Commun. Math. Sci.
Vol. 13, No. 7, pp. 1905–1936

GLOBAL WEAK SOLUTIONS TO 1D COMPRESSIBLE EULER EQUATIONS WITH RADIATION*

XAVIER BLANC† AND BERNARD DUCOMET‡

Abstract. We consider the Cauchy problem for the equations of one-dimensional motion of a compressible inviscid gas coupled with radiation through a radiative transfer equation. Assuming suitable hypotheses on the transport coefficients and the data, we prove that the problem admits a weak solution. More precisely, we show that a sequence of approximate solutions constructed by a generalized Glimm scheme admits a subsequence converging to an entropic solution of the problem.

Key words. Compressible, one-dimensional symmetry, radiative transfer.

AMS subject classifications. 35Q30, 76N10.

1. Introduction

The purpose of radiation hydrodynamics is to include the effects of radiation in the hydrodynamical framework. When the equilibrium holds between the matter and the radiation, a simple way to do that is to include local radiative terms into the state functions and the transport coefficients. On the other hand, radiation is described by photons which are massless particles traveling at the speed of light, c, characterized by their frequency $\nu \in \mathbb{R}_+$, their energy $E = h\nu$ (where h is the Planck's constant), and their momentum $\vec{p} = \frac{\hbar\nu}{c}\,\vec{\Omega}$, where $\vec{\Omega} \in S^2$ is a vector of the 2-unit sphere. Statistical mechanics allows us to describe macroscopically an assembly of massless photons of energy E and momentum $\vec{p}$ by using a distribution function, the radiative intensity $I(x,t,\vec{\Omega},\nu)$. Using this intensity, one can derive global quantities by integrating with respect to the angular and frequency variables: the spectral radiative energy density $E_R(x,t)$ per unit volume is then $E_R(x,t) := \frac{1}{c}\int\int I(x,t,\vec{\Omega},\nu)\; d\Omega\; d\nu$, and the spectral radiative flux is $\vec{F}_R(x,t) = \int\int \vec{\Omega}\; I(x,t,\vec{\Omega},\nu)\; d\Omega\; d\nu$.

In the absence of radiation, the hydrodynamical system is derived from the fundamental conservation laws (mass, momentum, and energy) by using the Boltzmann's equation satisfied by the $f_m(x,\vec{v},t)$ and Chapman–Enskog's expansion. One then gets formally the compressible Euler system for matter. When radiation is taken into account at a macroscopic level, supplementary source terms appear coupling matter variables to radiative intensity I which is supposed to satisfy a transport equation, the so called radiative transfer equation, an integro-differential equation discussed by Chandrasekhar in [4].

Supposing that the matter is at local thermodynamical equilibrium (LTE) and in the non-relativistic framework (the velocity of matter is much less than the velocity of light; i.e. $\vec{v}^2 \ll c^2$), the coupled system satisfied by the density ρ, the velocity $\vec{u}$, the

*Received: May 14, 2013; accepted (in revised form): April 18, 2015. Communicated by Yaguang Wang.
†Univ. Paris Diderot, Sorbonne Paris Cité, Laboratoire Jacques-Louis Lions, UMR 7598, UPMC, CNRS, F-75205 Paris, France (blanc@ann.jussieu.fr). http://www.ann.jussieu.fr/~blanc/.
‡CEA, DAM, DIF, F-91297 Arpajon, France (bernard.ducomet@cea.fr).

temperature ϑ, and the radiative intensity I in $\mathbb{R}^3$ reads [3, 26, 27]

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho\vec{v}) = 0, \\ \partial_t(\rho\vec{v}) + \mathrm{div}_x(\rho\vec{v}\otimes\vec{v}) + \nabla_x p = -\vec{\mathcal{S}_F}, \\ \partial_t(\rho E) + \mathrm{div}_x((\rho E + p)\vec{v}) = -\mathcal{S}_E, \\ \dfrac{1}{c}\partial_t I + \vec{\Omega}\cdot\nabla_x I = \mathcal{S}, \end{cases} \tag{1.1}$$

where $E = \frac{1}{2}\,\vec{v}^2 + e$ is the total energy, e the internal energy, p is the pressure, and $\vec{S}_F$ and S_E are the radiative force and energy source terms described below.

Let us describe the various coupling terms in the right-hand sides of (1.1) (see [25]).

The radiative source splits into two parts $\mathcal{S} = S_{a,e} + S_s$ where the first contribution,

$$S_{a,e}(x,t,\vec{\Omega},\nu) = -\sigma_a I\left(x,t,\vec{\Omega},\nu\right) + \sigma_a B(x,t,\nu),$$

is the absorption-emission contribution, and the second one,

$$S_s(x,t,\vec{\Omega},\nu) = -\sigma_s I\left(x,t,\vec{\Omega},\nu\right) + \frac{\sigma_s}{\pi}\int_{S^2} I\left(x,t,\vec{\Omega}',\nu\right)\, d\vec{\Omega}',$$

is the scattering contribution.

The radiative energy is

$$\mathcal{S}_E(x,t) := \int_{\mathbb{R}_+}\int_{S^2} S(x,t,\vec{\Omega},\nu)\, d\vec{\Omega}\, d\nu.$$

The radiative flux is

$$\vec{\mathcal{S}_F}(x,t) := \frac{1}{c}\int_{\mathbb{R}_+}\int_{S^2} \vec{\Omega}\; S(x,t,\vec{\Omega},\nu)\, d\vec{\Omega}\, d\nu.$$

In the radiative transfer equation (the last equation of (1.1)), the functions σ_a and σ_s appearing in the radiative source $\mathcal{S}$ describe in a phenomenological way the absorption-emission and scattering properties (frequency and angular transitions) of the interaction photon-matter, and the function B describes the thermodynamical equilibrium distribution.

Let us note that the foundations of the previous system have been described by Pomraning [27] and by Mihalas and Weibel-Mihalas [26] in the full framework of special relativity (oversimplified in the previous considerations). The coupled system (1.1) has been recently investigated by Lowrie, Morel, and Hittinger [25] and Buet and Després [3] with special attention to asymptotic regimes, and by Dubroca and Feugeas [7], Lin [20], and Lin, Coulombel, and Goudon [21] for numerical aspects. Concerning the existence of solutions, a proof of local-in-time existence, and blow-up of solutions has been proposed by Zhong and Jiang [33] (see also the recent papers by Jiang and Wang [15, 16] for a 1D related “Euler–Boltzmann” model). Moreover, a simplified version of the system has been investigated by Golse and Perthame [11].

As our goal is to prove global existence of solutions for the system (1.1). We restrict our study to the mono-dimensional geometry. The system (1.1) becomes

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho v) + \partial_x(\rho v^2 + p) = -\mathcal{S}_F, \\ \partial_t(\rho E) + \partial_x((\rho E + p)v)) = -\mathcal{S}_E, \\ \dfrac{1}{c}\partial_t I + \omega \partial_x I = \mathcal{S}, \end{cases} \tag{1.2}$$

where $E = \frac{1}{2}\, v^2 + e$ is the total energy with $e(\rho, S)$ the internal energy and $p(\rho, S)$ the pressure.

In order to simplify the study of the fluid part of the system, we can use ρ, v, and S as new variables (where ϑ and S are the temperature and the entropy of matter), and using the thermodynamical identity $\vartheta dS = dE + pd\left(\frac{1}{\rho}\right)$, we rewrite (1.1) as

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t v + v\partial_x v + \dfrac{1}{\rho}\, \partial_x p = -\dfrac{1}{\rho}\, \mathcal{S}_F, \\ \partial_t S + v\partial_x S = \dfrac{1}{\rho\vartheta}\, (v\mathcal{S}_F - \mathcal{S}_E). \end{cases} \tag{1.3}$$

The transfer equation is

$$\frac{1}{c}\, \partial_t I + \omega \partial_x I = \mathcal{S}, \tag{1.4}$$

for $(x, t) \in \mathbb{R} \times \mathbb{R}_+$ where the pressure is $p = p(\rho, S)$.

It will be convenient to use in the sequel the vector notation $U := \begin{pmatrix} \rho \\ v \\ S \end{pmatrix}$.

Then (1.3) becomes

$$\partial_t U + f_U \partial_x U = g,$$

with

$$f_U(U) := \begin{pmatrix} v & \rho & 0 \\ \frac{p_\rho}{\rho} & v & \frac{p_S}{\rho} \\ 0 & 0 & v \end{pmatrix},$$

and

$$g(I, U) := \begin{pmatrix} 0 \\ -\dfrac{1}{\rho}\, \mathcal{S}_F \\ \dfrac{1}{\rho\vartheta}\, (v\mathcal{S}_F - \mathcal{S}_E) \end{pmatrix}. \tag{1.5}$$

After an elementary computation, putting $c^2 = p_\rho$, one gets the three eigenpairs for $f_U(U)$:

$$\lambda_1(U) = v - c, \qquad r_1(U) = \begin{pmatrix} \rho \\ -c \\ 0 \end{pmatrix},$$

$$\lambda_2(U) = v, \qquad r_2(U) = \begin{pmatrix} p_S \\ 0 \\ -c^2 \end{pmatrix},$$

and

$$\lambda_3(U) = v + c, \qquad r_3(U) = \begin{pmatrix} \rho \\ c \\ 0 \end{pmatrix}.$$

The corresponding pairs of Riemann invariants are

$$Z_1^{(1)} = S, \;\; Z_1^{(2)} = v + \int^{\rho} \frac{c(w,S)}{w}\, dw,$$

$$Z_2^{(1)} = v, \;\; Z_2^{(2)} = p(\rho,S),$$

and

$$Z_3^{(1)} = S, \;\; Z_3^{(2)} = v - \int^{\rho} \frac{c(w,S)}{w}\, dw.$$

In (1.2), the radiative intensity $I = I(x,t,\nu,\omega)$ also depends on two extra variables: the radiation frequency $\nu \in \mathbb{R}_+$ and the angular variable $\omega \in S^1 := [-1,1]$.

The absorption-emission and scattering terms are

$$\mathcal{S}_{a,e}(x,t,\nu,\omega) = \sigma_a(t,\nu,\rho,\vartheta)\left[B(\nu,\vartheta) - I(x,t,\nu,\omega)\right], \tag{1.6}$$

and

$$\mathcal{S}_s(x,t,\nu,\omega) = \sigma_s(t,\nu,\rho,\vartheta)\;\left[\tilde{I}(x,t,\nu) - I(x,t,\nu,\omega)\right], \tag{1.7}$$

where $\tilde{I}(x,t,\nu) := \frac{1}{2}\int_{-1}^{1} I(x,t,\nu,\omega)\, d\omega$.

The function $B(\nu,\vartheta)$, which depends on the temperature and the frequency, describes the equilibrium state:

$$B(\nu,\vartheta) = 2h\nu^3 c^{-2}\;\left(e^{\frac{h\nu}{k_B\vartheta}} - 1\right)^{-1}, \tag{1.8}$$

where k_B is the Boltzmann constant and h is the Planck's constant and corresponds to the Planck equilibrium distribution of photons in a cavity at temperature ϑ (black body).

The coefficients σ_a and σ_s are positive, but their evaluation is a difficult problem of quantum mechanics, and their general form is not known (an expression of σ_a used for stars of moderate mass is given by the Kramers formula $\sigma_a(\nu,\vartheta) = \dfrac{C(\vartheta)}{\nu^3}\;\left(1 - e^{-\frac{h\nu}{k_B\vartheta}}\right)$ where C is a positive function).

In the following, we will also assume that σ_a and σ_s are positive and bounded from above: there exists $\overline{\sigma_a}(t), \overline{\sigma_s}(t) > 0$ in $L^1(\mathbb{R}^+) \cap C^0(\mathbb{R}^+)$ (that is, integrable and continuous) such that

$$0 \leqslant \sigma_a(t,\nu,\rho,\vartheta) \leqslant \overline{\sigma_a}(t),$$

and

$$0 \leqslant \sigma_s(t,\nu,\rho,\vartheta) \leqslant \overline{\sigma_s}(t).$$

REMARK 1.1. The above assumptions are very restrictive. In particular, they do not allow σ_a and σ_s to depend only on ρ, ϑ, and ν, as is physically relevant. The meaning of this, which is made precise below in (1.21)–(1.24), is that the coupling between radiation and hydrodynamics should be weak in the limit $t\to\infty$. This limitation is not satisfactory and is closely linked with the method we use here to prove our result.

We define the radiative energy

$$E_R := \frac{1}{c}\int_{-1}^{1}\int_0^{\infty} I(x,t,\nu,\omega)\ d\nu\ d\omega, \tag{1.9}$$

the radiative flux

$$F_R := \int_{-1}^{1}\int_0^{\infty} \omega I(x,t,\nu,\omega)\ d\nu\ d\omega, \tag{1.10}$$

and the radiative pressure

$$P_R := \frac{1}{c}\int_{-1}^{1}\int_0^{\infty} \omega^2 I(x,t,\nu,\omega)\ d\nu\ d\omega. \tag{1.11}$$

Finally, the radiative energy source is

$$\mathcal{S}_E := \int_{-1}^{1}\int_0^{\infty} \mathcal{S}(x,t,\nu,\omega)\ d\nu\ d\omega, \tag{1.12}$$

and the radiative force is

$$\mathcal{S}_F := \frac{1}{c}\int_{-1}^{1}\int_0^{\infty} \omega\mathcal{S}(x,t,\nu,\omega)\ d\nu\ d\omega. \tag{1.13}$$

From Equation (1.4) and the definitions (1.9)–(1.13), one also derives, after integrating in the frequency and angular variables, the equations

$$\begin{cases} \partial_t E_R + \partial_x F_R = \mathcal{S}_E, \\ \dfrac{1}{c^2}\ \partial_t F_R + \partial_x P_R = \mathcal{S}_F. \end{cases} \tag{1.14}$$

We consider, finally, the Cauchy problem

$$\begin{cases} \partial_t U(x,t) + \partial_x(f(U(x,t))) = g(U(x,t),x,t), \\ \dfrac{1}{c}\ \partial_t I(x,t,\nu,\omega) + \omega\partial_x I(x,t,\nu,\omega) = \mathcal{S}(U(x,t),I(x,t,\nu,\omega)), \end{cases} \tag{1.15}$$

with initial conditions

$$U|_{t=0} = U_0(x) = \left| \begin{array}{ll} U_\infty & \text{for } \ x < -N, \\ U^0(x) & \text{for } \ |x| \leqslant N, \\ U_\infty & \text{for } \ x > N, \end{array}\right. \tag{1.16}$$

and for any $\nu \in \mathbb{R}_+$

$$I|_{t=0} = I_0(x,\omega,\nu) \left| \begin{array}{ll} I_\infty(\omega,\nu) & \text{for } \ x < -N, \\ I^0(x,\omega,\nu) & \text{for } \ |x| \leqslant N, \\ I_\infty(\omega,\nu) & \text{for } \ x > N, \end{array}\right. \tag{1.17}$$

with $N>0$, U^0, and I^0 are measurable functions, U_∞ is a constant state, and $I_\infty = B(\nu,\vartheta_\infty)$ where ϑ_∞ is the temperature associated with U_∞.

Denoting by $\eta := \rho^{-1}$ the specific volume, pressure $p(\rho,S)$, internal energy $e(\rho,S)$, and temperature $\vartheta(\rho,S)$ are related by the thermodynamical relations

$$p = -\partial_\eta e \quad \text{and} \quad \vartheta = \partial_S e. \tag{1.18}$$

We assume that state functions e and p (resp. σ_a and σ_s) are C^2 (resp. C^1) functions of their arguments, and we suppose that e satisfies the following stability conditions:

$$\begin{cases} e(\eta,S)>0, \quad \partial_\eta e(\eta,S)<0, \quad \partial_S e(\eta,S)>0, \\ \partial^2_{\eta\eta} e(\eta,S)>0, \quad \partial^2_{\eta S} e(\eta,S)<0, \quad \partial^3_{\eta\eta\eta} e(\eta,S)<0, \\ \quad \lim\limits_{S\to+\infty} e(\eta,S) = +\infty, \quad \lim\limits_{S\to-\infty} \partial_\eta e(\eta,S) = 0. \end{cases} \tag{1.19}$$

These conditions imply that for any pair (η,p), there is a unique $S := S(\eta,p)$ such that $\partial_\eta e(\eta,S) = -p$, and we denote $\varepsilon(\eta,p) := e(\eta,S(\eta,p))$. After Smith [30], we assume that

$$\lim_{\eta\to 0} \varepsilon(\eta,p) = 0, \quad \partial_\eta p\big|_\varepsilon \leqslant \frac{p^2}{2\varepsilon}. \tag{1.20}$$

Here, $\partial_\eta p\big|_\varepsilon$ denotes the partial derivative of $p = p(\eta,\varepsilon)$ as a function of $\eta = 1/\rho$ and $\varepsilon = e$.

Finally, we give the conditions we need on σ_a and σ_s: we assume that

$$\exists h_a \in C^0\left[(0,\infty)^2\right], \quad \sup_{\nu\in\mathbb{R}_+} \sigma_a(\nu,\rho,\vartheta) \leqslant h_a(\rho,\vartheta)\overline{\sigma_a}(t), \tag{1.21}$$

$$\exists h_s \in C^0\left[(0,\infty)^2\right], \quad \sup_{\nu\in\mathbb{R}_+} \sigma_s(\nu,\rho,\vartheta) \leqslant h_s(\rho,\vartheta)\overline{\sigma_s}(t), \tag{1.22}$$

$$\exists \tilde{h}_a \in C^0\left[(0,\infty)^2\right], \quad \sup_{\nu\in\mathbb{R}_+} \left(|\partial_\rho \sigma_a(\nu,\rho,\vartheta)| + |\partial_\vartheta \sigma_a(\nu,\rho,\vartheta)|\right) \leqslant \tilde{h}_a(\rho,\vartheta)\overline{\sigma_a}(t), \tag{1.23}$$

and

$$\exists \tilde{h}_s \in C^0\left[(0,\infty)^2\right], \quad \sup_{\nu\in\mathbb{R}_+} \left(|\partial_\rho \sigma_s(\nu,\rho,\vartheta)| + |\partial_\vartheta \sigma_s(\nu,\rho,\vartheta)|\right) \leqslant \tilde{h}_s(\rho,\vartheta)\overline{\sigma_s}(t). \tag{1.24}$$

Here, C^0 means continuous, and $\overline{\sigma_a}$ and $\overline{\sigma_s}$ are supposed to be in $L^1(\mathbb{R}^+) \cap C^0(\mathbb{R}^+)$. A simple argument then proves the following lemma.

Lemma 1.2. *Assume that* (1.21) *is satisfied and that g is defined by* (1.5). *Then, there exists $h_0 \in C^0[(0,\infty)\times[0,\infty)]$ such that, for any $I \geqslant 0$ in $L^1_{\omega,\nu}([-1,1]\times\mathbb{R}^+)$ and any $U \in \mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+$,*

$$\int_0^\infty \int_{-1}^1 |\mathcal{S}(I,U)| d\omega d\nu \leqslant h_0\left(|U|, \|I\|_{L^1_{\omega,\nu}([-1,1]\times\mathbb{R}^+)}\right)(\overline{\sigma_a}(t) + \overline{\sigma_s}(t)). \tag{1.25}$$

Moreover, there exists $h_1 \in C^0[(0,\infty)\times[0,\infty)]$ such that, for any $I \geqslant 0$ in $L^1(\mathbb{R}_+\times[-1,1])$ and any $U \in \mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+$,

$$|g(I,U)| \leqslant h_1\left(|U|, \|I\|_{L^1_{\omega,\nu}([-1,1]\times\mathbb{R}^+)}\right)(\overline{\sigma_a}(t) + \overline{\sigma_s}(t)). \tag{1.26}$$

Finally, if in addition (1.23) *is satisfied, there exists* $h_2 \in C^0\left[(0,\infty)^2\times[0,\infty)^2\right]$ *such that, for any* $I_1, I_2 \geqslant 0$ *in* $L^1(\mathbb{R}_+\times[-1,1])$ *and any* $(U_1,U_2)\in(\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+)^2$,

$$|g(I_1,U_1)-g(I_2,U_2)| \leqslant h_2\left(|U_1|,|U_2|,\|I_1\|_{L^1},\|I_2\|_{L^1_{\omega,\nu}([-1,1]\times\mathbb{R}^+)}\right) \cdot\left[|U_1-U_2|+\|I_1-I_2\|_{L^1_{\omega,\nu}([-1,1]\times\mathbb{R}^+)}\right](\overline{\sigma_a}(t)+\overline{\sigma_s}(t)). \quad (1.27)$$

Proof. Recall that, according to (1.7), $\mathcal{S}=\sigma_a(B-I)+\sigma_s\left(\tilde{I}-I\right)$. Hence, applying (1.21) and (1.22),

$$\int_0^\infty\int_{-1}^1|\mathcal{S}|(x,t,\nu,\omega)d\omega d\nu \leqslant \int_0^\infty\int_{-1}^1\overline{\sigma_a}(t)h_a(\rho,\vartheta)|B-I|d\omega d\nu + \int_0^\infty\int_{-1}^1\overline{\sigma_s}(t)h_s(\rho,\vartheta)|\tilde{I}-I|d\omega d\nu.$$

Next, we use the fact that $\int_0^\infty B d\nu = a\vartheta^4$, for some pure constant $a>0$, finding

$$\int_0^\infty\int_{-1}^1|\mathcal{S}|(x,t,\nu,\omega)d\omega d\nu \leqslant \overline{\sigma_a}(t)h_a(\rho,\vartheta)\left(a\vartheta^4+\|I\|_{L^1_{\omega,\nu}([-1,1]\times\mathbb{R}^+)}\right) + 2\overline{\sigma_s}(t)h_s(\rho,\vartheta)\|I\|_{L^1_{\omega,\nu}([-1,1]\times\mathbb{R}^+)}.$$

This gives (1.25). Next, we have

$$|g| \leqslant \frac{1}{\rho}\int_0^\infty\int_{-1}^1|\mathcal{S}| + 2\frac{|v|}{\rho\vartheta}\int_0^\infty\int_{-1}^1|\mathcal{S}|.$$

Hence (1.26) follows. As for (1.27), the same kind of proof applies. □

2. The approximating scheme

The idea is first to freeze the unknown U in the second equation of (1.15) which allows us to find I as the solution of a linear Boltzmann equation. Then plugging I into the quasilinear hyperbolic part (first Equation (1.15)), we get U by solving this system by using a discrete scheme mixing Glimm–Liu scheme for the conservative part and a fractional step method for the source term, using ideas of T.-P. Liu [23], Hong and LeFloch [12], and Dafermos and Hsiao [6].

2.1. An iterative method In order to achieve this program, we first consider the family $(U^\ell,I^\ell)\equiv(\rho^\ell,v^\ell,S^\ell,I^\ell)$ defined inductively for $\ell>1$ by

$$\begin{cases} \partial_t U^\ell(x,t)+\partial_x f(U^\ell(x,t)) = g^\ell, \\ \dfrac{1}{c}\ \partial_t I^\ell(x,t,\nu,\omega)+\omega\partial_x I^\ell(x,t,\nu,\omega)=\mathcal{S}^\ell, \end{cases} \quad (2.1)$$

for $(x,t)\in\mathbb{R}\times\mathbb{R}_+$, where

$$g^\ell = \begin{pmatrix} 0 \\ -\dfrac{1}{\rho^\ell}\ \mathcal{S}^\ell_F \\ \dfrac{1}{\rho^\ell\vartheta^\ell}\ \left(v^\ell\mathcal{S}^\ell_F-\mathcal{S}^\ell_E\right) \end{pmatrix},$$

with

$$\mathcal{S}_E^\ell := \int_{-1}^{1}\int_0^\infty \mathcal{S}(U^{\ell-1},I^\ell)\ d\nu\ d\omega,\quad \mathcal{S}_F^\ell := \frac{1}{c}\int_{-1}^{1}\int_0^\infty \omega\ \mathcal{S}(U^{\ell-1},I^\ell)\ d\nu\ d\omega,$$

for

$$\mathcal{S}(V,W) = \sigma_a(t,\nu,V)[B(\nu,V)-W] + \sigma_s(t,\nu,V)\ [\tilde{W}-W],$$

and

$$\mathcal{S}^\ell = \mathcal{S}(U^{\ell-1},I^\ell),$$

with initial conditions

$$U^\ell\big|_{t=0} = U_0(x), \tag{2.2}$$

and

$$I^\ell\big|_{t=0} = I_0(x,\nu,\omega). \tag{2.3}$$

We define the sequence $\{(U^\ell,I^\ell)\}_{\ell\geqslant 1}$ as follows:

1. Solving first the linear Boltzmann equation

$$\frac{1}{c}\ \partial_t I^1(x,t,\nu,\omega) + \omega\partial_x I^1(x,t,\nu,\omega) = \mathcal{S}^1,$$

with

$$I^1\big|_{t=0} = I_0(x,\nu,\omega),$$

and

$$\mathcal{S}^1 = \mathcal{S}(U_0,I^1),$$

gives $I^1(x,t,\nu,\omega)$ for $x\in\mathbb{R}$ and $0<t\leqslant\Delta t$.

2. Then solving the hyperbolic system

$$\partial_t U^1(x,t) + \partial_x(f(U^1(x,t))) = g^1,$$

with

$$U^1\big|_{t=0} = U_0(x),$$

defines $U^1(x,t)$ for $x\in\mathbb{R}$ and $0<t\leqslant\Delta t$.

3. Supposing now that for any $\ell>1$ we know$(U^{\ell-1},I^{\ell-1})$ for $x\in\mathbb{R}$ and $(\ell-2)\Delta t< t\leqslant(\ell-1)\Delta t$. We solve the linear Boltzmann equation

$$\frac{1}{c}\ \partial_t I^\ell(x,t,\nu,\omega) + \omega\partial_x I^\ell(x,t,\nu,\omega) = \mathcal{S}(\tilde{U}^\ell,I^\ell),$$

with source

$$\tilde{U}^\ell(x,t) = U^{\ell-1}(x,(\ell-1)\Delta t),$$

and initial data

$$I^\ell\big|_{t=(\ell-1)\Delta t} = I^{\ell-1}(x,(\ell-1)\Delta t,\nu,\omega),$$

which gives $I^\ell(x,t,\nu,\omega)$ for $x\in\mathbb{R}$ and $(\ell-1)\Delta t<t\leqslant\ell\Delta t$.

4. Plugging $I^\ell(x,t,\nu,\omega)$ into the right-hand side, we solve the hyperbolic system

$$\partial_t U^\ell(x,t) + \partial_x (f(U^\ell(x,t))) = g^\ell,$$

with

$$U^\ell\big|_{t=\ell\Delta t} = U^{\ell-1}(x,\ell\Delta t),$$

which defines $U^\ell(x,t)$ for $x\in\mathbb{R}$ and $(\ell-1)\Delta t < t \leqslant \ell\Delta t$.

2.2. The Riemann problem for the radiative transfer equation We consider, for $(x,t,\omega,\nu)\in\mathbb{R}\times[0,T]\times[-1,1]\times\mathbb{R}_+$, the problem

$$\begin{cases} \partial_t I(x,t,\nu,\omega) + c\omega\partial_x I(x,t,\nu,\omega) = c\mathcal{S} \quad \text{for } t>t_0, \\ I|_{t=0} \equiv I^{in}(x,\omega,\nu) = \left| \begin{array}{ll} I_L(\omega,\nu) & \text{for } x<x_0 \\ I_R(\omega,\nu) & \text{for } x>x_0 \end{array} \right., \end{cases} \tag{2.4}$$

where $t_0>0$,

$$\mathcal{S} = \sigma_a(\nu,U)\,[B(\nu,U) - I(x,t,\nu,\omega)] + \sigma_s(\nu,U)\ \left[\tilde{I}(x,t,\nu) - I(x,t,\nu,\omega)\right],$$

$$\tilde{I}(x,t,\nu) := \frac{1}{2}\int_{-1}^{1} I(x,t,\nu,\omega)\ d\omega,$$

$$B(\nu,U) = 2h\nu^3 c^{-2}\ \left(e^{\frac{h\nu}{k_B\vartheta}} - 1\right)^{-1},$$

and

$$U(x,t) = \left| \begin{array}{ll} U_L & \text{for } x<x_0 \\ U_R & \text{for } x>x_0 \end{array} \right. .$$

The following standard result holds (see [8] and [10]).

Proposition 2.1. *Suppose that* $I^{in}\in L^\infty(\mathbb{R}\times[-1,1]\times\mathbb{R}_+)$ *and that*

$$(x,t,\omega,\nu) \to a(x,t,\omega,\nu) := \sigma_a(\nu,\omega,\rho(x,t),\vartheta(x,t)) + \sigma_s(\nu,\omega,\rho(x,t),\vartheta(x,t)) \in L^\infty(\mathbb{R}\times[0,T]\times[-1,1]\times\mathbb{R}_+),$$

where $\sigma_a \geqslant 0$ *and* $\sigma_s \geqslant 0$.

Problem (2.4) has a unique generalized solution $I\in L^\infty(\mathbb{R}\times[0,T]\times[-1,1]\times\mathbb{R}_+)$.

Moreover, suppose that

$$(x,t,\omega,\nu) \to Q(x,t,\omega,\nu) := \sigma_a(\nu,\omega,\rho(x,t),\vartheta(x,t))B(\nu,\vartheta(x,t)) \in L^\infty(\mathbb{R}\times[0,T]\times[-1,1]\times\mathbb{R}_+).$$

The following bound holds:

$$I(x,t,\omega,\nu) \leqslant \|I^{in}\|_{L^\infty(\mathbb{R}\times[-1,1]\times\mathbb{R}_+)} + T\|Q\|_{L^\infty(\mathbb{R}\times[0,T]\times[-1,1]\times\mathbb{R}_+)}. \tag{2.5}$$

Proof. Although this classical result is well-known, it is not easy to find it in the literature. We therefore provide a proof, borrowed from [10], which uses only elementary arguments. Applying the method of characteristics to the transport equation (2.4), and using the notation

$$\mathcal{A}(x,t,\nu) := c(\sigma_a(\nu,U(t,x)) + \sigma_s(\nu,U(t,x))),$$

$$\mathcal{K}I(x,t,\omega,\nu) := \frac{c}{2}\sigma_s(\nu,U(t,x)) \int_{-1}^{1} I(x,t,\omega,\nu)\ d\omega,$$

and

$$\mathcal{Q}(x,t,\nu) := c\sigma_a(\nu,U(t,x))B(\nu,U(t,x)),$$

for any $(x,t,\omega,\nu) \in (\mathbb{R}\times[0,T]\times[-1,1]\times\mathbb{R}_+)$ and for $I \in L^\infty(\mathbb{R}\times[0,T]\times[-1,1]\times\mathbb{R}_+)$, one checks the formula

$$I = \mathcal{F}[I^{in},Q] + \mathcal{T}I, \tag{2.6}$$

where

$$\begin{aligned}\mathcal{F}[I^{in},Q](x,t,\omega,\nu) &= I^{in}(x - c\omega t,\omega,\nu)e^{-\int_0^t \mathcal{A}(x+c(\tau-t)\omega,\tau,\omega,\nu)\ d\tau} \\ &\quad + \int_0^t \mathcal{Q}(x + c(s-t)\omega, s,\omega,\nu)\, e^{-\int_s^t \mathcal{A}(x+c(\tau-t)\omega,\tau,\omega,\nu)\ d\tau}\ ds,\end{aligned}$$

and

$$\mathcal{T}I(x,t,\omega,\nu) = \int_0^t \mathcal{K}I\,(x + c(s-t)\omega, s,\omega,\nu)\, e^{-\int_s^t \mathcal{A}(x+c(\tau-t)\omega,\tau,\omega,\nu)\ d\tau}\ ds.$$

One also verifies at the same time that

$$I(x,t,\omega,\nu) = I^{in}(x - c\omega t,\omega,\nu) + \int_0^t (\mathcal{K}I + \mathcal{Q} - \mathcal{A}I)(x + c(s-t)\omega, s,\omega,\nu)\ ds. \tag{2.7}$$

Considering (2.6) as a fixed point equation, one is led to show the convergence of the series

$$I := \sum_{n\geqslant 0} \mathcal{T}^n \mathcal{F}[I^{in},Q]. \tag{2.8}$$

We denote $\mathcal{U} = \{U(x,t), x\in\mathbb{R}, t\in\mathbb{R}_+\}$. One has first

$$\|\mathcal{F}[I^{in},Q]\|_{L^\infty(\mathbb{R}\times[-1,1]\times\mathcal{U})} \leqslant \|I^{in}(\cdot,\omega,\nu)\|_{L^\infty(\mathbb{R}\times[-1,1]\times\mathcal{U})} + T\|\mathcal{Q}\|_{L^\infty(\mathbb{R}\times[-1,1]\times\mathcal{U})}.$$

Moreover, we have, for any $J \in L^\infty(\mathbb{R}\times[0,T]\times[-1,1]\times\mathbb{R}^+)$,

$$|\mathcal{K}J(x,t,\omega,\nu)| \leqslant \underbrace{\left(c \sup_{\nu>0,U\in\mathcal{U}} \sigma_s(\nu,U)\right)}_{M(\mathcal{U})} \sup_{\omega'\in[-1,1]} |J(x,t,\omega',\nu)|.$$

Hence, we have

$$
\begin{aligned}
|\mathcal{T}^n J|(x,t,\omega,\nu) &\leqslant \int_0^t M(\mathcal{U}) \sup_{\omega'\in[-1,1]} \left|\mathcal{T}^{n-1}J(x+c(s-t)\omega',s,\omega',\nu)\right| ds \\
&\leqslant M(\mathcal{U}) \int_0^t \sup_{(x',\omega',\nu')\in\mathbb{R}\times[-1,1]\times\mathbb{R}^+} \left|\mathcal{T}^{n-1}J(x',s,\omega',\nu')\right| ds \\
&\leqslant M(\mathcal{U})^n \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} \sup_{(x',\omega',\nu')\in\mathbb{R}\times[-1,1]\times\mathbb{R}^+} |J(x',t_n,\omega',\nu')| dt_n \dots dt_1 \\
&\leqslant \frac{M(\mathcal{U})^n t^n}{n!} \|J\|_{L^\infty(\mathbb{R}\times[0,T]\times[-1,1]\times\mathbb{R}_+)}.
\end{aligned}
$$

Applying this inequality to $J=\mathcal{F}\left[I^{in},Q\right]$ and summing over n, we have

$$
\begin{aligned}
&\sum_{n\geqslant 0} \left\|\mathcal{T}^n \mathcal{F}\left[I^{in},Q\right]\right\|_{L^\infty(\mathbb{R}\times[0,T]\times[-1,1]\times\mathbb{R}_+)} \\
&\leqslant \sum_{n\geqslant 0} \frac{M(\mathcal{U})^n T^n}{n!} \left\|\mathcal{F}\left[I^{in},Q\right]\right\|_{L^\infty(\mathbb{R}\times[0,T]\times[-1,1]\times\mathbb{R}_+)}.
\end{aligned}
$$

Hence, the sum (2.8) is normally convergent in $L^\infty(\mathbb{R}\times[0,T]\times[-1,1]\times\mathbb{R}_+)$. Moreover, if $I^{in}\geqslant 0$ and $Q\geqslant 0$, then $\mathcal{F}\left[I^{in},Q\right]\geqslant 0$. Since, in addition, $J\geqslant 0$ implies $\mathcal{T}(J)\geqslant 0$, we infer that each term of the series (2.8) is non-negative, so

$$
\left(I^{in}\geqslant 0,\ Q\geqslant 0\right) \quad\Rightarrow\quad I\geqslant 0. \tag{2.9}
$$

Now, defining $Z=\|I^{in}\|_{L^\infty(\mathbb{R}\times[-1,1]\times\mathbb{R}_+)}+t\|Q\|_{L^\infty(\mathbb{R}\times[0,T]\times\mathbb{R}_+)}-I$, we clearly have

$$
\begin{aligned}
\partial_t Z + c\omega\partial_x Z - \mathcal{K}Z + \mathcal{A}Z = &-Q + \|Q\|_{L^\infty(\mathbb{R}\times[0,T]\times\mathbb{R}_+)} \\
&-\mathcal{K}\left(\|I^{in}\|_{L^\infty(\mathbb{R}\times[-1,1]\times\mathbb{R}_+)} + t\|Q\|_{L^\infty(\mathbb{R}\times[0,T]\times\mathbb{R}_+)}\right) \\
&+\mathcal{A}\left(\|I^{in}\|_{L^\infty(\mathbb{R}\times[-1,1]\times\mathbb{R}_+)} + t\|Q\|_{L^\infty(\mathbb{R}\times[0,T]\times\mathbb{R}_+)}\right) \\
\geqslant &\, 0,
\end{aligned}
$$

where we have used the fact that $\|I^{in}\|_{L^\infty(\mathbb{R}\times[-1,1]\times\mathbb{R}_+)}+t\|Q\|_{L^\infty(\mathbb{R}\times[0,T]\times\mathbb{R}_+)}$ is independent of ω. Hence, applying (2.9) to Z, we infer $Z\geqslant 0$; that is, (2.5) holds. ∎

2.3. The generalized Riemann problem for radiative hydrodynamics. Given two states (U_L,I_L) and (U_R,I_R) and a point (x_0,t_0), we now consider the generalized Riemann problem $GR(x_0,t_0;U_L,I_L,U_R,I_R)$

$$
\begin{cases}
\partial_t U + \partial_x f(U) = g(I,U) \quad \text{for } t>t_0,\ \ x\in\mathbb{R}, \\
\partial_t I + c\omega\partial_x I = c\mathcal{S}(I,U) \quad \text{for } t>t_0,\ \ x\in\mathbb{R},\ \ (\omega,\nu)\in\times[-1,1]\times\mathbb{R}_+ \\
(U(x,t_0),I(x,t_0)) = \left| \begin{array}{ll} (U_L,I_L) & \text{for } x<x_0, \\ (U_R,I_R) & \text{for } x>x_0. \end{array} \right.
\end{cases} \tag{2.10}
$$

After [12] we treat the hydrodynamical part of (2.10) as a perturbation of the classical Riemann problem $CRU(x_0,t_0;U_L,U_R)$

$$
\begin{cases}
\partial_t U + \partial_x f(U) = 0 \quad \text{for } t>t_0,\ \ x\in\mathbb{R} \\
U(x,0) = \left| \begin{array}{ll} U_L & \text{for } x<x_0, \\ U_R & \text{for } x>x_0, \end{array} \right.
\end{cases} \tag{2.11}
$$

for $(x,t)\in\mathbb{R}\times\mathbb{R}_+$.

From general results (see Serre [29] and references therein), one knows that $CRU(x_0,t_0;U_L,U_R)$ has a self-similar solution $W_C(\xi,x_0,t_0;U_L,U_R)$ with $\xi=\frac{x-x_0}{t-t_0}$, provided that the quantity $|U_R-U_L|$ is small enough, which consists of at most 4 constant states U_i $i\leqslant 4$, separated by shock waves, contact discontinuities, or rarefaction waves. We say that $CR(x_0,t_0;U_L,U_R)$ is solved by the elementary waves (U_{i-1},U_i), with $i=1,\dots,4$, if each U_i belongs to the i-wave curve $\mathcal{W}_i(U_{i-1})$ issued from the state U_{i-1} in phase space, and (U_{i-1},U_i) is called an i-wave of $CR(x_0,t_0;U_L,U_R)$.

If the i-characteristic field is genuinely nonlinear, then $\mathcal{W}_i(U_{i-1})$ consists of two parts: the i-rarefaction curve and the i-shock issued from U_{i-1}. If the i-characteristic field is linearly degenerate, then $\mathcal{W}_i(U_{i-1})$ consists of a C^2 curve of i-contact discontinuities.

Denoting by $\varepsilon_i\equiv\varepsilon_i(U_L,U_R;t_0,x_0)$ the strength of the i-wave (U_{i-1},U_i) along the i-curve, one can assume that, if the i-characteristic field is genuinely nonlinear, then $\varepsilon_i\geqslant 0$ for an i-rarefaction curve and $\varepsilon_i\leqslant 0$ for an i-shock. The global strength of $W_C(\xi,x_0,t_0;U_L,U_R)$ is then the vector $\varepsilon=(\varepsilon_1,\varepsilon_2,\varepsilon_3)$.

We also denote by $\sigma_i^-=\lambda_i(U_{i-1},t_0,x_0)$ and $\sigma_i^+=\lambda_i(U_i,t_0,x_0)$ the lower and upper speeds of the i- wave (U_{i-1},U_i) when it is a rarefaction and just by σ_i the speed of (U_{i-1},U_i) if it is an i-shock or an i-contact discontinuity.

We also treat the radiative part of (2.10) as a perturbation of the linear problem $CRI(x_0,t_0;I_L,I_R)$

$$\begin{cases}\partial_t I(x,t,\nu,\omega)+c\omega\partial_x I(x,t,\nu,\omega)=0 \quad \text{for } t>t_0,\\ I\big|_{t=0}\equiv I^{in}(x,\omega,\nu)=\left|\begin{array}{ll} I_L(\omega,\nu) & \text{for } x<x_0\\ I_R(\omega,\nu) & \text{for } x>x_0\end{array}\right.,\end{cases} \tag{2.12}$$

the explicit solution of which is

$$I_C(x,t,\omega,\nu)=I^{in}(x-c\omega t,\omega,\nu),$$

and one observes that it becomes as $I_C(\xi,\omega,\nu,x_0,t_0;I_L,I_R)$.

In this trivial case, there is only one (linearly degenerate) field, and one can decide that the strength of the unique simple wave is

$$\varepsilon_r\equiv\varepsilon_r(I_L,I_R,\omega,\nu;t_0,x_0)=I_R-I_L.$$

A straightforward extension of [19] shows now that the generalized Riemann problem $GR(x_0,t_0;U_L,I_L,U_R,I_R)$ also has a piecewise smooth solution locally similar to $(W_C(\xi,x_0,t_0;U_L,U_R),I_C(\xi,\omega,\nu,x_0,t_0;I_L,I_R))$.

After [12], we define our approximate solution (W_G,I_G) of the generalized Riemann problem $GR(x_0,t_0;U_L,I_L,U_R,I_R)$ by the perturbative expansion

$$\begin{aligned} W_G(t,x;x_0,t_0;U_L,U_R)&=W_C(\xi,x_0,t_0;U_L,U_R)\\ &\quad+(t-t_0)g(I_C(t,\omega,\nu,t_0,x_0;I_L,I_R),W_C(t,x_0,t_0;U_L,U_R)),\end{aligned} \tag{2.13}$$

for $t>t_0$ and $x\in\mathbb{R}$, where the correction is small when $t-t_0$ is small, and in the same way

$$\begin{aligned}&I_G(t,x,\omega,\nu;x_0,t_0;I_L,I_R)\\ &=I_C(\xi,\omega,\nu,x_0,t_0;I_L,I_R)+c(t-t_0)\mathcal{S}(I_C(t,\omega,\nu,t_0,x_0;I_L,I_R),W_C(t,x_0,t_0;U_L,U_R)).\end{aligned} \tag{2.14}$$

We will use the notation $W_G(t,x)$ (resp $I_G(t,x,\omega,\nu)$) for $W_G(t,x;x_0,t_0;U_L,U_R)$ (resp. $I_G(t,x,\omega,\nu;x_0,t_0;I_L,I_R)$.

DEFINITION 2.2. *For the rest of the paper, we assume that* $(\rho_\infty,v_\infty,S_\infty,I_\infty)$ *is a steady state for the system* (1.2) *and that* $\mathcal{U}$ *is a small neighborhood of* $(\rho_\infty,v_\infty,S_\infty,I_\infty)$ *in* $\mathbb{R}^4$.

Given space and time steps Δt and Δx satisfying the CFL condition

$$\frac{\Delta t}{\Delta x}\max\left[\max_{i=1,2,3}\left(\sup_{u\in\mathcal{U}}|\lambda_i(u)|\right),c\right]\leqslant 1, \tag{2.15}$$

we first show that (W_G,I_G) actually approximates GR.

PROPOSITION 2.3. *Suppose* $\phi\in X:=C^1(\mathbb{R}_+\times\mathbb{R},\mathbb{R}^3)$ *and* $\psi\in Y:=C^1(\mathbb{R}_+\times\mathbb{R},\mathbb{R})$ *are two compactly supported functions in the strip* $(0,\Delta t)$. *Then for any* $(t_0,x_0)\in\mathbb{R}_+\times\mathbb{R}$, $(U_L,I_L),(U_R,I_R)\in\mathcal{U}^2$, *and for any* Δt *and* Δx *satisfying the condition (2.15):*

1. *the function* $W_G(t,x;x_0,t_0;U_L,U_R)$ *satisfies*

$$\begin{aligned}&\int_{t_0}^{t_0+\Delta t}\int_{x_0-\Delta x}^{x_0+\Delta x}\{W_G\partial_t\phi+f(W_G)\partial_x\phi+g(I_G,W_G)\phi\}\;dx\;dt\\&=\int_{x_0-\Delta x}^{x_0+\Delta x}W_G(t_0+\Delta t,\cdot)\phi(t_0+\Delta t,\cdot)\;dx-\int_{x_0-\Delta x}^{x_0+\Delta x}W_G(t_0,\cdot)\phi(t_0,\cdot)\;dx\\&\quad+\int_{t_0}^{t_0+\Delta t}f(W_G(\cdot,x_0+\Delta x))\phi(\cdot,x_0+\Delta x)\;dt\\&\quad-\int_{t_0}^{t_0+\Delta t}f(W_G(\cdot,x_0-\Delta x))\phi(\cdot,x_0-\Delta x)\;dt\\&\quad+O(1)\left(\Delta t^2\left(\Delta t+\Delta x+|U_R-U_L|\right)\right)\|\phi\|_X.\end{aligned}\tag{2.16}$$

2. *the function* $I_G(t,x,\omega,\nu;x_0,t_0)$ *satisfies*

$$\begin{aligned}&\int_{t_0}^{t_0+\Delta t}\int_{x_0-\Delta x}^{x_0+\Delta x}\{I_G\partial_t\psi+c\omega I_G\partial_x\psi+c\mathcal{S}(I_G,W_G)\psi\}\;dx\;dt\\&=\int_{x_0-\Delta x}^{x_0+\Delta x}I_G(t_0+\Delta t,\cdot)\psi(t_0+\Delta t,\cdot)\;dx-\int_{x_0-\Delta x}^{x_0+\Delta x}I_G(t_0,\cdot)\psi(t_0,\cdot)\;dx\\&\quad+\int_{t_0}^{t_0+\Delta t}c\omega I_G(\cdot,x_0+\Delta x)\psi(\cdot,x_0+\Delta x)\;dt\\&\quad-\int_{t_0}^{t_0+\Delta t}c\omega I_G(\cdot,x_0-\Delta x)\psi(\cdot,x_0-\Delta x)\;dt\\&\quad+O(1)(\Delta t)^2\left(\Delta t+\Delta x\right)\;\|\psi\|_Y.\end{aligned}\tag{2.17}$$

Proof. Since we essentially follow the proof of Hong and LeFloch (see Proposition 2.1 in [12]), we will sketch the arguments only emphasizing the structure of radiation sources through I.

Assuming that $(t_0,x_0)=(0,0)$ and defining the auxiliary function $M(t,x):= W_G\partial_t\phi+f(W_G)\partial_x\phi+g(I_G,W_G)\phi$, we call I_1 the subset of indices i such that $\forall i\in I_1$ the corresponding i-wave is either a shock or a contact discontinuity and I_2 the subset of indices i such that $\forall i\in I_2$ the corresponding i-wave is a rarefaction wave (clearly

$I_1 \cup I_2 = \{1,2,3\}$). We have the following decomposition of $\int\int M\ dx\ dt$ on the elementary cell $[0,\Delta t]\times[-\Delta x,\Delta x]$:

$$\int_0^{\Delta t}\int_{-\Delta x}^{\Delta x} M(t,x)\ dx\ dt = \sum_{i\in I_1}\int_0^{\Delta t}\int_{\sigma_i^+ t}^{\sigma_{i+1}^- t} M(t,x)\ dx\ dt$$
$$+\int_0^{\Delta t}\int_{-\Delta x}^{\sigma_1^- t} M(t,x)\ dx\ dt + \int_0^{\Delta t}\int_{\sigma_3^+ t}^{\Delta x} M(t,x)\ dx\ dt$$
$$+\sum_{i\in I_2}\int_0^{\Delta t}\int_{\sigma_i^- t}^{\sigma_i^+ t} M(t,x)\ dx\ dt =: \sum_{k=1}^{4} J_k.$$

In a region $\{(x,t)\ :\ \sigma_i^+ t \leqslant x \leqslant \sigma_{i+1}^- t\}$ where $W_C \equiv U_i$ and $I_C = I_i(\omega,\nu)$ are constant

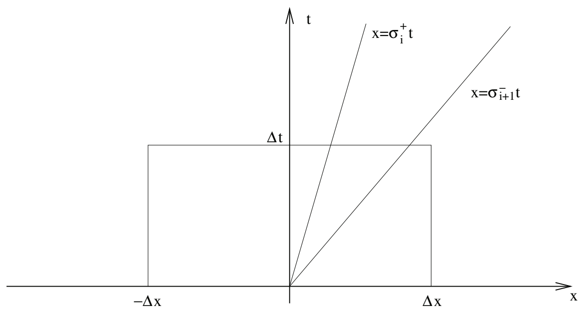

FIG. 2.1. *Representation of a part of the Riemann solution. Note that due to CFL condition* (2.15)*, lines* $x = \sigma_i^{\pm} t$ *never intersect the vertical boundary of the cell* $[-\Delta x,\Delta x]\times[0,\Delta t]$.

states, from (2.13) and (2.14) one has

$$W_G(t,x) = U_i + tg(I_C,U_i),$$

and

$$I_G(t,x,\omega,\nu) = I_i(\omega,\nu) + ct\mathcal{S}(I_C,U_i).$$

Then for $i = 1,2$,

$$\partial_t W_G + \partial_x f(W_G) - g(I_G,W_G) = g(I_C,U_i) - g(I_G,W_G).$$

Multiplying by ϕ and integrating by parts, we find

$$\int_0^{\Delta t}\int_{\sigma_i^+ t}^{\sigma_{i+1}^- t} M(t,x)\ dx\ dt = \int_{\sigma_i^+\Delta t}^{\sigma_{i+1}^-\Delta t}\phi(\Delta t,x)W_G(\Delta t,x)\ dx$$
$$+\int_0^{\Delta t}\left\{f(W_G(\Delta t,\sigma_{i+1}^-\Delta t)\phi(\Delta t,\sigma_{i+1}^-\Delta t) - f(W_G(\Delta t,\sigma_i^+\Delta t)\phi(\Delta t,\sigma_i^+\Delta t)\right\}\ dt$$
$$-\int_0^{\Delta t}\left\{\sigma_{i+1}^-\phi(t,\sigma_{i+1}^- t)W_G(t,\sigma_{i+1}^- t) - \sigma_i^+\phi(t,\sigma_i^+ t)W_G(t,\sigma_i^+ t)\right\}\ dt$$
$$-\int_0^{\Delta t}\int_{\sigma_i^+ t}^{\sigma_{i+1}^- t}\left(g(I_C(t),U_i) - g(I_G,W_G)\right)\phi(t,x)\ dx\ dt.$$

Applying Lemma 1.2, we have

$$\begin{aligned}|g(I_C,U_i)-g(I_G,W_G)| &\leqslant h_2(|U_i|,|W_G|,\|I_C\|_{L^1_{\omega,\nu}([-1,1]\times\mathbb{R}^+)},\|I_G\|_{L^1_{\omega,\nu}([-1,1]\times\mathbb{R}^+)})\\ &\quad\times\Big[t|g(I_C,U_i)|+ct\|\mathcal{S}(I_C,U_i)\|_{L^1_{\omega,\nu}([-1,1]\times\mathbb{R}^+)}\Big]\\ &\leqslant th_2(|U_i|,|W_G|,\|I_C\|_{L^1_{\omega,\nu}([-1,1]\times\mathbb{R}^+)},\|I_G\|_{L^1_{\omega,\nu}([-1,1]\times\mathbb{R}^+)})\\ &\quad\times\Big[h_1(|U_i|,\|I_C\|_{L^1_{\omega,\nu}([-1,1]\times\mathbb{R}^+)})+ch_0(|U_i|,\|I_C\|_{L^1_{\omega,\nu}([-1,1]\times\mathbb{R}^+)})\Big]\\ &\leqslant Ct,\end{aligned}$$

where C depends only on $\mathcal{U}$, h_0, h_1, and h_2. Hence, since σ_i^+ and σ_i^- are bounded, according to (2.15),

$$\left|\int_0^{\Delta t}\int_{\sigma_i^+ t}^{\sigma_{i+1}^- t}(g(I_C,U_i)-g(I_G,W_G))\,\phi(t,x)\;dx\;dt\right|\leqslant C\,(\Delta t)^3\,\|\phi\|_{C^0},$$

where C depends only on g and $\mathcal{U}$. Thus, we finally get

$$\begin{aligned}\int_0^{\Delta t}\int_{\sigma_i^+ t}^{\sigma_{i+1}^- t}M(t,x)\;dx\;dt &= \int_{\sigma_i^+\Delta t}^{\sigma_{i+1}^-\Delta t}\phi(\Delta t,x)W_G(\Delta t,x)\;dx\\ &+\int_0^{\Delta t}\left\{f(W_G(\Delta t,\sigma_{i+1}^-\Delta t)\phi(\Delta t,\sigma_{i+1}^-\Delta t)-f(W_G(\Delta t,\sigma_i^+\Delta t)\phi(\Delta t,\sigma_i^+\Delta t)\right\}\;dt\\ &-\int_0^{\Delta t}\left\{\sigma_{i+1}^-\phi(t,\sigma_{i+1}^- t)W_G(t,\sigma_{i+1}^- t)-\sigma_i^+\phi(t,\sigma_i^+ t)W_G(t,\sigma_i^+ t)\right\}\;dt\\ &+O\left((\Delta t)^3\right)\|\phi\|_X.\end{aligned}\tag{2.18}$$

At the same time

$$\begin{aligned}J_2&=\int_0^{\Delta t}\int_{-\Delta x}^{\sigma_1^- t}M(t,x)\;dx\;dt\\ &=\int_{-\Delta x}^{\sigma_1^-\Delta t}\phi(\Delta t,x)W_G(\Delta t,x)\;dx-\int_{-\Delta x}^{0}\phi(0,x)W_G(0,x)\;dx\\ &\quad+\int_0^{\Delta t}\left\{f(W_G(t,\sigma_1^- t)\phi(t,\sigma_1^- t)-f(W_G(t,-\Delta x)\phi(t,-\Delta x)\right\}\;dt\\ &\quad-\int_0^{\Delta t}\sigma_1^-\phi(t,\sigma_1^- t)W_G(t,\sigma_1^- t)\;dt\\ &\quad-\int_0^{\Delta t}\int_{-\Delta x}^{\sigma_1^- t}(g(I_C,U_i)-g(I_G,W_G))\,\phi(t,x)\;dx\;dt.\end{aligned}$$

Using Lemma 1.2 again, the last two terms are bounded by $O\Big((\Delta t)^3+(\Delta t)^2\Delta x\Big)\|\phi\|_X$. Hence we get

$$\begin{aligned}&\int_0^{\Delta t}\int_{-\Delta x}^{\sigma_1^- t}M(t,x)\;dx\;dt\\ &=\int_{-\Delta x}^{\sigma_1^-\Delta t}\phi(\Delta t,x)W_G(\Delta t,x)\;dx-\int_{-\Delta x}^{0}\phi(0,x)W_G(0,x)\;dx\end{aligned}$$

$$+\int_0^{\Delta t} \left\{ f(W_G(t,\sigma_1^- t)\phi(t,\sigma_1^- t) - f(W_G(t,-\Delta x)\phi(t,-\Delta x) \right\} \, dt$$
$$-\int_0^{\Delta t} \sigma_1^- \phi(t,\sigma_1^- t) W_G(t,\sigma_1^- t) \, dt + O\left((\Delta t)^3 + (\Delta t)^2 \Delta x \right) \|\phi\|_X. \tag{2.19}$$

Finally,

$$J_3 = \int_0^{\Delta t} \int_{\sigma_3^+ t}^{\Delta x} M(t,x) \, dx \, dt = \int_{\sigma_3^+ \Delta t}^{\Delta x} \phi(\Delta t,x) W_G(\Delta t,x) \, dx - \int_0^{\Delta x} \phi(0,x) W_G(0,x) \, dx$$
$$+\int_0^{\Delta t} \left\{ f(W_G(t,\Delta x)\phi(t,\Delta x) - f(W_G(t,\sigma_3^+ t)\phi(t,\sigma_3^+ t) \right\} \, dt$$
$$-\int_0^{\Delta t} \sigma_3^+ \phi(t,\sigma_3^+ t) W_G(t,\sigma_3^+ t) \, dt$$
$$-\int_0^{\Delta t} \int_{\sigma_3^+ t}^{\Delta x} \left(g(I_C,U_i) - g(I_G,W_G) \right) \phi(t,x) \, dx \, dt,$$

and, with the help of Lemma 1.2, the last two terms are bounded by $O\left((\Delta t)^3 + (\Delta t)^2 \Delta x \right) \|\phi\|_X$. So

$$\int_0^{\Delta t} \int_{\sigma_3^+ t}^{\Delta x} M(t,x) \, dx \, dt = \int_{\sigma_3^+ \Delta t}^{\Delta x} \phi(\Delta t,x) W_G(\Delta t,x) \, dx - \int_0^{\Delta x} \phi(0,x) W_G(0,x) \, dx$$
$$+\int_0^{\Delta t} \left\{ f(W_G(t,\Delta x)\phi(t,\Delta x) - f(W_G(t,\sigma_3^+ t)\phi(t,\sigma_3^+ t) \right\} \, dt$$
$$-\int_0^{\Delta t} \sigma_3^+ \phi(t,\sigma_3^+ t) W_G(t,\sigma_3^+ t) \, dt + O\left((\Delta t)^3 + (\Delta t)^2 \Delta x \right) \|\phi\|_X. \tag{2.20}$$

Next, we assume that $W_C(t,x)$ consists of aa i-rarefaction wave in the fan region $\left\{ (x,t) \, : \, \sigma_i^- t < x < \sigma_i^+ t \right\}$, so that W_G becomes

$$W_G(t,x) = \tilde{W}_C(\zeta) + t g(I_C(t), W_C(t)),$$

where $\zeta = \frac{x}{t}$ and $\tilde{W}_C$ is the i-rarefaction wave. Following [12], we have

$$\partial_t W_G + \partial_x (f(W_G)) - g(I_G,W_G) = \frac{1}{t} \left(Df(W_G) - \frac{x}{t} \right) \tilde{W}_C' \left(\frac{x}{t} \right) + g(I_C,W_C) - g(I_G,W_G).$$

Hence, using the fact that $\left(Df(\tilde{W}_C(\xi)) - \xi \right) \tilde{W}_C'(\xi) = 0$,

$$\partial_t W_G + \partial_x (f(W_G)) - g(I_G,W_G) = (Df(W_G) - Df(W_C)) \partial_x W_C$$
$$+ g(I_C,W_C) - g(I_G,W_G).$$

All terms in the right-hand side of the above equality are dealt with exactly as before, except for the first one, for which we use the fact that $\|f\|_{C^2(\mathcal{U})}$ is bounded, and

$$|(Df(W_G) - Df(W_C)) \partial_x W_C| \leqslant \|f\|_{C^2(\mathcal{U})} \Delta t |g(I_C,W_C)| |\partial_x W_C|$$
$$\leqslant C \Delta t |\partial_x W_C|,$$

where C depends only on f, g, and $\mathcal{U}$. Hence,

$$\left|\int_0^{\Delta t}\int_{\sigma_i^+\Delta t}^{\sigma_i^-\Delta t}(Df(W_G)-Df(W_C))\partial_x W_C\phi\right|\leqslant C\Delta t\int_0^{\Delta t}TV(W_C)dt\|\phi\|_X.$$

For a rarefaction wave, the total variation is non-increasing, so the above term is bounded by $O(\Delta t^2|U_R-U_L|)\|\phi\|_X$. Hence, we have

$$\begin{aligned}&\int_0^{\Delta t}\int_{\sigma_i^- t}^{\sigma_i^+ t}M(t,x)\;dx\;dt\\ &=\int_{\sigma_i^+\Delta t}^{\sigma_i^-\Delta t}\phi(\Delta t,x)W_G(\Delta t,x)\;dx\\ &-\int_0^{\Delta t}\left\{f(W_G(t,\sigma_i^- t)\phi(t,\sigma_i^- t)-f(W_G(t,\sigma_i^+ t)\phi(t,\sigma_i^+ t)\right\}\;dt\\ &+\int_0^{\Delta t}\left\{\sigma_i^-\phi(t,\sigma_{i+1}^- t)W_G(t,\sigma_i^- t)-\sigma_i^+\phi(t,\sigma_i^+ t)W_G(t,\sigma_i^+ t)\right\}\;dt\\ &+O\left(\Delta t^2\left(\Delta t+|U_R-U_L|\right)\right)\|\phi\|_X.\end{aligned}\tag{2.21}$$

Collecting all the previous estimates (2.18), (2.19), (2.20), and (2.21), proves (2.16).

Defining, in the same spirit, the auxiliary function $N(t,x):=I_G\partial_t\psi+c\omega I_G\partial_x\psi+c\mathcal{S}(I_G,W_G)\psi$, we have the following decomposition of $\int\int N\;dx\;dt$ on the elementary cell $[0,\Delta t]\times[-\Delta x,\Delta x]$:

$$\int_0^{\Delta t}\int_{-\Delta x}^{\Delta x}N(t,x)\;dx\;dt=\int_0^{\Delta t}\int_{-\Delta x}^{\sigma t}M(t,x)\;dx\;dt+\int_0^{\Delta t}\int_{\sigma t}^{\Delta x}M(t,x)\;dx\;dt,$$

where $\sigma=c\omega$.

Since $I_G(t,x,\omega,\nu)=I_C(\omega,\nu)+ct\mathcal{S}(I_C,W_C)$, we get

$$\partial_t I_G+c\omega\partial_x I_G-c\mathcal{S}(I_G,W_G)=c\mathcal{S}(I_C,U_i)-c\mathcal{S}(I_G,W_G).$$

Multiplying by ψ and integrating by parts, we find

$$\begin{aligned}\int_0^{\Delta t}\int_{-\Delta x}^{\sigma t}N(t,x)\;dx\;dt&=\int_{-\Delta x}^{\sigma\Delta t}\psi(\Delta t,x)I_G(\Delta t,x)\;dx-\int_{-\Delta x}^{0}\psi(0,x)I_G(0,x)\;dx\\ &+\int_0^{\Delta t}\sigma\left\{I_G(t,\sigma t)\psi(t,\sigma t)-I_G(t,-\Delta x)\psi(t,-\Delta x)\right\}\;dt\\ &-c\int_0^{\Delta t}\int_{-\Delta x}^{\sigma t}(\mathcal{S}(I_C,U_i)-\mathcal{S}(I_G,W_G))\psi(t,x)\;dx\;dt.\end{aligned}$$

Using the previous arguments, the last two terms are bounded by $O\Big((\Delta t)^3+(\Delta t)^2|I_R-I_L|\Big)\|\psi\|_Y$, and we get

$$\begin{aligned}\int_0^{\Delta t}\int_{-\Delta x}^{\sigma t}N(t,x)\;dx\;dt&=\int_{-\Delta x}^{\sigma\Delta t}\psi(\Delta t,x)I_G(\Delta t,x)\;dx-\int_{-\Delta x}^{0}\psi(0,x)I_G(0,x)\;dx\\ &+\int_0^{\Delta t}\sigma\left\{I_G(t,\sigma t)\psi(t,\sigma t)-I_G(t,-\Delta x)\psi(t,-\Delta x)\right\}\;dt\end{aligned}$$

$$-\int_0^{\Delta t} \sigma\phi(t,\sigma t) W_G(t,\sigma t)\, dt + O\left((\Delta t)^3 + (\Delta t)^2 \Delta x\right) \|\psi\|_Y. \tag{2.22}$$

Finally,

$$\begin{aligned}\int_0^{\Delta t}\int_{\sigma t}^{\Delta x} N(t,x)\, dx\, dt = &\int_{\sigma\Delta t}^{\Delta x} \psi(\Delta t,x) I_G(\Delta t,x)\, dx - \int_0^{\Delta x} \psi(0,x) I_G(0,x)\, dx \\ &+ \int_0^{\Delta t} \sigma\{I_G(t,\Delta x)\psi(t,\Delta x) - I_G(t,\sigma t)\psi(t,\sigma t)\}\, dt \\ &- \int_0^{\Delta t}\int_{\sigma t}^{\Delta x} (\mathcal{S}(I_C,U_i) - \mathcal{S}(I_G,W_G))\psi(t,x)\, dx\, dt,\end{aligned}$$

and

$$\left|\int_0^{\Delta t}\int_{\sigma t}^{\Delta x} (\mathcal{S}(I_C,U_i) - \mathcal{S}(I_G,W_G))\psi(t,x)\, dx\, dt\right| = O\left((\Delta t)^3 + (\Delta t)^2 |I_R - I_L|\right) \|\psi\|_X,$$

so

$$\begin{aligned}\int_0^{\Delta t}\int_{\sigma t}^{\Delta x} N(t,x)\, dx\, dt = &\int_{\sigma\Delta t}^{\Delta x} \psi(\Delta t,x) I_G(\Delta t,x)\, dx - \int_0^{\Delta x} \psi(0,x) I_G(0,x)\, dx \\ &+ \int_0^{\Delta t} \sigma\{I_G(t,\Delta x)\psi(t,\Delta x) - I_G(t,\sigma t)\psi(t,\sigma t)\}\, dt \\ &- \int_0^{\Delta t} \sigma\phi(t,\sigma t) W_G(t,\sigma t)\, dt + O\left((\Delta t)^3 + (\Delta t)^2 |I_R - I_L|\right) \|\psi\|_Y.\end{aligned} \tag{2.23}$$

Collecting estimates (2.22) and (2.23) proves (2.17) ∎

3. Interaction of waves

Following [9] (see also [24] for a complete presentation), we define the wave interaction potential involving two solutions of the classical Riemann problem of respective strengths $\alpha := (\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ and $\beta := (\beta_1,\beta_2,\beta_3,\beta_4)$, by

$$D(\alpha,\beta) := \sum_{i>j} |\alpha_i||\beta_j| + \sum_k Q_k(\alpha_k,\beta_k), \tag{3.1}$$

where

$$Q_i(\alpha_i,\beta_i) = |\alpha_i||\beta_i| - \alpha_i^+\beta_i^+,$$

with $z^+ = \sup(z,0)$, if the i-field is genuinely non linear, and

$$Q_i(\alpha_i,\beta_i) = 0,$$

if the i-field is linearly degenerate.

The following result due to Glimm [9] describes wave interaction estimates.

Proposition 3.1 (Glimm). *Let U_L, U_M, and U_R be three constant states in $\mathcal{U}$, and let (α,β,γ) be the strengths of the solutions of the Riemann problems $CRU(x_0,t_0;U_L,U_M)$, $CRU(x_0,t_0;U_M,U_R)$, and $CRU(x_0,t_0;U_L,U_R)$.*

The following properties hold:

1.

$$|\gamma-\alpha-\beta|=O(1)D(\alpha,\beta). \tag{3.2}$$

2. If V_L and V_R are two other constant states in $\mathcal{U}$ and δ is the strength of the solution of the Riemann problem $CRU(x_0,t_0;V_L,V_M)$,

$$D(\gamma,\delta)=D(\alpha,\delta)+D(\beta,\delta)+O(1)|\delta|D(\alpha,\beta). \tag{3.3}$$

3. The mapping $(U_L,U_R,t_0,x_0)\to\alpha:\mathcal{U}\times\mathcal{U}\times\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}^3$ is a C^2 function of its arguments. Moreover, for (U_L,U_R) and (U'_L,U'_R) in $\mathcal{U}\times\mathcal{U}$ and (t_0,x_0) and (t'_0,x'_0) in $\mathbb{R}_+\times\mathbb{R}$, one has

$$|\alpha'-\alpha|=O(1)\{|\alpha|(|U'_L-U_L|+|U'_R-U_R|)+(|U'_R-U'_L-(U_R-U_L)|)\}, \tag{3.4}$$

where $\alpha=\alpha(U_L,U_R;t_0,x_0)$ and $\alpha'=\alpha(U'_L,U'_R;t'_0,x'_0)$.

4. For (U_L,U_R), (V_L,V_R), (U'_l,U'_R), and (V'_L,V'_R) in $\mathcal{U}\times\mathcal{U}$ and for (t_1,x_1), (t_2,x_2), (t'_1,x'_1), and (t'_2,x'_2) in $\mathbb{R}_+\times\mathbb{R}$, one has

$$\begin{aligned}D(\alpha',\beta')&=D(\alpha,\beta)+O(1)|\alpha||(V'_R-V'_L)+(V_R-V_L)|\\&+O(1)|\beta||(U'_R-U'_L)+(U_R-U_L)|\\&+O(1)|\alpha||\beta|(|U'_L-U_L|+|U'_R-U_R|+|V'_L-V_L|+|V'_R-V_R|)\\&+O(1)|U'_R-U'_L-(U_R-U_L)|\cdot|V'_R-V'_L-(V_R-V_L)|,\end{aligned} \tag{3.5}$$

where $\alpha=\alpha(U_L,U_R;t_1,x_1)$, $\alpha'=\alpha(U'_L,U'_R;t'_1,x'_1)$, $\beta=\beta(V_L,V_R;t_2,x_2)$, and $\beta'=\beta(V'_L,V'_R;t'_2,x'_2)$.

5. Let U_L, U_L+u_L, U_R, U_R+u_R, and U_M be constant states in $\mathcal{U}$, and let (α,β,γ) be the strengths of the solutions of the Riemann problems $CRU(x_0-\Delta x,t_0;U_L,U_M)$, $CRU(x_0+\Delta x,t_0;U_M,U_R)$, and $CRU(x_0,t_0+\Delta t;U_L+u_L,U_R+u_R)$. We have

$$|\gamma|=|\alpha|+|\beta|+O(1)D(\alpha,\beta)+O(1)(|\alpha|+|\beta|)(|u_L|+|u_R|)+O(1)(|u_R-u_L|). \tag{3.6}$$

6. For the previous α, β, and γ, for (V_L,V_R) in $\mathcal{U}\times\mathcal{U}$, and for (t_1,x_1) in $\mathbb{R}_+\times\mathbb{R}$, one has

$$\begin{aligned}D(\gamma,\delta)&=D(\alpha,\delta)+D(\beta,\delta)+O(1)|\delta|D(\alpha,\beta)\\&+O(1)|\delta||u_R-u_L|+O(1)|\delta|(|\alpha|+|\beta|)(|u_L|+|u_R|).\end{aligned} \tag{3.7}$$

The first four items of the previous result were proved by Glimm [9] for genuinely nonlinear fields and extended by Tai Ping Liu [24] for linearly degenerate fields. The two last items were proved by Hong and LeFloch [12] in a more general context (in [12], the right-hand side f depends on x and t).

REMARK 3.2. Let I_L, I_M, and I_R be three constant (radiative) states. The strengths of the solutions of the Riemann problems $CRI(x_0,t_0;I_L,I_M)$, $CRI(x_0,t_0;I_M,I_R)$, and $CRI(x_0,t_0;I_L,I_R)$ satisfy (non interacting framework)

$$\gamma_r=\alpha_r+\beta_r. \tag{3.8}$$

Then we have the following consequence of Proposition 3.1.

PROPOSITION 3.3. *Let us denote by u_L and u_R the numbers*

$$u_L := -\Delta t \; g(I_L, U_L) \quad \textit{for } x < 0, \tag{3.9}$$

and

$$u_R := -\Delta t \; g(I_R, U_R) \quad \textit{for } x > 0. \tag{3.10}$$

The following properties hold (α, β, and γ are defined in Proposition 3.1, item 5, and $\overline{\sigma_a}$ and $\overline{\sigma_s}$ are defined in Lemma 1.2):

$$\begin{aligned} |\gamma| = {} & |\alpha| + |\beta| + O(1) D(\alpha,\beta) + O(1)(|\alpha| + |\beta|)(\overline{\sigma_a}(t_0) + \overline{\sigma_s}(t_0))\Delta t \\ & \times \Big(|U_L| + |U_R| + \|I_L\|_{L^1_{\omega,\nu}([-1,1]\times\mathbb{R}^+)} \|I_R\|_{L^1_{\omega,\nu}([-1,1]\times\mathbb{R}^+)} \Big) \\ & + O(1)(\overline{\sigma_a}(t_0) + \overline{\sigma_s}(t_0))\Delta t \Big(|U_L - U_R| + \|I_L - I_R\|_{L^1_{\omega,\nu}([-1,1]\times\mathbb{R}^+)} \Big), \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} D(\gamma,\delta) = {} & D(\alpha,\delta) + D(\beta,\delta) + O(1)|\delta| D(\alpha,\beta) + O(1)|\delta|(|\alpha| + |\beta|)(\overline{\sigma_a}(t_0) + \overline{\sigma_s}(t_0))\Delta t \\ & \times \Big(|U_L| + |U_R| + \|I_L\|_{L^1_{\omega,\nu}([-1,1]\times\mathbb{R}^+)} \|I_R\|_{L^1_{\omega,\nu}([-1,1]\times\mathbb{R}^+)} \Big) \\ & + O(1)(\overline{\sigma_a}(t_0) + \overline{\sigma_s}(t_0))\Delta t \Big(|U_L - U_R| + \|I_L - I_R\|_{L^1_{\omega,\nu}([-1,1]\times\mathbb{R}^+)} \Big). \end{aligned} \tag{3.12}$$

Proof. Applying (3.6), we first get

$$|\gamma| = |\alpha| + |\beta| + O(1)(|\alpha| + |\beta|)(|u_L| + |u_R|) + O(1)|u_L - u_R|. \tag{3.13}$$

Now let us estimate the terms $|u_L| + |u_R|$. For this purpose, we apply (1.26) of Lemma 1.2. This gives us

$$|u_L| \leqslant \Delta t |g(I_L, U_L)| \leqslant \Delta t h_1(|U_L)|, \|I_L\|_{L^1_{\omega,\nu}(\mathbb{R}^+\times[-1,1])})(\overline{\sigma_a}(t_0) + \overline{\sigma_s}(t_0)). \tag{3.14}$$

Similarly,

$$|u_R| \leqslant \Delta t h_1(|U_R)|, \|I_R\|_{L^1_{\omega,\nu}(\mathbb{R}^+\times[-1,1])})(\overline{\sigma_a}(t_0) + \overline{\sigma_s}(t_0)). \tag{3.15}$$

Next, we bound the term $|u_L - u_R|$. For this purpose, we apply (1.27) of Lemma 1.2, finding

$$\begin{aligned} |u_L - u_R| & = \Delta t |g(I_L, U_L) - g(I_R, U_R)| \\ & \leqslant h_2 \Big(|U_L|, |U_R|, \|I_L\|_{L^1_{\omega,\nu}([-1,1]\times\mathbb{R}^+)}, \|I_R\|_{L^1_{\omega,\nu}([-1,1]\times\mathbb{R}^+)} \Big) \\ & \qquad \times \Big[|U_L - U_R| + \|I_L - I_R\|_{L^1_{\omega,\nu}([-1,1]\times\mathbb{R}^+)} \Big] (\overline{\sigma_a}(t) + \overline{\sigma_s}(t)). \end{aligned} \tag{3.16}$$

Inserting estimates (3.14), (3.15), and (3.16) into (3.13), we find (3.11). Estimate (3.12) is obtained in the same way. □

4. The generalized Glimm scheme

We approximate the problem (1.15)–(1.17) by using a two-step scheme relying on Glimm's method as follows.

Let $\mathcal{U}$ be a small neighborhood of $(\rho_\infty, v_\infty, S_\infty, I_\infty)$ in $\mathbb{R}^4$ (see Definition 2.2). Given space and time steps Δt and Δx satisfying the mixed CFL condition

$$\frac{\Delta t}{\Delta x}\ \max\left\{c, \max_{i=1,2,3}\left(\sup_{u\in\mathcal{U}}|\lambda_i(u)|\right)\right\}\leqslant 1 \tag{4.1}$$

and a sequence $a\equiv(a_\ell)_{\ell\geqslant 0}$ of real numbers equidistributed in the interval $(-1,1)$, let us define (see [9, 22]) the discretization of the (t,x) half-plane by the *mesh points*

$$A_n^\ell = (t_\ell, x_n)\equiv(\ell\Delta t, n\Delta x)\quad \text{for } \ell\in\mathbb{N},\ n\in\mathbb{Z},\ n+\ell \text{ odd}. \tag{4.2}$$

Connecting nearest-neighbor mesh points by segments defines a partition of the plane into diamond-like regions.

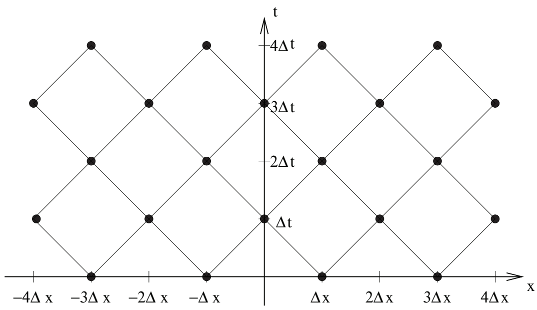

FIG. 4.1. *The mesh points defined by* (4.2).

DEFINITION 4.1. *Consider a mesh* $\left(A_n^\ell\right)_{\ell\in\mathbb{N},n\in\mathbb{Z}}$ *as defined by* (4.2). *We call a* mesh curve L *an unbounded piecewise linear curve lying on diamond boundaries of* $\left(A_n^\ell\right)_{\ell\in\mathbb{N},n\in\mathbb{Z}}$ *which satisfies the following: for any point* $A_n^\ell\in L$*, the next point is either* $A_{n+1}^{\ell-1}$ *or* $A_{n+1}^{\ell+1}$ *(see Figure 4.2).* Any mesh curve L divides the half-plane $t\geqslant 0$ into two

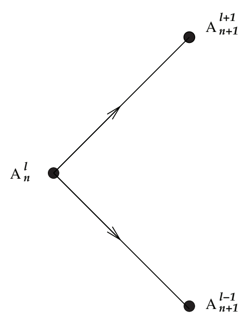

FIG. 4.2. *Given a point in a mesh curve, the next point should be one of its two nearest right neighbors.*

components, L^+ and L, where L^- contains the axis $t=0$. We define an order on the mesh curves: $L_1 > L_2$ if any point of L_1 either is on L_2 or is contained in L_2^+. We say that L_1 is an *immediate successor* of L_2 if $L_1 > L_2$ and if each mesh point of L_1 except one is on L_2.

Let us approximate the initial data (1.16) by

$$U_{\Delta x}(x,0) = U_0(n\Delta x), \text{ for } x \in [(n-1)\Delta x,(n+1)\Delta x], \quad n \text{ odd}, \tag{4.3}$$

which allows us to construct, for $0 < t \leqslant \Delta t$, a solution W_C of each classical Riemann problem with initial data $U_{\Delta x}$ and, due to the CFL condition, to define the solution of the Cauchy problem (2.11) in the strip $\{(x,t) \in \mathbb{R}\times]0,\Delta t[\}$.

Approximating the initial data (1.17) as well by

$$I_{\Delta x}(x,0,\omega,\nu) = I_0(n\Delta x,\omega,\nu) \text{ for } x \in [(n-1)\Delta x,(n+1)\Delta x], \quad n \text{ odd}, \tag{4.4}$$

allows us to construct, for $0 < t \leqslant \Delta t$, a solution I_C of each classical Riemann problem with initial data $I_{\Delta x}$ and, due to the CFL condition, to define the solution of the Cauchy problem (2.12) in the strip $\{(x,t) \in \mathbb{R}\times]0,\Delta t[\}$.

Then using (2.13) and (2.14), we can construct, for $0 < t \leqslant \Delta t$, the approximate solutions W_G and I_G for each generalized Riemann problem (2.10) with the previous initial data $U_{\Delta x}$ and $I_{\Delta x}$, defining the solution $(I_{\Delta x}(x,t,\omega,\nu),U_{\Delta x}(x,t))$ in the strip $\{(x,t) \in \mathbb{R}\times]0,\Delta t[\}$.

Supposing now that $(I_{\Delta x}(x,t,\omega,\nu),U_{\Delta x}(x,t))$ is well-defined in the strip $\{(x,t) \in \mathbb{R}\times[(\ell-1)\Delta t,\ell\Delta t[\}$. We set

$$U_{\Delta x}(x,\ell\Delta t) := U_{\Delta x}((n+a_\ell)\Delta x,\ell\Delta t - 0), \tag{4.5}$$

$$I_{\Delta x}(x,\ell\Delta t,\omega,\nu) := I_{\Delta x}((n+a_\ell)\Delta x,\ell\Delta t - 0,\omega,\nu), \tag{4.6}$$

for $x \in [(n-1)\Delta x,(n+1)\Delta x]$, $n+\ell$ odd, $\omega \in [-1,1]$, and $\nu \in \mathbb{R}_+$.

Then we solve, in the same stroke, first the corresponding classical Riemann problem with initial data $U_{\Delta x}(x,\ell\Delta t)$, the transport problem with initial data $I_{\Delta x}(x,\ell\Delta t,\omega,\nu)$, and the corresponding generalized problems with initial data $U_{\Delta x}(x,\ell\Delta t)$ and $I_{\Delta x}(x,\ell\Delta t,\omega,\nu)$ which allows to get $(I_{\Delta x}(x,t,\omega,\nu),U_{\Delta x}(x,t))$ well defined in the strip $\{(x,t) \in \mathbb{R}\times[\ell\Delta t,(\ell+1)\Delta t]\}$. This process inductively defines $(I_{\Delta x}(x,t,\omega,\nu),U_{\Delta x}(x,t))$ in the half-plane $\{(x,t) \in \mathbb{R}\times\mathbb{R}_+\}$.

Using the simplified notation $U_{\ell,n} := U_{\Delta x}(\ell\Delta t,n\Delta x)$ for $\ell+n$ odd and $I_{\ell,n} := I_{\Delta x}(\ell\Delta t,n\Delta x,\omega,\nu)$ for $\ell+n$ odd, we have, after (2.13) and (2.14), the value of the solution of the generalized Riemann (GR) problem

$$U_{\ell,n} = V_{\ell,n} + \Delta t \; g(\mathcal{I}_{\ell,n};V_{\ell,n}),$$

and

$$I_{\ell,n} = \mathcal{I}_{\ell,n} + c\Delta t \; \mathcal{S}(\mathcal{I}_{\ell,n};V_{\ell,n}),$$

where $V_{\ell,n}$ is the value of the solution of the classical Riemann (CR) problem (2.11):

$$CRU(n\Delta x,(\ell-1)\Delta t;U_{\ell-1,n-1},U_{\ell-1,n+1}),$$

taken at the sample point $(\ell\Delta t,(n+a_\ell)\Delta x)$, i.e.

$$V_{\ell,n} := W_C\left(a_\ell\,\frac{\Delta x}{\Delta t},n\Delta x,(\ell-1)\Delta t;U_{\ell-1,n-1},U_{\ell-1,n+1}\right),$$

and $\mathcal{I}_{\ell,n}$ is the value of the solution of the problem

$$CRI(n\Delta x,(\ell-1)\Delta t;I_{\ell-1,n-1},I_{\ell-1,n+1}),$$

taken at the same sample point $(\ell\Delta t,(n+a_\ell)\Delta x)$, i.e.

$$\mathcal{I}_{\ell,n}:=I_C\left(a_\ell\,\frac{\Delta x}{\Delta t},(\ell-1)\Delta t,n\Delta x;I_{\ell-1,n-1},I_{\ell-1,n+1}\right).$$

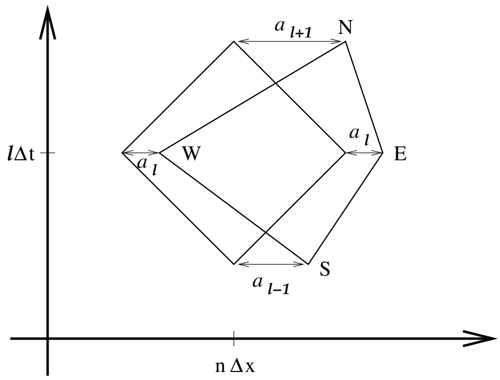

FIG. 4.3. *The diamond region $D_{\ell,n}$.*

We now define, as in [31], the *diamond* region $D_{\ell,n}$ as the convex hull of the points $S:=((\ell-1)\Delta t,(n+a_{\ell-1})\Delta x)$, $W:=(\ell\Delta t,(n-1+a_\ell)\Delta x)$, $E:=(\ell\Delta t,(n+1+a_\ell)\Delta x)$, and $N:=((\ell+1)\Delta t,(n+a_{\ell+1})\Delta x)$, together with the "GR" values $U_S=U_{\ell-1,n}$, $U_W:=U_{\ell,n-1}$, $U_E:=U_{\ell,n+1}$, and $U_N:=U_{\ell+1,n}$ and the "CR" values $V_S=V_{\ell-1,n}$, $V_W:=V_{\ell,n-1}$, $V_E:=V_{\ell,n+1}$, and $V_N:=V_{\ell+1,n}$.

An i-wave $(U_{i-1}(t),U_i(t))$ in the approximate Riemann problem $U_{\Delta x}$ is defined as

$$(U_{i-1}(t),U_i(t))=(U_{i-1},U_i)+(t-n\Delta t)\ g(\mathcal{I}_{\ell,n},V_{\ell,n}).$$

The strength of this i-wave is not clearly defined, so we set it equal to the strength of the wave (U_{i-1},U_i) as in [12]:

$$\varepsilon_i=\varepsilon_i(U_{i-1},U_i).$$

Let J_ℓ be a mesh curve located in the strip $\{(t,x);\ \ell\Delta t\leqslant t\leqslant(\ell+1)\Delta t\}$, and let $TV(U_{\Delta x}(t,x);J_\ell)$ (resp. $TV(V_{\Delta x}(t,x);J_\ell)$) be the total variation of $U_{\Delta x}$ (resp. $V_{\Delta x}$) on J_ℓ. If an i-wave $(U_{i-1}(t),U_i(t))$ of $U_{\Delta x}(t,x)$ originating in $(\ell\Delta t,n\Delta x)$ intersects J_ℓ, then the corresponding classical $i-$wave (U_{i-1},U_i) of $V_{\Delta x}(t,x)$ also intersects J_ℓ.

Following the ideas of Hong and LeFloch, we expect that the sum of the strength of the elementary waves in $U_{\Delta x}$ which cross a mesh J can be considered as a measure equivalent to the total variation of $U_{\Delta x}$: the radiative source g will remain small at any time provided they actually are at $t=0$ and provided that the transport coefficients are smooth enough.

After formula (2.13),

$$(U_{i-1}(t),U_i(t))=(U_{i-1},U_i)+\Delta t\ g(I_{\ell,n},(U_{i-1},U_i)).$$

Then

$$|TV(U_{\Delta x}(t,x);J_\ell) - TV(V_{\Delta x}(t,x);J_\ell)| \leqslant \Delta t \sum_{n\in\mathbb{Z}} TV(g(\mathcal{I}_{\ell,n},V_{\ell,n});J_\ell). \tag{4.7}$$

Let us assume that $(U,I)\in\mathcal{U}$ are such that $TV(U)$ and $\|TV(I)\|_{L^1(\mathbb{R}_+\times[-1,1])}$ are small, i.e.

$$TV\{U\} + \|TV\{I\}\|_{L^1(\mathbb{R}_+\times[-1,1])} \leqslant \varepsilon, \tag{4.8}$$

for $\varepsilon > 0$ small enough, and that the hypotheses of Proposition 3.3 are satisfied.

Let us estimate the term $TV(g(\mathcal{I}_{\ell,n},V_{\ell,n});J_\ell)$ in the right-hand side of (4.7). As in Proposition 3.3, we must consider the two nonzero components of g.

We then have the estimates

$$TV\{g_2\} := TV\left\{\int_{\mathbb{R}_+\times[-1,1]} \mathcal{S}(I,U_i)\ d\omega\ d\nu\right\},$$

and

$$TV\{g_3\} := TV\left\{\int_{\mathbb{R}_+\times[-1,1]} \omega\mathcal{S}(I,U_i)\ d\omega\ d\nu\right\}.$$

As the transport coefficients do not depend on ω, one gets first

$$TV\{g_2\} \leqslant TV\left\{\int_{\mathbb{R}_+} \sigma_a(U_i,\nu)B(\nu,U_i)\ d\nu\right\} + TV\left\{\int_{\mathbb{R}_+\times[-1,1]} \sigma_a(U_i,\nu)I\ d\omega\ d\nu\right\},$$

so using the identity $\int_{\mathbb{R}_+} B(\nu,U_i)\ d\nu = a\vartheta_{\ell,n}^4$ where a is a pure positive constant, we have, using (1.21) and (1.22),

$$\begin{aligned} &TV\{g_2\} \\ &\leqslant a\left(4\sup_{U\in\mathcal{U}}\left(|h_a(U)|\vartheta^3\right) + \sup_{U\in\mathcal{U}}\left(|\tilde{h}_a(U)|\vartheta^4\right)\right)\cdot TV\{U_i\}\overline{\sigma_a}(t) \\ &\quad + \|TV\{I\}\|_{L^1(\mathbb{R}_+\times[-1,1])}\sup_{U\in\mathcal{U}}|h_a(U)|\overline{\sigma_a}(t) + \|I\|_{L^1(\mathbb{R}_+\times[-1,1])}\sup_{U\in\mathcal{U}}|\tilde{h}_a(U)|TV\{U_i\}\overline{\sigma_a}(t). \end{aligned}$$

At the same time,

$$\begin{aligned} TV\{g_3\} &= TV\left\{\int_{\mathbb{R}_+\times[-1,1]} (\sigma_a(V_{\ell,n},\nu) + \sigma_s(V_{\ell,n},\nu))\mathcal{I}_{\ell,n}\ d\omega\ d\nu\right\} \\ &\leqslant \left(\sup_{U\in\mathcal{U}}|h_a(U)|\,\|TV\{\mathcal{I}_{\ell,n}\}\|_{L^1(\mathbb{R}_+\times[-1,1])} + \sup_{U\in\mathcal{U}}|\tilde{h}_a(U)|\|I\|_{L^1(\mathbb{R}_+\times[-1,1])}TV(V_{\ell,n})\right)\overline{\sigma_a}(t) \\ &+ \left(\sup_{U\in\mathcal{U}}|h_s(U)|\,\|TV\{\mathcal{I}_{\ell,n}\}\|_{L^1(\mathbb{R}_+\times[-1,1])} + \sup_{U\in\mathcal{U}}|\tilde{h}_s(U)|\|I\|_{L^1(\mathbb{R}_+\times[-1,1])}TV(V_{\ell,n})\right)\overline{\sigma_s}(t). \end{aligned}$$

When (U_{i-1},U_i) is a rarefaction wave, the same computation shows that a similar estimate holds. Hence, summing over all the elementary waves crossing J_ℓ, plugging into (4.7), and taking into account the CFL condition (4.1), we get

$$
\begin{aligned}
&|TV(U_{\Delta x};J_\ell)-TV(V_{\Delta x};J_\ell)| \\
&\leqslant O(\Delta t)\left(TV\{U_{\Delta x};J_\ell\}+TV\left\{\|I\|_{L^1(\mathbb{R}_+\times[-1,1])}\right\}\right).
\end{aligned}
$$

After (4.8), the right-hand side is small, and we conclude that the total variations of $U_{\Delta x}$ and $V_{\Delta x}$ are equivalent on any mesh curve J_k.

We define the strength $\varepsilon_{in}(D_{\ell,n})$ of waves coming into $D_{\ell,n}$ for $\ell+n$ even by

$$
\varepsilon_{in}(D_{\ell,n})=|\varepsilon(V_W,U_S;(\ell-1)\Delta t,(n-1)\Delta x)|+|\varepsilon(U_S,V_E;(\ell-1)\Delta t,(n+1)\Delta x)|, \quad (4.9)
$$

and as well the strength $\varepsilon_{out}(D_{\ell,n})$ of waves leaving $D_{\ell,n}$ by

$$
\varepsilon_{out}(D_{\ell,n})=|\varepsilon(U_W,V_N;\ell\Delta t,n\Delta x)|+|\varepsilon(V_N,U_E;\ell\Delta t,n\Delta x)|=|\varepsilon(U_W,U_E;\ell\Delta t,n\Delta x)|. \quad (4.10)
$$

We define analogous quantities for the radiative waves coming into $D_{\ell,n}$ for $\ell+n$ even by

$$
\begin{aligned}
\varepsilon_{r,in}(D_{\ell,n})=&\int_0^\infty\int_0^1|\varepsilon_r(\mathcal{I}_W,I_S;(\ell-1)\Delta t,(n-1)\Delta x)|\,d\omega\,d\nu \\
&+\int_0^\infty\int_{-1}^0|\varepsilon_r(I_S,\mathcal{I}_E;(\ell-1)\Delta t,(n+1)\Delta x)|\,d\omega\,d\nu, \qquad (4.11)
\end{aligned}
$$

and as well the strength $\varepsilon_{r,out}(D_{\ell,n})$ of waves leaving $D_{\ell,n}$ by

$$
\begin{aligned}
\varepsilon_{r,out}(D_{\ell,n})=&\int_0^\infty\int_{-1}^0|\varepsilon_r(I_W,\mathcal{I}_N;\ell\Delta t,n\Delta x)|\,d\omega\,d\nu \\
&+\int_0^\infty\int_0^1|\varepsilon_r(\mathcal{I}_N,I_E;\ell\Delta t,n\Delta x)|\,d\omega\,d\nu. \qquad (4.12)
\end{aligned}
$$

We also define the potential $\mathcal{P}(D_{\ell,n})$ of interaction in the diamond $D_{\ell,n}$ for $\ell+n$ even by

$$
\mathcal{P}(D_{\ell,n}):=D(\varepsilon(V_W,U_S,(\ell-1)\Delta t,(n-1)\Delta x),\varepsilon(U_S,V_E,(\ell-1)\Delta t,(n+1)\Delta x)),
$$

where $D(\cdot,\cdot)$ is defined by (3.1).

Given a mesh curve J, one can split the waves crossing J into two groups: the "left-incoming" waves (or type I waves) $(V_{\ell,n-1},U_{\ell-1,n})$ crossing the WS-type segments of J, and the "right-outgoing" waves (or type II waves) $(U_{\ell-1,n},V_{\ell,n+1})$ crossing the SE-type segments of J. Then we define the linear functional

$$
\begin{aligned}
\mathcal{L}(J):=&\sum_{type\ I\ waves}|\varepsilon(V_{\ell,n-1},U_{\ell-1,n};(\ell-1)\Delta t,(n-1)\Delta x)| \\
&+\sum_{type\ II\ waves}|\varepsilon(U_{\ell-1,n},V_{\ell,n+1};(\ell-1)\Delta t,(n+1)\Delta x)| \\
&+\sum_{type\ I\ rad.\ waves}\int_0^\infty\int_0^1|\varepsilon_r(\mathcal{I}_{\ell,n-1},I_{\ell-1,n};(\ell-1)\Delta t,(n-1)\Delta x)|\,d\omega\,d\nu \\
&+\sum_{type\ II\ rad.\ waves}\int_0^\infty\int_{-1}^0|\varepsilon_r(I_{\ell-1,n},\mathcal{I}_{\ell,n-1};(\ell-1)\Delta t,(n+1)\Delta x)|\,d\omega\,d\nu. \qquad (4.13)
\end{aligned}
$$

Next, applying Glimm's strategy [9] (see also [28]), we correct $\mathcal{L}(J)$ by a quadratic term

$$\mathcal{Q}(J) := \sum_{(\alpha,\beta)} D(\alpha,\beta), \tag{4.14}$$

and we consider the final Glimm functional

$$\mathcal{F}(J) := \mathcal{L}(J) + K\mathcal{Q}(J), \tag{4.15}$$

where $K>0$.

Note that due to the linear character of CRI, one does not need any quadratic correction for the radiative part.

Our aim is now to prove that $\mathcal{F}(J)$ is uniformly bounded for any mesh curve J provided that the constant K is large enough. As $\mathcal{F}(J)$ is equivalent to the total variation norm in $BV(\mathbb{R})$, we will conclude that $TV(U_{\Delta x}) \leqslant CTV(U_0)$ which implies the convergence of the scheme by using a compactness argument.

PROPOSITION 4.2. *Let J_1 be a mesh curve, J_2 an immediate successor of J_1, and D_{ℓ_0,n_0} the diamond region determined by J_1 and J_2. The Glimm functionals $\mathcal{L}(J_{1,2})$ and $\mathcal{Q}(J_{1,2})$ satisfy*

$$\mathcal{L}(J_1) - \mathcal{L}(J_2) = O(1)\left\{\mathcal{P}(D_{\ell_0,n_0}) + \varepsilon_{in}(D_{\ell_0,n_0})\ \Delta t\left(\overline{\sigma_a}((\ell_0+1)\Delta t) + \overline{\sigma_s}((\ell_0+1)\Delta t)\right)\right\}, \tag{4.16}$$

and

$$\begin{aligned}\mathcal{Q}(J_1) - \mathcal{Q}(J_2) = &-\mathcal{P}(D_{\ell_0,n_0}) + O(1)\mathcal{L}(J_1)\mathcal{P}(D_{\ell_0,n_0})\\ &+ O(1)\mathcal{L}(J_1)\varepsilon_{in}(D_{\ell_0,n_0})\ \Delta t\left(\overline{\sigma_a}((\ell_0+1)\Delta t) + \overline{\sigma_s}((\ell_0+1)\Delta t)\right),\end{aligned} \tag{4.17}$$

where the constants C_j are defined in Proposition 3.3.

Proof. From the previous definitions (4.9), (4.10), and (4.13), we have

$$\mathcal{L}(J_2) - \mathcal{L}(J_1) = \varepsilon_{out}(D_{\ell_0,n_0}) - \varepsilon_{in}(D_{\ell_0,n_0}) + \varepsilon_{r,out}(D_{\ell_0,n_0}) - \varepsilon_{r,in}(D_{\ell_0,n_0}).$$

Using estimates (3.11) and (3.12) in Proposition 3.3, with $x_0 = n_0\Delta x$, $t_0 = (\ell_0 - 1)\Delta t$, $U_L = V_W$, $U_M = U_S$, $U_R = V_E$, $\mu_L = U_W - V_W$, and $\mu_R = U_E - V_E$, we find

$$\begin{aligned}\varepsilon_{out}(D_{\ell_0,n_0}) = &\ \varepsilon_{in}(D_{\ell_0,n_0}) + O(1)\mathcal{P}(D_{\ell_0,n_0})\\ &+ O(1)\varepsilon_{in}(D_{\ell_0,n_0})\Delta t\left(\overline{\sigma_a}((\ell_0+1)\Delta t) + \overline{\sigma_s}((\ell_0+1)\Delta t)\right),\end{aligned}$$

which gives (4.16).

Since hydrodynamical and radiative elementary waves do not interact, (4.17) directly follows from Proposition 3.3 ∎

In the spirit of [12], we have the following lemma.

LEMMA 4.3. *Let ℓ_0 be a positive integer, and suppose that J_1 (resp. J_2) is a mesh curve such that all the mesh points on J_1 (resp. J_2) belong either to the line $t = (\ell_0 - 1)\Delta t$ or to the line $t = \ell_0\Delta t$ (resp. the line $t = \ell_0\Delta t$ or to the line $t = (\ell_0+1)\Delta t$).*

Suppose there exists a positive constant $M>0$ sufficiently small such that

$$\mathcal{L}(J_1) \leqslant M. \tag{4.18}$$

Then provided that the constant K in (4.15) is large enough, the following estimate holds:

$$\mathcal{F}(J_2)\leqslant\mathcal{F}(J_1)+O(1)\Delta t(\overline{\sigma_a}((\ell_0+1)\Delta t)+\overline{\sigma_s}((\ell_0+1)\Delta t))\sum_{n_0\in\mathbb{Z}}\varepsilon_{in}(D_{\ell_0,n_0}),\tag{4.19}$$

where $O(1)$ depends on M and K.

Proof. Fixing $n_0\in\mathbb{Z}$, we multiply (4.17) by K and add it to (4.16). We obtain

$$\begin{aligned}\mathcal{F}(J_2)-\mathcal{F}(J_1)=&-K\sum_{n_0\in\mathbb{Z}}\mathcal{P}(D_{\ell_0,n_0})+O(1)[1+K\mathcal{L}(J_1)]\sum_{n_0\in\mathbb{Z}}\mathcal{P}(D_{\ell_0,n_0})\\&+O(1)[1+K\mathcal{L}(J_1)]\sum_{n_0\in\mathbb{Z}}(\varepsilon_{in}(D_{\ell_0,n_0})\Delta t(\overline{\sigma_a}((\ell_0+1)\Delta t)+\overline{\sigma_s}((\ell_0+1)\Delta t))).\end{aligned}$$

Since $\sum_{n_0\in\mathbb{Z}}\mathcal{P}(D_{\ell_0,n_0})=\mathcal{Q}(J_1)$, this implies

$$\begin{aligned}\mathcal{F}(J_2)-\mathcal{F}(J_1)&=-K\mathcal{Q}(J_1)+O(1)[1+K\mathcal{L}(J_1)]\mathcal{Q}(J_1)\\&\quad+O(1)\Delta t\left(\sum_{n_0\in\mathbb{Z}}\varepsilon_{in}(D_{\ell_0,n_0})\right)(\overline{\sigma_a}((\ell_0+1)\Delta t)+\overline{\sigma_s}((\ell_0+1)\Delta t))\\&=\mathcal{Q}(J_1)[K(O(1)\mathcal{L}(J_1)-1)+O(1)]\\&\quad+O(1)\Delta t\left(\sum_{n_0\in\mathbb{Z}}\varepsilon_{in}(D_{\ell_0,n_0})\right)(\overline{\sigma_a}((\ell_0+1)\Delta t)+\overline{\sigma_s}((\ell_0+1)\Delta t))\\&\leqslant\mathcal{Q}(J_1)[K(O(1)M-1)+O(1)]\\&\quad+O(1)\Delta t\left(\sum_{n_0\in\mathbb{Z}}\varepsilon_{in}(D_{\ell_0,n_0})\right)(\overline{\sigma_a}((\ell_0+1)\Delta t)+\overline{\sigma_s}((\ell_0+1)\Delta t)).\end{aligned}$$

Choosing $M>0$ sufficiently small and $K>0$ sufficiently large, the first term of the right-hand side is negative, so (4.19) is proved. ∎

THEOREM 4.4. *Let $(U_\infty,I_\infty)\in\mathcal{U}$ be a constant state such that $I_\infty(\nu,\omega)=B(\nu,\vartheta_\infty)$ where the temperature ϑ_∞ corresponds to U_∞. Assume that the initial data $(U_0,I_0)\in\mathcal{U}$ is such that*

$$\|U_0-U_\infty\|_{L^\infty(\mathbb{R})},\quad \|I_0-I_\infty\|_{L^\infty_x(\mathbb{R},L^1_{\omega,\nu}([-1,1]\times\mathbb{R}_+))},$$

$$TV(U_0),\quad TV\left(\|I_0\|_{L^\infty([-1,1]\times\mathbb{R}_+)}\right)\tag{4.20}$$

are sufficiently small. Assume also that the norms

$$\int_0^\infty\overline{\sigma_a}(t)dt,\quad \int_0^\infty\overline{\sigma_s}(t)dt,\tag{4.21}$$

are sufficiently small. Then, the approximate solutions $(U_{\Delta x},I_{\Delta x})$ are bounded uniformly in $L^\infty(\mathbb{R}^+\times\mathbb{R})$ and in $BV(\mathbb{R}^+\times\mathbb{R})$:

$$\begin{aligned}&\|U_{\Delta x}-U_\infty\|_{L^\infty([0,T]\times\mathbb{R})}+\|I_{\Delta x}-I_\infty\|_{L^\infty([0,T]\times\mathbb{R}),L^1([-1,1]\times\mathbb{R}^+)}\\&\leqslant O(1)\left(\|U_0-U_\infty\|_{L^\infty(\mathbb{R})}+\|I_0-I_\infty\|_{L^\infty_x(\mathbb{R},L^1_{\omega,\nu}([-1,1]\times\mathbb{R}_+))}+C\right),\end{aligned}\tag{4.22}$$

$$TV(U_{\Delta x})+\|TV(I_{\Delta x})\|_{L^1_{\omega,\nu}([-1,1]\times\mathbb{R}_+)}\leqslant O(1)\Big(TV(U_0)+\|TV(I_0)\|_{L^1_{\omega,\nu}([-1,1]\times\mathbb{R}_+)}+C\Big), \tag{4.23}$$

where

$$C=\int_0^\infty (\overline{\sigma_a}(t)+\overline{\sigma_s}(t))\,dt, \tag{4.24}$$

and M *is described in* (4.18). *Furthermore,* $(U_{\Delta x},I_{\Delta x})$ *is Lipschitz continuous in time. That is, for any* $t_1,t_2\in[0,T]$,

$$\int_{\mathbb{R}}|U_{\Delta x}(t_1,x)-U_{\Delta x}(t_2,x)|dx+\left\|\int_{\mathbb{R}}|I_{\Delta x}(t_1,x)-I_{\Delta x}(t_2,x)|\,dx\right\|_{L^1_{\omega,\nu}([-1,1]\times\mathbb{R}_+)}$$
$$\leqslant O(1)\,(|t_1-t_2|+\Delta t)\Big(TV(U_0)+\|TV(I_0)\|_{L^1_{\omega,\nu}([-1,1]\times\mathbb{R}_+)}+C\Big). \tag{4.25}$$

Proof. We first prove that condition (4.18) holds, under the assumptions (4.20) and (4.21), by induction on ℓ_0. For this purpose, we denote by J_{ℓ_0} a mesh curve which satisfies the hypotheses of Lemma 4.3. That is, any point of J_{ℓ_0} belongs either to the line $t=(\ell_0-1)\Delta t$ or to the line $t=\ell_0\Delta t$. For $\ell_0=0$, we clearly have

$$\mathcal{F}(J_0)\leqslant O(1)\bigg(TV(U_0)+\|TV(I_0)\|_{L^1_{\omega,\nu}([-1,1]\times\mathbf{R}^+)}$$
$$+K\Big[TV(U_0)+\|TV(I_0)\|_{L^1_{\omega,\nu}([-1,1]\times\mathbf{R}^+)}\Big]^2\bigg), \tag{4.26}$$

where the term $O(1)$ depends only on the coefficients of the system. Hence, if $TV(U_0)+\|TV(I_0)\|_{L^1_{\omega,\nu}([-1,1]\times\mathbf{R}^+)}$ is sufficiently small, then $\mathcal{F}(J_0)\leqslant M$ hence $\mathcal{L}(J_0)\leqslant M$, where M is defined in Lemma 4.3.

Next, we assume that

$$\forall \ell\leqslant \ell_0-1,\quad \mathcal{L}(J_\ell)\leqslant M. \tag{4.27}$$

Applying Lemma 4.3, we infer

$$\mathcal{F}(J_{\ell_0})\leqslant\mathcal{F}(J_{\ell_0-1})+O(1)\Delta t\sum_{n_0\in\mathbb{Z}}(\overline{\sigma_a}((\ell_0+1)\Delta t)+\overline{\sigma_s}((\ell_0+1)\Delta t))\,\varepsilon_{\rm in}(D_{\ell_0-1,n_0}).$$

Repeating this argument for each ℓ and summing from $\ell=0$ to $\ell=\ell_0-1$, we find

$$\mathcal{F}(J_{\ell_0})\leqslant\mathcal{F}(J_0)+O(1)\Delta t\sum_{n_0\in\mathbb{Z}}\sum_{\ell=0}^{\ell_0-1}(\overline{\sigma_a}((\ell+1)\Delta t)+\overline{\sigma_s}((\ell+1)\Delta t))\,\varepsilon_{\rm in}(D_{\ell,n_0}).$$

Then, we point out that

$$\sum_{n_0\in\mathbb{Z}}\varepsilon_{\rm in}(D_{\ell,n_0})=\mathcal{L}(J_\ell).$$

Hence,

$$\mathcal{F}(J_{\ell_0})\leqslant\mathcal{F}(J_0)+O(1)\Delta t\sum_{\ell=0}^{\ell_0-1}\left(\mathcal{L}(J_\ell)\,(\overline{\sigma_a}((\ell+1)\Delta t)+\overline{\sigma_s}((\ell+1)\Delta t))\right).$$

Together with (4.26) and (4.27), this implies

$$\mathcal{F}(J_{\ell_0}) \leqslant O(1)\Big(1+KTV(U_0)+\|TV(I_0)\|_{L^1_{\omega,\nu}([-1,1]\times\mathbf{R}^+)}\Big)TV(U_0) + \|TV(I_0)\|_{L^1_{\omega,\nu}([-1,1]\times\mathbf{R}^+)} + O(1)M\Delta t\sum_{\ell=0}^{\ell_0-1}(\overline{\sigma_a}((\ell+1)\Delta t)+\overline{\sigma_s}((\ell+1)\Delta t)).$$

Applying (4.24), this gives

$$\mathcal{F}(J_{\ell_0}) \leqslant O(1)\Big[\Big(1+KTV(U_0)+\|TV(I_0)\|_{L^1_{\omega,\nu}([-1,1]\times\mathbf{R}^+)}\Big)TV(U_0) + \|TV(I_0)\|_{L^1_{\omega,\nu}([-1,1]\times\mathbf{R}^+)} + CM\Big].$$

Hence,

$$\mathcal{L}(J_{\ell_0}) \leqslant O(1)\Big[\Big(1+KTV(U_0)+\|TV(I_0)\|_{L^1_{\omega,\nu}([-1,1]\times\mathbf{R}^+)}\Big)TV(U_0) + \|TV(I_0)\|_{L^1_{\omega,\nu}([-1,1]\times\mathbf{R}^+)} + CM\Big],$$

where the term $O(1)$ is independent of C, M, and K, and $TV(U_0)+\|TV(I_0)\|_{L^1_{\omega,\nu}([-1,1]\times\mathbf{R}^+)}$. Hence, we can choose $TV(U_0)+\|TV(I_0)\|_{L^1_{\omega,\nu}([-1,1]\times\mathbf{R}^+)}$ small enough to have

$$O(1)\Big(1+KTV(U_0)+\|TV(I_0)\|_{L^1_{\omega,\nu}([-1,1]\times\mathbf{R}^+)}\Big)TV(U_0) + \|TV(I_0)\|_{L^1_{\omega,\nu}([-1,1]\times\mathbf{R}^+)} \leqslant M/2,$$

and

$$O(1)CM \leqslant M/2.$$

Thus,

$$\mathcal{L}(J_{\ell_0}) \leqslant M.$$

This ends the induction proof.

We thus know that the sequence $\mathcal{L}(J_{\ell_0})$ is bounded independently of ℓ_0 and of Δt. Hence, since the functional $\mathcal{L}(J_{\ell_0})$ is equivalent to the total variation of $U_{\Delta x}$, we deduce that the total variation of $U_{\Delta x}(t,x)$ is bounded independently of t and Δx. This proves (4.23). The proofs of (4.22) and (4.25) are dealt with exactly as in [9]. ∎

We are now in position to prove our main result:

THEOREM 4.5. *Under the hypotheses of Theorem 4.4, if the sequence $(a_\ell)_{\ell\geqslant 0}$ is equidistributed, the sequence $(U_{\Delta x}, I_{\Delta x})$ defined by* (4.3)–(4.6) *converges in L^1_{loc}, up to extraction of a subsequence, to a function $(U,I)=(U(x,t),I(x,t))$ which is a solution of* (1.2) *such that U is an entropy solution of the first three equations of* (1.2).

Proof. Let $(U_{\Delta x}, I_{\Delta x})$ be the solution defined by the generalized Glimm scheme (4.3)–(4.6). Applying Theorem 4.4 and Helly's Theorem, there exists a subsequence of $(U_{\Delta x}, I_{\Delta x})$ converging to $(U,I)\in L^1_{\text{loc}}$ which is in BV.

Let θ be a test function in $\mathcal{D}(\mathbb{R}^+ \times \mathbb{R})$, and define

$$R(U_{\Delta x}, I_{\Delta x}, \theta) = \int_{\mathbb{R}^+} \int_{\mathbb{R}} [U_{\Delta x} \partial_t \theta + f(U_{\Delta x}) \partial_x \theta + g(U_{\Delta x}, I_{\Delta x}) \theta] \, dx dt \\ + \int_{\mathbb{R}} U_{\Delta x}(0,x) \theta(0,x) dx.$$

The fact that (U,I) is a weak solution of (1.2) is exactly equivalent to $R(U,I,\theta)=0$ for any smooth θ. The dominated convergence theorem implies that

$$\lim_{\Delta x \to 0} |R(U_{\Delta x}, I_{\Delta x}, \theta) - R(U,I,\theta)| = 0.$$

Hence, we are left to prove that

$$\lim_{\Delta x \to 0} R(U_{\Delta x}, I_{\Delta x}, \theta) = 0. \tag{4.28}$$

We split the integral defining $R(U_{\Delta x}, I_{\Delta x}, \theta)$ into a sum of integrals over the sets $Q_{\ell_0, n_0} = [\ell_0 \Delta t, (\ell_0+1)\Delta t] \times [(n_0-1)\Delta x, (n_0+1)\Delta x]$ for $\ell_0 \in \mathbb{N}$ and $n_0 + \ell_0 \in 2\mathbb{Z}$:

$$R(U_{\Delta x}, I_{\Delta x}, \theta) = \sum_{\ell_0=0}^{\infty} \sum_{n_0+\ell_0 \in 2\mathbb{Z}} \int_{\ell_0 \Delta t}^{(\ell_0+1)\Delta t} \int_{(n_0-1)\Delta x}^{(n_0+1)\Delta x} U_{\Delta x} \partial_t \theta + f(U_{\Delta x}) \partial_x \theta + g(U_{\Delta x}, I_{\Delta x}) \theta.$$

Applying Proposition 2.3, we thus have

$$\begin{aligned} &R(U_{\Delta x}, I_{\Delta x}, \theta) \\ &= O(1) \sum_{\ell_0=0}^{\infty} \sum_{n_0+\ell_0 \in 2\mathbb{Z}} \left(\Delta t^2 + \Delta x^2\right) \left(\Delta x + \Delta t + |U_{\ell_0, n_0+1} - U_{\ell_0, n_0-1}|\right) \|\theta\|_{C^1(Q_{\ell_0,n_0})} \\ &\quad + \sum_{\ell_0=0}^{\infty} \sum_{n_0+\ell_0 \in 2\mathbb{Z}} \left(\int_{(n_0-1)\Delta x}^{(n_0+1)\Delta x} U_{\Delta x}\left((\ell_0+1)\Delta t-, x\right) \theta\left((\ell_0+1)\Delta t, x\right) dx \right. \\ &\quad \left. - \int_{(n_0-1)\Delta x}^{(n_0+1)\Delta x} U_{\Delta x}\left(\ell_0 \Delta t+, x\right) \theta\left(\ell_0 \Delta t+, x\right) dx \right) + \int_{\mathbb{R}} U_{\Delta x}(0,x) \theta(0,x) dx \\ &\quad + \sum_{\ell_0=0}^{\infty} \sum_{n_0+\ell_0 \in 2\mathbb{Z}} \left(\int_{\ell_0 \Delta t}^{(\ell_0+1)\Delta t} f\left(t, (n_0+1)\Delta x, U_{\Delta x}(t, (n_0+1)\Delta x-)\right) dt \right. \\ &\qquad \left. - \int_{\ell_0 \Delta t}^{(\ell_0+1)\Delta t} f\left(t, (n_0-1)\Delta x, U_{\Delta x}(t, (n_0-1)\Delta x+)\right) dt \right). \end{aligned}$$

We denote by R_1 the first line of the right-hand side, by R_2 the second and third lines, and by R_3 the fourth and fifth lines: $R(U_{\Delta x}, I_{\Delta x}, \theta) = R_1 + R_2 + R_3$. We deal with each term separately: for R_1, we apply (4.23), and find that

$$\begin{aligned} &|R_1| \leqslant C \Delta x \\ &+ C \sum_{\ell_0=0}^{\infty} \sum_{n_0+\ell_0 \in 2\mathbb{Z}} \left(\Delta x^2 + \Delta t^2\right) \left(1 + TV(U_0) + \|TV(I_0)\|_{L^1_{\omega,\nu}([-1,1]\times \mathbf{R}^+)}\right) \|\theta\|_{C^1(Q_{\ell_0,n_0})}. \end{aligned}$$

Hence,

$$|R_1| \leqslant C \Delta x. \tag{4.29}$$

Turning to R_2, we have

$$R_2 = -\sum_{\ell_0=0}^{\infty} \sum_{n_0+\ell_0\in 2\mathbb{Z}} \int_{(n_0-1)\Delta x}^{(n_0+1)\Delta x} [U_{\Delta x}](\ell_0\Delta t,x)\theta(\ell_0\Delta t,x)dx,$$

where $[\cdot]$ denotes the jump of a function: $[U_{\Delta x}](\ell_0\Delta t,x) = U_{\Delta x}(\ell_0\Delta t+,x) - U_{\Delta x}(\ell_0\Delta t-,x)$. We apply a result by Liu [22], which implies that, since the sequence $(a_n)_{n\in\mathbb{Z}}$ is equidistributed,

$$\lim_{\Delta x\to 0}\left(\sum_{\ell_0=0}^{\infty} \sum_{n_0+\ell_0\in 2\mathbb{Z}} \int_{(n_0-1)\Delta x}^{(n_0+1)\Delta x} [U_{\Delta x}](\ell_0\Delta t,x)\theta(\ell_0\Delta t,x)dx\right) = 0.$$

Hence,

$$\lim_{\Delta x\to 0} R_2 = 0. \tag{4.30}$$

Turning to R_3, we have, according to the definition of $U_{\Delta x}$ and since g is smooth,

$$\begin{aligned}
|R_3| &\leqslant C\sum_{\ell_0=0}^{\infty} \sum_{n_0+\ell_0\in 2\mathbb{Z}} \int_{\ell_0\Delta t}^{(\ell_0+1)\Delta t} |U_{\Delta x}(t,(n_0+1)\Delta t+) - U_{\Delta x}(t,(n_0-1)\Delta t-)|\,\|\theta\|_{C^1(Q_{\ell_0,n_0})}dt \\
&\leqslant C\sum_{\ell_0=0}^{\infty} \sum_{n_0+\ell_0\in 2\mathbb{Z}} \int_{\ell_0\Delta t}^{(\ell_0+1)\Delta t} t|g(\ell_0\Delta t,(n_0+2)\Delta x, U_{\ell_0,n_0+1}) \\
&\qquad\qquad - g(\ell_0\Delta t, n_0\Delta x, U_{\ell_0,n_0+1})|\,dt\|\theta\|_{C^1(Q_{\ell_0,n_0})} \\
&\leqslant C\sum_{\ell_0=0}^{\infty} \sum_{n_0+\ell_0\in 2\mathbb{Z}} \int_{\ell_0\Delta t}^{(\ell_0+1)\Delta t} t\Delta x dt\|\theta\|_{C^1(Q_{\ell_0,n_0})}.
\end{aligned}$$

Hence,

$$|R_3| \leqslant C\Delta x. \tag{4.31}$$

We then collect (4.29)–(4.31) and find that (4.28) holds. That is to say, U is a weak solution of (1.2). As far as the radiative part is concerned, the preceding proof applies in a simpler way since we do not use any Riemann problems for this part.

It remains to prove that (U,I) is an entropy solution. That is, for any entropy pair (S,F) and any non-negative test function θ, we have

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} S(U)\partial_t\theta + F(U)\partial_x\theta + P(U)\theta + \int_{\mathbb{R}} S(U_0)\theta(0,x)dx \geqslant 0,$$

with

$$P = \partial_U S g(t,x,U) + \partial_x F(t,x,U).$$

The proof of this fact follows exactly the same lines as that of (4.28), applying Proposition 2.3. ∎

Acknowledgements. The authors are grateful to Prof. F. Golse for interesting discussions on the subject.

REFERENCES

[1] A. Bourgeade, Ph. LeFloch, and P.A. Raviart, *An asymptotic expansion for the solution of the generalized Riemann problem. Part 2: application to the equations of gas dynamics*, Ann. I.H.P. Analyse non linéaire, 6, 437–480, 1989.

[2] A. Bressan, *Hyperbolic Systems of Conservation Laws*, Oxford University Press, 2000.

[3] C. Buet and B. Després, *Asymptotic analysis of fluid models for the coupling of radiation and hydrodynamics*, J. Quant. Spectrosc. Ra., 85, 385–418, 2004.

[4] S. Chandrasekhar, *Radiative transfer*, Dover Publications, Inc., New York, 1960.

[5] C. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*, Springer Verlag, Berlin, Heidelberg, New York, 2000.

[6] C. Dafermos and L. Hsiao, *Hyperbolic systems of balance laws with inhomogeneity and dissipation*, Indiana Univ. Math. J., 31, 471–491, 1982.

[7] B. Dubroca and J.-L. Feugeas, *Etude théorique et numérique d'une hiérarchie de modèles aux moments pour le transfert radiatif*, C.R. Acad. Sci. Paris, Série I, 329, 915–920, 1999.

[8] G. Frosali, C.V.M. van der Mee, and S.L. Paveri-Fontana, *Conditions for runaway phenomena in the kinetic theory of particle swarms*, J. Math. Phys., 30(5), 1177–1186, 1989.

[9] J. Glimm, *Solutions in the large for nonlinear hyperbolic systems of conservation laws*, Commun. Pure Appl. Math., 18, 697–715, 1965.

[10] F. Golse and G. Allaire, *Transport et Diffusion*, Lecture Notes, Ecole polytechnique, 2010.

[11] F. Golse and B. Perthame, *Generalized solutions of the radiative transfer equations in a singular case*, Commun. Math. Phys., 106, 211–239, 1986.

[12] J.M. Hong and Ph. LeFloch, *A version of the Glimm method based on generalized Riemann problems*, Port. Math., 64, 199–236, 2007.

[13] L. Hörmander, *Lectures on Nonlinear Hyperbolic Differential Equations*, Springer Verlag, Berlin, Heidelberg, New York, 1997.

[14] L. Hsiao, *Quasilinear Hyperbolic Systems and Dissipative Mechanisms*, World Scientific, Singapore, New Jersey, London, Hong Kong, 1997.

[15] P. Jiang and D. Wang, *Formation of singularities of solutions to the three-dimensional Euler–Boltzmann equations in radiation hydrodynamics*, Nonlinearity, 23(4), 2010.

[16] P. Jiang and D. Wang, *Global weak solutions to the Euler-Boltzmann equations in radiation hydrodynamics*, Quart. Appl. Math., 70(1), 25-44, 2012.

[17] Ph.G. LeFloch, *Hyperbolic Conservation of Conservation Laws*, Birkhäuser Verlag, Basel, Boston, Berlin, 2002.

[18] Ph.G. LeFloch and T.P. Liu, *Existence theory for nonlinear hyperbolic systems*, Forum Math., 5, 261–280, 1993.

[19] Ph.G. LeFloch and P.A. Raviart, *An asymptotic expansion for the solution of the generalized Riemann problem. Part 1: general theory*, Ann. I.H.P. Analyse non linéaire, 5, 179–209, 1988.

[20] C. Lin, *Mathematical analysis of radiative transfer models*, PhD Thesis, 2007.

[21] C. Lin, J.-F. Coulombel, and T. Goudon, *Shock profiles for non-equilibrium radiating gases*, Physica D, 218, 83–94, 2006.

[22] T.-P. Liu, *The deterministic version of the Glimm scheme*, Commun. Math. Phys., 57, 135–148, 1977.

[23] T.-P. Liu, *Quasilinear hyperbolic systems*, Commun. Math. Phys., 68, 141–172, 1979.

[24] T.-P. Liu, *Admissible solutions of hyperbolic conservation laws*, Memoirs AMS, 30(240), 1981.

[25] R.B. Lowrie, J.E. Morel, and J.A. Hittinger, *The coupling of radiation and hydrodynamics*, The Astrophysical Journal, 521, 432–450, 1999.

[26] D. Mihalas and B. Weibel-Mihalas, *Foundations of radiation hydrodynamics*, Oxford University Press, New York, 1984.

[27] G.C. Pomraning, *Radiation hydrodynamics*, Dover Publications, Inc., Mineola, New York, 2005.

[28] M. Schatzman, *Continuous Glimm functionals and uniqueness of solutions of the Riemann problem*, Indiana Univ. Math. J., 34, 533–589, 1985.

[29] D. Serre, *Systèmes de lois de conservation I*, Diderot Editeur, Arts et Sciences, Paris, New-York, Amsterdam, 1996.

[30] R.G. Smith, *The Riemann problem in gas dynamics*, Transactions of the AMS, 249, 1–50, 1979.

[31] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Springer Verlag, Berlin, Heidelberg, New York, 1983.

[32] Ya.B. Zel'dovich and Yu.P. Raiser, *Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena*, Dover Publications, Inc., Mineola, New York, 2002.

[33] X. Zhong and S. Jiang, *Local existence and finite-time blow-up in multidimensional radiation hydrodynamics*, J. Math. fluid Mech., 9, 543–564, 2007.