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GLOBAL STRONG SOLUTIONS OF THE COMPRESSIBLE NEMATIC LIQUID CRYSTAL FLOW WITH THE CYLINDER SYMMETRY*

QIANG TAO†, JINCHENG GAO‡, AND ZHENG-AN YAO§

Abstract. In this paper, we consider the well-posedness of the compressible nematic liquid crystal flow with the cylinder symmetry in $\mathbb{R}^3$. By establishing a uniform pointwise positive lower and upper bounds of the density, we derive the global existence and uniqueness of strong solution and show the long time behavior of the global solution. Our results do not need the smallness of the initial data. Furthermore, a regularity result of global strong solution is given as well.

Key words. Nematic liquid crystal flow, global strong solution, long time behavior, regularity.

AMS subject classifications. 35B40, 35Q35, 76N10.

1. Introduction

In this paper, we shall study global strong solutions to the compressible hydrodynamic flow of liquid crystals with cylindrical symmetry in $\mathbb{R}^3$. We restrict ourselves to the flows between two circular coaxial cylinders and assume that the corresponding solutions depend only on the radial variable r in $\Omega := \{r \in \mathbb{R}^+, 0 < a < r < b < \infty\}$ and the time variable $t \in \mathbb{R}^+ = [0,\infty)$. Precisely, under the cylindrical symmetric transformation:

$$\mathbf{u} = \left(u\frac{x_1}{r} - v\frac{x_2}{r},\ u\frac{x_2}{r} + v\frac{x_1}{r}, w\right),\ r = \sqrt{x_1^2 + x_2^2},\ u = u(r,t),\ v = v(r,t),\ w = w(r,t),$$

we consider the following three-dimensional compressible nematic liquid crystal flow:

$$\rho_t + \operatorname{div}(\rho\mathbf{u}) = 0, \tag{1.1}$$

$$(\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla(P(\rho)) = \mu\Delta\mathbf{u} + (\lambda+\mu)\nabla\operatorname{div}\mathbf{u} - \nu\operatorname{div}\left(\nabla\mathbf{n}\odot\nabla\mathbf{n} - \frac{|\nabla\mathbf{n}|^2}{2}\mathbb{I}_3\right), \tag{1.2}$$

$$\mathbf{n}_t + (\mathbf{u}\cdot\nabla)\mathbf{n} = \theta(\Delta\mathbf{n} + |\nabla\mathbf{n}|^2\mathbf{n}). \tag{1.3}$$

Here $\rho : \Omega\times[0,\infty) \to \mathbb{R}^1$ denotes the density, $\mathbf{u} : \Omega\times[0,\infty) \to \mathbb{R}^3$ denotes the velocity field and $\mathbf{n} : \Omega\times[0,\infty) \to \mathcal{S}^2$ denotes the optical axis vector of the liquid crystal which is a unit vector $|\mathbf{n}| = 1$, where u is the component of the velocity vector $\mathbf{u}$ along the radial variable $r (r \in \Omega)$, v is the angular component of $\mathbf{u}$, and w is the axial component of $\mathbf{u}$. The pressure is $P = A\rho^\gamma$, where $\gamma > 1$ and A is a positive constant, λ and μ are the bulk and shear viscosity coefficients, respectively, which satisfy the physical conditions $\mu > 0$ and $2\mu + 3\lambda \geq 0$. $\mathbb{I}_3$ denotes the identity matrix of order 3. $\otimes$ and $\odot$ represent the tensor products which satisfy

$$\mathbf{u}\otimes\mathbf{u} = (u^i u^j)_{3\times3}, \quad \nabla\mathbf{n}\odot\nabla\mathbf{n} = (\mathbf{n}_{x_i}\cdot\mathbf{n}_{x_j})_{3\times3}.$$

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†Corresponding author. College of Mathematics and Computational Science, Shenzhen University, P.R. China (taoq060@126.com).

‡School of Mathematics and Computational Science, Sun Yat-sen University, P.R. China (gaojc1998@163.com).

§School of Mathematics and Computational Science, Sun Yat-sen University, P.R. China (mcsyao@mail.sysu.edu.cn).

The reduced system is now of the form (see [1] and [2] for the case of the compressible Navier–Stokes equations):

$$\rho_t + (\rho u)_r + \frac{\rho u}{r} = 0, \tag{1.4}$$

$$\rho\left(u_t + uu_r - \frac{v^2}{r}\right) + P_r - \kappa\left(u_r + \frac{u}{r}\right)_r - \frac{\nu}{2}\left(|\mathbf{n}_r|^2\right)_r - \nu\frac{|\mathbf{n}_r|^2}{r} = 0, \tag{1.5}$$

$$\rho\left(v_t + uv_r + \frac{uv}{r}\right) - \mu\left(v_r + \frac{v}{r}\right)_r = 0, \tag{1.6}$$

$$\rho(w_t + uw_r) - \mu\left(w_{rr} + \frac{w_r}{r}\right) = 0, \tag{1.7}$$

$$\mathbf{n}_t + u\mathbf{n}_r - \theta\mathbf{n}_{rr} - \theta|\mathbf{n}_r|^2\mathbf{n} - \theta\frac{\mathbf{n}_r}{r} = 0, \tag{1.8}$$

where $\rho = \rho(r,t)$, $\mathbf{n} = \mathbf{n}(r,t)$, $\kappa = 2\mu + \lambda$, the positive constants ν and θ are competitive between kinetic and potential energy, and microscopic elastic relaxation time, respectively.

In the domain $\Omega \times [0,\infty)$, we consider the initial-boundary value problem given by (1.4)–(1.8) and

$$(\rho,u,v,w,\mathbf{n})|_{t=0} = (\rho_0,u_0,v_0,w_0,\mathbf{n}_0)(r), \quad r \in \Omega, \tag{1.9}$$

$$(u,v,w)|_{r=a,b} = 0, \quad \mathbf{n}_r|_{r=a,b} = 0, \quad t > 0, \tag{1.10}$$

where $\mathbf{n}_0 : \Omega \to \mathcal{S}^2$.

Liquid crystals are substances that exhibit a phase of matter that has properties between those of a conventional liquid, and those of a solid crystal [3]. The continuum theory of liquid crystals was established by Ericksen [4] and Leslie [5] during the period of 1958 through 1968. Since then, the mathematical theory is still progressing and the study of the full Ericksen–Leslie model presents relevant mathematical difficulties. When the fluid containing nematic liquid crystal materials is at rest, we have the well-known Ossen–Frank theory for static nematic liquid crystals, see the pioneering work by Hardt–Lin–Kinderlehrer [6] on the analysis of energy minimal configurations of nematic liquid crystals. In general, the motion of fluid always takes place. The so-called Ericksen–Leslie system is a macroscopic continuum description of the time evolution of the materials under the influence of both the flow velocity field u and the macroscopic description of the microscopic orientation configurations $\mathbf{n}$ of rod-like liquid crystals.

It is well-known that when $\mathbf{n}$ is a constant vector field, the system (1.1)–(1.3) becomes a classical compressible isentropic Navier–Stokes equation, which is an extremely important equation to describe compressible fluids. There are many important works on the qualitative theory of this equation (see, for example, Lions [8], Feireisl [7] and references therein).

When the density function ρ is a positive constant, then (1.1)–(1.3) becomes the hydrodynamic flow equation of incompressible liquid crystals. Lin [9] first derived a simplified Ericksen–Leslie equations modeling the liquid crystal flows in 1989. Later, Lin and Liu [10, 11] made some important analytic studies, such as the existence of weak and strong solutions and the partial regularity of suitable solutions of the simplified Ericksen–Leslie system, under the assumption that the liquid crystal director field is of either varying length by Leslie's terminology or variable degree of orientation by Ericksen's terminology.

However, when the fluid is compressible, the Ericksen–Leslie system becomes more complicated and there are only a few analytic results available yet. This is now attracting increasing research attention. The authors in [12, 13] first consider the solutions to the initial-boundary value problem of (1.1)–(1.3) with nonnegative initial density. They obtained the global existence and uniqueness for classical, weak or strong solutions in 1-D case. Then, Huang–Wang–Wen [14, 15] showed the existence of the local-in-time strong solutions to the initial value or initial-boundary value problem in 3-D and established a series of blow-up criterion of strong solutions. Inspired by [16], for the radially symmetric initial data and nonnegative initial density, Huang–Ding [17] proved the global existence and uniqueness of global strong solutions to the problem of (1.1)–(1.3) in dimension $N(N\geq 2)$. In [18], the global existence and uniqueness of strong solution to the Cauchy problem are obtained in critical Besov spaces. Recently, based on the arguments in [19] for the compressible Navier–Stokes equations, the global well-posedness of classical solutions were proved in [20] for the initial data were sufficiently smooth and were suitably small in some energy-norm. At the same time, Wu and Tan [21] established the existence of global weak solutions to the Cauchy problem by using Suen and Hoff's method [22].

In the present paper, we shall discuss the well-posedness of the compressible nematic liquid crystal flow with the cylinder symmetry. In the last decades, there are many works addressing the cylinder symmetry motion of a viscous and heat-conductive polytropic idea gas. Frid and Shelukhin [2] discussed the vanishing shear viscosity and obtained the global existence of solutions to the compressible fluids for the flows with the cylinder symmetry. This model was dealt with in [1]. Hoff and Jenssen [23] studied the spherically and cylindrically symmetric nonbarotropic flows with large data and forces, and established the global existence of weak solutions to the compressible non-barotropic Navier–Stokes equations. In [24] and [29] Qin and Jiang showed the global existence and exponential stability of solutions in H^1, H^2, and H^4 for the compressible Navier–Stokes equations with the cylinder symmetry. Moreover, we also mention that the boundary layer problem for Navier–Stokes equations with the cylinder symmetry were considered in [25, 26, 27]. Our aim in this paper is to study the existence and uniqueness, the long time behavior and the regularity of global strong solutions for the initial-boundary value problem (1.4)–(1.10). It's worth mentioning that since we will study the large-time behavior of global solutions, all the estimates should be uniform, that is, they should be independent of any length of time. The main difficulty here is to establish uniform pointwise positive lower and upper bounds of the density. By the energy law, we first show some uniform estimates of derivative for $\mathbf{n}$. Then, motivated by [28] and [29] for the compressible Navier–Stokes equations, we establish the uniform lower and upper bounds of the density ρ without the smallness of the initial total energy. On the other hand, we prove a regularity result in Euler coordinates.

Throughout this paper, we denote the velocity vector $\mathbf{v}=(u,v,w)$ with $\mathbf{v}_0=(u_0,v_0,w_0)$ and without loss of generality, let $A=\theta=\nu=1$. Furthermore, we need the following notation which will be used.

Notation.

(1) For $p\geq 1$, denote $L^p=L^p(\Omega)$ as the L^p space with the norm $\|\cdot\|_{L^p}$. For $k\geq 1$ and $p\geq 1$, denote $W^{k,p}=W^{k,p}(\Omega)$ the standard Sobolev, whose norm is denoted as $\|\cdot\|_{W^{k,p}}$ and $H^k=W^{k,2}$.

(2) For any $T>0$ and $k\geq 1$, denote $W_2^{2k,k}=W_2^{2k,k}(\Omega\times(0,T)):=\{u,D_r^\alpha D_t^\beta u\in L^2(\Omega\times(0,T))$, for any α and β such that $|\alpha|+2\beta\leq 2k\}$.

(3) For simplicity, we will use the abbreviation $\|\cdot\|=\|\cdot\|_{L^2}$.

(4) $\|(f_1,f_2,f_3)\|^2=\|f_1\|^2+\|f_2\|^2+\|f_3\|^2$ for functions f_1,f_2,f_3 belonging to L^2.

(5) Subscripts t and r denote the (partial) derivatives with respect to t and r, respectively.

Then, the main results can be stated as follows.

THEOREM 1.1. *Assume that the initial data satisfies* $(\rho_0,\mathbf{v}_0,\mathbf{n}_0)\in H^2\times(H_0^1\cap H^2)\times H^3$ *with* $0<c_0\le\rho_0$ *and* $|\mathbf{n}_0|=1$ *in* $\bar{\Omega}$ *for some constant* c_0. *Then for any* $T>0$, *there exist two positive constants* C_* *and* C^* *which are independent of* T *and a unique global strong solution* $(\rho,\mathbf{v},\mathbf{n})$ *of the initial boundary value problem* (1.4)–(1.10), *such that*

$$\begin{aligned}
&\rho\in L^\infty(0,T;H^2),\quad \rho_t\in L^\infty(0,T;H^1),\\
&\mathbf{v}\in L^\infty(0,T;H_0^1\cap H^2)\cap L^2(0,T;H^3),\quad \mathbf{v}_t\in L^\infty(0,T;L^2)\cap L^2(0,T;H_0^1)\\
&\mathbf{n}\in L^\infty(0,T;H^3)\cap W_2^{4,2}(\Omega\times(0,T]),\quad \mathbf{n}_t\in L^\infty(0,T;H^1)\cap L^2(0,T;H^2)\\
&0<C_*\le\rho(r,t)\le C^*,\quad |\mathbf{n}|=1,\ \forall(r,t)\in\Omega\times[0,T].
\end{aligned}$$

THEOREM 1.2. *Assume that the assumptions of Theorem 1.1 are satisfied. Then the global strong solution of problem* (1.4)–(1.10) *has the following asymptotic property:*

$$\|(\rho-\bar{\rho},\mathbf{v},\mathbf{n}-\bar{\mathbf{n}})(t)\|_{L^\infty}+\|(\rho_r,\mathbf{v}_r,\mathbf{n}_r)(t)\|\to 0,\quad \text{as } t\to\infty,$$

where $\bar{\rho}$ *is a certain positive constant and* $\bar{\mathbf{n}}$ *is a certain constant vector with* $|\bar{\mathbf{n}}|=1$.

THEOREM 1.3. *Assume that the initial data satisfies* $(\rho_0,\mathbf{v}_0,\mathbf{n}_0)\in H^4\times H_0^4\times H^4$ *with* $0<c_0\le\rho_0$ *and* $|\mathbf{n}_0|=1$ *in* $\bar{\Omega}$. *Then there exists a unique global solution* $(\rho,\mathbf{v},\mathbf{n})$ *to the initial boundary value problem* (1.4)–(1.10) *verifying that for any* $T>0$

$$\begin{aligned}
&\|\rho(t)\|_{H^4}^2+\|\mathbf{v}(t)\|_{H^4}^2+\|\mathbf{n}(t)\|_{H^4}^2+\|\mathbf{v}_t(t)\|_{H^2}^2+\|\mathbf{n}_t(t)\|_{H^2}^2+\|\mathbf{v}_{tt}(t)\|^2+\|\mathbf{n}_{tt}(t)\|^2\\
&+\int_0^T(\|\mathbf{v}_r\|_{H^4}^2+\|\mathbf{n}_r\|_{H^4}^2+\|\mathbf{v}_{tr}\|_{H^2}^2+\|\mathbf{n}_{tr}\|_{H^2}^2+\|\mathbf{v}_{ttr}\|^2+\|\mathbf{n}_{ttr}\|^2)(s)ds\le C_T,
\end{aligned}$$

where C_T *is a positive constant depending on* a, b, κ, μ, γ, *initial data and* T.

REMARK 1.1. It is clear that the solution $(\rho,\mathbf{v},\mathbf{n})$ obtained in Theorem 1.3 is in fact a classical solution verifying that

$$\|(\rho,\mathbf{v},\mathbf{n})\|_{C^{3+\frac12}\times C^{3+\frac12}\times C^{3+\frac12}}\le C_T.$$

The paper is organized as follows. In Section 2, we first investigate the local existence and uniqueness of the solution for problem (1.4)–(1.10). In Section 3, by some energy estimates, the global existence and long time behavior of strong solution are derived. In Section 4, we prove a regularity result in Euler coordinates.

2. Local existence

In this section, we first study the local (in time) solution of (1.4)–(1.10) since the equations are nonlinear. Our main result is as follows:

THEOREM 2.1. *Suppose that the initial data satisfies* $c_0\le\rho_0\in H^2$ *for some* $c_0>0$, $\mathbf{v}_0\in H_0^1\cap H^2$, $\mathbf{n}_0\in H^3$ *with* $|\mathbf{n}_0|=1$ *in* $\bar{\Omega}$. *Then there exist a positive time* $T_0>0$ *and a unique strong solution* $(\rho,\mathbf{v},\mathbf{n})$ *to the initial boundary value problem (1.4)–(1.10), such that*

$$\begin{aligned}
&0<\rho\in L^\infty(0,T_0;H^2),\quad \rho_t\in L^\infty(0,T_0;H^1),\\
&\mathbf{v}\in L^\infty(0,T_0;H_0^1\cap H^2)\cap L^2(0,T_0;H^3),\quad \mathbf{v}_t\in L^\infty(0,T_0;L^2)\cap L^2(0,T_0;H_0^1)
\end{aligned}$$

$\mathbf{n}\in L^\infty(0,T_0;H^3)\cap W_2^{4,2}(\Omega\times(0,T_0]),\quad \mathbf{n}_t\in L^\infty(0,T_0;H^1)\cap L^2(0,T_0;H^2)$ *and* $|\mathbf{n}|=1$ *in* $\Omega\times(0,T_0]$.

Proof. In order to get the local existence and unique result for the strong solution of (1.4)–(1.10), we need to employ the Schauder fixed point theorem. The idea is given by [16] and [17]. For completeness, we outline the proof here. Firstly, we introduce a new variable $\sigma=r\rho$ and rewrite problem (1.4)–(1.8) as an equivalent one

$$\sigma_t+(\sigma u)_r=0, \tag{2.1}$$

$$(\sigma u)_t+(\sigma u^2)_r-\sigma\frac{v^2}{r}+rP_r=\kappa r(u_r+\frac{u}{r})_r-\frac{r}{2}(|\mathbf{n}_r|^2)_r-\frac{|\mathbf{n}_r|^2}{r}, \tag{2.2}$$

$$(\sigma v)_t+(\sigma uv)_r+\sigma\frac{uv}{r}=\mu r(v_r+\frac{v}{r})_r, \tag{2.3}$$

$$(\sigma w)_t+(\sigma uw)_r=\mu r(w_{rr}+\frac{w_r}{r}), \tag{2.4}$$

$$\mathbf{n}_t+u\mathbf{n}_r=\mathbf{n}_{rr}+|\mathbf{n}_r|^2\mathbf{n}+\frac{\mathbf{n}_r}{r}, \tag{2.5}$$

with the following initial and boundary conditions:

$$(\sigma,\mathbf{v},\mathbf{n})|_{t=0}=(r\rho_0,\mathbf{v}_0,\mathbf{n}_0),\quad r\in[a,b], \tag{2.6}$$

$$\mathbf{v}(a,t)=\mathbf{v}(b,t)=0,\quad \mathbf{n}_r(a,t)=\mathbf{n}_r(b,t)=0,\quad t\in(0,\infty). \tag{2.7}$$

We now turn to study the local existence of a unique strong solution to (2.1)–(2.7) by using a standard fixed point argument. Let us consider the following linearized system:

$$\sigma_t+(\sigma z)_r=0, \tag{2.8}$$

$$(\sigma u)_t+(\sigma zu)_r-\sigma\frac{v^2}{r}+rP_r=\kappa r(u_r+\frac{u}{r})_r-\frac{r}{2}(|\mathbf{n}_r|^2)_r-\frac{|\mathbf{n}_r|^2}{r}, \tag{2.9}$$

$$(\sigma v)_t+(\sigma zv)_r+\sigma\frac{zv}{r}=\mu r(v_r+\frac{v}{r})_r, \tag{2.10}$$

$$(\sigma w)_t+(\sigma zw)_r=\mu r(w_{rr}+\frac{w_r}{r}), \tag{2.11}$$

$$\mathbf{n}_t+z\mathbf{n}_r=\mathbf{n}_{rr}+|\mathbf{m}_r|^2\mathbf{n}+\frac{\mathbf{n}_r}{r}, \tag{2.12}$$

supplemented by the initial and boundary conditions (2.6)–(2.7), where $(z,\mathbf{m})$ are known smooth functions which satisfy the boundary conditions

$$z(a,t)=z(b,t)=0 \text{ and } \mathbf{m}_r(a,t)=\mathbf{m}_r(b,t)=0,\ t>0.$$

In addition, we have the fact that $\sigma(r,0)\geq a/c_0>0$. Then, define

$$R_{T_0}=\left\{(z,\mathbf{m})\left|\begin{array}{l}\|z\|^2_{H_0^1\cap H^2}+\|z_t\|^2_{L^2}+\displaystyle\int_0^{T_0}\|z\|^2_{H^3}+\|z_t\|^2_{H^1}dt\leq M_1,\\ \|\mathbf{m}\|^2_{H^3}+\|\mathbf{m}_t\|^2_{H^1}+\displaystyle\int_0^{T_0}\|\mathbf{m}_t\|^2_{H^2}dt\leq M_2,\\ z(r,0)=u_0(r),\mathbf{m}(r,0)=\mathbf{n}_0(r),r\in[a,b],\end{array}\right.\right\},$$

where the values of $M_1>1$, $M_2>1$ and T_0 will be decided later.

Notice that (2.8) is a hyperbolic equation. Therefore, the existence of a unique strong solution of (2.8) is well-known. Moreover, the solution satisfies (see [30]), if we choose T_0 sufficiently small,

$$\sup_{0\le t\le T_0}(\|\sigma(t)\|_{H^2}+M_1^{-\frac{1}{2}}\|\sigma_t(t)\|_{H^1}+\|\sigma^{-1}(t)\|_{L^\infty})\le C$$

which the constant $C>0$ depending only on a,b and the initial data, but independent on M_1 and M_2. At the same time, one can find that (2.9)–(2.12) are all linear parabolic equations for the variables their own. Thus, by the classical theory of parabolic equations, we have the existence of a unique strong solution $(\mathbf{v},\mathbf{n})$.

Next, we define a map Φ by

$$\Phi: R_{T_0}\to R_{T_0},\ (z,\mathbf{m})\mapsto (u,\mathbf{n}).$$

Furthermore, by the method of energy estimate, one can obtain some priori estimates for $\mathbf{n}$, v,w and u one by one, which show $(u,\mathbf{n})$ belongs to R_{T_0} for some large positive constants M_1 and M_2 and an appropriate small T_0.

Finally, the Schauder's fixed point theorem implies the local existence for the strong solution of (1.4)–(1.10). This can be done in a similar way as in [16] and [17]. And the uniqueness of the solution can be proved by the standard method similar to [13]. This completes the proof of Theorem 2.1. ∎

3. Global existence and long time behavior

In this section we shall complete the proof of Theorem 1.1 and Theorem 1.2. To this end, we first need a priori estimates. Low order estimates have to be uniform, that is, they should be independent of T. While high order estimates may depend on time.

3.1. A priori estimates.

LEMMA 3.1 (see [31]). *There exists a positive constant C_0, such that the following inequality holds for all $\mathbf{f}\in H_0^1(R_0,\infty)$ and $r^\alpha\mathbf{f},r^\alpha\mathbf{f}_r\in L^2(R_0,\infty)$:*

$$\|r^\beta\mathbf{f}\|_{L^p(R_0,\infty)}\le C_0\|r^\alpha\mathbf{f}_r\|_{L^2(R_0,\infty)}^b\|r^\alpha\mathbf{f}\|_{L^2(R_0,\infty)}^{1-b} \tag{3.1}$$

if and only if the following relations hold:

$$\begin{aligned}&\frac{1}{p}+\beta=\frac{1}{2}+\alpha-b,\\&\alpha-\sigma\ge 0,\ \textit{if } b>0,\\&\alpha-\sigma\le 1,\ \textit{if } b>0,\ \textit{and } \alpha-\frac{1}{2}=\frac{1}{p}+\beta,\end{aligned}$$

where $p>0,0\le b\le 1,\alpha>-\frac{1}{2},\beta>-\frac{1}{p}$ and $\beta=b\sigma+(1-b)\alpha$.

LEMMA 3.2. *For any $0\le t<T$, it holds that*

$$\begin{aligned}&\int_a^b r\Big(\frac{\rho\mathbf{v}^2}{2}+\frac{\rho^\gamma}{\gamma-1}+\rho+\frac{|\mathbf{n}_r|^2}{2}\Big)(t)dr\\&+\int_0^t\int_a^b r\left[\kappa\Big(u_r^2+\frac{u^2}{r^2}\Big)+\mu\Big(v_r^2+\frac{v^2}{r^2}\Big)+\mu w_r^2+(\mathbf{n}_{rr}+|\mathbf{n}_r|^2\mathbf{n})^2+\frac{|\mathbf{n}_r|^2}{r^2}\right]drds=E_0,\end{aligned} \tag{3.2}$$

where

$$E_0=\int_a^b r\left(\frac{\rho_0\mathbf{v_0}^2}{2}+\frac{\rho_0^\gamma}{\gamma-1}+\rho_0+\frac{|\mathbf{n_0}_r|^2}{2}\right)dr$$

denotes the total energy of the initial data.

Proof. Multiplying (1.5)–(1.7) by u, v and w, respectively, adding the results and integrating the resulting equations over (a,b), we have

$$\begin{aligned}&\frac{d}{dt}\int_a^b r(\frac{\rho\mathbf{v}^2}{2}+\frac{\rho^\gamma}{\gamma-1})dr+\int_a^b r\left[\kappa(u_r^2+\frac{u^2}{r^2})+\mu(v_r^2+\frac{v^2}{r^2})+\mu w_r^2\right]dr\\&=-\int_a^b[ru(\mathbf{n}_{rr}\cdot\mathbf{n}_r)+u|\mathbf{n}_r|^2]dr.\end{aligned}\tag{3.3}$$

Then multiplying (1.8) by $r(\mathbf{n}_{rr}+|\mathbf{n}_r|^2\mathbf{n})$, integrating in (a,b) and using the fact that

$$(\mathbf{n}_t+u\mathbf{n}_r)\cdot|\mathbf{n}_r|^2\mathbf{n}=0,\ \frac{\mathbf{n}_r}{r}\cdot|\mathbf{n}_r|^2\mathbf{n}=0,$$

we get

$$\int_a^b ru\mathbf{n}_{rr}\cdot\mathbf{n}_r dr-\int_a^b\mathbf{n}_t\cdot\mathbf{n}_r dr-\frac{1}{2}\frac{d}{dt}\int_a^b r|\mathbf{n}_r|^2dr=\int_a^b r(\mathbf{n}_{rr}+|\mathbf{n}_r|^2\mathbf{n})^2dr.\tag{3.4}$$

On the other hand, multiplying (1.8) by $\mathbf{n}_r$ and integrating over (a,b), we have

$$\int_a^b\mathbf{n}_t\cdot\mathbf{n}_r dr+\int_a^b u|\mathbf{n}_r|^2dr=\int_a^b\frac{|\mathbf{n}_r|^2}{r}dr.\tag{3.5}$$

Combining (3.3)–(3.5), we derive

$$\begin{aligned}&\frac{d}{dt}\int_a^b r(\frac{\rho\mathbf{v}^2}{2}+\frac{\rho^\gamma}{\gamma-1}+\frac{|\mathbf{n}_r|^2}{2})dr+\int_a^b r\left[\kappa(u_r^2+\frac{u^2}{r^2})+\mu(v_r^2+\frac{v^2}{r^2})+\mu w_r^2\right]dr\\&+\int_a^b\left[r(\mathbf{n}_{rr}+|\mathbf{n}_r|^2\mathbf{n})^2+\frac{|\mathbf{n}_r|^2}{r}\right]dr=0,\end{aligned}\tag{3.6}$$

which together with (1.4) give (3.2). ∎

In order to obtain further estimates, we need to introduce the following Sobolev inequalities for radially symmetric functions ([16]):

$$\|\rho\|_{L^\infty}^2\le C\|\rho\|_{H^1}\le C\int_a^b r(\rho^2+\rho_r^2)dr,\tag{3.7}$$

$$\|\mathbf{v}\|_{L^\infty}^2\le C\int_a^b r(\mathbf{v}_r^2+\frac{\mathbf{v}^2}{r^2})dr,\tag{3.8}$$

$$\|\mathbf{n}_r\|_{L^\infty}^2\le C\|\mathbf{n}_r\|_{H^1}\le C\int_a^b r(|\mathbf{n}_r|^2+|\mathbf{n}_{rr}|^2)dr,.\tag{3.9}$$

LEMMA 3.3. *For any $0\le t<T$, it holds that*

$$\int_0^t\int_a^b r(|\mathbf{n}_t|^2+|\mathbf{n}_{rr}|^2)drds\le C,\tag{3.10}$$

where the positive constant C depending only on a,b and E_0.

Proof. By Lemma 3.1, we have

$$\|r^{1/2}\mathbf{n}_r\|_{L^4}^4\le C_0\|r^{1/2}\mathbf{n}_r\|_{L^2}^3\|r^{1/2}\mathbf{n}_{rr}\|_{L^2}.\tag{3.11}$$

With the help of $|\mathbf{n}|=1$ and (3.11), we get

$$\begin{aligned}\int_a^b r|\mathbf{n}_{rr}|^2dr=&\int_a^b r(\mathbf{n}_{rr}+|\mathbf{n}_r|^2\mathbf{n})^2dr+\int_a^b r|\mathbf{n}_r|^4dr\\ \le&\int_a^b r(\mathbf{n}_{rr}+|\mathbf{n}_r|^2\mathbf{n})^2dr+\frac{1}{a}\int_a^b r^2|\mathbf{n}_r|^4dr\\ \le&\int_a^b r(\mathbf{n}_{rr}+|\mathbf{n}_r|^2\mathbf{n})^2dr+C\left(\int_a^b r|\mathbf{n}_r|^2dr\right)^{\frac{3}{2}}\left(\int_a^b r|\mathbf{n}_{rr}|^2dr\right)^{\frac{1}{2}}\\ \le&\int_a^b r(\mathbf{n}_{rr}+|\mathbf{n}_r|^2\mathbf{n})^2dr+\frac{1}{2}\int_a^b r|\mathbf{n}_{rr}|^2dr+C\left(\int_a^b r|\mathbf{n}_r|^2dr\right)^3.\end{aligned}$$

Then, integrating the above inequality over $(0,t)$, we have

$$\begin{aligned}&\int_0^t\int_a^b r|\mathbf{n}_{rr}|^2drds\\ \le&2\int_0^t\int_a^b r(\mathbf{n}_{rr}+|\mathbf{n}_r|^2\mathbf{n})^2drds+C\int_0^t\left[\left(\int_a^b\frac{|\mathbf{n}_r|^2}{r}dr\right)\left(\int_a^b r|\mathbf{n}_r|^2dr\right)^2\right]ds\\ \le&2E_0+CE_0^2\int_0^t\int_a^b\frac{|\mathbf{n}_r|^2}{r}drds\\ \le&C.\end{aligned}$$

Next, multiplying (1.8) by $r\mathbf{n}_t$ and integrating over $(0,t)\times(a,b)$, we have

$$\begin{aligned}\int_0^t\int_a^b r|\mathbf{n}_t|^2drds\le&C\left(\int_0^t\int_a^b(ru^2|\mathbf{n}_r|^2+r|\mathbf{n}_{rr}|^2+r|\mathbf{n}_r|^4+\frac{|\mathbf{n}_r|^2}{r})drds\right)\\ \le&C\int_0^t\left(\|u\|_{L^\infty}^2\int_a^b r|\mathbf{n}_r|^2dr\right)ds+C\\ \le&C\int_0^t\int_a^b r(u_r^2+\frac{u^2}{r^2})drds+C\\ \le&C.\end{aligned}$$

Thus the lemma is proved. ∎

LEMMA 3.4. *For any $0\le t<T$, it holds that*

$$\int_a^b r|\mathbf{n}_{rr}|^2(t)dr+\int_0^t\int_a^b r(|\mathbf{n}_{tr}|^2+|\mathbf{n}_{rrr}|^2)drds\le C, \tag{3.12}$$

where the positive constant C depending only on a, b, E_0, and $\|\mathbf{n}_0\|_{H^2}$.

Proof. By Lemma 3.1, we have

$$\|r^{1/2}\mathbf{n}_r\|_{L^6}^6\le C_0\|r^{1/2}\mathbf{n}_r\|_{L^2}^4\|r^{1/2}\mathbf{n}_{rr}\|_{L^2}^2\le C_0E_0\|r^{1/2}\mathbf{n}_r\|_{L^2}^2\|r^{1/2}\mathbf{n}_{rr}\|_{L^2}^2. \tag{3.13}$$

Now, differentiating (1.8) with respect to r, one can get

$$\mathbf{n}_{tr}+u_r\mathbf{n}_r+u\mathbf{n}_{rr}=\mathbf{n}_{rrr}+|\mathbf{n}_r|^2\mathbf{n}_r+2(\mathbf{n}_r\cdot\mathbf{n}_{rr})\mathbf{n}+\frac{\mathbf{n}_{rr}}{r}-\frac{\mathbf{n}_r}{r^2}. \tag{3.14}$$

Multiplying (3.14) by $r\mathbf{n}_{tr}$ and then integrating it over (a,b), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_a^b r|\mathbf{n}_{rr}|^2dr+\int_a^b r|\mathbf{n}_{tr}|^2dr\\
=&\int_a^b [r|\mathbf{n}_r|^2(\mathbf{n}_r\cdot\mathbf{n}_{tr})+2r(\mathbf{n}_r\cdot\mathbf{n}_{rr})(\mathbf{n}\cdot\mathbf{n}_{tr})]dr-\int_a^b \frac{\mathbf{n}_r}{r}\cdot\mathbf{n}_{tr}dr\\
&-\int_a^b [ru_r(\mathbf{n}_r\cdot\mathbf{n}_{tr})+ru(\mathbf{n}_{rr}\cdot\mathbf{n}_{tr})]dr\\
\le&\frac{1}{2}\int_a^b r|\mathbf{n}_{rt}|^2dr+C\int_a^b r\left(|\mathbf{n}_r|^6+|\mathbf{n}_r|^2|\mathbf{n}_{rr}|^2+\frac{|\mathbf{n}_r|^2}{r^4}+u_r^2|\mathbf{n}_r|^2+u^2|\mathbf{n}_{rr}|^2\right)dr\\
\le&\frac{1}{2}\int_a^b r|\mathbf{n}_{tr}|^2dr+C\int_a^b r|\mathbf{n}_r|^6dr+C(\|\mathbf{n}_r\|_{L^\infty}^2+\|u\|_{L^\infty}^2)\int_a^b |\mathbf{n}_{rr}|^2dr\\
&+C\|\mathbf{n}_r\|_{L^\infty}^2\int_a^b ru_r^2dr+C\int_a^b \frac{|\mathbf{n}_r|^2}{r}dr.
\end{aligned}$$

By virtue of (3.13), (3.8), and (3.9), we have

$$\begin{aligned}
&\frac{d}{dt}\int_a^b r|\mathbf{n}_{rr}|^2dr+\int_a^b r|\mathbf{n}_{tr}|^2dr\\
\le&C\left[\int_a^b r(|\mathbf{n}_r|^2+|\mathbf{n}_{rr}|^2)dr+\int_a^b r(u_r^2+\frac{u^2}{r^2})dr\right]\int_a^b r|\mathbf{n}_{rr}|^2dr\\
&+C\int_a^b ru_r^2dr+C\int_a^b \frac{|\mathbf{n}_r|^2}{r}dr.
\end{aligned}$$

Combining this with Lemma 3.2, Lemma 3.3 and the Gronwall's inequality, implies that

$$\int_a^b r|\mathbf{n}_{rr}|^2(t)dr+\int_0^t\int_a^b r|\mathbf{n}_{tr}|^2drds\le C. \tag{3.15}$$

Then, we infer from (3.14) and (3.15) that

$$\int_0^t\int_a^b r|\mathbf{n}_{rrr}|^2drds\le C,$$

which together with (3.15) gives (3.12). □

In order to show the long time behavior of the solutions, the crucial step is to establish uniform pointwise positive the lower and upper bounds of the density. The difficulties arise from the dependence on the time and spatial variables of the coefficients, but which can be overcome in our approach by modifying the methods of [28] and [29] for the compressible Navier–Stokes equations. The main idea of above references was introduced by Kazhikhov in [32] to deal with the one-dimensional case.

Lemma 3.5. *There exist positive constants C_* and C^*, such that*

$$C_*\le\rho(r,t)\le C^*,\ (r,t)\in(a,b)\times(0,T), \tag{3.16}$$

where the positive constant C_* and C^* depending only on a,b, E_0, κ,γ and $\|\rho_0\|_{L^\infty}$.

Proof. We first introduce Lagrangian coordinates $(y,\tilde{\tau})$ defined by

$$y=\int_a^r r\rho(r,t)dr,\ \tilde{\tau}=t.$$

Without danger of confusion, we denote $(y,\tilde{\tau})$ by (y,t). Notice that Ω is transformed into $(0,L)$ via

$$L=\int_a^b r\rho(r,t)dr=\int_a^b r\rho_0(r)dr,\quad \forall t\geq 0.$$

We use $\tau=1/\rho$ to denote the specific volume. Thus, (1.4)–(1.8) in Eulerian coordinates can be written in Lagrangian coordinate in the new variables $(y,t)\in(0,L)\times(0,T)$ as follows:

$$\tau_t=(ru)_y, \tag{3.17}$$

$$u_t=r[\frac{\kappa(ru)_y}{\tau}]_y-rP_y+\frac{v^2}{r}-\frac{r}{2}[(\frac{r\mathbf{n}_y}{\tau})^2]_y-\frac{r\mathbf{n}_y^2}{\tau}, \tag{3.18}$$

$$v_t=\mu r[\frac{(rv)_y}{\tau}]_y-\frac{uv}{r}, \tag{3.19}$$

$$w_t=\mu r[\frac{(rw)_y}{\tau}]_y+\frac{\mu\tau w}{r^2}, \tag{3.20}$$

$$\mathbf{n}_t=\frac{r}{\tau}[\frac{r\mathbf{n}_y}{\tau}]_y+\frac{r^2\mathbf{n}_y^2\mathbf{n}}{\tau^2}+\frac{\mathbf{n}_y}{\tau}. \tag{3.21}$$

subject to the following initial and boundary conditions:

$$\begin{aligned}&\tau(y,0)=\tau_0(y),\quad \mathbf{v}(y,0)=\mathbf{v}_0(y),\quad \mathbf{n}(y,0)=\mathbf{n}_0(y),\qquad y\in(0,L)\\ &\mathbf{v}(0,t)=\mathbf{v}(L,t)=\mathbf{0},\quad \mathbf{n}_y(0,t)=\mathbf{n}_y(L,t)=\mathbf{0},\qquad t\in(0,T).\end{aligned}$$

Now, we only have to show that $C^{*-1}\leq\tau(y,t)\leq C_*^{-1}$, $(y,t)\in(0,L)\times(0,T)$. Let

$$\sigma(y,t)=\frac{\kappa(ru)_y}{\tau}-P-\frac{1}{2}(\frac{r\mathbf{n}_y}{\tau})^2+\int_0^y\left(\frac{v^2-u^2}{r^2}-\frac{\mathbf{n}_y^2}{\tau}\right)dx,\quad \text{and}$$

$$\phi(y,t)=\int_0^y\frac{u_0}{r_0}dx+\int_0^t\sigma(y,s)ds,$$

with $r_0(y)=\eta^{-1}(y)$ and $\eta(r)=\int_a^r r\rho_0 dr$. Here, we note that if $\inf\{\rho_0(r):r\in(a,b)\}>0$, then η is invertible. From (3.18), we have

$$\left(\frac{u}{r}\right)_t=\sigma_y.$$

Thus,

$$\phi_y=\frac{u_0}{r_0}+\int_0^t\sigma_y(y,s)ds=\frac{u}{r},\quad \phi_t=\sigma.$$

With the help of (3.17), we get

$$(\phi\tau)_t=\sigma\tau+\phi(ru)_y$$

$$=\kappa(ru)_y-P\tau-\frac{1}{2}(\frac{r^2\mathbf{n}_y^2}{\tau})+\tau\int_0^y\left(\frac{v^2-u^2}{r^2}-\frac{\mathbf{n}_y^2}{\tau}\right)dx+(\phi ru)_y-u^2.$$

Integrating the above equality over $(0,L)\times(0,T)$, we obtain

$$\begin{aligned}
&\int_0^L\phi\tau dy\\
=&\int_0^L\phi_0\tau_0dy-\int_0^t\int_0^L\left(P\tau+u^2+\frac{1}{2}(\frac{r^2\mathbf{n}_y^2}{\tau})\right)dyds\\
&+\int_0^t\int_0^L\left[\tau\int_0^y\left(\frac{v^2-u^2}{r^2}-\frac{\mathbf{n}_y^2}{\tau}\right)dx\right]dyds\\
=&\int_0^L\phi_0\tau_0dy-\int_0^t\int_0^L\left(\frac{u^2+v^2}{2}+P\tau\right)dyds+\frac{b^2}{2}\int_0^t\int_0^L\left(\frac{v^2-u^2}{r^2}-\frac{\mathbf{n}_y^2}{\tau}\right)dyds.
\end{aligned}$$

On the other hand, there exists a point $x_0(t)\in(0,L)$ such that

$$\tau^*\phi(x_0(t),t)=\int_0^L\phi(y,t)\tau(y,t)dy$$

where $\tau^*=\int_0^L\tau(y,t)dy=\frac{b^2-a^2}{2}$. Thus,

$$\begin{aligned}
&\int_0^t\sigma(x_0(t),s)ds\\
=&\phi(x_0(t),t)-\int_0^{x_0(t)}\frac{u_0}{r_0}dx\\
=&\frac{1}{\tau^*}\int_0^L\phi\tau dy-\int_0^{x_0(t)}\frac{u_0}{r_0}dx\\
=&\frac{1}{\tau^*}\left[\int_0^L\phi_0\tau_0dy-\int_0^t\int_0^L\left(\frac{u^2+v^2}{2}+P\tau\right)dyds+\frac{b^2}{2}\int_0^t\int_0^L\left(\frac{v^2-u^2}{r^2}-\frac{\mathbf{n}_y^2}{\tau}\right)dyds\right]\\
&-\int_0^{x_0(t)}\frac{u_0}{r_0}dx.
\end{aligned}\tag{3.22}$$

We observe from (3.17) that

$$\left(\frac{u}{r}\right)_t=\sigma_y=\kappa(\log\tau)_{ty}-P_y-\frac{1}{2}\left((\frac{r\mathbf{n}_y}{\tau})^2\right)_y+\frac{v^2-u^2}{r^2}-\frac{\mathbf{n}_y^2}{\tau}.$$

Integrating the above equality over $(x_0(t),y)\times(0,t)$ by the definition of σ, we obtain

$$\begin{aligned}
&\kappa\log\tau(y,t)-\int_0^tP(y,s)ds\\
=&\kappa\log\tau_0(y)+\int_{x_0(t)}^y\left(\frac{u}{r}-\frac{u_0}{r_0}\right)dx+\int_0^t\sigma(x_0(t),s)ds
\end{aligned}$$

$$-\int_0^t\int_0^y\left(\frac{v^2-u^2}{r^2}-\frac{\mathbf{n}_y^2}{\tau}\right)dxds+\frac{1}{2}\int_0^t\frac{r^2\mathbf{n}_y^2}{\tau^2}ds.$$

It follows from (3.22) that

$$\begin{aligned}
&\kappa\log\tau(y,t)-\int_0^t P(y,s)ds\\
&=\kappa\log\tau_0(y)+\int_{x_0(t)}^y\frac{u}{r}dx-\int_0^y\frac{u_0}{r_0}dx-\int_0^t\int_0^y\left(\frac{v^2-u^2}{r^2}-\frac{\mathbf{n}_y^2}{\tau}\right)dxds+\frac{1}{2}\int_0^t\frac{r^2\mathbf{n}_y^2}{\tau^2}ds\\
&+\frac{1}{\tau^*}\left[\int_0^L\phi_0\tau_0dy-\int_0^t\int_0^L\left(\frac{u^2+v^2}{2}+P\tau\right)dyds+\frac{b^2}{2}\int_0^t\int_0^L\left(\frac{v^2-u^2}{r^2}-\frac{\mathbf{n}_y^2}{\tau}\right)dyds\right].
\end{aligned}$$

Therefore, we have the following representation:

$$\frac{1}{\tau}\exp\left\{\frac{1}{\kappa}\int_0^t P(y,s)ds\right\}=\frac{B(t)}{D(y,t)},\tag{3.23}$$

where

$$B(t)=\exp\left\{\frac{1}{\kappa\tau^*}\int_0^t\int_0^L P\tau dyds\right\},\tag{3.24}$$

$$\begin{aligned}
D(y,t)=&\tau_0(y)\exp\left\{\frac{1}{\kappa}\left(\frac{1}{\tau^*}\int_0^L\phi_0\tau_0dy+\int_{x_0(t)}^y\frac{u}{r}dx-\int_0^y\frac{u_0}{r_0}dx\right)\right.\\
&+\frac{1}{\kappa}\left[-\frac{1}{\tau^*}\int_0^t\int_0^L\frac{u^2+v^2}{2}dyds+\frac{b^2}{2\tau^*}\int_0^t\int_0^L\left(\frac{v^2-u^2}{r^2}-\frac{\mathbf{n}_y^2}{\tau}\right)dyds\right.\\
&\left.\left.-\int_0^t\int_0^y\left(\frac{v^2-u^2}{r^2}-\frac{\mathbf{n}_y^2}{\tau}\right)dxds+\frac{1}{2}\int_0^t\frac{r^2\mathbf{n}_y^2}{\tau^2}ds\right]\right\}.
\end{aligned}\tag{3.25}$$

Moreover, since

$$\frac{d}{dt}\exp\left\{\frac{\gamma}{\kappa}\int_0^t P(y,s)ds\right\}=\frac{\gamma}{\kappa}\left[\rho\exp\left\{\frac{1}{\kappa}\int_0^t P(y,s)ds\right\}\right]^\gamma=\frac{\gamma}{\kappa}\left(\frac{B}{D}\right)^\gamma,$$

we have

$$\exp\left\{\frac{1}{\kappa}\int_0^t P(y,s)ds\right\}=\left(1+\frac{\gamma}{\kappa}\int_0^t\left(\frac{B}{D}\right)^\gamma ds\right)^{\frac{1}{\gamma}},$$

which implies that

$$\tau(y,t)=\frac{D}{B}\left(1+\frac{\gamma}{\kappa}\int_0^t\left(\frac{B}{D}\right)^\gamma ds\right)^{\frac{1}{\gamma}}.\tag{3.26}$$

Therefore, we represent $\tau(y,t)$ only in terms of $B(t)$ and $D(y,t)$. Then, we can show the uniform bounds of τ by (3.26) and some estimates of $B(t)$ and $D(y,t)$. Note that

$$\int_0^y \frac{|u|}{r}dx = \int_a^b \rho|u|dr \le \frac{1}{a}\int_a^b r\rho dr + \frac{1}{a}\int_a^b r\rho|u|^2 dr \le C.$$

and

$$\begin{aligned}
&\Bigg| -\frac{1}{\tau^*}\int_0^t\int_0^L \frac{u^2+v^2}{2}dyds + \frac{b^2}{2\tau^*}\int_0^t\int_0^L \left(\frac{v^2-u^2}{r^2} - \frac{\mathbf{n}_y^2}{\tau}\right)dyds \\
&-\int_0^t\int_0^y \left(\frac{v^2-u^2}{r^2} - \frac{\mathbf{n}_y^2}{\tau}\right)dxds + \frac{1}{2}\int_0^t \frac{r^2\mathbf{n}_y^2}{\tau^2}ds| \Bigg| \\
\le& C\int_0^t\int_0^L \left[(u^2+v^2) + \frac{\mathbf{n}_y^2}{\tau}\right]dyds + C\int_0^t \frac{r^2\mathbf{n}_y^2}{\tau^2}ds \\
=& C\int_0^t\int_a^b \left[r\rho(u^2+v^2) + r\rho^2\mathbf{n}_y^2\right]drds + C\int_0^t r^2\rho^2\mathbf{n}_y^2 ds \\
\le& C\int_0^t\int_a^b \left[r\rho(u^2+v^2) + \frac{|\mathbf{n}_r|^2}{r}\right]drds + C\int_0^t |\mathbf{n}_r|^2 ds \\
\le& C\left[\int_0^t (\|u\|_{L^\infty}^2 + \|v\|_{L^\infty}^2)\left(\int_a^b r\rho dr\right)ds + \int_0^t\int_a^b r(|\mathbf{n}_r|^2 + |\mathbf{n}_{rr}|^2)drds\right] \\
\le& C,
\end{aligned}$$

where we have used the embedding relation (3.8) and (3.9). Hence, it follows from (3.25) that

$$C_1 \le D(y,t) \le C_2, \tag{3.27}$$

where C_1, C_2 are positive constants and independent of T. Now, let

$$B(t) = \exp\left\{\int_0^t A(s)ds\right\}$$

with $A(t) = \frac{1}{\kappa\tau^*}\int_0^L P\tau dy$. Applying Lemma 3.2, it is easy to see

$$A(t) = \frac{1}{\kappa\tau^*}\int_0^L \rho^{\gamma-1}dy = \frac{1}{\kappa\tau^*}\int_a^b r\rho^\gamma dr \le C_3.$$

By virtue of conservation of mass and Hölder's inequality,

$$\int_a^b r\rho_0 dr = \int_a^b r\rho dr \le \left(\frac{b^2-a^2}{2}\right)^{\frac{\gamma-1}{\gamma}}\left(\int_a^b r\rho^\gamma dr\right)^{\frac{1}{\gamma}},$$

namely,

$$A(t) \ge C_4.$$

Thus, for any $0 \le s < t$, we have

$$\frac{B(s)}{B(t)} = \exp\left\{-\int_s^t A(\xi)d\xi\right\}.$$

Moreover,

$$e^{-C_3(t-s)} \leq \frac{B(s)}{B(t)} \leq e^{-C_4(t-s)}, \quad \text{and} \quad 1 \leq B(t) \leq e^{C_3 t}. \tag{3.28}$$

Then, we deduce from (3.26)–(3.28) that

$$\begin{aligned}
\tau(y,t) &= D\left[\frac{1}{B^\gamma} + \frac{\gamma}{\kappa}\int_0^t \left(\frac{B(s)}{B(t)}\right)^\gamma \frac{1}{D^\gamma} ds\right]^{\frac{1}{\gamma}} \\
&\leq C_2\left(1 + C\int_0^t e^{-\gamma C_4(t-s)} ds\right)^{\frac{1}{\gamma}} \\
&= C_2\left(1 + C\left(\frac{1}{\gamma C_4} - \frac{1}{\gamma C_4} e^{-\gamma C_4 t}\right)\right)^{\frac{1}{\gamma}} \\
&\leq C_2\left(1 + \frac{C}{\gamma C_4}\right)^{\frac{1}{\gamma}} := C_*^{-1}.
\end{aligned} \tag{3.29}$$

On the other hand,

$$\begin{aligned}
\tau(y,t) &\geq C\left(\int_0^t \left(\frac{B(s)}{B(t)}\right)^\gamma ds\right)^{\frac{1}{\gamma}} \\
&\geq C\left(\int_0^t e^{-\gamma C_3(t-s)} ds\right)^{\frac{1}{\gamma}} \\
&= C\left(\frac{1}{\gamma C_3} - \frac{1}{\gamma C_3} e^{-\gamma C_3 t}\right)^{\frac{1}{\gamma}}.
\end{aligned} \tag{3.30}$$

It follows that there exists a time $t_0 > 0$ such that as $t \geq t_0$,

$$\tau(y,t) \geq C_5,$$

where the positive constant C_5 is independent of T. Also from (3.26)–(3.28), we get for $0 < t \leq t_0$,

$$\tau(y,t) \geq D(y,t)B^{-1}(t) \geq C_1 e^{-C_3 t} \geq C_1 e^{-C_3 t_0} := C_6.$$

Finally, taking $C^{*-1} = \min\{C_5, C_6\}$, we get

$$\tau(y,t) \geq C^{*-1}.$$

This, combing with (3.29), yields the desired estimate (3.16) and the lemma is proved. ∎

In the sequel we derive uniform estimates of derivatives for ρ and $\mathbf{v}$.

LEMMA 3.6. *For any $0 \leq t < T$, it holds that*

$$\int_a^b \rho \left|\left(\frac{1}{\rho}\right)_r\right|^2 dr + \int_0^t \int_a^b \rho_r^2 dr ds \leq C, \tag{3.31}$$

where the positive constant C depending only on a, b, κ, γ and initial values.

Proof. It follows from (1.4) that

$$\begin{aligned}
&\frac{d}{dt}\int_a^b \rho\left|\left(\frac{1}{\rho}\right)_r\right|^2 dr\\
=&\int_a^b \rho_t\left|\left(\frac{1}{\rho}\right)_r\right|^2 dr+2\int_a^b \rho\left(\frac{1}{\rho}\right)_r\left(\frac{1}{\rho}\right)_{rt} dr\\
=&-\int_a^b (\rho u)_r\left|\left(\frac{1}{\rho}\right)_r\right|^2 dr-\int_a^b \frac{\rho u}{r}\left|\left(\frac{1}{\rho}\right)_r\right|^2 dr\\
&+2\int_a^b \rho\left(\frac{1}{\rho}\right)_r\left(\left[\frac{1}{\rho}(u_r+\frac{u}{r})\right]_r-\left(\frac{1}{\rho}\right)_{rr}u-\left(\frac{1}{\rho}\right)_r u_r\right)dr\\
=&-\int_a^b (\rho u)_r\left|\left(\frac{1}{\rho}\right)_r\right|^2 dr-\int_a^b \frac{\rho u}{r}\left|\left(\frac{1}{\rho}\right)_r\right|^2 dr+2\int_a^b \left(\frac{1}{\rho}\right)_r(u_r+\frac{u}{r})_r dr\\
&+2\int_a^b \left|\left(\frac{1}{\rho}\right)_r\right|^2(\rho u_r+\frac{\rho u}{r})dr+\int_a^b \rho_r u\left|\left(\frac{1}{\rho}\right)_r\right|^2 dr-\int_a^b \rho u_r\left|\left(\frac{1}{\rho}\right)_r\right|^2 dr\\
=&2\int_a^b \left(\frac{1}{\rho}\right)_r(u_r+\frac{u}{r})_r dr+\int_a^b \frac{\rho u}{r}\left|\left(\frac{1}{\rho}\right)_r\right|^2 dr. \qquad (3.32)
\end{aligned}$$

Multiplying (1.5) by $\left(\frac{1}{\rho}\right)_r$ and using (1.4), we have

$$\begin{aligned}
&\frac{d}{dt}\int_a^b \rho u\left(\frac{1}{\rho}\right)_r dr\\
=&\int_a^b \rho u_t\left(\frac{1}{\rho}\right)_r dr+\int_a^b \rho_t u\left(\frac{1}{\rho}\right)_r dr+\int_a^b \rho u\left(\frac{1}{\rho}\right)_{rt} dr\\
=&\kappa\int_a^b \left(\frac{1}{\rho}\right)_r(u_r+\frac{u}{r})_r dr-\int_a^b \rho u u_r\left(\frac{1}{\rho}\right)_r dr+\int_a^b \frac{\rho v^2}{r}\left(\frac{1}{\rho}\right)_r dr-\int_a^b P_r\left(\frac{1}{\rho}\right)_r dr\\
&-\int_a^b \left(\frac{(|\mathbf{n}_r|^2)_r}{2}+\frac{|\mathbf{n}_r|^2}{r}\right)\left(\frac{1}{\rho}\right)_r dr+\int_a^b \rho_t u\left(\frac{1}{\rho}\right)_r dr+\int_a^b \rho u\left(\frac{1}{\rho}\right)_{rt} dr.
\end{aligned}$$

By virtue of (3.32), we get

$$\begin{aligned}
&\frac{\kappa}{2}\frac{d}{dt}\int_a^b \rho\left|\left(\frac{1}{\rho}\right)_r\right|^2 dr+\gamma\int_a^b \rho^{\gamma-3}\rho_r^2 dr-\frac{d}{dt}\int_a^b \rho u\left(\frac{1}{\rho}\right)_r dr\\
=&\frac{\kappa}{2}\int_a^b \frac{\rho u}{r}\left|\left(\frac{1}{\rho}\right)_r\right|^2 dr+2\int_a^b \rho u u_r\left(\frac{1}{\rho}\right)_r dr+\int_a^b \left(\rho_r u^2+\frac{\rho u^2}{r}-\frac{\rho v^2}{r}\right)\left(\frac{1}{\rho}\right)_r dr\\
&-\int_a^b \frac{\rho_t}{\rho^2}(\rho u)_r dr+\int_a^b \left(\frac{(|\mathbf{n}_r|^2)_r}{2}+\frac{|\mathbf{n}_r|^2}{r}\right)\left(\frac{1}{\rho}\right)_r dr\\
=&\frac{\kappa}{2}\int_a^b \frac{\rho u}{r}\left|\left(\frac{1}{\rho}\right)_r\right|^2 dr+\int_a^b \left(u_r^2+\frac{u u_r}{r}\right)dr+\int_a^b \left(\mathbf{n}_r\mathbf{n}_{rr}+\frac{|\mathbf{n}_r|^2}{r}-\frac{\rho v^2}{r}\right)\left(\frac{1}{\rho}\right)_r dr\\
\le&\epsilon\int_a^b \left|\left(\frac{1}{\rho}\right)_r\right|^2 dr+C(\epsilon)\left(\|u\|_{L^\infty}^2\int_a^b \left|\left(\frac{1}{\rho}\right)_r\right|^2 dr+\|v\|_{L^\infty}^2\int_a^b \rho v^2 dr\right)\\
&+2\int_a^b \left(u_r^2+\frac{u^2}{r^2}\right)dr+\frac{1}{2}\int_a^b (|\mathbf{n}_r|^2+|\mathbf{n}_{rr}|^2)dr+\int_a^b |\mathbf{n}_r|^2\left|\left(\frac{1}{\rho}\right)_r\right|^2 dr.
\end{aligned}$$

Then, integrating over $(0,t)$, by choosing $\epsilon>0$ small enough and using Lemma 3.2, Lemma 3.3 and Lemma 3.5, we arrive at

$$\begin{aligned}
&\int_a^b \rho\left|\left(\frac{1}{\rho}\right)_r\right|^2 dr+\int_0^t\int_a^b \rho^{\gamma-3}\rho_r^2 drds\\
\le& C+C\int_a^b \rho u\left(\frac{1}{\rho}\right)_r dr+C\int_0^t (\|u\|_{L^\infty}^2+\|\mathbf{n}_r\|_{L^\infty}^2)\int_a^b\left|\left(\frac{1}{\rho}\right)_r\right|^2 drds\\
\le& C+C\int_a^b \rho u^2 dr+\frac{1}{2}\int_a^b \rho\left|\left(\frac{1}{\rho}\right)_r\right|^2 dr+C\int_0^t (\|u\|_{L^\infty}^2+\|\mathbf{n}_r\|_{L^\infty}^2)\int_a^b\left|\left(\frac{1}{\rho}\right)_r\right|^2 drds,
\end{aligned}$$

which, by the Gronwall's inequality, implies (3.31). The proof is now complete. □

Obviously, we can get the following corollary by directly using Lemma 3.5 and Lemma 3.6.

COROLLARY 3.7. *For any $0\le t<T$, it holds that*

$$\int_a^b \rho_r^2(t)dr\le C, \tag{3.33}$$

where the positive constant C depending only on a,b, κ, γ and initial values.

LEMMA 3.8. *For any $0\le t<T$, it holds that*

$$\int_a^b\left(u_r^2+\frac{u^2}{r^2}+v_r^2+\frac{v^2}{r^2}+w_r^2\right)(t)dr+\int_0^t\int_a^b \mathbf{v}_t^2 drds\le C, \tag{3.34}$$

where the positive constant C depending only on a,b, κ, μ, γ and initial values.

Proof. Multiplying (1.5)–(1.7) by u_t,v_t,w_t, respectively, integrating the resulting equations over (a,b) and then using integration by parts, we get

$$\begin{aligned}
&\int_a^b \rho\mathbf{v}_t^2 dr+\frac{\kappa}{2}\frac{d}{dt}\int_a^b (u_r^2+\frac{u^2}{r^2})dr+\frac{\mu}{2}\frac{d}{dt}\int_a^b\left(v_r^2+\frac{v^2}{r^2}+w_r^2\right)dr\\
=&\kappa\int_a^b \frac{u_tu_r}{r}dr+\mu\int_a^b\left(\frac{v_tv_r}{r}+\frac{w_tw_r}{r}\right)dr+\int_a^b\left(\frac{\rho v^2u_t}{r}-\frac{\rho uvv_t}{r}\right)dr-\gamma\int_a^b \rho^{\gamma-1}\rho_r u_t dr\\
&-\int_a^b \rho u(u_ru_t+v_rv_t+w_rw_t)dr-\int_a^b (\mathbf{n}_r\cdot\mathbf{n}_{rr})u_t dr-\int_a^b \frac{|\mathbf{n}_r|^2}{r}u_t dr.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_a^b \mathbf{v}_t^2 dr+\frac{d}{dt}\int_a^b\left(u_r^2+\frac{u^2}{r^2}+v_r^2+\frac{v^2}{r^2}+w_r^2\right)dr\\
\le&\frac{1}{2}\int_a^b \mathbf{v}_t^2 dr+C\left(\int_a^b \frac{u_r^2+v_r^2+w_r^2}{r^2}dr+\int_a^b u^2(u_r^2+v_r^2+w_r^2)dr\right.\\
&\left.+\int_a^b v^4 dr+\int_a^b u^2v^2 dr+\int_a^b \rho_r^2 dr+\int_a^b |\mathbf{n}_{rr}|^2 dr+\int_a^b |\mathbf{n}_r|^2 dr\right),
\end{aligned}$$

which implies

$$\int_a^b \mathbf{v}_t^2 dr+\frac{d}{dt}\int_a^b\left(u_r^2+\frac{u^2}{r^2}+v_r^2+\frac{v^2}{r^2}+w_r^2\right)dr$$

$$\leq C(\|u\|_{L^\infty}^2+\|v\|_{L^\infty}^2)\int_a^b\left(u_r^2+\frac{u^2}{r^2}+v_r^2+\frac{v^2}{r^2}+w_r^2\right)dr$$
$$+C\left(\int_a^b\frac{u_r^2+v_r^2+w_r^2}{r^2}dr+\int_a^b\rho_r^2dr+\int_a^b|\mathbf{n}_{rr}|^2dr+\int_a^b|\mathbf{n}_r|^2dr\right).$$

Integrating over $(0,t)$, with the help of previous lemmas and Gronwall's inequality, we arrive at (3.34). This completes the proof. ∎

Making use of (1.4)–(1.8) and (3.33)–(3.34), we deduce that

Corollary 3.9. *For any $0\leq t<T$, it holds that*

$$\int_a^b(\rho_t^2+\mathbf{n}_t^2+G^2)(t)dr\int_0^t\int_a^b(\mathbf{v}_{rr}^2+G_r^2)drds\leq C, \tag{3.35}$$

where $G=\kappa(u_r+\frac{u}{r})-P$ and the positive constant C depending only on a, b, κ, μ, γ, and initial values.

Lemma 3.10. *For any $0\leq t<T$, it holds that*

$$\int_0^t\left|\int_a^b\rho_r\rho_{rt}dr\right|ds\leq C, \tag{3.36}$$

where the positive constant C depending only on a, b, κ, μ, γ, and initial values.

Proof. Differentiating (1.4) with respect to r, multiplying the resulting equation by ρ_r, integrating over (a,b) and then employing integration by parts, we have

$$\left|\int_a^b\rho_r\rho_{rt}dr\right|$$
$$=\left|-\frac{3}{2}\int_a^b\rho_r^2u_rdr-\int_a^b\rho\rho_ru_{rr}dr-\int_a^b\frac{\rho_r^2u}{r}dr-\int_a^b\frac{\rho\rho_ru_r}{r}dr+\int_a^b\frac{\rho\rho_ru}{r^2}dr\right|$$
$$\leq C\left(\int_a^b\rho_r^2dr+\int_a^b\rho_r^2u_r^2dr+\int_a^bu_{rr}^2dr+\int_a^b\rho_r^2u^2dr+\int_a^b\rho^2(u^2+u_r^2)dr\right).$$

Integrating over $(0,t)$ and applying previous lemmas, we obtain

$$\int_0^t\left|\int_a^b\rho_r\rho_{rt}dr\right|ds\leq C+C\int_0^t(\|u\|_{L^\infty}^2+\|u_r\|_{L^\infty}^2)\int_a^b\rho_r^2drds\leq C,$$

which completes the proof. ∎

3.2. Higher-order estimates.

Lemma 3.11. *For any $0\leq t<T$, it holds that*

$$\int_a^b(\rho\mathbf{v}_t^2+\mathbf{v}_{rr}^2+G_r^2)(t)dr+\int_0^t\int_a^b(\mathbf{v}_{tr}^2+G_{rr}^2)drds\leq C_T, \tag{3.37}$$

where the positive constant C_T depending on C and T.

Proof. Differentiating (1.5) with respect to t, multiplying the resulting equation by u_t and integrating over (a,b), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_a^b \rho u_t^2 dr + \kappa \int_a^b u_{tr}^2 dr \\
=& -\frac{1}{2}\int_a^b \rho_t u_t^2 dr - \kappa \int_a^b \frac{u_{tr}u_t}{r} dr + \int_a^b (\rho^\gamma)_t u_{tr} dr \\
&+ \int_a^b \Big(\frac{\rho_t v^2 u_t}{r} - \rho_t u u_r u_t - \rho u_t^2 u_r - \rho u u_{tr} u_t + \frac{2\rho v v_t u_t}{r}\Big) dr \\
&+ \int_a^b \left[(\mathbf{n}_r\cdot\mathbf{n}_{tr})u_{tr} - \frac{2(\mathbf{n}_r\cdot\mathbf{n}_{tr})u_t}{r}\right] dr,
\end{aligned}$$

which with (1.4) gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_a^b \rho u_t^2 dr + \kappa \int_a^b u_{tr}^2 dr \\
=&\frac{1}{2}\int_a^b \rho_r u u_t^2 dr - \frac{1}{2}\int_a^b \rho u_r u_t^2 dr + \frac{1}{2}\int_a^b \frac{\rho u u_t^2}{r} dr - \kappa \int_a^b \frac{u_{tr}u_t}{r} dr \\
&-\gamma \int_a^b \rho^{\gamma-1}\rho_r u u_{tr} dr - \gamma \int_a^b \rho^\gamma u_r u_{tr} dr - \gamma \int_a^b \frac{\rho^\gamma u u_{tr}}{r} dr \\
&+ \int_a^b \Big(\rho_r u^2 u_r u_t + \rho u u_r^2 u_t + \frac{\rho u^2 u_r u_t}{r} - \frac{\rho_r u v^2 u_t}{r} - \frac{\rho u_r v^2 u_t}{r} - \frac{\rho u v^2 u_t}{r^2}\Big) dr \\
&- \int_a^b \Big(\rho u_t^2 u_r + \rho u u_{tr} u_t - \frac{2\rho v v_t u_t}{r}\Big) dr + \int_a^b \left[(\mathbf{n}_r\cdot\mathbf{n}_{tr})u_{tr} - \frac{2(\mathbf{n}_r\cdot\mathbf{n}_{tr})u_t}{r}\right] dr \\
\le&\frac{\kappa}{2}\int_a^b u_{tr}^2 dr + C(1+\|u\|_{L^\infty}^2)\int_a^b \rho u_t^2 dr + C\|v\|_{L^\infty}^2 \int_a^b \rho v_t^2 dr \\
&+ C\int_a^b (u^2 + |\mathbf{n}_{tr}|^2) dr + C(\|u\|_{L^\infty}^2 + \|u\|_{L^\infty}^4\|u_r\|_{L^\infty}^2 + \|u\|_{L^\infty}^2\|v\|_{L^\infty}^4)\int_a^b \rho_r^2 dr \\
&+ C(1+\|u\|_{L^\infty}^4 + \|u\|_{L^\infty}^2\|u_r\|_{L^\infty}^2 + \|v\|_{L^\infty}^4)\int_a^b u_r^2 dr + C\|u\|_{L^\infty}^2\|v\|_{L^\infty}^2\int_a^b v^2 dr.
\end{aligned}$$

Then, from lemmas 3.2–3.8, we infer

$$\begin{aligned}
&\frac{d}{dt}\int_a^b \rho u_t^2 dr + \kappa \int_a^b u_{tr}^2 dr \\
\le& C + C\|u_r\|_{L^\infty}^2 + C(1+\|u\|_{L^\infty}^2)\int_a^b \rho u_t^2 dr + C\|v\|_{L^\infty}^2 \int_a^b \rho v_t^2 dr. \qquad (3.38)
\end{aligned}$$

Similarly to (3.38), differentiating (1.6) and (1.7) with respect to t, and multiplying the resulting equation by v_t and w_t, respectively, we derive

$$\begin{aligned}
&\frac{d}{dt}\int_a^b \rho v_t^2 dr + \mu \int_a^b v_{tr}^2 dr \\
\le& C + C\|v_r\|_{L^\infty}^2 + C(1+\|u\|_{L^\infty}^2)\int_a^b \rho v_t^2 dr + C(\|v\|_{L^\infty}^2 + \|v_r\|_{L^\infty}^2)\int_a^b \rho u_t^2 dr. \qquad (3.39)
\end{aligned}$$

and

$$\begin{aligned}&\frac{d}{dt}\int_a^b \rho w_t^2 dr+\mu\int_a^b w_{tr}^2 dr\\ \le& C+C\|w_r\|_{L^\infty}^2+C(1+\|u\|_{L^\infty}^2)\int_a^b \rho w_t^2 dr+C\|w_r\|_{L^\infty}^2\int_a^b \rho u_t^2 dr.\end{aligned} \tag{3.40}$$

Adding (3.38)–(3.40), we obtain

$$\begin{aligned}&\frac{d}{dt}\int_a^b \rho \mathbf{v}_t^2 dr+\int_a^b \mathbf{v}_{tr}^2 dr\\ \le& C+C\|\mathbf{v}_r\|_{L^\infty}^2+C(1+\|u\|_{L^\infty}^2+\|v\|_{L^\infty}^2+\|v_r\|_{L^\infty}^2+\|w_r\|_{L^\infty}^2)\int_a^b \rho \mathbf{v}_t^2 dr.\end{aligned} \tag{3.41}$$

In fact, by (1.5)–(1.7), we have

$$\begin{aligned}&\int_a^b \rho \mathbf{v}_t^2(s) dr\\ \le& C\int_a^b \frac{1}{\rho}\left[\left((u_r+\frac{u}{r})_r-(\rho^\gamma)_r+\frac{1}{2}(|\mathbf{n}_r|^2)_r+\frac{|\mathbf{n}_r|^2}{r}\right)^2+(v_r+\frac{v}{r})_r^2+(w_{rr}+\frac{w_r}{r})^2\right]dr\\ &+C\int_a^b \rho u^2\mathbf{v}_r^2 dr+C\int_a^b\left(\frac{\rho u^2v^2}{r^2}+\frac{\rho v^4}{r^2}\right)dr\\ \to& C(\rho_0,\mathbf{v}_0,\mathbf{n}_0)<\infty, \quad \text{as } s\to 0.\end{aligned}$$

Integrating (3.41) over $(0,t)$, it follows from the Gronwall's inequality that

$$\int_a^b \rho \mathbf{v}_t^2(t) dr+\int_0^t\int_a^b \mathbf{v}_{tr}^2 drds\le C_T. \tag{3.42}$$

Thanks to (1.5)–(1.7), (3.42) and the definition of G, (3.37) holds. □

LEMMA 3.12. *For any $0\le t<T$, it holds that*

$$\int_a^b (\mathbf{n}_{tr}^2+\mathbf{n}_{rrr}^2)(t) dr+\int_0^t\int_a^b (\mathbf{n}_{trr}^2+\mathbf{n}_{rrrr}^2) drds\le C_T, \tag{3.43}$$

where the positive constant C_T depending on C and T.

Proof. Differentiating (1.8) with respect to r, multiplying the resulting equation by $\mathbf{n}_{trrr}$ and integrating over (a,b), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_a^b \mathbf{n}_{rrr}^2 dr+\int_a^b \mathbf{n}_{trr}^2 dr\\ =&\frac{d}{dt}\int_a^b\left[(u\mathbf{n}_r)_r-(|\mathbf{n}_r|^2\mathbf{n})_r-(\frac{\mathbf{n}_r}{r})_r\right]\mathbf{n}_{rrr} dr\\ &-\int_a^b\left[(u\mathbf{n}_r)_{tr}-(|\mathbf{n}_r|^2\mathbf{n})_{tr}-(\frac{\mathbf{n}_r}{r})_{tr}\right]\mathbf{n}_{rrr} dr\\ \le&\frac{d}{dt}\int_a^b\left[(u\mathbf{n}_r)_r-(|\mathbf{n}_r|^2\mathbf{n})_r-(\frac{\mathbf{n}_r}{r})_r\right]\mathbf{n}_{rrr} dr+\frac{1}{2}\int_a^b \mathbf{n}_{trr}^2 dr\end{aligned}$$

$$+C\int_a^b (1+\mathbf{n}_{rrr}^2+\mathbf{n}_{tr}^2+u_{tr}^2)dr,$$

where previous lemmas are used. Integrating the above inequality over $(0,t)$, one can get

$$\begin{aligned}&\int_a^b \mathbf{n}_{rrr}^2 dr+\int_0^t\int_a^b \mathbf{n}_{trr}^2 drds\\ \le& C(u_0,\mathbf{n}_0,T)+\int_a^b \Big[(u\mathbf{n}_r)_r-(|\mathbf{n}_r|^2\mathbf{n})_r-(\frac{\mathbf{n}_r}{r})_r\Big]^2 dr+C\int_0^t\int_a^b \mathbf{n}_{rrr}^2 drds\\ \le& C(u_0,\mathbf{n}_0,T)+C\int_0^t\int_a^b \mathbf{n}_{rrr}^2 drds,\end{aligned}$$

which, by employing the Gronwall's inequality, gives

$$\int_a^b \mathbf{n}_{rrr}^2(t)dr+\int_0^t\int_a^b \mathbf{n}_{trr}^2 drds\le C_T. \tag{3.44}$$

Finally, (1.8) and (3.44) imply (3.43), which completes this lemma. □

LEMMA 3.13. *For any* $0\le t<T$*, it holds that*

$$\int_a^b (\rho_{tr}^2+\rho_{rr}^2)(t)dr+\int_0^t\int_a^b \mathbf{v}_{rrr}^2 drds\le C_T, \tag{3.45}$$

where the positive constant C_T *depending on* C *and* T*.*

Proof. In order to show (3.44), we first need to give a $W^{1,4}$ estimation for the density. To this end, we have to take the derivative of (1.4) with respect to r and then multiply the resulting equation by $|\rho_r|^2\rho_r$ and integrate over (a,b). Thus, we get

$$\begin{aligned}&\frac{1}{4}\frac{d}{dt}\int_a^b |\rho_r|^4 dr\\ =&-\int_a^b |\rho_r|^2\rho_r\rho_{rr}u dr-\int_a^b |\rho_r|^4(2u_r+\frac{u}{r})dr-\int_a^b |\rho_r|^2\rho_r\rho(u_{rr}+\frac{u_r}{r}-\frac{u}{r^2})dr\\ =&\frac{1}{4}\int_a^b |\rho_r|^2\rho_r u_r dr-\int_a^b |\rho_r|^4(2u_r+\frac{u}{r})dr-\frac{1}{\kappa}\int_a^b |\rho_r|^2\rho_r\rho(G+P)_r dr\\ \le& C(1+\|u_r\|_{L^\infty}^2+\|u\|_{L^\infty}^2)\int_a^b |\rho_r|^4 dr+C\|u_r\|_{L^\infty}^2\int_a^b |\rho_r|^2 dr+C\|G_r\|_{L^\infty}^2\int_a^b |\rho_r|^2 dr\\ \le& C(1+\|u_r\|_{L^\infty}^2+\|u\|_{L^\infty}^2)\int_a^b |\rho_r|^4 dr+C\|u_r\|_{L^\infty}^2+C\|G_r\|_{L^\infty}^2.\end{aligned}$$

By the Gronwall's inequality and the embedding theorem, we obtain

$$\int_a^b |\rho_r|^4(t)dr\le C_T. \tag{3.46}$$

Now, we are ready to prove (3.44). Differentiating (1.4) with respect to r twice, multiplying the by ρ_{rr} and integrating over (a,b), we have

$$\frac{1}{2}\frac{d}{dt}\int_a^b \rho_{rr}^2 dr$$

$$
\begin{aligned}
&= -\int_a^b (\rho u)_{rrr}\rho_{rr}dr - \int_a^b (\frac{\rho u}{r})_{rr}\rho_{rr}dr\\
&= -\frac{5}{2}\int_a^b \rho_{rr}^2 u_r dr - 3\int_a^b \rho_r\rho_{rr}u_{rr}dr - \int_a^b \rho_{rr}^2\frac{u}{r}dr - 2\int_a^b \rho_{rr}\frac{\rho_r u_r}{r}dr + 2\int_a^b \rho_{rr}\frac{\rho_r u}{r^2}dr\\
&\quad - \int_a^b \rho\rho_{rr}(u_{rrr} + \frac{u_{rr}}{r} - \frac{2u_r}{r^2} + \frac{2u}{r^3})dr\\
&= -\frac{5}{2}\int_a^b \rho_{rr}^2 u_r dr - 3\int_a^b \rho_r\rho_{rr}u_{rr}dr - \int_a^b \rho_{rr}^2\frac{u}{r}dr - 2\int_a^b \rho_{rr}\frac{\rho_r u_r}{r}dr + 2\int_a^b \rho_{rr}\frac{\rho_r u}{r^2}dr\\
&\quad - \frac{1}{\kappa}\int_a^b \rho\rho_{rr}(G_{rr} + P_{rr})dr\\
&\le C(1 + \|u\|_{L^\infty}^2 + \|u_r\|_{L^\infty}^2)\int_a^b \rho_{rr}^2 dr + C\|\rho_r\|_{L^\infty}^2\int_a^b (u_{rr}^2 + u_r^2)dr + C\|u\|_{L^\infty}^2\int_a^b \rho_r^2 dr\\
&\quad + C\int_a^b G_{rr}^2 dr + C\int_a^b P_{rr}^2 dr\\
&\le C + C(1 + \|u\|_{L^\infty}^2 + \|u_r\|_{L^\infty}^2)\int_a^b \rho_{rr}^2 dr + C\int_a^b G_{rr}^2 dr + C\int_a^b |\rho_r|^4 dr.
\end{aligned}
$$

Therefore, Lemma 3.11, (3.46) and the Gronwall's inequality imply

$$\int_a^b \rho_{rr}^2 dr \le C_T,$$

which, together with (1.4) and the definition of G, gives (3.45). The proof is complete. ∎

3.3. Global existence. The global existence of unique strong solution to the problem (1.4)–(1.10) can be established in terms of the local existence and the priori estimates.

Proof. (Proof of Theorem 1.1.) Based on all the lemmas in Section 3.1 and 3.2, it is not hard to get

$$
\begin{aligned}
&\|\rho\|_{L^1\cap H^2}^2 + \|\rho_t\|_{L^2}^2 + \|\mathbf{v}\|_{H_0^1\cap H^2}^2 + \|\mathbf{n}\|_{H^3}^2 + \|\mathbf{n}_t\|_{H^1}^2 + \|\mathbf{v}_t\|^2\\
&\quad + \int_0^t \Big(\|\mathbf{v}\|_{H^3}^2 + \|\mathbf{v}_t\|_{H_0^1}^2 + \|\mathbf{n}\|_{H^4}^2 + \|\mathbf{n}_t\|_{H^2}^2\Big)ds \le C_T,
\end{aligned}
\tag{3.47}
$$

where the positive constant C_T depend on a, b, κ, μ, γ, T, and initial data.

Therefore, by Theorem 2.1 and (3.47), we conclude that the solution obtained in Theorem 2.1 exists in $[0,\infty]$. Thus the proof of Theorem 1.1 is complete. ∎

3.4. Long time behavior. In this part, we will show the long time behavior of the global strong solution to problem (1.4)–(1.10).

Proof. (Proof of Theorem 1.2.) Since the strong solution is global existence and from Lemma 3.2 and 3.6, it is easy to find

$$\int_0^\infty (\|\rho_r\|^2 + \|\mathbf{v}_r\|^2 + \|\mathbf{n}_r\|^2)dt \le C. \tag{3.48}$$

By virtue of the uniform estimates (3.10), (3.34)–(3.36) and the identities

$$\int_a^b \mathbf{v}_r\cdot\mathbf{v}_{tr}dr=-\int_a^b \mathbf{v}_t\cdot\mathbf{v}_{rr}dr,\ \int_a^b \mathbf{n}_r\cdot\mathbf{n}_{tr}dr=-\int_a^b \mathbf{n}_t\cdot\mathbf{n}_{rr}dr,$$

we see that

$$\begin{aligned}&\int_0^\infty\left(\left|\frac{d}{dt}\|\rho_r\|^2\right|+\left|\frac{d}{dt}\|\mathbf{v}_r\|^2\right|+\left|\frac{d}{dt}\|\mathbf{n}_r\|^2\right|\right)dt\\ =&2\int_0^\infty\left(\left|\int_a^b\rho_r\rho_{rt}dr\right|+\left|\int_a^b\mathbf{v}_t\mathbf{v}_{rr}dr\right|+\left|\int_a^b\mathbf{n}_t\mathbf{n}_{rr}dr\right|\right)dt\\ \le&2\int_0^\infty\left|\int_a^b\rho_r\rho_{rt}dr\right|dt+\int_0^\infty\int_a^b\mathbf{v}_{rr}^2drdt+\int_0^\infty\int_a^b\mathbf{v}_t^2drdt\\ &+\int_0^\infty\int_a^b\mathbf{n}_{rr}^2drdt+\int_0^\infty\int_a^b\mathbf{n}_t^2drdt\\ \le&C. \end{aligned}\tag{3.49}$$

Thus, (3.48) and (3.49) give that

$$\|\rho_r\|^2+\|\mathbf{v}_r\|^2+\|\mathbf{n}_r\|^2\to 0,\quad \text{as } t\to\infty.\tag{3.50}$$

It holds from Gagliardo–Nirenberg–Sobolev inequality that

$$\begin{aligned}&\|\rho-\bar\rho\|_{L^\infty}^2+\|\mathbf{v}\|_{L^\infty}^2+\|\mathbf{n}-\bar{\mathbf{n}}\|_{L^\infty}^2\\ \le&C(\|\rho-\bar\rho\|\|(\rho-\bar\rho)_r\|+\|\mathbf{v}\|\|\mathbf{v}_r\|+\|\mathbf{n}-\bar{\mathbf{n}}\|\|(\mathbf{n}-\bar{\mathbf{n}})_r\|)\end{aligned}$$

which together with (3.2), (3.16) and (3.50) implies the theorem. ∎

4. Regularity

In this section we shall establish the regularity in $H^4\times H_0^4\times H^4$. The proof of Theorem 1.3 can be divided into the following several lemmas.

LEMMA 4.1. *Assume that the initial data satisfies* $(\rho_0,\mathbf{v}_0,\mathbf{n}_0)\in H^4\times H_0^4\times H^4$ *with* $0<c_0\le\rho_0$ *and* $|\mathbf{n}_0|=1$ *in* $\bar\Omega$. *Then,*

$$\|\mathbf{v}_{tt}(r,0)\|^2+\|\mathbf{n}_{tt}(r,0)\|^2+\|\mathbf{v}_{tr}(r,0)\|^2+\|\mathbf{v}_{trr}(r,0)\|^2+\|\mathbf{n}_{trr}(r,0)\|^2\le C_T.\tag{4.1}$$

$$\begin{aligned}&\|\mathbf{v}_{tt}(t)\|^2+\|\mathbf{n}_{tt}(t)\|^2+\|\mathbf{n}_{trr}(t)\|^2\\ &+\int_0^t\left(\|\mathbf{v}_{ttr}(s)\|^2+\|\mathbf{v}_{trr}(s)\|^2+\|\mathbf{n}_{ttr}(s)\|^2+\|\mathbf{n}_{trrr}(s)\|^2\right)ds\le C_T.\end{aligned}\tag{4.2}$$

Proof. It follows from the definition of P, (1.4) and (3.47) that

$$\|P_r(t)\|^2\le C\|\rho(t)\|_{H^1}^2,\ \|P_{rr}(t)\|^2\le C_T\|\rho(t)\|_{H^2}^2,\ \|P_{rrr}(t)\|^2\le C_T\|\rho(t)\|_{H^3}^2,\tag{4.3}$$

$$\|P_t(t)\|^2\le C(\|\rho(t)\|_{H^1}^2+\|u(t)\|_{H^1}^2),\ \|P_{tr}(t)\|^2\le C_T(\|\rho(t)\|_{H^2}^2+\|u(t)\|_{H^2}^2).\tag{4.4}$$

Multiplying (1.5)–(1.7) by $\frac{1}{\rho}$, we have

$$u_t+uu_r-\frac{v^2}{r}+\frac{1}{\rho}P_r-\frac{\kappa}{\rho}(u_r+\frac{u}{r})_r+\frac{1}{2\rho}(|\mathbf{n}_r|^2)_r+\frac{|\mathbf{n}_r|^2}{r\rho}=0,\tag{4.5}$$

$$v_t+uv_r+\frac{uv}{r}-\frac{\mu}{\rho}(v_r+\frac{v}{r})_r=0, \tag{4.6}$$

$$w_t+uw_r-\frac{\mu}{\rho}(w_{rr}+\frac{w_r}{r})=0. \tag{4.7}$$

It is clear that

$$\|u_t(t)\|^2\le C_T(\|\rho\|_{H^1}^2+\|u(t)\|_{H^2}^2+\|v(t)\|^2+\|\mathbf{n}(t)\|_{H^2}^2), \tag{4.8}$$

$$\|v_t(t)\|^2\le C_T\|v(t)\|_{H^2}^2,\ \|w_t(t)\|^2\le C_T\|w(t)\|_{H^2}^2. \tag{4.9}$$

Differentiating (4.5) with respect to r, using (3.47) and the embedding theorem, we obtain

$$\|u_{tr}(t)\|^2\le C_T\big(\|\rho(t)\|_{H^2}^2+\|u(t)\|_{H^3}^2+\|v(t)\|_{H^1}^2+\|\mathbf{n}(t)\|_{H^3}^2\big) \tag{4.10}$$

or

$$\|u_{rrr}(t)\|^2\le C_T\big(\|\rho(t)\|_{H^2}^2+\|u(t)\|_{H^2}^2+\|v(t)\|_{H^1}^2+\|\mathbf{n}(t)\|_{H^3}^2+\|u_{tr}(t)\|^2\big). \tag{4.11}$$

Differentiating (4.5) and (1.8) with respect to r twice, using (3.47) and the embedding theorem, we get

$$\|u_{trr}(t)\|^2\le C_T\big(\|\rho(t)\|_{H^3}^2+\|u(t)\|_{H^4}^2+\|v(t)\|_{H^2}^2+\|\mathbf{n}(t)\|_{H^4}^2\big), \tag{4.12}$$

$$\|\mathbf{n}_{trr}(t)\|^2\le C_T\big(\|u(t)\|_{H^2}^2+\|\mathbf{n}(t)\|_{H^4}^2\big) \tag{4.13}$$

or

$$\|u_{rrrr}(t)\|^2\le C_T\big(\|\rho(t)\|_{H^3}^2+\|u(t)\|_{H^3}^2+\|v(t)\|_{H^2}^2+\|\mathbf{n}(t)\|_{H^4}^2+\|u_{trr}(t)\|^2\big), \tag{4.14}$$

$$\|\mathbf{n}_{rrrr}(t)\|^2\le C_T\big(\|u(t)\|_{H^2}^2+\|\mathbf{n}(t)\|_{H^3}^2+\|\mathbf{n}_{trr}(t)\|^2\big). \tag{4.15}$$

Differentiating (4.5) and (1.8) with respect to t, respectively, using (3.47), (4.4), (4.8)–(4.10), (4.12)–(4.13) and the embedding theorem, we have

$$\begin{aligned}\|u_{tt}(t)\|^2\le& C_T\big(\|u_t(t)\|^2+\|u_{tr}(t)\|^2+\|u(t)\|_{H^2}^2+\|u_{trr}(t)\|^2\\&+\|v_t(t)\|^2+\|\rho(t)\|_{H^2}^2+\|\mathbf{n}(t)\|_{H^2}^2+\|\mathbf{n}_{tr}(t)\|^2+\|\mathbf{n}_{trr}(t)\|^2\big)\\ \le& C_T\big(\|\rho(t)\|_{H^3}^2+\|u(t)\|_{H^4}^2+\|\mathbf{n}(t)\|_{H^4}^2+\|v(t)\|_{H^2}^2\big),\end{aligned} \tag{4.16}$$

$$\begin{aligned}\|\mathbf{n}_{tt}(t)\|^2\le& C_T\big(\|u_t(t)\|^2+\|\mathbf{n}_t(t)\|^2+\|\mathbf{n}(t)\|_{H^1}^2+\|\mathbf{n}_{tr}(t)\|^2+\|\mathbf{n}_{trr}(t)\|^2\big)\\ \le& C_T\big(\|u(t)\|_{H^2}^2+\|\mathbf{n}(t)\|_{H^4}^2+\|v(t)\|^2\big)\end{aligned} \tag{4.17}$$

or

$$\begin{aligned}\|u_{trr}(t)\|^2\le& C_T\big(\|u_{tr}(t)\|^2+\|u(t)\|_{H^2}^2+\|u_{tt}(t)\|^2\\&+\|\rho(t)\|_{H^2}^2+\|\mathbf{n}(t)\|_{H^2}^2+\|\mathbf{n}_{tr}(t)\|^2+\|\mathbf{n}_{trr}(t)\|^2+\|v(t)\|_{H^2}^2\big),\end{aligned} \tag{4.18}$$

$$\|\mathbf{n}_{trr}(t)\|^2\le C_T\big(\|u(t)\|_{H^2}^2+\|v(t)\|^2+\|\mathbf{n}_t(t)\|^2+\|\mathbf{n}(t)\|_{H^2}^2+\|\mathbf{n}_{tr}(t)\|^2+\|\mathbf{n}_{tt}(t)\|^2\big). \tag{4.19}$$

Thus, we obtain

$$\|u_{tt}(r,0)\|^2+\|\mathbf{n}_{tt}(r,0)\|^2+\|u_{tr}(r,0)\|^2+\|u_{trr}(r,0)\|^2+\|\mathbf{n}_{trr}(r,0)\|^2\leq C_T \tag{4.20}$$

by combining (4.10), (4.12)–(4.13) and (4.16)–(4.17).

Similarly, from (4.6) and (4.7)we can obtain the estimates for v and w, which give (4.1). Moreover,

$$\|v_{trr}(t)\|^2\leq C_T\big(\|v_t(t)\|_{H^1}^2+\|v(t)\|_{H^2}^2+\|v_{tt}(t)\|^2+\|u_t(t)\|^2+\|u(t)\|^2+\|\rho(t)\|_{H^1}^2\big), \tag{4.21}$$

$$\|w_{trr}(t)\|^2\leq C_T\big(\|w_t(t)\|_{H^1}^2+\|w(t)\|_{H^2}^2+\|w_{tt}(t)\|^2+\|u_t(t)\|^2+\|u(t)\|^2+\|\rho(t)\|_{H^1}^2\big). \tag{4.22}$$

Let $B=P_r-\kappa(u_r+\frac{u}{r})_r+\frac{1}{2}(|\mathbf{n}_r|^2)_r+\frac{|\mathbf{n}_r|^2}{r}$ and differentiating (4.5) with respect to t twice, multiplying the resulting equation by u_{tt} and integrating over (a,b), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_a^b u_{tt}^2dr=&-\int_a^b(uu_r)_{tt}u_{tt}dr+\int_a^b(\frac{v^2}{r})_{tt}u_{tt}dr\\&-\int_a^b(\frac{1}{\rho})_{tt}Bu_{tt}dr+2\int_a^b(\frac{1}{\rho})_tB_tu_{tt}dr-\int_a^b\frac{1}{\rho}B_{tt}u_{tt}dr\\=&I_1+I_2+I_3+I_4+I_5.\end{aligned} \tag{4.23}$$

By (3.47) and the embedding theorem, we have

$$\begin{aligned}I_1=&-\int_a^b\big(u_{tt}^2u_r+2u_tu_{tr}u_{tt}+uu_{ttr}u_{tt}\big)dr\\=&-\int_a^bu_{tt}^2u_rdr+\frac{1}{2}\int_a^bu_{ttr}u_t^2dr-\int_a^buu_{ttr}u_{tt}dr\\\leq&\epsilon_1\int_a^bu_{ttr}^2dr+C_T\int_a^bu_{tt}^2dr+C_T\int_a^bu_t^2dr+C_T\int_a^bu_{tr}^2dr.\end{aligned} \tag{4.24}$$

$$I_2=2\int_a^b\frac{1}{r}\big(vv_{tt}+v_t^2\big)u_{tt}dr\leq C_T\int_a^bu_{tt}^2dr+C_T\int_a^b\big(v_{tt}^2+v_t^2+v_{tr}^2\big)dr. \tag{4.25}$$

$$\begin{aligned}I_3=&\int_a^b(\frac{\rho_{tt}}{\rho^2}-\frac{2\rho_t^2}{\rho^3})Bu_{tt}dr\\=&-\int_a^b\frac{1}{\rho^2}\big(\rho_{tr}u+\rho_ru_t+\rho_tu_r+\rho u_{tr}+\frac{\rho_tu}{r}+\frac{\rho u_t}{r}\big)Bu_{tt}dr-2\int_a^b\frac{\rho_t^2}{\rho^3}Bu_{tt}dr\\\leq&C_T+C_T\int_a^bu_{tt}^2dr+C_T\int_a^bu_{rr}^2dr+C_T\int_a^bu_{tr}^2dr\\&+\kappa\int_a^b\frac{1}{\rho^2}\big(\rho_{tr}u+\rho_ru_t+\rho u_{tr}+\frac{\rho u_t}{r}\big)u_{rr}u_{tt}dr.\end{aligned} \tag{4.26}$$

For the last term of the right-hand side of (4.26), we have

$$\kappa\int_a^b\frac{1}{\rho^2}\Big(\rho_{tr}u+\rho_ru_t+\frac{\rho u_t}{r}\Big)u_{rr}u_{tt}dr\leq C_T\int_a^b(\rho_{tr}^2+u_t^2)u_{rr}^2dr+\int_a^bu_{tt}^2dr$$

$$\leq C_T\|u_{rr}\|_{L^\infty}^2\int_a^b(\rho_{tr}^2+u_t^2)dr+\int_a^b u_{tt}^2dr$$
$$\leq C_T\left(\int_a^b u_{rr}^2dr+\int_a^b u_{rrr}^2dr\right)+\int_a^b u_{tt}^2dr. \quad (4.27)$$

Then differentiating (4.5) with respect to r implies

$$\begin{aligned}
&\kappa\int_a^b\frac{1}{\rho}u_{tr}u_{rr}u_{tt}dr\\
\leq&C_T\left(\int_a^b u_{tt}^2dr+\int_a^b u_{rr}^2dr+\int_a^b\rho_{rr}^2u_{rr}^2dr\right)+\kappa^2\int_a^b\frac{1}{\rho^2}u_{rrr}u_{rr}u_{tt}dr\\
&+\kappa\int_a^b\left(\frac{\kappa}{r\rho^2}-\frac{u}{\rho}-\frac{\kappa\rho_r}{\rho^3}\right)u_{rr}^2u_{tt}dr-\kappa\int_a^b\frac{1}{\rho^2}(\mathbf{n}_r\cdot\mathbf{n}_{rrr})u_{rr}u_{tt}dr\\
\leq&C_T\int_a^b u_{tt}^2dr+C_T\int_a^b u_{rr}^2dr+C_T\int_a^b\rho_{rr}^2u_{rr}^2dr+C_T\int_a^b u_{rr}^4dr\\
&+C_T\int_a^b\mathbf{n}_{rrr}^2u_{rr}^2dr-\frac{\kappa^2}{2}\int_a^b\left(\frac{1}{\rho^2}\right)_r u_{rr}^2u_{tt}dr-\frac{\kappa^2}{2}\int_a^b\frac{1}{\rho^2}u_{rr}^2u_{ttr}dr\\
\leq&\epsilon_1\int_a^b u_{ttr}^2dr+C_T\int_a^b u_{tt}^2dr+C_T\int_a^b u_{rr}^2dr+C_T\|u_{rr}\|_{L^\infty}^2\int_a^b(\rho_{rr}^2+u_{rr}^2+\mathbf{n}_{rrr}^2)dr\\
\leq&\epsilon_1\int_a^b u_{ttr}^2dr+C_T+C_T\int_a^b u_{tt}^2dr+C_T\int_a^b u_{rrr}^2dr.
\end{aligned}\quad (4.28)$$

Thus, we get

$$I_3\leq\epsilon_1\int_a^b u_{ttr}^2dr+C_T+C_T\int_a^b u_{tt}^2dr+C_T\int_a^b u_{tr}^2dr+C_T\int_a^b u_{rrr}^2dr \quad (4.29)$$

by combining (4.26)–(4.28).

$$\begin{aligned}
I_4=&-2\int_a^b\frac{\rho_t}{\rho^2}B_tu_{tt}dr\\
\leq&C_T+C_T\int_a^b\rho_{tr}^2dr+C_T\int_a^b u_{tt}^2dr+2\kappa\int_a^b\frac{\rho_t}{r\rho^2}u_{tr}u_{tt}dr+2\kappa\int_a^b\frac{\rho_t}{\rho^2}u_{trr}u_{tt}dr\\
&+C_T\int_a^b\mathbf{n}_{trr}^2dr-2\int_a^b\frac{\rho_t}{\rho^2}(\mathbf{n}_{tr}\cdot\mathbf{n}_{rr})u_{tt}dr-4\int_a^b\frac{\rho_t}{r\rho^2}(\mathbf{n}_r\cdot\mathbf{n}_{tr})u_{tt}dr\\
\leq&C_T+C_T\left(\int_a^b u_{tt}^2dr+\int_a^b u_{tr}^2dr+\int_a^b\mathbf{n}_{tr}^2dr+\int_a^b\mathbf{n}_{trr}^2dr\right)+2\kappa\int_a^b\frac{\rho_t}{\rho^2}u_{trr}u_{tt}dr.
\end{aligned}\quad (4.30)$$

Differentiating (4.5) with respect to t, we obtain

$$\kappa u_{trr}=\rho u_{tt}+\rho(uu_r)_t-\rho(\frac{v^2}{r})_t-\frac{\rho_t}{\rho}B+P_{tr}-\frac{\kappa u_{tr}}{r}+\frac{\kappa u_t}{r^2}+\frac{1}{2}(|\mathbf{n}_r|^2)_{tr}+\frac{|\mathbf{n}_r|_t^2}{r}.$$

Therefore,

$$2\kappa\int_a^b\frac{\rho_t}{\rho^2}u_{trr}u_{tt}dr\leq C_T+C_T\int_a^b u_{tt}^2dr+C_T\int_a^b u_{tr}^2dr+C_T\int_a^b\mathbf{n}_{tr}^2dr+C_T\int_a^b\mathbf{n}_{trr}^2dr,$$

which implies

$$I_4 \le C_T + C_T \int_a^b u_{tt}^2 dr + C_T \int_a^b u_{tr}^2 dr + C_T \int_a^b \mathbf{n}_{tr}^2 dr + C_T \int_a^b \mathbf{n}_{trr}^2 dr. \tag{4.31}$$

Using integration by parts, we get

$$\begin{aligned} I_5 = & -\int_a^b \frac{1}{\rho}\Big(P_{ttr} - \kappa u_{ttrr} - \kappa \frac{u_{ttr}}{r} + \kappa \frac{u_{tt}}{r^2} + \mathbf{n}_{ttr}\cdot\mathbf{n}_{rr} + 2\mathbf{n}_{tr}\cdot\mathbf{n}_{trr} \\ & + \mathbf{n}_r \cdot \mathbf{n}_{ttrr} + \frac{2\mathbf{n}_{tr}^2}{r} + \frac{2\mathbf{n}_r\cdot\mathbf{n}_{ttr}}{r}\Big) u_{tt} dr \\ \le & -\int_a^b \frac{1}{\rho} P_{ttr} u_{tt} dr - \kappa \int_a^b \frac{1}{\rho} u_{ttr}^2 dr + \kappa \int_a^b \frac{\rho_r}{\rho^2} u_{tt} u_{ttr} dr \\ & + \epsilon_1 \int_a^b u_{ttr}^2 dr + C_T \int_a^b u_{tt}^2 dr + \int_a^b \mathbf{n}_{ttr}^2 dr + C_T \int_a^b n_{tr}^4 dr \\ & - 2\int_a^b \frac{1}{\rho}(\mathbf{n}_{tr}\cdot\mathbf{n}_{trr}) u_{tt} dr + \int_a^b \frac{1}{\rho} u_{ttr}(\mathbf{n}_r \cdot \mathbf{n}_{ttr}) dr + \int_a^b \frac{1}{\rho} u_{tt}(\mathbf{n}_{rr}\cdot\mathbf{n}_{ttr}) dr. \end{aligned} \tag{4.32}$$

Notice that

$$\int_a^b \frac{1}{\rho}(\mathbf{n}_{tr}\cdot\mathbf{n}_{trr}) u_{tt} dr = -\int_a^b \frac{1}{\rho}(\mathbf{n}_{tr}\cdot\mathbf{n}_{trr}) u_{tt} dr - \int_a^b \frac{1}{\rho} \mathbf{n}_{tr}^2 u_{ttr} dr + \int_a^b \frac{\rho_r}{\rho^2} \mathbf{n}_{tr}^2 u_{ttr} dr$$

which implies

$$\begin{aligned} -2\int_a^b \frac{1}{\rho}(\mathbf{n}_{tr}\cdot\mathbf{n}_{trr}) u_{tt} dr &= \int_a^b \frac{1}{\rho}\mathbf{n}_{tr}^2 u_{ttr} dr - \int_a^b \frac{\rho_r}{\rho^2}\mathbf{n}_{tr}^2 u_{ttr} dr \\ &\le \epsilon_1 \int_a^b u_{ttr}^2 dr + C_T \int_a^b \mathbf{n}_{tr}^4 dr + C_T \int_a^b u_{tt}^2 dr, \end{aligned}$$

and

$$\int_a^b \mathbf{n}_{tr}^4 dr \le \|\mathbf{n}_{tr}\|_{L^\infty}^2 \int_a^b \mathbf{n}_{tr}^2 dr \le C_3 + C_3 \int_a^b \mathbf{n}_{trr}^2 dr.$$

For the first term of the right-hand side of (4.32), it is not hard to show that

$$\begin{aligned} P_{tt} &= \gamma(\gamma-1)\rho^{\gamma-2}\rho_t^2 + \gamma\rho^{\gamma-1}\rho_{tt} \\ &= \gamma(\gamma-1)\rho^{\gamma-2}\rho_t^2 - \gamma\rho^{\gamma-1}[(\rho u)_{tr} + (\frac{\rho u}{r})_t]. \end{aligned}$$

Then, we have

$$\begin{aligned} -\int_a^b \frac{1}{\rho} P_{ttr} u_{tt} dr &= \int_a^b \frac{1}{\rho} P_{tt} u_{ttr} dr - \int_a^b \frac{\rho_r}{\rho^2} P_{tt} u_{tt} dr \\ &\le \epsilon_1 \int_a^b u_{ttr}^2 dr + C_T + C_T \int_a^b u_{tt}^2 dr + C_T \int_a^b u_{tr}^2 dr. \end{aligned}$$

Therefore,

$$I_5 \le -\kappa \int_a^b \frac{1}{\rho} u_{ttr}^2 dr + C_T + 3\epsilon_1 \int_a^b u_{ttr}^2 dr + \tilde{C} \int_a^b \mathbf{n}_{ttr}^2 dr$$

$$+C_T\int_a^b u_{tt}^2 dr+C_T\int_a^b u_{tr}^2 dr+C_T\int_a^b \mathbf{n}_{trr}^2 dr. \tag{4.33}$$

Substituting (4.24)–(4.25), (4.29), (4.31) and (4.33) into (4.23), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_a^b u_{tt}^2 dr+\kappa\int_a^b \frac{1}{\rho}u_{ttr}^2 dr \le & 5\epsilon_1\int_a^b u_{ttr}^2 dr+C_T+C_T\int_a^b \left(v_{tt}^2+v_{tr}^2\right)dr\\ &+C_T\int_a^b u_{tt}^2 dr+C_T\int_a^b u_{tr}^2 dr+C_T\int_a^b u_{rrr}^2 dr\\ &+C_T\int_a^b \mathbf{n}_{trr}^2 dr+\tilde{C}\int_a^b \mathbf{n}_{ttr}^2 dr. \end{aligned}\tag{4.34}$$

Similarly, we can get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_a^b v_{tt}^2 dr+\mu\int_a^b \frac{1}{\rho}v_{ttr}^2 dr \le & \epsilon_2\int_a^b v_{ttr}^2 dr+C_T+C_T\int_a^b u_{tt}^2 dr\\ &+C_T\int_a^b v_{tt}^2 dr+C_T\int_a^b v_{tr}^2 dr+C_T\int_a^b v_{rrr}^2 dr. \end{aligned}\tag{4.35}$$

and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_a^b w_{tt}^2 dr+\mu\int_a^b \frac{1}{\rho}w_{ttr}^2 dr \le & \epsilon_3\int_a^b w_{ttr}^2 dr+C_T+C_T\int_a^b u_{tt}^2 dr\\ &+C_T\int_a^b w_{tt}^2 dr+C_T\int_a^b w_{tr}^2 dr+C_T\int_a^b w_{rrr}^2 dr. \end{aligned}\tag{4.36}$$

Adding the above three inequalities and taking $\epsilon_i,\ i=1,\ 2,\ 3$ small enough, we have

$$\begin{aligned}\frac{d}{dt}\|\mathbf{v}_{tt}(t)\|^2+\|\mathbf{v}_{ttr}(t)\|^2 \le & C_T+C_T\|\mathbf{v}_{tt}(t)\|^2+C_T\|\mathbf{v}_{tr}(t)\|^2+C_T\|\mathbf{v}_{rrr}(t)\|^2\\ &+C_T\|\mathbf{n}_{trr}(t)\|^2+\tilde{C}\|\mathbf{n}_{ttr}(t)\|^2. \end{aligned}\tag{4.37}$$

Thus, by (3.47), we arrive at

$$\|\mathbf{v}_{tt}(t)\|^2+\int_0^t \|\mathbf{v}_{ttr}(s)\|^2 ds \le C_T+C_T\int_0^t \|\mathbf{v}_{tt}(s)\|^2 ds+\tilde{C}\int_0^t \|\mathbf{n}_{ttr}(s)\|^2 ds. \tag{4.38}$$

On the other hand, Differentiating (1.8) with respect to t twice, multiplying the resulting equation by $\mathbf{n}_{tt}$ and integrating over (a,b), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_a^b \mathbf{n}_{tt}^2 dr+\int_a^b \mathbf{n}_{ttr}^2 dr \le & 3\epsilon_4\int_a^b \mathbf{n}_{ttr}^2 dr+\int_a^b u_{tt}^2 dr+C_T\int_a^b \mathbf{n}_{tt}^2 dr\\ &+C_T\int_a^b \mathbf{n}_{tr}^2\mathbf{n}_t^2 dr+C_T\int_a^b \mathbf{n}_{tr}^4 dr+C_T\int_a^b \mathbf{n}_{tr}^2 u_t^2 dr. \end{aligned}\tag{4.39}$$

Taking ϵ_4 small enough and using (3.47) and the embedding theorem, we get

$$\|\mathbf{n}_{tt}(t)\|^2+\int_0^t \|\mathbf{n}_{ttr}(s)\|^2 ds \le C_T+C_T\int_0^t \|u_{tt}(s)\|^2 ds+C_T\int_0^t \|\mathbf{n}_{tt}(s)\|^2 ds. \tag{4.40}$$

From (4.38) and (4.40),

$$\begin{aligned}&\|\mathbf{v}_{tt}(t)\|^2+\|\mathbf{n}_{tt}(t)\|^2+\int_0^t\|\mathbf{v}_{ttr}(s)\|^2ds+\int_0^t\|\mathbf{n}_{ttr}(s)\|^2ds\\ \le&C_T+C_T\int_0^t(\|\mathbf{v}_{tt}(s)\|^2+\|\mathbf{n}_{tt}(s)\|^2)ds,\end{aligned}$$

which by Gronwall's inequality, (4.1) and (4.19), gives

$$\|\mathbf{v}_{tt}(t)\|^2+\|\mathbf{n}_{tt}(t)\|^2+\|\mathbf{n}_{trr}(t)\|^2+\int_0^t\|\mathbf{v}_{ttr}(s)\|^2ds+\int_0^t\|\mathbf{n}_{ttr}(s)\|^2ds\le C_T. \tag{4.41}$$

Differentiating (1.8) with respect to r and t, we deduce

$$\|\mathbf{n}_{trrr}(t)\|^2\le C_T\big(\|\mathbf{n}_t(t)\|_{H^1}^2+\|\mathbf{n}(t)\|_{H^2}^2+\|\mathbf{n}_{trr}(t)\|^2+\|\mathbf{n}_{ttr}(t)\|^2+\|u_t(t)\|_{H^1}^2\big), \tag{4.42}$$

which, together with (4.18)–(4.19), (4.21)–(4.22) and (4.42), gives (4.2). The proof is complete. □

LEMMA 4.2. *Under the assumption of Lemma 4.1, it holds that*

$$\|\rho_{rrr}(t)\|^2+\int_0^t\Big(\|\rho_{trr}(s)\|^2+\|\mathbf{v}_{rrrr}(s)\|^2\Big)ds\le C_T. \tag{4.43}$$

Proof. Differentiating (1.4) with respect to r three times, multiplying the resulting equation by ρ_{rrr} and integrating over (a,b), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_a^b\rho_{rrr}^2dr=&-\int_a^b\rho_{rrrr}\rho_{rrr}udr-\int_a^b\rho\rho_{rrr}u_{rrrr}dr-4\int_a^b\rho_{rrr}^2u_rdr\\&-6\int_a^b\rho_{rr}\rho_{rrr}u_{rr}dr-4\int_a^b\rho_r\rho_{rrr}u_{rrr}dr-\int_a^b\left(\frac{\rho u}{r}\right)_{rrr}\rho_{rrr}dr\\=&-\frac{7}{2}\int_a^b\rho_{rrr}^2u_rdr-\int_a^b\rho\rho_{rrr}u_{rrrr}dr-6\int_a^b\rho_{rr}\rho_{rrr}u_{rr}dr\\&-4\int_a^b\rho_r\rho_{rrr}u_{rrr}dr-\int_a^b\left(\frac{\rho u}{r}\right)_{rrr}\rho_{rrr}dr\end{aligned} \tag{4.44}$$

leading to

$$\frac{d}{dt}\int_a^b\rho_{rrr}^2dr\le C_T+C_T\int_a^b\rho_{rrr}^2dr+C_T\int_a^b u_{rrrr}^2dr+C_T\int_a^b u_{rrr}^2dr.$$

Integrating over $(0,t)$, we get

$$\|\rho_{rrr}(t)\|^2\le C_T+C_T\int_0^t\|\rho_{rrr}(s)\|^2ds+C_T\int_0^t\|u_{rrrr}(s)\|^2ds. \tag{4.45}$$

It follows from (3.47), (4.2), (4.11) and (4.14) that

$$\int_0^t\|u_{rrrr}(s)\|^2ds\le C_T+C_T\int_0^t\|\rho_{rrr}(s)\|^2ds. \tag{4.46}$$

Then (4.45)–(4.46) and the Gronwall's inequality imply that

$$\|\rho_{rrr}(t)\|^2 + \int_0^t \|u_{rrrr}(s)\|^2 ds \le C_T, \tag{4.47}$$

which, along with conservation of mass, gives

$$\int_0^t \|\rho_{trr}(s)\|^2 ds \le C_T. \tag{4.48}$$

Following the same process we can get the estimates for v and w and together with (4.47)–(4.48), we arrive at (4.43). ∎

LEMMA 4.3. *Under the assumption of Lemma 4.1, it holds that*

$$\|\mathbf{v}_{tr}(t)\|^2 + \|\mathbf{v}_{trr}(t)\|^2 + \|\mathbf{v}_{rrr}(t)\|^2 + \|\mathbf{v}_{rrrr}(t)\|^2 + \|\mathbf{n}_{rrrr}(t)\|^2 + \int_0^t \|\mathbf{v}_{trrr}(s)\|^2 ds \le C_T. \tag{4.49}$$

Proof. Differentiating (4.5) with respect to t and r, multiplying the resulting equation by u_{tr} and integrating over (a,b), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_a^b u_{tr}^2 dr \le & C_T + C_T\int_a^b u_{tr}^2 dr + C_T\int_a^b u_{trr}^2 dr + C_T\int_a^b u_t^2 u_{rr}^2 dr + C_T\int_a^b u_{rrr}^2 dr \\ & + C_T\int_a^b P_{rr}^2 dr + C_T\int_a^b \mathbf{n}_{tr}^2 dr + C_T\int_a^b \mathbf{n}_{trr}^2 dr + C_T\int_a^b v_{tr}^2 dr \\ & + C_T\int_a^b \mathbf{n}_{trrr}^2 dr + C_T\int_a^b \mathbf{n}_{tr}^2\mathbf{n}_{rrr}^2 dr - \int_a^b \frac{P_{trr}}{\rho} u_{tr} dr + \kappa\int_a^b \frac{u_{trrr}}{\rho} u_{tr} dr.\end{aligned} \tag{4.50}$$

or

$$\begin{aligned}\|u_{trrr}(t)\|^2 \le & C_T + C_T(\|u_t(t)\|_{H^2}^2 + \|u(t)\|_{H^3}^2 + \|u_{ttr}(t)\|^2 + \|v_t(t)\|_{H^1}^2 \\ & + \|\rho(t)\|_{H^2}^2 + \|\rho_t(t)\|_{H^2}^2 + \|\mathbf{n}(t)\|_{H^3}^2 + \|\mathbf{n}_t(t)\|_{H^3}^2).\end{aligned} \tag{4.51}$$

For the last terms of the right-hand side of (4.50), we have

$$-\int_a^b \frac{P_{trr}}{\rho} u_{tr} dr + \kappa\int_a^b \frac{u_{trrr}}{\rho} u_{tr} dr = J_1 + J_2, \tag{4.52}$$

where

$$\begin{aligned} J_1 &= -\frac{P_{tr}}{\rho} u_{tr}\Big|_a^b + \frac{\kappa u_{trr}}{\rho} u_{tr}\Big|_a^b \\ J_2 &= -\int_a^b \frac{\rho_r}{\rho^2} P_{tr} u_{trr} dr + \int_a^b \frac{P_{tr}}{\rho} u_{trr} dr - \kappa\int_a^b \frac{u_{trr}^2}{\rho} dr + \kappa\int_a^b \frac{\rho_r}{\rho^2} u_{trr} u_{tr} dr.\end{aligned}$$

By virtue of (3.47) and using the embedding theorem, we deduce that

$$J_1 \le C_1\int_a^b \left(\rho_{tr}^2 + \rho_{trr}^2 + u_{tr}^2 + u_{rrr}^2 + u_{trr}^2 + u_{ttr}^2 + \mathbf{n}_{trrr}^2\right) dr$$

and

$$J_2 \le C_T + C_T \int_a^b \left(u_{tr}^2 + u_{trr}^2\right) dr.$$

Therefore,

$$\frac{d}{dt}\int_a^b u_{tr}^2 dr \le C_T + C_T\int_a^b u_{tr}^2 dr + C_T\int_a^b u_{trr}^2 dr + C_T\int_a^b u_{ttr}^2 dr + C_T\int_a^b u_{rrr}^2 dr$$
$$+ C_T\int_a^b \mathbf{n}_{trr}^2 dr + + C_T\int_a^b \rho_{trr}^2 dr + C_T\int_a^b v_{tr}^2 dr + C_T\int_a^b \mathbf{n}_{trrr}^2 dr, \quad (4.53)$$

which, by Gronwall's inequality and (4.11), yields

$$\|u_{tr}(t)\|^2 + \|u_{rrr}(t)\|^2 \le C_T. \quad (4.54)$$

Thus, combined with (4.2), (4.14)–(4.15), (4.18), (4.43), (4.51) and (4.54), implies

$$\|u_{trr}(t)\|^2 + \|u_{rrrr}(t)\|^2 + \|\mathbf{n}_{rrrr}(t)\|^2 + \int_0^t \|u_{trrr}(s)\|^2 ds \le C_T. \quad (4.55)$$

In view of (4.6) and (4.7), we can obtain the estimates for v and w by the same way, which, together with (4.55) bring us to (4.49), which completes the proof. □

LEMMA 4.4. *Under the assumption of Lemma 4.1, it holds that*

$$\|\rho_{rrrr}(t)\|^2 + \int_0^t \left(\|\mathbf{n}_{rrrrr}(s)\|^2 + \|\mathbf{v}_{rrrrr}(s)\|^2\right) ds \le C_T. \quad (4.56)$$

Proof. Differentiating (4.5) and (1.8) with respect to r three times and integrating over (a,b), we get

$$\|u_{rrrrr}(t)\|^2 \quad (4.57)$$
$$\le C_T\|u_{trrr}(t)\|^2 + C_T\|v(t)\|_{H^3}^2 + C_T\|u(t)\|_{H^4}^2 + C_T\|\rho(t)\|_{H^4}^2 + C_T\|\mathbf{n}(t)\|_{H^5}^2. \quad (4.58)$$

and

$$\|\mathbf{n}_{rrrrr}(t)\|^2 \le C_T\|\mathbf{n}_{trrr}(t)\|^2 + C_T\|u(t)\|_{H^3}^2 + C_T\|\mathbf{n}(t)\|_{H^4}^2. \quad (4.59)$$

It is clear that (4.57)–(4.59), with the help of (3.47) and Lemma 4.1-4.3, imply

$$\int_0^t \|\mathbf{n}_{rrrrr}(t)\|^2 ds \le C_T. \quad (4.60)$$

and

$$\int_0^t \|u_{rrrrr}(s)\|^2 ds \le C_T + C_T\int_0^t \|\rho_{rrrr}(s)\|^2 ds. \quad (4.61)$$

Differentiating (1.4) with respect to r four times, multiplying the resulting equation by ρ_{rrrr} and integrating over (a,b), we have

$$\frac{1}{2}\frac{d}{dt}\int_a^b \rho_{rrrr}^2 dr = -\int_a^b \rho_{rrrrr}\rho_{rrrr} u dr - \int_a^b \rho\rho_{rrrr} u_{rrrrr} dr - 5\int_a^b \rho_{rrrr}^2 u_r dr$$

$$
\begin{aligned}
&-10\int_a^b \rho_{rrr}\rho_{rrrr}u_{rr}dr-10\int_a^b \rho_{rr}\rho_{rrrr}u_{rrr}dr\\
&-5\int_a^b \rho_r\rho_{rrrr}u_{rrrr}dr-\int_a^b\left(\frac{\rho u}{r}\right)_{rrrr}\rho_{rrrr}dr\\
\leq &C_T+C_T\int_a^b \rho_{rrrr}^2dr+C_T\int_a^b u_{rrrrr}^2dr.
\end{aligned}
\tag{4.62}
$$

By (4.61) and Gronwall's inequality, we obtain

$$
\|\rho_{rrrr}(t)\|^2+\int_0^t\|u_{rrrrr}(s)\|^2ds\leq C_T. \tag{4.63}
$$

Observe that, similar to the estimates for u, we can get the estimates for v and w, which yields (4.56). Completing the proof of Lemma 4.4. □

Proof. (Proof of Theorem 1.3.) We obtain the result by directly using Lemma 4.1–4.4. Thus the proof of Theorem 1.3 is complete. □

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