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SUBSONIC POTENTIAL FLOWS IN GENERAL SMOOTH BOUNDED DOMAINS*

CHAO CHEN†

Abstract. In this paper, we study the existence and uniqueness of subsonic potential flows in general smooth bounded domains when the normal component of the momentum on the boundary is prescribed. It is showed that if the Bernoulli constant is given larger than a critical number, there exists a unique subsonic potential flow. Moreover, as the Bernoulli constants decrease to the critical number, the subsonic flows converge to a subsonic-sonic flow.

Key words. Steady Euler equations, subsonic flows, subsonic-sonic flows, potential flows, Bernoulli constant.

AMS subject classifications. 35J25, 35J70, 35Q35, 76H05.

1. Introduction and main results

As a natural starting point for multidimensional gas flows, the study of steady fluid has experienced a lot of development in the past sixty years. The main feature of the steady Euler equations is that they may be hyperbolic or hyperbolic-elliptic. An important approximate model is the potential flow, which describes flows without vorticity. As is well known on the subject, the equations are hyperbolic-elliptic if the maximum of the velocity is less than the speed of sound while the equations themselves are hyperbolic when the flows are supersonic. Many physical observations indicate that the supersonic flows may give rise to shock waves and other phenomena, [29, 30, 31]. Therefore, smooth solutions are generally not expected in supersonic flows. On the other hand, since 1950s, tremendous progress has been made in the study for subsonic potential flows. Shiffman in [32], Bers in [3], Finn, and Gilbarg in [21, 22], and Dong and Biao in [12] studied subsonic potential flows past a profile. Roughly speaking, in their work, the flows are subsonic if the speed at infinity is not larger than a critical value. Recently, Xie and Xin in [39, 40], Du, Xin, and Yan in [18] gave a positive answer to the existence and uniqueness of the subsonic-sonic potential flows in a infinitely long nozzle. The critical mass flux was used to make the flows lie in the subsonic region instead of the speed at infinity. Besides flows in unbounded domains, there are also some results on finite nozzles with special geometries. Two dimensional subsonic irrotational flows in finitely long flat nozzles were considered in [15]. Subsonic flows in three dimensional finitely long flat nozzles were studied in [5, 38].

As a limit flow of subsonic flows, the subsonic-sonic flow was studied in [6, 9, 25] via compensated compactness method. In another aspect, the existence of subsonic-sonic flows with sonic boundary was obtained by Wang and Xin in [34, 35].

When the flows have non-zero vorticity, steady Euler flows were investigated in [1, 4, 5, 7, 10, 13, 14, 16, 17, 19, 33, 38, 41] and references therein. As an important constituent of subsonic flows, the existence of transonic flows which are small perturbations of background flows was also studied in [8, 11, 27, 28, 36, 42, 43] and references therein.

Since previous work mostly focused on unbounded domains or bounded domains with special geometries, an interesting question is whether subsonic potential flows

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†School of Mathematics and Computer Science, Fujian Normal University, Fuzhou, Fujian 350108, P.R. China (chenchao_math@sina.cn, chenchao@fjnu.edu.cn).
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exist in general bounded domains. Hence, in this paper, we will give a complete answer on potential flows in general bounded domains when prescribing the normal component of the momentum on the boundary. In our result, the flows are subsonic if the Bernoulli constant is given larger than a critical number. Furthermore, they approach subsonic-sonic as the Bernoulli constant decreases to the critical value.

The steady isentropic compressible ideal flows are governed by the following Euler equations:

$$\operatorname{div}(\rho\mathbf{u})=0, \tag{1.1}$$

$$\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u})+\nabla p=0, \tag{1.2}$$

where ρ, $\mathbf{u}=(u_1,u_2,\dots,u_n)$, p are the density, velocity, and pressure, respectively. We assume that $p=p(\rho)$ satisfies

$$p=\rho^\gamma \quad \text{for} \quad \gamma>1. \tag{1.3}$$

The equations (1.1) and (1.2) are hyperbolic-elliptic in the subsonic region ($|\mathbf{u}|^2<p'(\rho)$) and hyperbolic in the supersonic region ($|\mathbf{u}|^2>p'(\rho)$) (cf. [41]), where the quantity $\sqrt{p'(\rho)}$ is called local sound speed.

The Bernoulli's law yields that the Bernoulli function B defined by $B=\frac{|\mathbf{u}|^2}{2}+\frac{\gamma}{\gamma-1}\rho^{\gamma-1}$ is invariant along each streamline, that is $\mathbf{u}\cdot\nabla B\equiv 0$.

If in addition, we consider the potential flow, that is $\mathbf{u}=\nabla\varphi$ for some potential function φ, the Bernoulli function is a constant in the whole domain Ω thereby. Thus we set such a constant $\bar{B}$.

Since the density can be uniquely determined by the relation $\rho=\left(\frac{\gamma-1}{\gamma}(\bar{B}-\frac{1}{2}|\nabla\varphi|^2)\right)^{\frac{1}{\gamma-1}}$, the Euler system (1.1)–(1.2) reduces to the potential equation

$$\operatorname{div}\left(\left(\frac{\gamma-1}{\gamma}\left(\bar{B}-\frac{1}{2}|\nabla\varphi|^2\right)\right)^{\frac{1}{\gamma-1}}\nabla\varphi\right)=0. \tag{1.4}$$

For simplicity, we set $\rho=\rho(|\nabla\varphi|^2)$ and the Equation (1.4) is

$$\operatorname{div}(\rho(|\nabla\varphi|^2)\nabla\varphi)=0. \tag{1.5}$$

We prescribe the normal component of the momentum on the boundary, i.e.,

$$\rho\mathbf{u}\cdot\mathbf{n}=f, \tag{1.6}$$

where $\mathbf{n}$ is the unit outer normal vector and f satisfies the compatibility condition

$$\int_{\partial\Omega} f\,dS=0. \tag{1.7}$$

Without loss of generality, we assume that there exists at least a point on $\partial\Omega$ such that $|f|>0$.

Our main theorem is stated in the following:

THEOREM 1.1. *Assume that Ω is a $C^{2,\alpha}$ smooth bounded domain in $\mathbb{R}^n$.*

(1) *For any given normal component of the momentum $f\in C^{1,\alpha}(\partial\Omega)$ satisfying (1.7), there exists a critical number B^*, such that for any $\bar{B}\in(B^*,+\infty)$, the problem*

(1.4) *supplemented with the boundary condition* (1.6) *has a* $C^{2,\alpha}(\bar{\Omega})$ *smooth subsonic potential flow.*

(**2**) *Furthermore, as* $\bar{B}\to B^*$, *the subsonic flows approach to a subsonic-sonic potential flow, which satisfies the Euler system*

$$\begin{cases} \operatorname{div}\left(\dfrac{\gamma-1}{\gamma}\left(B^*-\dfrac{|\mathbf{u}|^2}{2}\right)^{\frac{1}{\gamma-1}}\mathbf{u}\right)=0, \\ \operatorname{curl}\mathbf{u}=0 \end{cases} \tag{1.8}$$

in the sense of distribution and the boundary condition (1.6) *in the sense of boundary trace.*

(**3**) *Assume that both* $(\rho_1,\nabla\varphi_1)$ *and* $(\rho_2,\nabla\varphi_2)$ *solves the original Euler system* (1.4) *with the same Bernoulli constant. If in addition,*

$$|\nabla\varphi_1|^2<p'(\rho_1), \quad |\nabla\varphi_2|^2\le p'(\rho_2), \tag{1.9}$$

then $\nabla\varphi_1\equiv\nabla\varphi_2$.

REMARK 1.2. The uniform state at infinity plays an important role in the work in the exterior domain [3, 32, 21, 22, 12] or infinitely long nozzles [39, 40]. Compared with these, the data on the boundary are arbitrary in our result. The key point is that the magnitude of the velocity is always finite while the sound speed goes to the infinity as the Bernoulli constant increases to the infinity.

REMARK 1.3. Compared with the results [37, 38] in finitely long nozzles, the domain we consider is general and our result holds for general multidimensional irrotational flows.

REMARK 1.4. In [5], the authors also considered the existence of subsonic-sonic potential flows in bounded domains. They lessened the normal momentum on the boundary to make sure the flows lie in the subsonic region. In our paper, we enlarge the Bernoulli constant to reach this instead. This makes the boundary data unchanged and seem more reasonable. On the other hand, since the equation itself depends on the Bernoulli constant, it is not easy to see how the Mach number changes as the Bernoulli constant increases, which is the main difficulty.

REMARK 1.5. In this paper, we impose the normal component of the momentum on the boundary. This is also a physical condition.

REMARK 1.6. Our main results are true for general isentropic gas with the pressure satisfying $c_1\rho^{\gamma-1}\le p'(\rho)\le c_2\rho^{\gamma-1}$ for large ρ with some positive constants c_1,c_2. The proof is virtually identical, so we omit it here.

The rest of the paper is organized as follows. In Section 2, the wellposedness of the truncated potential flows is studied for fixed Bernoulli constants; this section is divided into two parts. We obtain a priori estimates for the solution to the truncated problem first and the existence is derived by the fixed point theorem in the second. In Section 3, we study the existence of the critical Bernoulli exponent and the limiting subsonic-sonic flows.

2. Truncated potential flows

As introduced in [39], for a given Bernoulli constant $\bar{B}$, we define the maximum density and the critical density by $\overline{\rho}(\bar{B})=\left(\frac{\gamma-1}{\gamma}\bar{B}\right)^{\frac{1}{\gamma-1}}$ and $\underline{\rho}(\bar{B})=\left(\frac{2(\gamma-1)}{\gamma(\gamma+1)}\bar{B}\right)^{\frac{1}{\gamma-1}}$ respectively. It is easy to compute that the flow is subsonic, sonic or supersonic if and only if $\underline{\rho}(\bar{B})<\rho(\le\overline{\rho}(\bar{B}))$, $\underline{\rho}(\bar{B})=\rho$ or $\underline{\rho}(\bar{B})>\rho$. This condition can be replaced by $|\mathbf{u}|^2<p'(\underline{\rho}(\bar{B}))$, $|\mathbf{u}|^2=p'(\underline{\rho}(\bar{B}))$, and $|\mathbf{u}|^2>p'(\underline{\rho}(\bar{B}))$ respectively.

Since $\operatorname{curl}\mathbf{u}=0$, there exists a function φ such that $\mathbf{u}=\nabla\varphi$. In order to fix the integral constant, we always choose $\phi\in\mathcal{L}^1=\{\phi\in L^1(\Omega),\int_\Omega\phi=0\}$. The continuity equation becomes

$$\operatorname{div}(\rho(|\nabla\varphi|^2)\nabla\varphi)=0, \tag{2.1}$$

where $\rho=\left(\frac{\gamma-1}{\gamma}(\bar{B}-\frac{1}{2}|\nabla\varphi|^2)\right)^{\frac{1}{\gamma-1}}$. We define $\mathcal{C}^{k,\alpha}(\Omega)=C^{k,\alpha}(\bar{\Omega})\cap\mathcal{L}^1$, where $C^{k,\alpha}(\bar{\Omega})$ is the standard Hölder space.

Here we study the potential flow when it is subsonic for $\rho>\underline{\rho}(\bar{B})$.

One of the main difficulties is that Equation (2.1) becomes degenerate elliptic as $\rho\to\underline{\rho}(\bar{B})$. Hence, we begin with the study on the well-posedness of the truncated potential flows in bounded domains, i.e.,

$$\begin{cases}\operatorname{div}(\rho_m(|\nabla\phi|^2)\nabla\phi)=0 & \text{in} \quad \Omega,\\ \rho_m(|\nabla\phi|^2)\nabla\phi\cdot\mathbf{n}=\sigma f & \text{on} \quad \partial\Omega,\end{cases} \tag{2.2}$$

where $\sigma\in[0,1]$ is a fixed constant, $\zeta_m(m\in\mathbb{N})$ is a given smooth increasing function satisfying

$$\zeta_m(s)=\begin{cases}s, & \text{if}\, s\le 1-\dfrac{3}{m},\\ 1-\dfrac{2}{m}, & \text{if}\, s\ge 1-\dfrac{1}{m}\end{cases}$$

and $\rho_m(s)=\rho(\zeta_m(\frac{s}{p'(\underline{\rho})})p'(\underline{\rho}))$.

According to our choice of ζ_m, ρ_m satisfies that

$$(I)\ \ \underline{\rho}(\bar{B})\le\rho_m\le\bar{\rho}(\bar{B}), \tag{2.3}$$

$$(II)\ \ \frac{1}{m}\underline{\rho}(\bar{B})|\xi|^2\le(\rho_m(|\mathbf{p}|^2)\delta_{ij}+2\rho_m'(|\mathbf{p}|^2)p_ip_j)\xi_i\xi_j\le\bar{\rho}(\bar{B})|\xi|^2. \tag{2.4}$$

Therefore, the truncated potential flow is strictly and uniformly elliptic for fixed m and large Bernoulli constant $\bar{B}$.

2.1. A priori estimates for truncated potential flows.

For simplicity, in this subsection, we omit the index m and σ in (2.2) and all the bounds C here are dependent on m, but independent of $\bar{B}$, if $\bar{B}$ is large enough.

PROPOSITION 2.1. *Any weak solution to the truncated problem (2.2) in $\mathcal{H}^1=\{\phi\in H^1,\int_\Omega\phi=0\}$ satisfies*

$$\|\nabla\varphi\|_{L^2(\Omega)}\le C\|f\|_{L^2(\partial\Omega)}, \quad \|\varphi\|_{L^\infty(\Omega)}\le C, \tag{2.5}$$

with some uniform constant C.

Proof. Suppose that φ is a weak solution to the truncated problem (2.2) in $\mathcal{H}^1$, i.e.,

$$\int_\Omega \rho(|\nabla\varphi|^2)\nabla\varphi\cdot\nabla\psi dx = \int_{\partial\Omega} f\psi dS \quad \text{for all} \quad \psi\in\mathcal{H}^1. \tag{2.6}$$

Moreover, (2.6) can be extended such that

$$\int_\Omega \rho(|\nabla\varphi|^2)\nabla\varphi\cdot\nabla\psi dx = \int_{\partial\Omega} f\psi dS \quad \text{for any} \quad \psi\in H^1(\Omega). \tag{2.7}$$

In fact, for $\tilde{\psi}=\psi-\frac{1}{|\Omega|}\int_\Omega\psi\in\mathcal{H}^1$, due to the condition (1.7) then

$$\int_\Omega \rho(|\nabla\varphi|^2)\nabla\varphi\cdot\nabla\psi dx = \int_{\partial\Omega} f\tilde{\psi}dS = \int_{\partial\Omega} f\psi - \frac{1}{|\Omega|}\int_\Omega\psi\int_{\partial\Omega} f = \int_{\partial\Omega} f\psi.$$

Now taking ψ with φ itself, and noting that ρ has positive lower bound give the H^1-norm of φ, i.e.,

$$\int_\Omega |\nabla\varphi|^2 dx \le C\int_{\partial\Omega} f^2 dS,\ \|\varphi\|_{H^1(\Omega)} \le C\|f\|_{L^2(\partial\Omega)}. \tag{2.8}$$

The L^∞-norm can be deduced by standard Moser's iteration.

Set $\bar{\varphi}_\beta=\min\{\beta,\max\{\varphi,1\}\}\in[1,\beta]$ and take $\psi=\bar{\varphi}_\beta^q\in H^1$ $(q\ge 2)$ in (2.7). Applying the trace theorem $\|\cdot\|_{L^1(\partial\Omega)}\le C\|\cdot\|_{W^{1,1}(\Omega)}$, one has that

$$\begin{aligned}\int_\Omega q\rho\bar{\varphi}_\beta^{q-1}|\nabla\bar{\varphi}_\beta|^2 = \int_{\partial\Omega} f\bar{\varphi}_\beta^q &\le C(\int_\Omega \bar{\varphi}_\beta^q + \int_\Omega q\bar{\varphi}_\beta^{q-1}|\nabla\bar{\varphi}_\beta|)\\ &\le C(q\int_\Omega\bar{\varphi}_\beta^q + \int_\Omega \epsilon q\bar{\varphi}_\beta^{q-1}|\nabla\bar{\varphi}_\beta|^2).\end{aligned}$$

or

$$\|\bar{\varphi}_\beta\|_{L^{3(q+1)}} \le C^{\frac{1}{q+1}}\|\bar{\varphi}_\beta^{\frac{q+1}{2}}\|_{H^1}^{\frac{2}{q+1}} \le C^{\frac{1}{q+1}}(q+1)^{\frac{2}{q+1}}\|\bar{\varphi}_\beta\|_{L^{q+1}}.$$

Once setting $q+1=3^\nu,\nu=1,2,3\cdots$ and letting $\nu\to\infty$, we have

$$\sup\varphi_\beta \le C\|\bar{\varphi}_\beta\|_{L^3} \le C(\sup\bar{\varphi}_\beta)^{\frac{1}{3}}\|\bar{\varphi}_\beta\|_{L^2}^{\frac{2}{3}}.$$

So $\sup\bar{\varphi}_\beta\le C\|\bar{\varphi}_\beta\|_{L^2}\le C$. Since the bound is independent of β, it yields that $\sup_\Omega\varphi\le C$ immediately. Similarly, $\inf_\Omega\underline{\varphi}^\beta\ge -C$ for $\underline{\varphi}^\beta=\max\{-\beta,\min\{\varphi,-1\}\}$ and $\inf_\Omega\varphi\ge -C$. Together with these two inequalities, it gives that $|\varphi|_{L^\infty(\Omega)}\le C$. □

The next proposition gives the $C^{2,\alpha}$-priori estimate of the solution.

Proposition 2.2. *Suppose that φ is a $C^{2,\alpha}$ solution to the truncated potential problem* (2.2). *The following prior estimate is obtained:*

$$\|\varphi\|_{C^{2,\alpha}(\bar{\Omega})} \le C. \tag{2.9}$$

Proof. First, since the truncated equation in (2.2) is strictly and uniformly elliptic, applying Ladyzhenskaya's work [26, Chapter 10.2] directly implies the $C^{1,\alpha}$ estimate, that is

$$\|\varphi\|_{C^{1,\alpha}(\bar{\Omega})} \le C. \tag{2.10}$$

Second, rewrite the truncated Equation (2.2) into the non-divergent form, i.e.,

$$\sum_{i,j=1}^{n} \frac{1}{\underline{\rho}(\bar{B})}(\rho(|\nabla\varphi|^2)\delta_{ij}+2\rho'(|\nabla\varphi|^2)\partial_i\varphi\partial_j\varphi)\partial_{ij}\varphi=0. \tag{2.11}$$

Note that the second order coefficients for such elliptic equation have bounded $C^\alpha(\bar{\Omega})$ norm. This yields that

$$\|\varphi\|_{C^{2,\alpha}(K)}\le C \tag{2.12}$$

for any compact subset $K\Subset\Omega$.

To obtain the higher regularity on the boundary, we need the Hölder estimate of the derivative for the divergence equation as follows.

LEMMA 2.3. *Suppose that $\mathcal{M}=(\mathcal{M}_{ij})$ is a given $C^\alpha(\bar{\Omega})$, positive definite matrix function, i.e., $\mathcal{M}\ge\lambda I$. Then the solution to the linear elliptic equation of the divergent form and cornormal boundary condition*

$$\begin{cases} \operatorname{div}(\mathcal{M}\nabla\phi)=g+\operatorname{div}\mathcal{F} & \text{in} \quad \Omega,\\ \mathcal{M}\nabla\phi\cdot\mathbf{n}=h & \text{on} \quad \partial\Omega. \end{cases} \tag{2.13}$$

satisfies the following local Hölder estimate.

For $x_0\in\partial\Omega$ and if $B_{2r}^+\triangleq B_{2r}\cap\Omega$($r$ is some fixed small constant), it holds that

$$\|\nabla\phi\|_{C^\alpha(\overline{B_r^+})}\le C_r\Big(\|\phi\|_{C^1(\overline{B_{2r}^+})}+\mathcal{G}^+\Big). \tag{2.14}$$

Here $\mathcal{G}^+=\|\mathcal{F}\|_{C^\alpha(\overline{B_{2r}^+})}+\|g\|_{C(\overline{B_{2r}^+})}+\|h\|_{C^\alpha(\partial\overline{B_{2r}^+})}$ and C_r only depends on λ, r, and $\|\mathcal{M}\|_{C^\alpha(\overline{B_{2r}^+})}$.

Proof. First let us begin with considering the simplest case for $\mathcal{M}=\bar{\mathcal{M}}$(constant), $\Omega=\{x\in\mathbb{R}^n,x_n>0\}$, and g, $\mathcal{F}$, h have compact support in $B_2\cap\{x_n\ge0\}$, i.e.,

$$\begin{cases} \operatorname{div}(\bar{\mathcal{M}}\nabla\phi)=g+\operatorname{div}\mathcal{F} & x_n>0,\\ \bar{\mathcal{M}}\nabla\phi\cdot\mathbf{e}_n=h & x_n=0. \end{cases}$$

with $\mathbf{e}_n=(0,0,\cdots,1)$. Without loss of generality, by a simple change of coordinates, it reduces to the standard Neumann problem defined on half plane.

$$\begin{cases} \triangle_z\phi=g+\operatorname{div}_z(L\mathcal{F}) & z_n>0,\\ \partial_{z_n}\bar{\phi}=\bar{c}h+(L\mathcal{F})\cdot\mathbf{e}_n & z_n=0, \end{cases}$$

where $x=Qz$ and Q, L, $\bar{c}$ are constants only depending on $\bar{\mathcal{M}}$. Similar to the Dirichlet boundary case, (see Chapter 4.5 [24]), it holds that

$$\|\nabla\phi\|_{C^\alpha(\{x_n\ge0\})}\le\bar{C}\Big(\|g\|_{C(\{x_n\ge0\})}+\|\mathcal{F}\|_{C^\alpha(\{x_n\ge0\})}+\|h\|_{C^\alpha(\{x_n=0\})}\Big).$$

It is important to point out that the constant $\bar{C}$ here only depends on the elliptic constant λ and $\sup_{i,j}\bar{\mathcal{M}}_{ij}$.

Next, choose some smooth truncated function η such that its support lies in $B_{2r}^+(x_0)$ and $\eta\equiv 1$ when it is inside $B_r^+(x_0)$. The constant r will be chosen later. The equation for $\eta\phi$ in B_{2r}^+ reads

$$\begin{aligned}&\operatorname{div}(\bar{\mathcal{M}}\nabla(\eta\phi))+\operatorname{div}((\mathcal{M}-\bar{\mathcal{M}})\nabla(\eta\phi))\\ =&\operatorname{div}(\eta\mathcal{F}+\phi\mathcal{M}\nabla\eta)+\eta g+(\mathcal{M}\nabla\phi-\mathcal{F})\cdot\nabla\eta,\quad \bar{\mathcal{M}}=\mathcal{M}(x_0).\end{aligned}$$

On the boundary, it holds that

$$\bar{\mathcal{M}}\nabla(\eta\phi)\cdot\mathbf{n}+(\mathcal{M}-\bar{\mathcal{M}})\nabla(\eta\phi)\cdot\mathbf{n}=\eta h+\phi\mathcal{M}\nabla\eta\cdot\mathbf{n}.$$

Due to the estimates for constant coefficients in half space and the strictly elliptic condition for $\mathcal{M}$, we have that

$$\|\nabla(\eta\phi)\|_{C^\alpha(\overline{B_{2r}^+})}\le \bar{C}\Big(\|(\mathcal{M}-\bar{\mathcal{M}})\nabla(\eta\phi)\|_{C^\alpha(\overline{B_{2r}^+})}+C_r\|\phi\|_{C^1(\overline{B_{2r}^+})}+\mathcal{G}^+\Big),$$

Note that

$$\begin{aligned}\|(\mathcal{M}-\bar{\mathcal{M}})\nabla(\eta\phi)\|_{C^\alpha}\le&\|\mathcal{M}-\bar{\mathcal{M}}\|_{C^0}\|\nabla(\eta\phi)\|_{C^\alpha}+\|\mathcal{M}-\bar{\mathcal{M}}\|_{C^\alpha}\|\nabla(\eta\phi)\|_{C^0}\\ \le&\bar{C}r^\alpha\|\nabla(\eta\phi)\|_{C^\alpha}+C_r\|\phi\|_{C^1}.\end{aligned}$$

Thus, for $\bar{C}r^\alpha<1$, this yields (2.14). □

Next, take the tangential derivative $\partial_{T_k}(k=1,2,\cdots,n-1)$ of the system of φ, i.e.,

$$\begin{cases}\dfrac{1}{\underline{\rho}(\bar{B})}\partial_i(\rho\partial_i w_k+2\rho'\partial_i\varphi\partial_j\varphi\partial_j w_k)=g+\operatorname{div}\mathcal{F} & \text{in}\quad B_{2r}^+,\\ \dfrac{1}{\underline{\rho}(\bar{B})}\rho\partial_n w_k+2\rho'\partial_n\varphi\partial_j\varphi\partial_j w_k=h & \text{on}\quad B_{2r}^+\cap\partial\Omega,\end{cases}\tag{2.15}$$

with $w_k=\partial_{T_k}\varphi$ and $\mathcal{G}^+=\|\mathcal{F}\|_{C^\alpha(\overline{B_{2r}^+})}+\|g\|_{C(\overline{B_{2r}^+})}+\|h\|_{C^\alpha(\partial\overline{B_{2r}^+})}\le C_r$. So using Lemma 2.3, we have that

$$\|w_k\|_{C^{1,\alpha}(\overline{B_r^+})}\le C_r\Big(\|w_k\|_{C^1(\bar\Omega)}+\|f\|_{C^{1,\alpha}(\partial\Omega)}\Big).$$

Once combined with the interior estimate showed in (2.12), this yields that

$$\|w_k\|_{C^{1,\alpha}(\bar\Omega)}\le C\Big(\|w_k\|_{C^1(\bar\Omega)}+\|f\|_{C^{1,\alpha}(\partial\Omega)}\Big).$$

From the equation $\partial_i(\rho\partial_i\varphi)=0$, i.e.,

$$\partial_{T_nT_n}\varphi=\frac{-1}{\rho+2\rho'(\partial_{T_n}\varphi)^2}\sum_{(i,j)\neq(n,n)}(\rho\delta_{ij}+2\rho'\partial_{T_i}\varphi\partial_{T_j}\varphi)\partial_{T_iT_j}\varphi+\mathcal{E},\tag{2.16}$$

where $T_n\equiv\mathbf{n}$ and $\|\mathcal{E}\|_{C^\alpha(\bar\Omega)}\le C$, we conclude that

$$\|\partial_{T_nT_n}\varphi\|_{C^\alpha(\overline{B_r^+})}\le C\sum_{(i,j)\neq(n,n)}\|\partial_{T_iT_j}\varphi\|_{C^\alpha(\bar\Omega)}+C$$

and then

$$\|\varphi\|_{C^{2,\alpha}(\bar\Omega)}\le C\Big(\|\varphi\|_{C^2(\bar\Omega)}+\|f\|_{C^{1,\alpha}(\partial\Omega)}\Big)$$

or

$$\|\varphi\|_{C^{2,\alpha}(\bar\Omega)}\le\Big(\|\varphi\|_{C(\bar\Omega)}+\|f\|_{C^{1,\alpha}(\partial\Omega)}\Big)\le C$$

This finishes the proof of Proposition 2.2. □

2.2. Existence of truncated potential flows and the continuous dependence on the Bernoulli constant. Back to our truncated flows (2.2), we set

$$F_m[\varphi]=\operatorname{div}(\rho_m\nabla\varphi) \quad \text{in } \Omega$$

and

$$G_m[\varphi]=\rho_m\nabla\varphi\cdot\mathbf{n} \quad \text{on } \partial\Omega.$$

Analogous to the Theorem 17.28 in [24] with slight modification, one has the existence theorem stated below for nonlinear oblique problem.

LEMMA 2.4. *Let* $0<\alpha<1$, $\mathcal{C}^{2,\alpha}(\bar\Omega)=\{u\in C^{2,\alpha}(\bar\Omega),\int_\Omega u\,dx=0\}$. *Set*

$$E=\{u\in\mathcal{C}^{2,\alpha}(\bar\Omega)|F_m[u]=0,G_m[u]=\sigma f \text{ for some } \sigma\in[0,1]\}.$$

Suppose that E *is bounded in* $C^{2,\alpha}(\bar\Omega)$. *Then the truncated problem* $F_m[\varphi]=0$ *in* Ω, $G_m[\varphi]=f$ *on* $\partial\Omega$ *is solvable in* $\mathcal{C}^{2,\alpha}(\bar\Omega)$ *for fixed* $\bar B\in(0,+\infty)$.

This gives the existence of truncated problem together with priori estimates obtained in the former propositions. In the final proposition of this section, it is showed that truncated potential solutions depend on the Bernoulli constant continuously.

PROPOSITION 2.5. *Suppose that* $\varphi_i(i=1,2)$ *are potential solutions to the truncated problem* (2.2) *with the Bernoulli constant* $B_i(i=1,2)$ *respectively. Then it holds that*

$$|\nabla\varphi_1-\nabla\varphi_2|\le C_m|B_1-B_2|. \tag{2.17}$$

Proof. Set $\Phi=\varphi_1-\varphi_2$ and $\mathcal{B}=B_1-B_2$. Then Φ solves the following oblique problem:

$$\sum_{i,j=1}^n\int_0^1\partial_i\Big((\hat\rho_m\delta_{ij}+2\hat\rho_m'\partial_i\hat\varphi\partial_j\hat\varphi)\partial_j\Phi\Big)dt=-\int_0^1\operatorname{div}((\frac{\partial\rho_m}{\partial B}(|\nabla\hat\varphi|^2,\hat B)\nabla\hat\varphi)\mathcal{B})dt \text{ in } \Omega, \tag{2.18}$$

$$\sum_{i,j=1}^n\int_0^1\mathbf{n}_i\Big((\hat\rho_m\delta_{ij}+2\hat\rho_m'\partial_i\hat\varphi\partial_j\hat\varphi)\partial_j\Phi\Big)dt=-\int_0^1(\frac{\partial\rho_m}{\partial B}(|\nabla\hat\varphi|^2,\hat B)\nabla\hat\varphi)\mathcal{B}dt\cdot\mathbf{n} \text{ on } \partial\Omega, \tag{2.19}$$

Due to Lemma 2.3, this yields the estimate (2.17). □

3. Subsonic-sonic potential flows

PROPOSITION 3.1. *There exist a* $B_m>0$, *such that for any* $\bar B>B_m$, *the solution to the truncated problem* (2.2) *also solve the original Euler system* (1.4) *and* (1.6).

Proof. It suffices to show that for large enough $\bar B$, the truncated function can be removed.

To see this, we first compute that $p'(\underline{\rho}(\bar B))=\gamma\underline{\rho}(\bar B)^{\gamma-1}$ and this yields that the sound speed goes to infinity as $\bar B\to+\infty$. In another aspect, due to the priori estimate of $\nabla\varphi$, the magnitude of the velocity is always bounded. Hence, there exists a $B_m>0$ large enough, such that for any $\bar B>B_m$, $|\nabla\varphi|^2<(1-\frac{3}{m})p'(\underline{\rho}(\bar B))$. Thus we are done.□

Furthermore, by the blow up argument, we can show the existence of the critical number B^* stated in Theorem 1.1.

Proof. Define the solution of truncated problem $F_m[\varphi]=0$ in Ω, $G_m[\varphi]=f$ on $\partial\Omega$ with Beroulli constant $\bar{B}$ to be $\varphi_m(\cdot;\bar{B})$ and $M_m(\bar{B})=\sup_{\bar{\Omega}}|\nabla\varphi_m(\cdot;\bar{B})|^2$. Since $\varphi_m\in C^{2,\alpha}(\bar{\Omega})$, Proposition 2.5 yields that $|M_m(B_1)-M_m(B_2)|\leq C_m|B_1-B_2|$. Thus, $M_m(B)$ is a continuous function of B.

Choose $B_m^*=\inf\{s\in(0,+\infty)|M_m(\tau)\leq(1-\frac{3}{m})p'(\underline{\rho}(\tau))$ for all $\tau\in(s,+\infty)\}$. Proposition 3.1 guarantees that $M_m(B)\leq(1-\frac{3}{m})p'(\underline{\rho}(B))$ at least for large enough B. Hence, B_m^* is well-defined.

Furthermore, B_m^* is decreasing and $M_m(B_m^*)=(1-\frac{3}{m})p'(\underline{\rho}(B_m^*))$ due to the continuity of M_m. Take $B^*=\lim_{m\to\infty}B_m^*$, and B^* is the critical number stated in Theorem 1.1. Then for any $B>B^*$, there exists B_m^* such that $B>B_m^*$. Due to the choice of B_m^*, the truncated function can be removed and the solution φ_m exactly solves the original potential flows (1.4) with boundary condition (1.6).

Moreover, the compensated compactness method implies that for such $B^*>0$, there exists a pair $(\rho,\mathbf{u})$ such that

$$\operatorname{div}(\rho\mathbf{u})=0,\operatorname{curl}\mathbf{u}=0,h(\rho)+\frac{1}{2}|\mathbf{u}|^2=B^*$$

in the sense of distribution and the boundary condition $\rho\mathbf{u}\cdot\mathbf{n}=f$ makes sense in the sense of boundary trace.

In fact, from the definition of B_m^*, $\mathbf{u}_m$ with respect to B_m^* is uniformly bounded in L^∞, i.e., $|\mathbf{u}_m|^2\leq p'(\underline{\rho}(B_m^*))\leq C$. Due to Young measure theory [20], there would exist a probability measure ν_x such that

$$F(\mathbf{u}_m(x))\to\bar{F}(x)=\int F(y)d\nu_x(y)\triangleq<\nu_x(y),F(y)>\ \text{ a.e. } x\in\Omega$$

for any continuous functions F. Thus, it suffices to show that ν_x is one δ-measure, which gives the pointwise convergence for $\mathbf{u}_m$. First we extend the definition of $\rho=\rho(|\mathbf{u}|^2)$ by $\rho\equiv\underline{\rho}(B)$ for $|\mathbf{u}|^2\geq p'(\underline{\rho}(B))$ and set $\rho_m=\rho(|\mathbf{u}_m|^2)$. Thus, ρ becomes a continuous function in $\mathbb{R}^3$. Then we consider $\mathbf{v}_m=\rho_m\mathbf{u}_m$. Applying Div-Curl Lemma [20] for $\mathbf{u}_m$ and $\mathbf{v}_m$ $(\operatorname{div}\mathbf{v}_m=0,\operatorname{curl}\mathbf{u}_m=\mathbf{0})$, we have

$$\rho_m|\mathbf{u}_m|^2\to\overline{\rho\mathbf{u}\cdot\mathbf{u}}=\overline{\rho\mathbf{u}}\cdot\mathbf{u}\quad\text{in the sense of distribution,}\tag{3.1}$$

where $\mathbf{u}$, $\overline{\rho\mathbf{u}}$ and $\overline{\rho\mathbf{u}\cdot\mathbf{u}}$ are the w-$*$ limit of $\mathbf{u}_m$, $\rho_m\mathbf{u}_m$, and $\rho_m|\mathbf{u}_m|^2$ respectively. Therefore, by a straight computation, this yields that

$$0=\langle\nu_x(\mathbf{y}_1)\otimes\nu_x(\mathbf{y}_2),(\rho(\mathbf{y}_1)\mathbf{y}_1-\rho(\mathbf{y}_2)\mathbf{y}_2)\cdot(\mathbf{y}_1-\mathbf{y}_2)\rangle.$$

Note that $(\rho(\mathbf{y}_1)\mathbf{y}_1-\rho(\mathbf{y}_2)\mathbf{y}_2)\cdot(\mathbf{y}_1-\mathbf{y}_2)>0$ on the subsonic branches if $\mathbf{y}_1\neq\mathbf{y}_2$. The only possibility for the identity holds when $\mathbf{y}_1=\mathbf{y}_2$. Hence, it holds that $\operatorname{supp}(\nu_x\otimes\nu_x)\subset\{(\mathbf{y}_1,\mathbf{y}_2):\mathbf{y}_1=\mathbf{y}_2\}$. That is to say, ν_x can only be a δ-measure and $\mathbf{u}_m\to\mathbf{u}$ a.e. Ω as m goes to infinity. Hence, using Lebesgue dominated convergent theorem immediately implies that $\mathbf{u}$ is a weak solution in the sense of distribution and the boundary condition makes sense in the sense of trace. □

The next proposition is about the uniqueness of the subsonic-sonic flows.

PROPOSITION 3.2. *Suppose that there are two solutions φ and $\tilde{\varphi}$ to the original potential problems* (1.4) *and* (1.6) *with the properties that*

$$|\nabla\varphi|^2<p'(\underline{\rho}(B)),\quad|\nabla\tilde{\varphi}|^2\leq p'(\underline{\rho}(B)).\tag{3.2}$$

Thus $\varphi \equiv \tilde{\varphi}$ *in* $\bar{\Omega}$.

Proof. Since φ and $\tilde{\varphi}$ are both solutions to the potential problem, $\Phi = \varphi - \tilde{\varphi}$ satisfies the equation

$$\sum_{i,j=1}^{n} \int_0^1 \partial_i \Big((\hat{\rho}\delta_{ij} + 2\hat{\rho}' \partial_1 \hat{\phi} \partial_j \hat{\phi}) \partial_j \Phi \Big) dt = 0 \quad \text{in } \Omega \tag{3.3}$$

with the boundary condition $\int_0^1 \hat{\rho} \nabla \Phi \cdot \mathbf{n} dt = 0$, where $\hat{\phi} = t\varphi + (1-t)\tilde{\varphi}$ and $\hat{\rho} = \rho(|\nabla \hat{\phi}|^2)$. From the basic Cauchy inequality, we conclude that

$$|\nabla \hat{\varphi}|^2 = |t\nabla\varphi + (1-t)\nabla\tilde{\varphi}|^2 \le (t|\nabla\varphi| + (1-t)|\nabla\tilde{\varphi}|)^2 < p'(\underline{\rho}(B)).$$

This implies the strictly elliptic condition for the Equation (3.3).

So integration by parts ensures that $\|\nabla \Phi\|_{L^2(\Omega)} = 0$ and $\varphi \equiv \tilde{\varphi}$ (at most up to a constant). □

Finally, we prove that the limit solution with the critical Bernoulli constant B^* is subsonic-sonic, i.e., $\max_{\bar{\Omega}} |\nabla \varphi^*| = p'(\underline{\rho}(B^*))$.

Proof. Note that the limit solution with the critical Bernoulli constant B^* is either subsonic or subsonic-sonic. If it is subsonic, i.e., $|\nabla\varphi^*| < p'(\underline{\rho}(B^*))$, one may choose some positive m such that $|\nabla\varphi^*| \le (1 - \frac{5}{m}) p'(\underline{\rho}(B^*))$. Let φ^δ be the solution to the truncated problem (2.2) with the Bernoulli constant $B^* - \delta > 0$. Proposition 2.5 implies that $|\nabla \varphi^\delta| \le (1 - \frac{4}{m}) p'(\underline{\rho}(B^* - \delta)) + C_m \delta$. Hence, for small $\delta > 0$,

$$|\nabla\varphi^\delta|^2 \le (1 - \frac{3}{m}) p'(\underline{\rho}(B^* - \delta)), \quad B_m^* \le B^* - \delta.$$

This contradicts with the definition of B^*. □

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