+\frac{1+iz}{\kappa_g}\varepsilon_g\varepsilon_v\rho_{gv}(1-e^{-\kappa_g\tau}),
$$

$$
\vartheta=-1-\frac{3}{2}iz+\frac{1}{2}z^2-(1+iz)^2\varepsilon_g\rho_{Sg}\frac{1}{\kappa_g}(1-e^{-\kappa_g\tau})-\frac{1}{2}\varepsilon_g^2\frac{(1+iz)^2}{\kappa_g^2}(1-e^{-\kappa_g\tau})^2.
$$

To solve the Riccati differential equation (4.17), we make the substitution

$$
\Lambda(\tau)=-\frac{2}{\varepsilon_v^2}\frac{I'(\tau)}{I(\tau)} \tag{4.18}
$$

and obtain a second order differential equation[10]

$$
\begin{cases}I''(\tau)+\eta I'(\tau)-\frac{\varepsilon_v^2}{2}\vartheta I(\tau)=0,\\ I'(0)=0.\end{cases} \tag{4.19}
$$

[10]The solution $\Lambda(\tau)$ will blow up in finite time due to the major quadratic term in Equation (4.17). Though the global solution of Equation (4.19) exists, Equation (4.19) is not well-posed and the singularity appears in the denominator of the substitution (4.18). The parameters (5.2) will not blow up for $\Lambda(\tau), \tau\in[0,1]$ by virtue of numerical tests.

The general solution of Equation (4.19) can be expressed by the confluent hypergeometric function $U(a,b,z)$ and the generalized Laguerre polynomial $L_n^a(\tau)$, but it is too complicated to show the expression here[11]. We use the Euler method to solve Equation (4.17) directly by numerical computation.

Then, plugging Equation (4.16) into Equation (4.15), we get

$$\begin{cases} \dot{\Pi}(\tau) = \delta - (1+iz)\theta_g(1-e^{-\kappa_g \tau}) + \kappa_v \theta_v \Lambda(\tau), \\ \Pi(0) = 0. \end{cases} \tag{4.20}$$

The theorem follows from equations (4.16), (4.17), (4.20), and (4.10). ☐

Remark 4.4. It is easily to find the following relationship from (4.2) and (4.4):

$$\omega(t) = E[\frac{dV}{Vdt}] = \mu(t) + \phi(t), \tag{4.21}$$

where

$$\begin{aligned} \phi(t) = {} & v(S\frac{\partial V}{\partial S} + \varepsilon_g \rho_{Sg}\frac{\partial V}{\partial g} + \varepsilon_v \rho_{Sv}\frac{\partial V}{\partial v})/V \\ & + \{\lambda[V(Se^x,g,v,t) - V] - \lambda^Q E^Q[V(Se^x,g,v,t) - V]\}/V. \end{aligned} \tag{4.22}$$

The rate $\omega(t)$ of the option equals to the underlying rate $\mu(t)$ plus a risk premium which compensates the risk of Brownian motions and jumps.

5. Decision making and numerical computation

With the help of option pricing, one can make investment decisions in two ways. Firstly, given a discount rate $\omega(t)$, the representative investor can compute the real option value through expression (4.5) and compare it directly with the market price. Secondly, if the real option price is given in the form of rates, like ROE, ROA, and so on, one can compute the IRP through (4.1) and (4.5) and make an investment decision by comparing the IRP with the real option rate. The two manners are the same as the famous NVP and IRR (inner rate return) methods in investment. We will perform the decision making process with numerical illustration.

Since the option pricing comparison is direct, we mainly illustrate the IRP method. Assume the market price of the real option is given by Equation (4.1); otherwise, the market is in arbitrage, and investment decision making is obvious. Let the market price of the real option be equal to expression (4.21). We can solve the expected inner risk premium $\phi(t)$. By comparing it with the real option rate, an investment decision can be made at once.

We perform Monte Carlo simulations for the endowment dynamics and the real option prices. The discrete scheme of the endowment system (3.1) is as follows:

$$\begin{cases} v(t+\Delta t) = v(t) + \kappa_v(\theta_v - v(t))\Delta t + \varepsilon_v\sqrt{v(t)}dW_v(t), \quad v_0 > 0. \\ g(t+\Delta t) = g(t) + \kappa_g(\theta_g - g(t))\Delta t + \varepsilon_g\sqrt{v(t)}dW_g(t), \quad g(0) = g_0, \\ \ln S(t+\Delta t) = \ln S(t) + (g(t) - \alpha - 0.5 * v(t))\Delta t + \sqrt{v(t)}dW_S(t) \\ \qquad + (e^x - 1)(N(t+\Delta t) - N(t)) - \lambda E(e^x - 1)\Delta t, \quad S_0 > 0, \end{cases} \tag{5.1}$$

where

$$\begin{cases} dW_S(t) = \varepsilon_1\sqrt{\Delta t}, \\ dW_g(t) = \rho_{Sg}dW_S(t) + \sqrt{1-\rho_{Sg}^2}\varepsilon_2\sqrt{\Delta t}, \\ dW_v(t) = \rho_{Sv}dW_S(t) + \sqrt{1-\rho_{Sv}^2}\varepsilon_3\sqrt{\Delta t}, \end{cases}$$

[11] One can rely on the *Mathematica* software to get the expression. $DSolve[\{y''[t] + (C_1 + C_2 Exp[-k_g t])y'[t] + (C_3 + C_4 Exp[-k_g t] + C_5 Exp[-2k_g t])y[t] == 0, y'[0] == 0\}, y[t], t]$.

Δt is the time interval, and ε_i are samples from the standard normal distribution. We have that $N(t)=j$, if $S_j \le t < S_{j+1}$. Furthermore, $S_j = \sum_{k=1}^{j} \tau_k$ where τ_k is the kth jump time interval which obeys an exponential distribution with parameter λ. The option prices are calculated by Monte Carlo simulation with (4.1). An antithetic variable technique is used to diminish computation errors. Every simulation trial involves calculating two values of the option price. The first value is calculated in the usual way; the second value is calculated by changing the sign of all the random samples from standard normal distributions.

Parameters are set as follows:

$$\begin{aligned}&\kappa_g = 0.3, \quad \theta_g = 0.13, \quad \varepsilon_g = 0.26, \quad \kappa_v = 1.2, \quad \theta_v = 0.04, \quad \varepsilon_v = 0.28, \quad \alpha = 0.2,\\ &\rho_{Sv} = -0.54, \quad \rho_{Sg} = 0.60, \quad \rho_{gv} = 0.44, \quad \lambda = 1, \quad x = -0.105, \quad \Delta t = 0.01, \quad T = 1,\\ &v_0 = 0.04, \quad g_0 = 0.13, \quad S_0 = 100, \quad \gamma = 1, \quad K = 100, 95, 105.\end{aligned} \tag{5.2}$$

Several paths of $g(t)$, $v(t)$, and $S(t)$ are illustrated in Figure 5.1 with parameters (5.2). Black and blue paths are generated by the sample pair ε and $-\varepsilon$ respectively. In the first column, two pairs of arbitrary paths are illustrated, and 20 pairs of arbitrary paths are shown in the second column. One can see directly that $v(t)$ is always greater than 0, but $g(t)$ is sometimes negative. These results illustrate that our newly added dynamic of $g(t)$ works and is in accordance with reality. Numerical results of the option prices are denoted by MC and listed in Table 5.1.

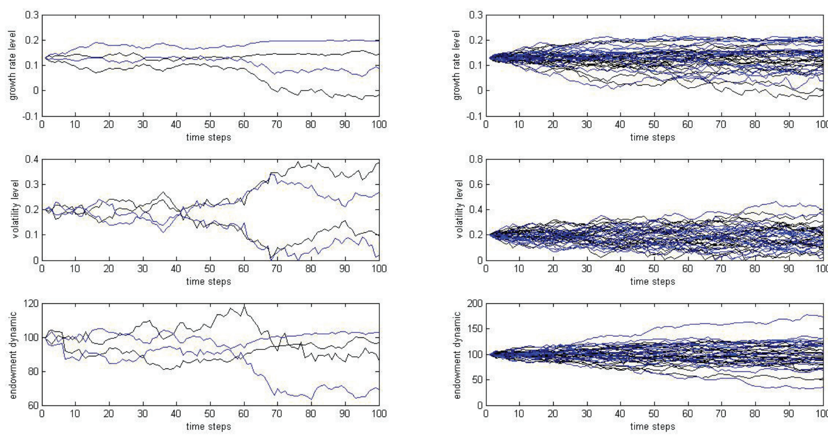

FIG. 5.1. *Simulation of the endowment dynamic.*

Then, taking the Monte Carlo option price as the market price, we evaluate the average inner risk premium $\bar{\phi}$ by the expression (4.5) by virtue of numerical computation. The analytical integration formula of the option price (4.5) can be calculated by Fast Fourier Transform; see Carr and Madan [30], Lewis [32, 33], and Pillay [34] for details. Here, we follow Lewis' methods and calculate the complex integration directly with *Mathematica*. We use the average $\bar{\mu} = \bar{g} - \bar{v} - \lambda E[(1-e^{-x})(e^{x}-1)]$ in the calculation, where $\bar{g}$ and $\bar{v}$ are calculated in Equation (2.2). Option prices and the average inner risk premium $\bar{\phi}$ are listed in Table 5.1. Thus, comparing the charge rate of the real option and the IRP, one can easily make investment decisions. For example, for the at-the-money call option in the first set of parameters, if an investment project charges less than 0.13, the investment is feasible theoretically.

One can see from Table 5.1 that the inner risk premium of the option is negative, which means investors would rather sacrifice some return in order to get the option. Besides, the risk premium of put is larger than call since the put option provides insurance. For the parameters (5.2), $\bar{\mu}_1 - \alpha_1 < 0$ which means the average real growth rate after consumption is negative while $\bar{\mu}_2 - \alpha_2 > 0$ is the opposite, so the option prices are quite different in the two sets of parameters. Though the risk premium of the option comes from three Brownian motions and a jump, the algorithm shows that $\bar{\phi}$ in Table 5.1 is relatively stable both in the call and put options. This is not easy even for financial options in Black–Scholes' world, since the β-risk of the option is more susceptible than the underlying and difficult to capture; see e.g. Coval and Shumway (2001) [35].

	Call option			Put option		
S_0	100	100	100	100	100	100
K	95	100	105	95	100	105
MC_1	8.80	6.61	4.76	5.95	7.96	10.33
$\bar{\mu}_1$		0.079			0.079	
α_1		0.20			0.20	
$\bar{\phi}_1$	-0.111	-0.130	-0.148	-0.224	-0.245	-0.265
MC_2	22.84	19.38	16.07	0.26	0.41	0.42
$\bar{\mu}_2$		0.229			0.229	
α_2		0.10			0.10	
$\bar{\phi}_2$	-0.121	-0.121	-0.121	-0.228	-0.228	-0.232

TABLE 5.1. *Prices and IRPs (inner risk premium) of the real options.*

In the computation of the integral expression, $z_i = \frac{1}{2}$ and $z_{rmax} = 100000$, and in the Monte Carlo simulations, 100000 pairs of paths are set. Option prices MC_1 and the risk premium $\bar{\phi}_1$ are calculated with parameters (5.2). Option prices MC_2 and the risk premium $\bar{\phi}_2$ are calculated with the changed parameters $\theta_g = 0.25, \theta_v = 0.01, g_0 = 0.25, v_0 = 0.01$, and $\alpha = 0.1$.

6. Discussions and conclusions

In the text, we design a real option model to help make decisions in real investment and management fields. To comply with the economic situation, we allow the underlying process to be non-Levy with stochastic growth rate and variance processes. The optimal consumption rate is solved under logarithmic utility and thus can be substituted in the discussion of real options. Since the market is incomplete without risk free assets, we can only price the real option under a pricing kernel, and we give two methods of decision making—option prices and IRP comparisons. Numerical illustrations verify the feasibility.

Mathematically, the endowment process can be trivially extended to slightly more complicated cases by introducing a stochastic jump size λ and time-varying discount rate α. We are temporarily unsure whether the optimal consumption conclusion (2.5) holds true for CRRA (constant relative risk aversion) type utility functions, but in this case, the risk aversion parameter γ will appear in the pricing kernel and the PIDE. This does not affect the concepts of real investment and management decision making in this paper.

Appendix A. Numerical computation of the analytical formulation (*Mathematica*).

```
(* The fundamental transform F *)
F[z_, kappag_, kappav_, tau_, thetag_, thetav_, lambda_, varepsilong_, varepsilonv_,
rhosg_, rhosv_, rhogv_, x_, alpha_, g0_, v0_] :=
Module[{Fval, theta, eta, delta, Gam, Lam, Pit, dt, i, step},
step = 8; dt = tau/step;
theta = Table[0, step]; eta = Table[0, step]; Lam = Table[0, step];
theta = -1 - 1.5 I z + 0.5 z^2 - (1 + I z)^2*varepsilong* rhosg*(1 - E^(-kappag
Range[0, tau, dt]))/kappag -0.5 * varepsilong^2*(1 + I z)^2*(1 - E^(-kappag Range[0,
tau, dt]))^2/kappag^2;
eta = kappav + (1 + I z) (varepsilonv*rhosv + varepsilong*varepsilonv* rhogv (1 -
E^(-kappag Range[0, tau, dt]))/kappag);
delta = (1 + I z)*(lambda*(1 - E^(-x)) (E^(x) - 1) + alpha) + I z lambda E^(-x)
(E^(x) - 1) + lambda E^(-x) (E^(-I z x) - 1);
Gam = -(1 + I z)*(1 - E^(-kappag tau))/kappag;
For [i = 1, i ≤ step - 1, i++,
Lam[[i + 1]] = Lam[[ i]] + (0.5*varepsilonv^2*Lam[[i]]^2 - eta[[i]]*Lam[[i]] -
theta[[i]])*dt;
];
Pit = delta tau - (1 + I z) thetag (tau + (E^(-kappag tau) - 1)/kappag) + kappav
thetav *Sum[Lam[[i]], i, 1, step]/step;
Fval = E^(Gam *g0 + Lam[[step]] *v0 + Pit);
Return[Fval]]
(* This code is used to find the excess return of the options.
cpflag=1 or (-1) for call (put) option. Option prices are prepared by Monte Carlo.
To avoid branch cut crossing, imaginary part must satisfy 0< zi <1. *)
Cvalue[S_, K_, zi_, kappag_, kappav_, tau_, thetag_, thetav_, lambda_, varepsilong_,
varepsilonv_, rhosg_, rhosv_, rhogv_, x_, alpha_, g0_, v0_,option_,cpflag_] :=
Module[{z, zr, val,mu, kmax,phi}, Clear[z]; z[zr_] := zi I + zr;
If[zi ≤ 1 || zi ≤ 0, Print["Illegal zi value"]; Abort[], Null];
Print["To avoid branch cut crossing, imaginary part 0< zi <1!"];
Print["Parameter zi=", zi];
Print["Call option if cpflag=1; Put option if cpflag=-1."];
Print["Parameter cpflag=", cpflag];
Print[" S=", S, "; K=", K, "; alpha=", alpha, "; g0=", g0, "; v0=", v0, "; tau=", tau];
Print[" kappag=", kappag, "; kappav=", kappav, "; thetag=", thetag, "; thetav=",
thetav];
Print[" lambda=", lambda, "; rho_sg=", rhosg, "; rho_sv=", rhosv, "; rho_gv=",
rhogv];
Print[" jump=", x, "; thetag=", thetag, "; thetav=", thetav];
kmax = 100000;(* stand for infinity in the integral *)
mu = Exp[-kappag tau] (g0 - thetag) + thetag - Exp[-kappav tau ] v0
- thetav (1 - Exp[-kappav tau]) - lambda (1 - Exp[-x]) (Exp [x] - 1);
val = N[ K /Pi * NIntegrate[ Re[Exp[-I z[zr] ( Exp[-kappag tau] (g0 - thetag) + thetag
- Exp[-kappav tau ] v0 - thetav (1 - Exp[-kappav tau]) - lambda (1 - Exp[-x]) (Exp[x] -
1) - alpha) tau] Exp[ I z[zr] Log[K/S]]/(z[zr]^2 - I z[zr])* F[z[zr], kappag, kappav, tau,
thetag, thetav, lambda, varepsilong, varepsilonv, rhosg, rhosv, rhogv, x, alpha, g0, v0]
```

```
], zr, 0, 10, kmax, MaxRecursion -> 40]];
Print["Average return of the underlying mu=", mu];
If[cpflag == 1, phi = (-Log[(S Exp[-alpha tau] - option)/val] - mu*tau)/tau;
Print["Infinity kmax=", kmax, ", excess return implied by call option=", phi]; Null]];
If[cpflag == -1, phi = (-Log[(K Exp[-mu tau] - option)/val] - mu*tau)/tau;
Print["Infinity kmax=", kmax, ", excess return implied by put option=", phi]; Null]];
]
In[99]:=Cvalue[100, 95, 0.5, 0.3, 1.2, 1, 0.13, 0.04, 1, 0.26, 0.28, 0.60, -0.54, 0.44,
-0.105, 0.20, 0.13, 0.04, 5.95, -1]
Out[99]:=To avoid branch cut crossing, imaginary part 0< zi <1!
Parameter zi=0.5
Call option if cpflag=1; Put option if cpflag=-1.
Parameter cpflag=-1
S=100; K=95; alpha=0.2; g0=0.13; v0=0.04; tau=1
kappag=0.3; kappav=1.2; thetag=0.13; thetav=0.04
lambda=1; rho_sg=0.6; rho_sv=-0.54; rho_gv=0.44
jump=-0.105; thetag=0.13; thetav=0.04
Average return of the underlying mu=0.0789649
Infinity kmax=100000, excess return implied by put option=-0.223924
```

Acknowledgement. We acknowledge helpful comments from Jerome Detemple, Ye Du, Wei Hu, Alan Lewis, Tao Li, Jun Liu, Qiang Liu, Zongxin Liu, Ronghua Luo, Jingtang Ma, John Wei, Hao Xing, Lung-an Ying, and Mingshan Zhou. Additionally, we greatly thank an anonymous referee for his constructive suggestions. This paper was revised during the first author's visit to the London School of Economics and Political Science hosted by Professor Xing Hao, and the second author's PHD study at the University of Wollongong. Any remaining inadequacies are ours alone.

REFERENCES

[1] F. Modigliani, *The life cycle hypothesis of saving, the demand for wealth and the supply of capital*, Soc. Res., 160–217, 1966.
[2] R.C. Merton, *Lifetime portfolio selection under uncertainty: The continuous-time case*, Rev. Econ. Stat., 51(3), 247–257, 1969.
[3] A. Molina, A. Kusiak, and V. Sanchez, *Handbook of Life Cycle Engineering: Concepts, Models and Technologies*, Springer, 1999.
[4] S.C. Myers, *Finance theory and financial strategy*, Interfaces, 14(1), 126–137, 1984.
[5] J.E. Hodder and H.E. Riggs, *Pitfalls in evaluating risky projects*, Harvard Bus. Rev., 63(1), 128–135, 1985.
[6] M. Amram, N. Kulatilaka, *Real Options: Managing Strategic Investment in an Uncertain World*, Boston: Harvard Business School Press, 1999.
[7] B.S. Bernanke, *Irreversibility, uncertainty, and cyclical investment*, Q. J. Econ., 98(1), 85–106, 1983.
[8] L. Trigeorgis, *Real Options: Managerial Flexibility and Strategy Inresource Allocation*, the MIT Press, 1996.
[9] D.S. Bates, *The market for crash risk*, J. Econ. Dyn. Control, 32(7), 2291–2321, 2008.
[10] D.T. Breeden, *An intertemporal asset pricing model with stochastic consumption and investment opportunities*, J. Financ. Econ., 7(3), 265–296, 1979.
[11] Jr, R.E. Lucas, *Asset prices in an exchange economy*, Econometrica, 1429–1445, 1978.
[12] V. Naik, M. Lee, *General equilibrium pricing of options on the market portfolio with discontinuous returns*, Rev. Financ. Stud., 3(4), 493–521, 1990.
[13] J. Pan, *The jump-risk premia implicit in options: Evidence from an integrated time-series study*, J. Financ. Econ., 63(1), 3–50, 2002.
[14] P. Santa-Clara and S. Yan, *Crashes, volatility, and the equity premium: Lessons from S&P 500*

options, Rev. Econ. Stat., 92(2), 435–451, 2010.
[15] F. Belo, *Production-based measures of risk for asset pricing*, J. Monetary Econ., 57(2), 146–163, 2010.
[16] W.A. Brock, *Asset prices in a production economy.* In The Economics of Information and Uncertainty, University of Chicago Press. 1–46, 1982.
[17] J.C. Cox, Jr, J.E. Ingersoll, and S.A. Ross, *An intertemporal general equilibrium model of asset prices*, Econometrica, 363–384, 1985.
[18] J.H. Cochrane, *Production-based asset pricing and the link between stock returns and economic fluctuations*, J. Financ., 46(1), 209–237, 1991.
[19] R.J. Balvers and D. Huang, *Productivity-based asset pricing: Theory and evidence*, J. Financ. Econ., 86(2), 405–445, 2007.
[20] U.J. Jermann, *Asset pricing in production economies*, J. Monetary Econ., 41(2), 257–275, 1998.
[21] J.E. Zhang, H. Zhao, and E.C. Chang, *Equilibrium asset and option pricing under jump diffusion*, Math. Financ., 22(3), (2012) 538-568.
[22] F. Black and M. Scholes, *The pricing of options and corporate liabilities*, J. Polit. Econ., 637–654, 1973.
[23] R.C. Merton, *Option pricing when underlying stock returns are discontinuous*, J. Financ. Econ., 3(1), 125–144, 1976.
[24] S.L. Heston, *A closed-form solution for options with stochastic volatility with applications to bond and currency options*, Rev. Financ. Stud., 6(2), 327-343, 1993.
[25] G. Bakshi, C. Cao, and Z. Chen, *Empirical performance of alternative option pricing models*, J. Financ., 52(5), 2003–2049, 1997.
[26] S.G. Kou and H. Wang, *Option pricing under a double exponential jump diffusion model*, Manage. Sci., 50(9), 1178–1192, 2004.
[27] K. Back, *Asset Pricing and Portfolio Choice Theory*, Oxford University Press, 2010.
[28] D. Duffie, J. Pan, and K. Singleton, *Transform analysis and asset pricing for affine jump-diffusions*, Econometrica, 68(6), 1343–1376, 2000.
[29] G. Chacko and S. Das, *Pricing interest rate derivatives: a general approach*, Rev. Financ. Stud., 15(1), 195–241, 2002.
[30] P. Carr and D. Madan, *Option pricing and the fast Fourier transform*, J. Comput. Finance, 2(4), 61–73, 1999.
[31] R. Kangro, K. Parna, and A. Sepp, *Pricing european-style options under jump diffusion processes with stochastic volatility: Applications of fourier transform*, Acta Comment. Univ. Tartu. Math., 8, 123–133, 2004.
[32] A. Lewis, *A simple option formula for general jump-diffusion and other exponential Levy processes*, available at SSRN 282110, 2001.
[33] A.L. Lewis, *Option Valuation Under Stochastic Volatility*, J. Financ., 2000.
[34] E. Pillay and J.G. O'Hara, *FFT based option pricing under a mean reverting process with stochastic volatility and jumps*, J. Comput. Appl. Math., 235(12), 3378–3384, 2011.
[35] J.D. Coval and T. Shumway, *Expected option returns*, J. Financ., 56(3), 983–1009, 2001.