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NON-UNIQUENESS AND PRESCRIBED ENERGY FOR THE CONTINUITY EQUATION*

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Abstract. In this note, we provide new non-uniqueness examples for the continuity equation by constructing infinitely many weak solutions with prescribed energy.

Key words. Transport and continuity equations, non-uniqueness, non-conservation of energy, renormalization, convex integration.

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1. Introduction

In this paper we consider the *continuity equation* for a bounded scalar function $u\colon \mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ with a bounded divergence-free vector field $\mathbf{b}\colon \mathbb{R}\times\mathbb{R}^d\to\mathbb{R}^d$:

$$\partial_t u+\operatorname{div}(u\mathbf{b})=0, \tag{1.1}$$

$$\operatorname{div}\mathbf{b}=0. \tag{1.2}$$

This equation appears in various problems of mathematical physics, in particular fluid mechanics and kinetic theory. In the smooth setting (and assuming suitable integrability) the *energy*,

$$\mathscr{E}(t):=\int_{\mathbb{R}^d} u^2(t,x)\,dx$$

of the solution u is conserved:

$$\frac{d}{dt}\mathscr{E}(t)=0. \tag{1.3}$$

Indeed, since $\mathbf{b}$ is divergence-free, by multiplying (1.1) with u, using the chain rule, and integrating over $\mathbb{R}^d$, one immediately obtains (1.3).

In many applications, one has to study (1.1) in a nonsmooth setting. Roughly speaking, since (1.1) is linear, the conservation of energy (1.3) implies uniqueness of weak solutions to the corresponding initial-value problem for (1.1). In fact, conservation of energy is a consequence of the so-called *renormalization property* which was proved in [14] for any vector field $\mathbf{b}$ with Sobolev regularity and later extended by Ambrosio in [6] to the case when b has bounded variation. We refer to [15, 3] for a detailed review of recent results in this direction.

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On the other hand, when the regularity of the vector field $\mathbf{b}$ is too low, the conservation of energy (1.3) fails in general. In a nonsmooth setting, several counterexamples to the uniqueness, and therefore to the conservation of energy, are known; see [5, 12, 13, 2, 1]. A similar phenomenon occurs in the context of the Euler equations. For example, in the papers [21, 22, 16], weak solutions of the Euler equations were constructed with compact support in space time.

In particular, the example in [13] gives a bounded vector field $\mathbf{b}$ and a bounded scalar field u, which satisfy (1.1) and (1.2), such that

$$\mathscr{E}(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ 1 & \text{for } t > 0. \end{cases} \tag{1.4}$$

In this paper, for any given nonnegative bounded function $E\colon \mathbb{R} \to \mathbb{R}$ which is continuous on an open interval and zero outside, we construct infinitely many pairs $(\mathbf{b}, u)$ satisfying (1.1) and (1.2) such that $\mathscr{E}(t) = E(t)$ for a.e. t. Thus, in contrast with (1.4), we provide more general profiles for the energy. Our results are also connected to the chain rule problem for the divergence operator; see [4, 7, 11].

We construct such pairs $(\mathbf{b}, u)$ using the method of convex integration, and our techniques are similar to the ones used in [16, 24]. The latter reference contains an appendix giving a general framework for convex integration, but, for the problem at hand, we need to consider a nonlinear constraint that depends on the points in the domain (as was the case, e.g., in [17], albeit in a different functional setting). For this reason we adapt the framework from [24] to this more general situation (see §2). We then apply this abstract framework to the specific situation of the continuity equation (see §3).

Finally, let us mention [10, 23, 8], where results were obtained by convex integration, respectively, that yield as a byproduct counterexamples to the energy conservation for continuity equations. However, in these works, the energy profile is always piecewise constant.

2. Differential inclusions with non-constant nonlinear constraint

We start with the so-called Tartar framework (cf. e.g. [16]). Consider a system of m linear partial differential equations

$$\sum_{i=1}^{D} A_i \partial_i z = 0 \tag{2.1}$$

in an open set $\mathscr{D} \subset \mathbb{R}^D$ where A_i are constant $m \times n$ matrices and $z\colon \mathscr{D} \to \mathbb{R}^n$. Consider a nonlinear constraint

$$z(y) \in K_y \tag{2.2}$$

for a.e. y in $\mathscr{D}$ where $K_y \subset \mathbb{R}^n$ is a compact set for any $y \in \mathscr{D}$. For any $y \in \mathscr{D}$, let $U_y := \operatorname{int}\operatorname{conv} K_y$, where with conv we denote the convex hull of the set K_y and with int we denote its interior. Let $\mathscr{U} \subset \mathscr{D}$ be a bounded open set.

Definition 2.1 (Subsolutions). *We say that $z \in L^2(\mathscr{D})$ is a* subsolution *of* (2.1) *and* (2.2) *if z is a weak solution of* (2.1) *in $\mathscr{D}$, z is continuous on $\mathscr{U}$,* (2.2) *holds for a.e. $y \in \mathscr{D} \setminus \mathscr{U}$, and*

$$z(y) \in U_y \tag{2.3}$$

for any $y \in \mathscr{U}$.

DEFINITION 2.2 (Localized plane waves/wave cone). *A set* $\Lambda \subset \mathbb{R}^n$ *is called a wave cone if there exists a constant* $C > 0$ *such that for any* $\bar{z} \in \Lambda$ *there exists a sequence* $w_k \in C_0^\infty(B_1(0);\mathbb{R}^n)$ *solving* (2.1) *in* $\mathbb{R}^D$ *such that*

- $\operatorname{dist}(w_k(x),[-\bar{z},\bar{z}]) \to 0$ *for all* $x \in B_1(0)$ *uniformly as* $k \to \infty$,
- $w_k \rightharpoonup 0$ *in* L^2 *as* $k \to \infty$,
- $\int |w_k|^2 \, dy > C|\bar{z}|^2$ *for all* $k \in \mathbb{N}$.

In the above definition we denoted the segment with endpoints $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ with $[x,y] := \operatorname{conv}\{x,y\}$. The functions w_k are called *localized plane waves*. We make the following assumptions:

ASSUMPTION 2.1 (Existence of the wave cone). *There exists a wave cone* Λ *that is dense in* $\mathbb{R}^n$.

Let $\mathscr{K}$ denote the set of all compact subsets of $\mathbb{R}^n$, endowed with the Hausdorff metric $d_{\mathcal{H}}$. It is well-known that $\mathscr{K}$ is a complete metric space.

ASSUMPTION 2.2 (Continuity of the nonlinear constraint). *The map* $f \colon \mathscr{U} \ni y \mapsto K_y \in \mathscr{K}$ *is continuous and bounded in the Hausdorff metric.*

Our main abstract result is the following:

THEOREM 2.1. *Suppose that assumptions 2.1 and 2.2 hold. Suppose that* z_0 *is a subsolution of* (2.1) *and* (2.2). *Then there exist infinitely many weak solutions* $z \in L^2(\mathscr{D})$ *of* (2.1) *which agree with* z_0 *a.e. on* $\mathscr{D} \setminus \mathscr{U}$ *and satisfy* (2.2) *for a.e.* $y \in \mathscr{D}$.

2.1. Geometric preliminaries. The next lemma shows that compact subsets of the interior of the convex hull of a compact set K are stable with respect to sufficiently small perturbations of K in the Hausdorff metric.

LEMMA 2.1. *Let* $K \subset \mathbb{R}^n$ *be a compact set. Then for any compact set* $C \subset \operatorname{int}\operatorname{conv} K$, *there exists* $\varepsilon > 0$ *such that for any compact set* $K' \subset \mathbb{R}^n$ *with* $d_{\mathcal{H}}(K,K') < \varepsilon$ *we have*

$$C \subset \operatorname{int}\operatorname{conv} K'.$$

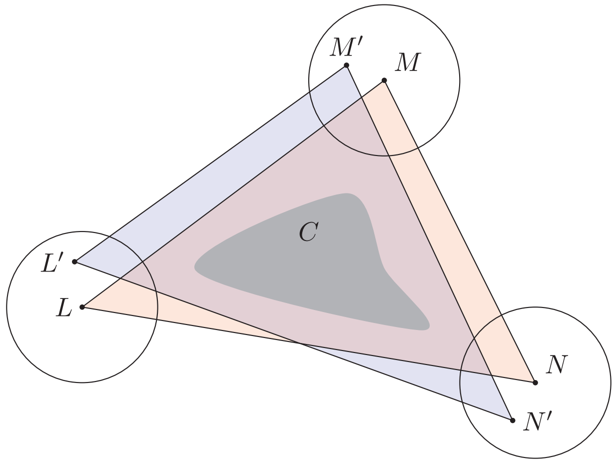

FIGURE 2.1. *An illustration of Lemma 2.1 in the case when* $K = \{L,M,N\}$ *and* $K' = \{L',M',N'\}$.

Proof. Since $\operatorname{int}\operatorname{conv} K$ is open, for any point $x \in C$ there exists a simplex S_x with vertices $\{v_i\}_{i=1..n+1} \subset \operatorname{conv} K$ such that x belongs to the inner open simplex,

$$I_x := \left\{ \sum_{i=1}^{n+1} \lambda_i v_i : \lambda_i \in \left(\frac{1}{2(n+1)}, \frac{2}{n+1} \right), \sum_{i=1}^{n+1} \lambda_i = 1,\, i = 1..n+1 \right\}.$$

Since C is a compact set and the inner simplices $\{I_x\}_{x\in C}$ cover C, we can extract a finite subcover $\{I_{x_k}\}_{k=1..m}$ of C. Let us fix $k \in 1..m$ and consider the simplex $S := S_{x_k}$ with vertices $\{v_i\}_{i=1..n+1} \subset \operatorname{conv} K$. Let $I := I_{x_k}$ denote the corresponding inner simplex. If $\varepsilon < \operatorname{dist}(\partial I, \partial S)$, then for any points $v_i' \in B_\varepsilon(v_i)$, $i = 1..n+1$, one has

$$I \subset \operatorname{conv}\{v_1', v_2', \dots, v_{n+1}'\}. \tag{2.4}$$

Observe that for any $\varepsilon > 0$ and $i = 1..n+1$, the ball $B_\varepsilon(v_i)$ contains a point $v_i' \in \operatorname{conv} K'$. Indeed, by Caratheodory's theorem, $v_i = \sum_{j=1}^{n+1} \mu_j z_j$ for some $z_j \in K$ and $\mu_j \in [0,1]$ with $\sum_{j=1}^{n+1} \mu_j = 1$. Since $d_{\mathcal{H}}(K, K') < \varepsilon$ there exist points $z_j' \in K'$ such that $z_j' \in B_\varepsilon(z_j)$ where $j = 1..n+1$. Let

$$v_i' := \sum_{j=1}^{n+1} \mu_j z_j',$$

then $|v_i - v_i'| \le \sum_{j=1}^{n+1} \mu_j |z_j - z_j'| < \varepsilon$. Hence by (2.4) we have $I \subset \operatorname{conv}\{v_1', v_2', \dots, v_{n+1}'\}$ provided that ε is small enough. But $v_i' \in \operatorname{conv} K'$, hence $I \subset \operatorname{conv} K'$. Since I is open, we can also write $I \subset \operatorname{int}\operatorname{conv} K'$. Since we have finitely many simplices, we can choose $\varepsilon > 0$ in such a way that the inclusion $I_{x_k} \subset \operatorname{int}\operatorname{conv} K'$ holds for any $k = 1..m$ (provided that $d_{\mathcal{H}}(K, K') < \varepsilon$). Then

$$C \subset \cup_{k=1..m} I_{x_k} \subset \operatorname{int}\operatorname{conv} K'.$$

∎

We will also need the following elementary lemma:

LEMMA 2.2. *Suppose that $z \in C(\mathscr{U}; \mathbb{R}^n)$ where $\mathscr{U} \subset \mathbb{R}^D$ is an open set. Suppose that for any $y \in \mathscr{U}$ we have a compact set $K_y \subset \mathbb{R}^n$ and the function $y \mapsto K_y$ is continuous in the Hausdorff metric. Then the function $F\colon y \mapsto \operatorname{dist}(z(y), K_y)$ is continuous on $\mathscr{U}$.*

Proof. One can prove directly that the function $(z, K) \mapsto \operatorname{dist}(z, K)$ is continuous on $\mathbb{R}^n \times \mathscr{K}$. The function $y \mapsto (z(y), K_y)$ is continuous in view of the assumptions. Hence the function F is continuous as a composition of continuous functions. ∎

In general, the distance from a point z to a compact set K does not control from below the distance from z to the boundary of $\operatorname{conv} K$. However the following lemma shows that there exists a segment inside $\operatorname{int}\operatorname{conv} K$ with midpoint z and length controlled from below by $\operatorname{dist}(z, K)$:

LEMMA 2.3 (Geometric lemma). *Let $K \subset \mathbb{R}^n$ be a compact set. For any $z \in \operatorname{int}\operatorname{conv} K$, there exists $\bar{z} \in \mathbb{R}^n$ such that*

- $[z - \bar{z}, z + \bar{z}] \subset \operatorname{int}\operatorname{conv} K$
- $|\bar{z}| \ge \frac{1}{2n} \operatorname{dist}(z, K)$

(This is exactly Lemma 5.3 from [18].)

2.2. Convex integration. The following lemma is the main building block of the convex integration scheme.

Lemma 2.4 (Perturbation lemma). *Suppose that assumptions 2.1 and 2.2 hold and that z is a subsolution of* (2.1) *and* (2.2) *such that*

$$\int_{\mathscr{U}} \operatorname{dist}^2(z(y),K_y)\,dy = \varepsilon > 0.$$

Then there exists $\delta=\delta(\varepsilon)>0$ and a sequence $\{z_k\}_{k\in\mathbb{N}}$ of subsolutions of (2.1) *and* (2.2) *such that*

- $z_k = z$ *on* $\mathscr{D}\setminus\mathscr{U}$ *for any* $k\in\mathbb{N}$
- $\int_{\mathscr{U}} |z-z_k|^2\,dy \geq \delta$ *for any* $k\in\mathbb{N}$
- $z_k \rightharpoonup z$ *in* $L^2(\mathscr{U})$ *as* $k\to\infty$.

Proof.
Step 1. Let $y\in\mathscr{U}$. Since $z(y)\in U_y$, we can apply Lemma 2.3 to obtain $\bar{z}_*(y)$ such that

$$[z(y)-\bar{z}_*(y), z(y)+\bar{z}_*(y)] \subset U_y,$$
$$|\bar{z}_*(y)| \geq \frac{1}{2n}\operatorname{dist}(z(y),K_y).$$

Since Λ is dense in $\mathbb{R}^n$ and U_y is open, we can find $\bar{z}(y)\in\Lambda$ such that

$$[z(y)-\bar{z}(y), z(y)+\bar{z}(y)] \subset U_y, \tag{2.5}$$
$$|\bar{z}(y)| \geq \frac{1}{4n}\operatorname{dist}(z(y),K_y). \tag{2.6}$$

Due to (2.5), there exists $\rho(y)>0$ such that

$$[z(y)-\bar{z}(y), z(y)+\bar{z}(y)] + \overline{B_{2\rho(y)}(0)} \subset U_y.$$

Hence, using Assumption 2.2, Lemma 2.1, and the continuity of z, we can find $R(y)>0$ such that

$$[z(x)-\bar{z}(y), z(x)+\bar{z}(y)] + \overline{B_{\rho(y)}(0)} \subset U_x \tag{2.7}$$

for all $x\in B_{R(y)}(y)\subset\mathscr{U}$. Moreover, in view of Lemma 2.2, we can choose $R(y)$ in such a way that

$$\operatorname{dist}(z(x),K_x) \leq 2\operatorname{dist}(z(y),K_y) \tag{2.8}$$

for all $x\in B_{R(y)}(y)$. Using Assumption 2.1 for any fixed $y\in\mathscr{U}$, we can construct a sequence $\{w_{y,k}\}_{k\in\mathbb{N}}\subset C_0^\infty(B_1(0))$ such that

- $w_{y,k}(x)\in[-\bar{z}(y),\bar{z}(y)]+B_{\rho(y)}(0)$ for all $x\in B_1(0)$ and $k\in\mathbb{N}$,
- $w_{y,k}\rightharpoonup 0$ in L^2 as $k\to\infty$,
- $\int |w_{y,k}|^2\,dx > C|\bar{z}(y)|^2$ for all $k\in\mathbb{N}$.

Step 2. Let $\varepsilon := \int_{\mathscr{U}} \operatorname{dist}^2(z(y),K_y)\,dy$. The balls $\{B_r(y) : y\in\mathscr{U}, r\in(0,R(y))\}$ cover $\mathscr{U}$, so using Vitali's covering theorem (see e.g. [9], Theorem 5.5.2) and the absolute continuity of the Lebesgue integral, we can find finitely many points $\{y_i\}_{i=1..N}\subset\mathscr{U}$ and radii $r_i\in(0,R(y_i))$ such that

$$\sum_{i=1}^N \int_{B_i} \operatorname{dist}^2(z(y),K_y)\,dy > \frac{1}{2}\varepsilon, \tag{2.9}$$

where the balls $B_i := B_{r_i}(y_i)$ are pairwise disjoint.

For each $i = 1..N$, let us introduce the scaled and translated perturbations $w_{i,k}(x) := w_{y_i,k}(\frac{x-y_i}{r_i})$. These functions belong to $C_0^\infty(B_i)$ and satisfy

(i) $w_{i,k}(x) \in [-\bar z(y_i), \bar z(y_i)] + B_{\rho(y_i)}(0)$ for all $x \in B_i$, $k \in \mathbb{N}$, $i = 1..N$;

(ii) $w_{i,k} \rightharpoonup 0$ in L^2 as $k \to \infty$ (for each fixed $i = 1..N$);

(iii) $\int |w_{i,k}|^2 \, dx > C|\bar z(y_i)|^2 \mathscr{L}^D(B_i)$ for all $k \in \mathbb{N}$.

In view of (i) and (2.7), we have $z(x) + w_{i,k}(x) \in U_x$ for all $x \in \mathscr{U}$ and $i = 1..N$, hence $z + w_{i,k} \in X_0$. Since the balls B_i are pairwise disjoint, the function

$$z_k := z + \sum_{i=1}^N w_{i,k}$$

also belongs to X_0.

Using successively (iii), (2.6), (2.8), and (2.9) we obtain:

$$\begin{aligned}
\int_{\mathscr{U}} |z - z_k|^2 \, dy &= \sum_{i=1}^N \int_{B_i} |w_{i,k}(y)|^2 \, dy \\
&\overset{\text{(iii)}}{>} C \sum_{i=1}^N |\bar z(y_i)|^2 \mathscr{L}^D(B_i) \\
&\overset{(2.6)}{\geq} \frac{C}{16n^2} \sum_{i=1}^N \operatorname{dist}^2(z(y_i), K_{y_i}) \mathscr{L}^D(B_i) \\
&= \frac{C}{16n^2} \sum_{i=1}^N \int_{B_i} \operatorname{dist}^2(z(y_i), K_{y_i}) \, dx \\
&\overset{(2.8)}{>} \frac{C}{32n^2} \sum_{i=1}^N \int_{B_i} \operatorname{dist}^2(z(x), K_x) \, dx \\
&\overset{(2.9)}{>} \frac{C}{64n^2} \varepsilon.
\end{aligned}$$

It remains to observe that, since N is finite and the points y_i are fixed, we have $z_k \rightharpoonup z$ in L^2 as $k \to \infty$. ∎

2.3. Proof of Theorem 2.1. We are now ready to prove our main abstract theorem.

Proof of Theorem 2.1. Let X_0 denote a set of all subsolutions of (2.1) and (2.2) which agree with z_0 on $\mathscr{D} \setminus \mathscr{U}$. Let X be the closure of X_0 in the weak topology of $L^2(\mathscr{U})$ endowed with the corresponding induced weak topology. Clearly any $z \in X$ solves (2.1) and satisfies (2.2) a.e. on $\mathscr{D} \setminus \mathscr{U}$.

For any $z \in X$, let us define

$$I(z) := \int_{\mathscr{U}} |z(y)|^2 \, dy.$$

This functional is a Baire-1 function on X. Indeed, for any $j \in \mathbb{N}$ let

$$I_j(z) := \int_{\{y \in \mathscr{U} : \operatorname{dist}(y, \partial U) > 1/j\}} |(\omega_{1/j} * z)(y)|^2 \, dy$$

where for any $\varepsilon > 0$ we denote by $\omega_\varepsilon(\cdot) = \varepsilon^{-D}\omega(\cdot/\varepsilon)$ the standard convolution kernel. Then for any $j \in \mathbb{N}$, the functional I_j is continuous on X, and for any $z \in X$, we have $I_j(z) \to I(z)$ as $j \to \infty$.

In view of Assumption 2.2, X is a *bounded* subset of $L^2(\mathscr{U})$. Since the weak topology is metrizable on the norm-bounded subsets of $L^2(\mathscr{U})$, we can consider X as a complete metric space with some metric d_X.

Then by Baire category theorem (see also Theorem 7.3 from [20]), the set

$$Y := \{z \in X : I \text{ is continuous at } z\}$$

is residual in X (and hence is infinite). We claim that $z \in Y$ implies $J(z) = 0$ where

$$J(z) := \int_{\mathscr{U}} \operatorname{dist}^2(z(y), K_y)\, dy.$$

Indeed, suppose that $J(z) = \varepsilon > 0$ for some $z \in Y$. Let $z_j \in X_0$ be a sequence such that $z_j \rightharpoonup z$ in $L^2(\mathscr{U})$ as $j \to \infty$. Since I is continuous at z, this implies that $I(z_j) \to I(z)$ and consequently $z_j \to z$ in $L^2(\mathscr{U})$ as $j \to \infty$.

Then in view of Assumption 2.2, we have $J(z_j) \to J(z)$ as $j \to \infty$, and hence, without loss of generality, we can assume that $J(z_j) > \varepsilon/2$ for all $j \in \mathbb{N}$.

Applying Lemma 2.4 to z_j for each $j \in \mathbb{N}$, we can find $\widetilde{z}_j \in X_0$ such that $d_X(\widetilde{z}_j, z_j) < 2^{-j}$ and $\int_{\mathscr{U}} |\widetilde{z}_j - z_j|^2\, dy \geq \delta > 0$ where $\delta = \delta(\varepsilon)$ is independent of j.

Since $d_X(\widetilde{z}_j, z) \leq d_X(\widetilde{z}_j, z_j) + d_X(z_j, z) \to 0$ as $j \to \infty$ we also have $\widetilde{z}_j \rightharpoonup z$ in L^2. Since z is a point of continuity of I, we also have $z_j \to z$ in $L^2(\mathscr{U})$ as $j \to \infty$. But then $\widetilde{z}_j - z_j \to 0$ in $L^2(\mathscr{U})$ which contradicts the construction of $\widetilde{z}_j$. ∎

3. Application to the continuity equation

In this section we apply our abstract framework to the case of the continuity equation.

THEOREM 3.1. *Suppose that $d \geq 2$. Let $E \colon \mathbb{R} \to \mathbb{R}$ be a non-negative bounded function which is continuous on some bounded open interval $I \subset \mathbb{R}$ and vanishes on $\mathbb{R} \setminus I$. Then there exist infinitely many bounded, compactly supported $u \colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and $\mathbf{b} \colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ which satisfy* (1.1) *and* (1.2) *in the sense of distributions and are such that*

$$\int_{\mathbb{R}^2} u^2(t,x)\, dx = E(t) \qquad \textit{for a.e. } t \in I.$$

REMARK 3.1. It is well-known that a representative of u can be chosen such that the map $t \mapsto u(t,\cdot)$ is continuous with values in L^2 equipped with the weak topology. Then the question arises whether the assertion in the theorem holds for *every*, and not just almost every, time t. We expect this to be true; indeed this should follow by methods similar to those of [17]. We will, however, not pursue this question further in this article.

REMARK 3.2. When $d = 2$ and f is a characteristic function of an interval, the statement of Theorem 3.1, essentially, follows from the example constructed in [13]. This particular case of Theorem 3.1 was also proved in [19] using the convex integration method.

REMARK 3.3. A similar problem can be addressed for more general equation of the form $\operatorname{div}(u\mathbf{B}) = 0$ instead of (1.1). For this equation the problem is stated as follows:

given a distribution g, is it possible to construct compactly supported bounded functions $u\colon \mathbb{R}^n \to \mathbb{R}$, $\mathbf{B}\colon \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\operatorname{div}(u\mathbf{B}) = 0, \quad \operatorname{div}\mathbf{B} = 0, \quad \operatorname{div}(u^2\mathbf{B}) = g?$$

This is related to the so-called *chain rule problem* for the divergence operator [4]. When $n=2$ such a construction is not possible for $g \neq 0$ in view of [7], but for $n \geq 3$ it is possible and is obtained using convex integration and rank-2 laminates in [11].

Let us put the continuity equation in the framework of the previous section. Fix a bounded open set $\Omega \subset \mathbb{R}^d$. Let $\mathscr{U} := I \times \Omega$ and

$$F(t,x) := \frac{E(t)}{\mathscr{L}^d(\Omega)} \mathbf{1}_\Omega(x),$$

where $\mathbf{1}_\Omega$ denotes the characteristic function of Ω.

We consider equations (1.1) and (1.2) as a linear system

$$\partial_t u + \operatorname{div}_x \mathbf{m} = 0, \tag{3.1}$$

$$\operatorname{div}_x \mathbf{b} = 0 \tag{3.2}$$

in $\mathscr{D} := \mathbb{R} \times \mathbb{R}^d$ with $u\colon \mathscr{D} \to \mathbb{R}$, $\mathbf{m}\colon \mathscr{D} \to \mathbb{R}^d$ and $\mathbf{b}\colon \mathscr{D} \to \mathbb{R}^d$ such that $z := (u, \mathbf{m}, \mathbf{b})$ satisfies the constraint

$$z(y) \in K_y := \begin{cases} \{(u,\mathbf{m},\mathbf{b}) : \mathbf{m} = u\mathbf{b}, \quad |\mathbf{b}| = 1, \quad u^2 = F(y)\} & \text{if } \; y \in \mathscr{U} \\ 0 & \text{if } \; y \in \mathscr{D} \setminus \mathscr{U} \end{cases} \tag{3.3}$$

for a.e. $y = (x,t) \in \mathscr{D}$.

Suppose that $z = (u, \mathbf{m}, \mathbf{b}) \in L^\infty(\mathscr{D})$ satisfies (3.1) and (3.2) in the sense of distributions and, moreover, (3.3) holds a.e. in $\mathscr{D}$. Then the couple $(u, \mathbf{b})$ satisfies the assertion of Theorem 3.1.

Let us check the assumption of Theorem 3.1.

Lemma 3.1. *Suppose that $A, B \subset \mathbb{R}^n$ are compact sets and $r > 0$ is such that*

- *for any $z \in A$ there exists $z' \in B \cap B_r(z)$*
- *for any $z \in B$ there exists $z' \in A \cap B_r(z)$*

Then $d_{\mathcal{H}}(A,B) < r$.

Proof. Suppose that $d_{\mathcal{H}}(A,B) \geq r$. Then without loss of generality, we can assume that there exists $z \in A$ such that for any $z' \in B$ we have $z \notin B_r(z')$. But then the ball $B_r(z)$ cannot contain any point of B which leads to a contradiction. ∎

Lemma 3.2. *If $F\colon \mathscr{U} \to \mathbb{R}$ is continuous, bounded, and non-negative then the map $y \mapsto K_y$ is continuous and bounded (w.r.t. $d_{\mathcal{H}}$) on $\mathscr{U}$.*

Proof . Let $f(y) := \sqrt{F(y)}$. Let us fix $y \in \mathscr{U}$. For any $\varepsilon > 0$ let $\delta > 0$ be such that $|f(y) - f(y')| < \varepsilon$ for any $y' \in B_\delta(y) \subset \mathscr{U}$. Let us prove that $d_{\mathcal{H}}(K_y, K_{y'}) < 2\varepsilon$ for all $y' \in B_\delta(y)$. For any $z \in K_y$, there exist $\sigma \in \{\pm 1\}$ and $\mathbf{b} \in \mathbb{R}^d$ with $|\mathbf{b}| = 1$ such that $z = (\sigma f(y), \sigma f(y)\mathbf{b}, \mathbf{b})$. Then $z' := (\sigma f(y'), \sigma f(y')\mathbf{b}, \mathbf{b})$ belongs to $K_{y'}$ and $|z - z'| \leq 2|f(y) - f(y')|$. Hence there exists $z' \in K_{y'} \cap B_{2\varepsilon}(z)$. Similarly, for any $z' \in K_{y'}$ there exist $\sigma \in \{\pm 1\}$ and $\mathbf{b} \in \mathbb{R}^d$ with $|\mathbf{b}| = 1$ such that $z' = (\sigma f(y'), \sigma f(y')\mathbf{b}, \mathbf{b})$. Then $z := (\sigma f(y), \sigma f(y)\mathbf{b}, \mathbf{b})$ belongs to K_y and $|z - z'| \leq 2|f(y) - f(y')|$. Hence there exists $z \in K_y \cap B_{2\varepsilon}(z')$. Therefore by Lemma 3.1 we have $d_{\mathcal{H}}(K_y, K_{y'}) < 2\varepsilon$. ∎

LEMMA 3.3. *Assumption 2.1 holds for the system* (3.1)–(3.3).

Proof . Let $\phi\colon \mathscr{D}\to\mathbb{R}$ be a non-negative smooth function such that $0\le\phi\le 1$ on $\mathscr{D}$, $\phi=0$ on $\mathscr{D}\setminus B_1(0)$, and $\phi=1$ on $B_{1/2}(0)$.

Part 1. Suppose that $d>2$. Let us show that Assumption 2.1 holds with $\Lambda=\mathbb{R}^{2d+1}$. Fix $\bar u\in\mathbb{R}$, $\bar{\mathbf{m}}\in\mathbb{R}^d$, and $\bar{\mathbf{b}}\in\mathbb{R}^d$ and let $\bar z=(\bar u,\bar{\mathbf{m}},\bar{\mathbf{b}})$. Since $d>2$, there exists a unit vector $\mathbf{n}\in\mathbb{R}^d$ such that $\mathbf{n}\cdot\bar{\mathbf{m}}=\mathbf{n}\cdot\bar{\mathbf{b}}=0$. Denote $\hat{\mathbf{n}}=(0,\mathbf{n})$, $\bar{\mathbf{a}}=(\bar u,\bar{\mathbf{m}})$. For any $k\in\mathbb{N}$, define $\bar{\mathbf{a}}_k\colon \mathscr{D}\to\mathbb{R}^{d+1}$ by

$$\bar{\mathbf{a}}_k(y):=\bar{\mathbf{a}}(\hat{\mathbf{n}}\cdot\nabla_y(\phi\Pi_k))-\hat{\mathbf{n}}(\bar{\mathbf{a}}\cdot\nabla_y(\phi\Pi_k))$$

where $y=(t,x)$ and

$$\Pi_k(y):=\frac{\sin(k\hat{\mathbf{n}}\cdot y)}{k}.$$

Observe that

$$\operatorname{div}_y\bar{\mathbf{a}}_k=(\bar{\mathbf{a}}\cdot\nabla_y)(\hat{\mathbf{n}}\cdot\nabla_y)(\phi\Pi_k)-(\hat{\mathbf{n}}\cdot\nabla_y)(\bar{\mathbf{a}}\cdot\nabla_y)(\phi\Pi_k)=0.$$

Let $(u_k,\mathbf{m}_k)$ denote the components of $\bar{\mathbf{a}}_k$; then by the equation above we have $\partial_t u_k+\operatorname{div}_x\mathbf{m}_k=0$.

Similarly, let

$$\mathbf{b}_k(t,x):=\bar{\mathbf{b}}(\mathbf{n}\cdot\nabla_x(\phi\Pi_k))-\mathbf{n}(\bar{\mathbf{b}}\cdot\nabla_x(\phi\Pi_k)).$$

Then arguing as above, $\operatorname{div}\mathbf{b}_k=0$.

Now we introduce $w_k:=(u_k,\mathbf{m}_k,\mathbf{b}_k)$. Then

$$w_k(y)=\bar z\phi\cos(k\hat{\mathbf{n}}\cdot y)+f\Pi_k$$

where f does not depend on k and vanishes on $B_{1/2}(0)$.

On the other hand,

$$\begin{aligned}\int_{\mathscr{D}}|w_k|^2\,dy\ge\int_{B_{1/2}(0)}|w_k|^2\,dy&=\int_{B_{1/2}(0)}|\bar z|^2\cos^2(k\hat{\mathbf{n}}\cdot y)\,dy\\&=\int_{B_{1/2}(0)}|\bar z|^2\frac{1+\cos(2k\hat{\mathbf{n}}\cdot y)}{2}\,dy\ge\frac{|\bar z|^2}{4}|B_{1/2}(0)|\end{aligned}\tag{3.4}$$

provided that k is sufficiently large.

Part 2. Suppose that $d=2$ and fix $\bar z=(\bar u,\bar{\mathbf{m}},\bar{\mathbf{b}})\in\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2$ with $\bar u\neq 0$. Let us look for a localized plane wave in the following form:

$$w_k=(\mathbf{a}_k,\mathbf{b}_k)$$

with

$$\begin{aligned}\mathbf{a}_k(y)&=\nabla_y\times\left(\phi\mathbf{A}\frac{\sin(k\mathbf{n}\cdot y)}{k}\right)\\ \mathbf{b}_k(t,x)&=\nabla_x^{\perp}\left(\phi\frac{\sin\big(k\mathbf{n}\cdot(t,x)\big)}{k}\right)\end{aligned}$$

where $\mathbf{n} = (n_t, \mathbf{n}_x) \in \mathbb{R} \times \mathbb{R}^2$ and $\mathbf{A} \in \mathbb{R}^3$ are to be chosen and $k \in \mathbb{N}$. Then, by construction,

$$\operatorname{div}_y \mathbf{a}_k = 0, \qquad \operatorname{div}_x \mathbf{b}_k = 0.$$

Then, we get

$$w_k = \hat{z}\phi \cos(k\mathbf{n} \cdot y) + f \frac{\sin(k\mathbf{n} \cdot y)}{k}$$

where $\hat{z} = (\mathbf{A} \times \mathbf{n}, \mathbf{n}_x^\perp)$ and f does not depend on k and vanishes on $B_{1/2}(0)$.

In order to have $\hat{z} = \bar{z}$, the vectors $\mathbf{A}$ and $\mathbf{n}$ must satisfy

$$\mathbf{A} \times \mathbf{n} = (\bar{u}, \bar{\mathbf{m}}),$$
$$\mathbf{n}_x^\perp = \bar{\mathbf{b}}.$$

From the second equation we immediately obtain that $\mathbf{n}_x = -\bar{\mathbf{b}}^\perp$. Since $\bar{u} \neq 0$ there exists n_t such that $\mathbf{n} \perp (\bar{u}, \bar{\mathbf{m}})$. Then, we can always find $\mathbf{A}$ such that the first equation is satisfied. It remains to observe that the estimate (3.4) also holds in the considered case. We thus have verified Assumption 2.1 for $\Lambda = \mathbb{R}^5 \setminus \{\bar{u} = 0\}$. ∎

Proof of Theorem 3.1. By symmetry of K_y for any $y \in \mathscr{U}$, we have $0 \in \operatorname{int} \operatorname{conv} K_y$. On the other hand, $K_y = \{0\}$ for any $y \in \mathscr{D} \setminus \mathscr{U}$. Therefore $u \equiv 0$, $\mathbf{m} \equiv 0$, and $\mathbf{b} \equiv 0$ is a subsolution of (3.1)–(3.3). Then the result follows from Lemma 2.2, Lemma 3.3, and Theorem 2.1. ∎

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