

NONLINEAR CONTINUUM THEORY OF SMECTIC-C LIQUID CRYSTALS*

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Abstract. Here we develop a model of smectic-C liquid crystals by forming their hydrostatic and hydrodynamic theories which are motivated by W.N. E's work [Arch. Rational Mech. Anal., 137 (1997)]. A simplified model is also presented. In order to prove the rationality of the model, we establish the energy dissipative relation of the new model. Meanwhile, we verify that the system can also be obtained using asymptotic analysis when both the fluid and layers are incompressible.

Key words. Continuum theory, smectic liquid crystals, smectic-C, hydrodynamic equations, kinetic theory.

AMS subject classifications. 35Q35, 35Q51, 76D07, 80A17.

1. Introduction

A fluid forms liquid crystals (LC) when its molecules are orientated with a certain order [5, 14, 15]. LC can be divided into two categories based on the methods of their generation: thermotropic LC and lyotropic LC. Thermotropic LC form nematic, cholesteric, and smectic phases due to the molecular orientation [14]. The molecules in nematic phase LC tend to be ordered, but the molecular weight is disorderly; see [24, 29, 30] and the references therein. We use a unit vector \mathbf{n} to denote the average direction of the molecular orientation in the nematic phase. The nematic phase will transform into the smectic phase if one further decreases the environmental temperature or increases the concentration of the liquid crystal. When this happens, the molecules are arranged layer by layer, and the direction of molecules is perpendicular to the surface of the liquid crystal layer (see Figure 1) [14]. The phase will translate from smectic-A to smectic-C phase when the temperature is further lowered. Due to the decreasing temperature, the symmetry of the crystal is destroyed. Molecules will not arrange along the normal direction of the LC layer but form a slope of angle θ between 20 and 35 degrees. The specific size of this angle is related to the material of the LC and the temperature. There are many works investigating these phenomena: [2, 9, 10, 14, 20, 21, 22] and



FIG. 1.1. *Smectic-A and Smectic-C*

the references therein. Weinan E [9, 10] has established a general nonlinear continuum theory for smectic-A liquid crystals applicable to situations with large deformation and

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non-trivial flow. It is an application of Ericksen–Leslie’s theory of the nematic phase [8, 11, 12, 13, 18, 23]. We are motivated to use similar methods to formulate a smectic-C fluid dynamics model because both smectic-A and smectic-C are liquid crystalline phases in which there is some degree of orientational order, but also some degree of positional order. We conclude that their continuum theories involve the same variables: molecular director, layer function, flow velocity, and hydrostatic pressure fields. What makes them different is that the molecular direction forms an angle with the normal direction of a layer in the smectic-C phase, whereas the molecular direction is perpendicular to the LC plane in the smectic-A phase. Consequently, a main feature of our work is the establishment of the equations for the molecular vector \mathbf{n} and the layer variable φ .

A continuum theory for the chiral smectic-C phase was proposed by Calderer and Joo [2]. They use the smectic energy of the form proposed by Chen and Lubensky [7] with the ferroelectric electrostatic energy in [2]. Here, the smectic energy terms we considered are parts of the Chen–Lubensky energy. It is a simply case. Moreover, our model can also be obtained by the work of Calderer and Joo [2] by setting the polarization and electric field to be zero. So the work [2] is more general. However, in [2], it is a Lagrange multiplier to add the energy for the condition $|\mathbf{n}|=1$. Here in our work, we state it as one of the hydrodynamic equations of (3.7)–(3.11). That means that we also obtain similar hydrodynamic equations, but through different ways. We think that it will state physical phenomena similar to the Calderer–Joo’s model [2]. Now we also hope to numerically verify this in the future since our model is simple. By setting $\mathbf{n}=(\cos\phi\sin\alpha,\sin\phi\sin\alpha,\cos\alpha)$, the last equation of (3.7)–(3.11) is naturally satisfied. Then we can calculate the unknown functions $\mathbf{u}, p, \varphi, \phi, \alpha$ to solve our full system. It will be easier to solve our full model than the full model in [2]. In addition, their main work was to study the backflow effect due to the spontaneous polarization of the liquid crystal [1, 6] after they develop the model, whereas ours is the proof of the rationality of our model by using asymptotic analysis. They also investigated smectic-A phase [3, 4]. To verify the rationality of the model, we refer to [16, 17, 19] and deduce the energy theorem of the model of smectic-C. Thus we also prove that the model is an energy dissipation system. This method is from our previous works [25, 26, 27, 28].

This paper is organized as follows. The hydrostatic theory, including the free energy, the permeation force, and the elastic stress, is established in Section 2. The hydrodynamic theory is developed in Section 3. The energy theorem, which highlights the dissipative nature of the system, is given in Section 4. The models are also obtained by using asymptotic analysis when both the fluid and the layers are incompressible in Section 5. Finally, we draw a conclusion and provide some discussion.

2. Hydrostatic theory

2.1. The free energy. The smectic-C liquid phase is mainly depicted by two variables. One is the molecular direction \mathbf{n} , and the other is the scalar function φ which characterizes the liquid layer position. The biggest difference between the smectic-C and smectic-A liquid phases is that the molecular orientation in the smectic-A phase tends to be the normal direction of the liquid crystal layer, whereas the molecular orientation in the smectic-C phase of the molecule forms an angle to the normal vector of the liquid crystal layer. If the angle is θ and $|\mathbf{n}|=1$, then this constraint can be imposed by requiring that $\mathbf{n}\cdot\nabla\varphi=|\nabla\varphi|\cos\theta$. There is no void space between the two layers in the continuous theory. In fact, the distance is approximately the projection of molecules in the normal direction of the liquid crystal layer; that is, $|\nabla\varphi|=\cos\theta$. At this time, the liquid crystal layers are incompressible. As a result, when $|\nabla\varphi|=\cos\theta$ and $\mathbf{n}\cdot\nabla\varphi=\cos^2\theta$, the smectic-C LC is at its most stable state. Therefore, the simplest form of the

free energy density for the smectic-C LC, which reflects angular and incompressibility characteristics, is as follows:

$$f_1 = \frac{1}{2}\xi_1(\mathbf{n} \cdot \nabla\varphi - \cos^2\theta)^2 + \frac{1}{4}\xi_2(|\nabla\varphi|^2 - \cos^2\theta)^2, \tag{2.1}$$

where ξ_1 and ξ_2 are the Lagrange multipliers related to the restrictive conditions. The nematic free energy density of the Oseen–Frank form is

$$f_2 = \frac{1}{2}K_1(\nabla \cdot \mathbf{n})^2 + \frac{1}{2}K_2(\mathbf{n} \cdot \nabla \times \mathbf{n})^2 + \frac{1}{2}K_3(\mathbf{n} \times \nabla \times \mathbf{n})^2, \tag{2.2}$$

where K_1, K_2 , and K_3 are the splay, twist, and bend elastic constants, respectively. For simplicity, we set $K_1 = K_2 = K_3 = K$. Then (2.2) can be written as

$$f_2 = \frac{1}{2}K(\nabla\mathbf{n} : \nabla\mathbf{n})^2. \tag{2.3}$$

Therefore, we get a free energy density

$$f = f_1 + f_2 = \frac{1}{2}\xi_1(\mathbf{n} \cdot \nabla\varphi - \cos^2\theta)^2 + \frac{1}{4}\xi_2(|\nabla\varphi|^2 - \cos^2\theta)^2 + \frac{1}{2}K(\nabla\mathbf{n} : \nabla\mathbf{n})^2. \tag{2.4}$$

Then, the free energy is $F(\varphi, \mathbf{n}) = \int f(\varphi, \mathbf{n})d^3x$.

Here we note that for the smectic-A phase, $\theta = 0$, $\cos\theta = 1$, and $\mathbf{n} = \nabla\varphi$. Thus, in the process of calculating the free energy, we take

$$f_1 = \frac{1}{2}\xi_1(|\nabla\varphi|^2 - 1)^2 + \frac{1}{4}\xi_2(|\nabla\varphi|^2 - 1)^2 = \xi(|\nabla\varphi|^2 - 1)^2,$$

where $\xi = \xi_1 + \frac{1}{2}\xi_2$. So we can get a free energy density of smectic-A:

$$f = f_1 + f_2 = \frac{1}{2}\xi(|\nabla\varphi|^2 - 1)^2 + \frac{1}{2}K(\nabla\mathbf{n} : \nabla\mathbf{n})^2. \tag{2.5}$$

Here we see that the free energy density of smectic-A (2.5) is consistent with the result in [9]. The free energy density (2.4) is more general than (2.5).

2.2. Permeation calculation. In order to establish the hydrostatic theory of liquid crystal dynamics (LCD), we need to consider two deformations:

1. when the molecular centers do not move the molecular orientations will form a molecular field (molecular reaction);
2. when the direction field of the molecules remain constant, the positions of the molecule will move to form the elastic force of the molecules.

In this section, we will discuss the molecular field and calculate the elastic tension. Similar to the nematic LCD, the molecular field can be used as the force field acting on the molecules. For the smectic-C phase, the direction field is determined by the liquid crystal layer. Therefore, we only need to discuss the force on the liquid crystal layer. Because the tangential force on the liquid crystal layer can not push the position of the liquid layer, it is the force in the normal direction of the layer that changes the position of the layer. We call this force the permeation force.

In order to calculate the permeation force, we only need to calculate the variation of the free energy functional with respect to the layer position, i.e. $\frac{\delta F}{\delta\varphi}$.

We know that \mathbf{n} is independent of $\delta\varphi$ and f_2 depends only on \mathbf{n} , so $\delta f_2 = 0$ since \mathbf{n} and $\nabla\varphi$ form the angle θ , but we observe through the normal direction of the layer. All

of the molecular orientations can form a taper. More precisely, the molecular direction in each layer is along any generatrix direction of the surface. First, we note that

$$\delta|\nabla\varphi| = \frac{1}{|\nabla\varphi|} \nabla\varphi \cdot \nabla\delta\varphi.$$

Set $\delta\varphi$ to be zero on the boundary, so that

$$\begin{aligned} \delta F &= \int (\delta f_1 + \delta f_2) d^3x = \int \delta f_1 d^3x \\ &= \int [\xi_1(\mathbf{n} \cdot \nabla\varphi - \cos^2\theta)\mathbf{n} \cdot \nabla\delta\varphi + \xi_2(|\nabla\varphi|^2 - \cos^2\theta)\nabla\varphi \cdot \nabla\delta\varphi] d^3x \\ &= - \int \nabla \cdot [\xi_1(\mathbf{n} \cdot \nabla\varphi - \cos^2\theta)\mathbf{n} + \xi_2(|\nabla\varphi|^2 - \cos^2\theta)\nabla\varphi] \cdot \delta\varphi d^3x. \end{aligned}$$

Therefore, the penetration g is

$$g = -\frac{\delta F}{\delta\varphi} = -\nabla \cdot \tau, \tag{2.6}$$

$$\tau = -\xi_1(\mathbf{n} \cdot \nabla\varphi - \cos^2\theta)\mathbf{n} - \xi_2(|\nabla\varphi|^2 - \cos^2\theta)\nabla\varphi. \tag{2.7}$$

Here we can see that the penetration of smectic-C includes the functions of the tilt angle of the molecular orientation and the position of the layer of the LC.

2.3. Calculating elastic stress. By definition, the elastic stress is the variation of the free energy with respect to the deformation of the material, keeping the layer position fixed. We assume that the material undergoes an infinitesimal deformation, but the relative configuration of the layers remains fixed:

$$\begin{aligned} \mathbf{r}' &= \mathbf{r} + \mathbf{u}(\mathbf{r}), \\ \varphi'(\mathbf{r}') &= \varphi(\mathbf{r}), \\ \mathbf{n}'(\mathbf{r}') &= \mathbf{n}(\mathbf{r}). \end{aligned}$$

Calculation yields (ignoring the higher-order terms)

$$\frac{\partial r'_k}{\partial r_j} = \delta_{kj} + \frac{\partial u_k}{\partial r_j}, \quad \frac{\partial r_j}{\partial r'_k} = \delta_{kj} - \frac{\partial u_j}{\partial r_k}, \tag{2.8}$$

and

$$\frac{\partial \varphi'}{\partial r'_j} = \frac{\partial \varphi}{\partial r_j} - \frac{\partial \varphi}{\partial r_k} \frac{\partial u_k}{\partial r_j}, \tag{2.9}$$

or

$$\nabla' \varphi' = (\mathbf{I} - \nabla \mathbf{u}) \nabla \varphi. \tag{2.10}$$

Hence

$$\delta \nabla \varphi = -\nabla \mathbf{u} \nabla \varphi, \tag{2.11}$$

where $(\nabla \mathbf{u})_{ij} = \frac{\partial u_j}{\partial r_i}$. Similarly, we have

$$\delta \nabla \mathbf{n} = -\nabla \mathbf{u} \nabla \mathbf{n}. \tag{2.12}$$

Further calculation results in,

$$|\nabla'\varphi'| = |\nabla\varphi| \left(1 - \left(\frac{\nabla\varphi}{|\nabla\varphi|} \right)^T \nabla\mathbf{u} \frac{\nabla\varphi}{|\nabla\varphi|} \right). \tag{2.13}$$

Therefore,

$$\delta|\nabla\varphi| = -|\nabla\varphi| \left(\frac{\nabla\varphi}{|\nabla\varphi|} \right)^T \nabla\mathbf{u} \frac{\nabla\varphi}{|\nabla\varphi|}. \tag{2.14}$$

Setting $\mathbf{l} = \frac{\nabla\varphi}{|\nabla\varphi|}$,

$$\begin{aligned} \delta|\nabla\varphi| &= -|\nabla\varphi| \left(\frac{\nabla\varphi}{|\nabla\varphi|} \right)^T \nabla\mathbf{u} \frac{\nabla\varphi}{|\nabla\varphi|} \\ &= -|\nabla\varphi| \mathbf{l}^T \nabla\mathbf{u} \mathbf{l} \\ &= -Tr[(|\nabla\varphi| \mathbf{l} \otimes \mathbf{l}) \nabla\mathbf{u}], \end{aligned}$$

and

$$\begin{aligned} \delta(\nabla\mathbf{n} : \nabla\mathbf{n}) &= 2(\delta\nabla\mathbf{n} : \nabla\mathbf{n}) \\ &= -2[(\nabla\mathbf{u} \nabla\mathbf{n}) : \nabla\mathbf{n}] \\ &= -Tr[2\nabla\mathbf{n}(\nabla\mathbf{n})^T \nabla\mathbf{u}]. \end{aligned}$$

By using these results, we can deduce the variation of the free energy. Integration by part yields

$$\begin{aligned} \delta F &= \int \delta \left[\frac{1}{2} \xi_1 (\mathbf{n} \cdot \nabla\varphi - \cos^2\theta)^2 + \frac{1}{4} \xi_2 (|\nabla\varphi|^2 - \cos^2\theta)^2 + \frac{1}{2} K (\nabla\mathbf{n} : \nabla\mathbf{n})^2 \right] d^3r \\ &= \int \left\{ -\xi_1 (\mathbf{n} \cdot \nabla\varphi - \cos^2\theta) \mathbf{n} \cdot \nabla\mathbf{u} \nabla\varphi - \xi_2 (|\nabla\varphi|^2 - \cos^2\theta) |\nabla\varphi| Tr[(\nabla\varphi \otimes \nabla\varphi) \nabla\mathbf{u}] \right. \\ &\quad \left. - K (\nabla\mathbf{n} : \nabla\mathbf{n}) Tr[2\nabla\mathbf{n}(\nabla\mathbf{n})^T \nabla\mathbf{u}] \right\} d^3r \\ &= \int Tr \left[-\xi_1 (\mathbf{n} \cdot \nabla\varphi - \cos^2\theta) \mathbf{n}^T \otimes \nabla\varphi - \xi_2 (|\nabla\varphi|^2 - \cos^2\theta) \nabla\varphi \otimes \nabla\varphi \right. \\ &\quad \left. - 2K (\nabla\mathbf{n} : \nabla\mathbf{n}) \nabla\mathbf{n}(\nabla\mathbf{n})^T \nabla\mathbf{u} \right] d^3r \\ &= \int Tr(\sigma^T \nabla\mathbf{u}) d^3r, \end{aligned}$$

where,

$$\begin{aligned} \sigma^T &= -\xi_1 (\mathbf{n} \cdot \nabla\varphi - \cos^2\theta) \mathbf{n}^T \otimes \nabla\varphi - \xi_2 (|\nabla\varphi|^2 - \cos^2\theta) \nabla\varphi \otimes \nabla\varphi \\ &\quad - 2K (\nabla\mathbf{n} : \nabla\mathbf{n}) \nabla\mathbf{n}(\nabla\mathbf{n})^T. \end{aligned}$$

This is the asymmetrical part of the elastic stress. We can obtain the elastic stress by adding the symmetrical part.

3. Hydrodynamics theory

In this section, we will obtain the smectic-C hydrodynamic model using the law of conservation of fluid dynamics equations. Using the parity of the reversible variable in time, we further simplify the equations. Here we refer to the works of the nematic hydrodynamic equations by Ericksen and Leslie [11, 12, 13, 18].

Now we introduce the hydrodynamic variables: the mass density ρ , the momentum density $\rho\mathbf{u}$, the energy density E , and the above variables φ and \mathbf{n} . Using the laws of conservation mass, momentum and energy, the fluid dynamics equations are as follows:

$$\begin{aligned} \rho_t + \nabla \cdot (\rho\mathbf{u}) &= 0, \\ (\rho\mathbf{u})_t + \nabla \cdot (\rho\mathbf{u} \otimes \mathbf{u} + \sigma) &= 0, \\ E_t + \nabla \cdot J_E &= 0, \\ (\rho S)_t + \nabla \cdot J_S &= \frac{D}{T}, \end{aligned}$$

where S is the entropy, T is the temperature, σ is the second tensor, J_E and J_S are relative energy and relative entropy, respectively, and $\frac{D}{T}$ is the entropy production term. The second law of thermodynamics states that the term D will disappear in the inverse process and should be positive in the positive process.

3.1. Validation model satisfies the second law of thermodynamics. We will use the main variables ρ (density), \mathbf{u} (velocity), E (energy density), φ (position function of layer), and \mathbf{n} (orientation of molecule). Below, we use material derivation, namely

$$\dot{f} = f_t + (\mathbf{u} \cdot \nabla)f. \tag{3.1}$$

Set

$$\dot{\varphi} + X = 0, (\nabla \dot{\mathbf{n}} : \nabla \mathbf{n}) + Y = 0.$$

Next we only need to determine X and Y . In physics, we have

$$\begin{aligned} E &= \frac{1}{2}\rho|\mathbf{u}|^2 + \rho e, \\ \dot{\rho} + \rho(\nabla \cdot \mathbf{u}) &= 0, \\ \rho\dot{\mathbf{u}} &= \nabla \cdot \sigma, \end{aligned}$$

where e is the internal energy. From the energy conservation in a volume Ω , we get

$$\frac{d}{dt} \int_{\Omega} E d^3x = \int_{\partial\Omega} (\sigma\mathbf{u} + \dot{\varphi}\tau) \cdot d\mathbf{s} - \int_{\partial\Omega} \mathbf{q} \cdot d\mathbf{s}, \tag{3.2}$$

where \mathbf{q} is the heat flux. The first term on the right-hand side is the rate of work done at the boundary $\partial\Omega$ by the stress to the fluid in Ω . The second term is the rate of work done at the boundary to the layers. The third term is the amount of heat taken out of the volume Ω . Equation (3.2) can be written as

$$E_t + \nabla \cdot (E\mathbf{u} + \mathbf{q} - \sigma\mathbf{u} - \dot{\varphi}\tau) = 0. \tag{3.3}$$

Since

$$\left(\frac{1}{2}\rho|\mathbf{u}|^2\right)_t + \nabla \cdot \left(\frac{1}{2}\rho|\mathbf{u}|^2\mathbf{u} - \sigma\mathbf{u}\right) = -Tr(\sigma\nabla\mathbf{u}), \tag{3.4}$$

inserting (3.4) into (3.3) and taking $E = \frac{1}{2}\rho|\mathbf{u}|^2 + \rho e$, we have

$$\rho\dot{e} + \nabla \cdot \mathbf{q} - \nabla \cdot (\dot{\varphi}\tau) = Tr(\sigma\nabla\mathbf{u}).$$

We use the second law of thermodynamics in the following form:

$$\rho\dot{S} + \nabla \cdot \left(\frac{\mathbf{q}}{T}\right) \geq 0.$$

By the definition of Helmholtz free-energy density, we have $F = e - TS$. So

$$\begin{aligned} \rho\dot{F} &= \rho(\dot{e} - T\dot{S} - S\dot{T}) \\ &= Tr(\sigma^T \nabla \mathbf{u}) + \nabla \cdot (\dot{\varphi}\tau) - \nabla \cdot \mathbf{q} - T\rho\dot{S} - \rho S\dot{T} \\ &\leq Tr(\sigma^T \nabla \mathbf{u}) + \nabla \cdot (\dot{\varphi}\tau) - \frac{1}{T}\mathbf{q} \cdot \nabla T - \rho S\dot{T}. \end{aligned}$$

Let $\alpha = (\nabla \mathbf{n} : \nabla \mathbf{n})$, $\beta = |\nabla \varphi|$. Here F is a function that depends on ρ, T, α , and β . Therefore,

$$\dot{F} = F_T \dot{T} + F_\rho \dot{\rho} + F_\alpha \dot{\alpha} + F_\beta \dot{\beta}.$$

Moreover,

$$\begin{aligned} \dot{\beta} &= \frac{1}{|\nabla \varphi|} \nabla \varphi \cdot (\nabla \varphi)', \\ \nabla \dot{\varphi} &= (\nabla \varphi)' + \nabla \mathbf{u} \nabla \varphi, \\ (\nabla \mathbf{n} : \nabla \mathbf{n})' &= 2(\nabla \mathbf{n})' : \nabla \mathbf{n}, \\ (\nabla \mathbf{n})' &= \nabla \dot{\mathbf{n}} - \nabla \mathbf{u} \nabla \mathbf{n}. \end{aligned}$$

Thus,

$$Tr(\sigma \nabla \mathbf{u}) - \frac{1}{T} \mathbf{q} \cdot \nabla T - \rho \dot{F} (S + F_T) - \rho F_\rho \dot{\rho} - \rho F_\beta \dot{\beta} - \rho F_\alpha \dot{\alpha} \geq 0.$$

Because \dot{T} is arbitrary, we set $S = -F_T$. So

$$\begin{aligned} 0 &\leq Tr(\sigma \nabla \mathbf{u}) - \frac{1}{T} \mathbf{q} \cdot \nabla T - \rho F_\rho \dot{\rho} - \rho \frac{F_\beta}{|\nabla \varphi|} \nabla \varphi \cdot (\nabla \varphi)' - \rho F_\alpha (\nabla \mathbf{n} : \nabla \mathbf{n})' \\ &= Tr[(\sigma + \rho F_\beta |\nabla \varphi| \mathbf{1} \otimes \mathbf{1}) \nabla \mathbf{u}] - \frac{1}{T} \mathbf{q} \cdot \nabla T + \rho^2 F_\rho (\nabla \cdot \mathbf{u}) + \dot{\varphi} \nabla \cdot (\rho F_\beta \mathbf{1}) \\ &\quad - 2\rho F_\alpha [(\nabla \dot{\mathbf{n}} - \nabla \mathbf{u} \nabla \mathbf{n}) : \nabla \mathbf{n}] \\ &= Tr[(\sigma + \rho F_\beta |\nabla \varphi| \mathbf{1} \otimes \mathbf{1} + 2\rho F_\alpha \nabla \mathbf{n} (\nabla \mathbf{n})^T) \nabla \mathbf{u}] \\ &\quad - \frac{1}{T} \mathbf{q} \cdot \nabla T + \rho^2 F_\rho (\nabla \cdot \mathbf{u}) + \dot{\varphi} \nabla \cdot (\rho F_\beta \mathbf{1}) - 2\rho F_\alpha (\nabla \dot{\mathbf{n}} : \nabla \mathbf{n}). \end{aligned}$$

Comparing this with that in Section 2 and replacing ρF in Section 2 by F , set

$$\begin{aligned} \sigma^e &= -\rho F_\beta |\nabla \varphi| \mathbf{1} \otimes \mathbf{1} - 2\rho F_\alpha \nabla \mathbf{n} (\nabla \mathbf{n})^T \\ &= -\xi_1 (\mathbf{n} \cdot \nabla \varphi - \cos^2 \theta) \mathbf{n}^T \otimes \nabla \varphi - \xi_2 (|\nabla \varphi|^2 - \cos^2 \theta) \nabla \varphi \otimes \nabla \varphi \\ &\quad - 2K (\nabla \mathbf{n} : \nabla \mathbf{n}) \nabla \mathbf{n} (\nabla \mathbf{n})^T, \\ g^I &= \nabla \cdot (\rho F_\beta \mathbf{1}) = \nabla \cdot [\xi_1 (\mathbf{n} \cdot \nabla \varphi - \cos^2 \theta) \mathbf{n} + \xi_2 (|\nabla \varphi|^2 - \cos^2 \theta) \nabla \varphi] = -\nabla \cdot \tau, \\ h^I &= -2\rho F_\alpha = -2K (\nabla \mathbf{n} : \nabla \mathbf{n}), \\ \sigma &= -pI + \sigma^e + \sigma^d, \end{aligned}$$

where p is pressure. Hence

$$\text{Tr}((\sigma^d)\nabla\mathbf{u}) + (\rho^2 F_\rho - p)\nabla \cdot \mathbf{u} - \frac{1}{T}\mathbf{q} \cdot \nabla T + \dot{\varphi}g^I + h^I(\nabla\dot{\mathbf{n}} : \nabla\mathbf{n}) \geq 0. \tag{3.5}$$

In most cases, the volume compression of fluid is negligible. The smectic-C LC can be regarded as an incompressible fluid, namely

$$\nabla \cdot \mathbf{u} = 0.$$

If not, we take p equal to the thermodynamic pressure, namely

$$p = \rho^2 F_\rho.$$

Therefore, (3.5) can be simplified as

$$\text{Tr}((\sigma^d)\nabla\mathbf{u}) - \frac{1}{T}\mathbf{q} \cdot \nabla T + \dot{\varphi}g^I + h^I(\nabla\dot{\mathbf{n}} : \nabla\mathbf{n}) \geq 0, \tag{3.6}$$

or

$$\text{Tr}((\sigma^d)\nabla\mathbf{u}) - \frac{1}{T}\mathbf{q} \cdot \nabla T - Xg^I - Yh^I \geq 0.$$

This can meet the needs of the model, which satisfies the second law of thermodynamics.

3.2. Simplifying model of smectic-C phase. The last step is to establish the relation between σ^d , \mathbf{q} , $\dot{\varphi}$, $(\nabla\dot{\mathbf{n}} : \nabla\mathbf{n})$ and $\nabla\mathbf{u}$, $-\frac{1}{T}\nabla T$, g^I , h^I . Therefore, we discuss the linear equations of the fluid. Most of the linear relations are as follows:

$$\begin{aligned} \sigma_{ij}^d &= A_{ijkl}^1 \frac{\partial u_l}{\partial r_k} + B_{ijk}^1 \frac{1}{T} \frac{\partial T}{\partial r_k} + C_{ij}^1 g^I + D_{ij}^1 h^I, \\ q_i &= A_{ijk}^2 \frac{\partial u_k}{\partial r_j} + B_{ij}^2 \frac{1}{T} \frac{\partial T}{\partial r_j} + C_i^2 g^I + D_i^2 h^I, \\ -X &= A_{jk}^3 \frac{\partial u_j}{\partial r_k} + B_i^3 \frac{1}{T} \frac{\partial T}{\partial r_i} + C^3 g^I + D^3 h^I, \\ -Y &= A_{jk}^4 \frac{\partial u_j}{\partial r_k} + B_i^4 \frac{1}{T} \frac{\partial T}{\partial r_i} + C^4 g^I + D^4 h^I. \end{aligned}$$

According to [9], we reduce

$$\begin{aligned} \sigma_{ij}^d &= A_{ijkl}^1 \frac{\partial u_l}{\partial r_k}, \\ q_i &= B_{ij}^2 \frac{1}{T} \frac{\partial T}{\partial r_j} + C_i^2 g^I + D_i^2 h^I, \\ -X &= B_i^3 \frac{1}{T} \frac{\partial T}{\partial r_i} + C^3 g^I + D^3 h^I, \\ -Y &= B_i^4 \frac{1}{T} \frac{\partial T}{\partial r_i} + C^4 g^I + D^4 h^I. \end{aligned}$$

We refer to [9, 14] and choose σ^d of the following form:

$$\sigma^d = \mu_1(\mathbf{n}^T D\mathbf{n})\mathbf{n} \otimes \mathbf{n} + \mu_4 D + \mu_5(D\mathbf{n} \otimes \mathbf{n} + \mathbf{n} \otimes D\mathbf{n}).$$

In most cases, we assume that the fluid is incompressible. Therefore, we obtain the smectic-C phase dynamic equations as follows:

$$\begin{aligned} \rho \dot{\mathbf{u}} &= \nabla \cdot \sigma, \\ \nabla \cdot \mathbf{u} &= 0, \\ E_t + \nabla \cdot (E\mathbf{u} + \mathbf{q} - \sigma\mathbf{u} - \dot{\varphi}\tau) &= 0, \\ \dot{\varphi} &= B_i^3 \frac{1}{T} \frac{\partial T}{\partial r_i} + C^3 g^I + D^3 h^I, \\ (\nabla \dot{\mathbf{n}} : \nabla \mathbf{n}) &= B_i^4 \frac{1}{T} \frac{\partial T}{\partial r_i} + C^4 g^I + D^4 h^I, \\ |\mathbf{n}| &= 1, \end{aligned}$$

where

$$\begin{aligned} \sigma &= -pI + \sigma^e + \sigma^d, \\ q_i &= B_{ij}^2 \frac{1}{T} \frac{\partial T}{\partial r_j} + C_i^2 g^I + D_i^2 h^I, \\ g^I &= \nabla \cdot (\rho F_\beta \mathbf{1}), \\ h^I &= -2\rho F_\alpha. \end{aligned}$$

Here, we mainly investigate the transformation of the fluid instead of the temperature. Moreover, there are phase transitions when the temperature changes. Therefore, we consider the isothermal conditions. Thus the model can be simplified as

$$\begin{aligned} \rho \dot{\mathbf{u}} &= \nabla \cdot \sigma, \\ \nabla \cdot \mathbf{u} &= 0, \\ \dot{\varphi} &= C^3 g^I + D^3 h^I, \\ (\nabla \dot{\mathbf{n}} : \nabla \mathbf{n}) &= C^4 g^I + D^4 h^I, \\ |\mathbf{n}| &= 1. \end{aligned}$$

In the following, we adopt the dimensionless isothermal model for simplicity. Without the loss of generality, we assume the coefficients C^3 , D^3 , C^4 , and D^4 in the terms g^I and h^I are as follows:

$$C^3 = a, \quad D^3 = 1, \quad C^4 = 1, \quad D^4 = -d,$$

where $a > 0$ and $d > 0$. Therefore the isothermal incompressible model is as follows:

$$\nabla \cdot \mathbf{u} = 0, \tag{3.7}$$

$$\rho \mathbf{u}_t + \rho \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \nabla \cdot (\sigma^e + \sigma^d), \tag{3.8}$$

$$\varphi_t + \mathbf{u} \cdot \nabla \varphi = a \nabla \cdot [\xi_1 (\mathbf{n} \cdot \nabla \varphi - \cos^2 \theta) \mathbf{n} + \xi_2 (|\nabla \varphi|^2 - \cos^2 \theta) \nabla \varphi] + 2K (\nabla \mathbf{n} : \nabla \mathbf{n}), \tag{3.9}$$

$$(\nabla \dot{\mathbf{n}} : \nabla \mathbf{n}) = \nabla \cdot [\xi_1 (\mathbf{n} \cdot \nabla \varphi - \cos^2 \theta) \mathbf{n} + \xi_2 (|\nabla \varphi|^2 - \cos^2 \theta) \nabla \varphi] - 2dK (\nabla \mathbf{n} : \nabla \mathbf{n}), \tag{3.10}$$

$$|\mathbf{n}| = 1, \tag{3.11}$$

where a and d are positive constants, and

$$\begin{aligned} \sigma^e = & -\xi_1(\mathbf{n} \cdot \nabla\varphi - \cos^2\theta)\mathbf{n}^T \otimes \nabla\varphi - \xi_2(|\nabla\varphi|^2 - \cos^2\theta)\nabla\varphi \otimes \nabla\varphi \\ & - 2K(\nabla\mathbf{n} : \nabla\mathbf{n})\nabla\mathbf{n}(\nabla\mathbf{n})^T, \end{aligned}$$

$$\sigma^d = \mu_1(\mathbf{n}^T D\mathbf{n})\mathbf{n} \otimes \mathbf{n} + \mu_4 D + \mu_5(D\mathbf{n} \otimes \mathbf{n} + \mathbf{n} \otimes D\mathbf{n}),$$

$$D = (d_{km}), d_{km} = \frac{1}{2} \left(\frac{\partial \mathbf{u}_k}{\partial \mathbf{r}_m} + \frac{\partial \mathbf{u}_m}{\partial \mathbf{r}_k} \right).$$

By comparing this model with the nematic model in [5, 14], we find the equations (3.7)–(3.11) are a simpler isothermal incompressible model. However, it is certainly more complex than the smectic-A model in [9] because of the tilt angle of the molecule orientation. Here, we see the tilt of the orientation, variation of the layer, and deformation of molecules from the equations of (3.9) and (3.10). Therefore, it is very rational. But this model can not predict the smectic-A to smectic-C phase transition since the angle θ is a given parameter.

REMARK 3.1. In the above hydrostatic theory, the smectic free energy density is

$$f_1 = \frac{1}{2}\xi_1(\mathbf{n} \cdot \nabla\varphi - \cos^2\theta)^2 + \frac{1}{4}\xi_2(|\nabla\varphi|^2 - \cos^2\theta)^2.$$

The second term is concerned with the incompressible LC layers. Therefore, using $|\nabla\varphi| = \cos\theta$, we can derive the isothermal dual incompressible model of smectic-C as (3.7)–(3.11) with $|\nabla\varphi| - \cos\theta = 0$. We call it the isothermal dual incompressible model.

4. Energy law of smectic-C phase

In this section, we will verify that the system (3.7)–(3.11), with certain initial values and boundary conditions, satisfies the energy law. We first add the initial conditions for \mathbf{u} , ϕ , and \mathbf{n} :

$$\mathbf{u}(x, 0) = \mathbf{u}_0(x), \phi(x, 0) = \phi_0(x), \mathbf{n}(x, 0) = \mathbf{n}_0(x), \forall x \in \Omega, \tag{4.1}$$

and the boundary conditions:

$$\mathbf{u}|_{\partial\Omega} = 0, \phi|_{\partial\Omega} = \phi_0(x)_{\partial\Omega}, \mathbf{n}|_{\partial\Omega} = \mathbf{n}_0, (x, t) \in \partial\Omega \times (0, \infty). \tag{4.2}$$

Similar to the method in [19, 28], we verify the energy law which highlights the dissipative nature of the system.

THEOREM 4.1. *If \mathbf{u} , ϕ , and \mathbf{n} are a smooth solution of the system (3.7)–(3.11) with the boundary and initial conditions (4.1)–(4.2), the following energy law holds:*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |\mathbf{u}|^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega} \xi_1 (\mathbf{n} \cdot \nabla\varphi - \cos^2\theta)^2 dx \\ & + \frac{1}{4} \frac{d}{dt} \int_{\Omega} \xi_2 (|\nabla\varphi|^2 - \cos^2\theta)^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega} K (\nabla\mathbf{n} : \nabla\mathbf{n})^2 dx \\ = & - \int_{\Omega} [\mu_1 (\mathbf{n}^T D\mathbf{n})^2 + \mu_4 D^2 + 2\mu_5 |D\mathbf{n}|^2] dx - \int_{\Omega} 4dK^2 (\nabla\mathbf{n} : \nabla\mathbf{n})^2 dx \\ & - \int_{\Omega} a \{ \nabla \cdot [\xi_1 (\mathbf{n} \cdot \nabla\varphi - \cos^2\theta) \cdot \mathbf{n}] + \nabla \cdot [\xi_2 (|\nabla\varphi|^2 - \cos^2\theta) \nabla\varphi] \}^2 dx. \end{aligned} \tag{4.3}$$

Proof. We multiply $\frac{|\mathbf{u}|^2}{2}$ to the mass conservation law and integrate it in Ω , considering the relation (3.8). It yields

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |\mathbf{u}|^2 dx \\
 &= \int_{\Omega} \left[\rho_t \cdot \frac{|\mathbf{u}|^2}{2} + (\rho \mathbf{u}_t) \cdot \mathbf{u} \right] dx \\
 &= \int_{\Omega} [-(\rho \mathbf{u} \cdot \nabla \mathbf{u}) \cdot \mathbf{u} + \nabla \cdot (-pI + \sigma^e + \sigma^d) \cdot \mathbf{u}] dx \\
 &= - \int_{\Omega} [\mu_1 (\mathbf{n}^T D \mathbf{n})^2 + \mu_4 D^2 + 2\mu_5 |D \mathbf{n}|^2] dx - \int_{\Omega} \nabla \cdot [\xi_1 (\mathbf{n} \cdot \nabla \varphi - \cos^2 \theta) \mathbf{n} \otimes \nabla \varphi] \cdot \mathbf{u} dx \\
 &\quad - \int_{\Omega} \nabla \cdot [\xi_2 (|\nabla \varphi|^2 - \cos^2 \theta) \nabla \varphi \otimes \nabla \varphi] \cdot \mathbf{u} dx - \int_{\Omega} \nabla \cdot [2K (\nabla \mathbf{n} : \nabla \mathbf{n}) \nabla \mathbf{n} (\nabla \mathbf{n})^T] \cdot \mathbf{u} dx.
 \end{aligned} \tag{4.4}$$

Next we use (3.9) and integrate by parts to deduce

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \int_{\Omega} \xi_1 (\mathbf{n} \cdot \nabla \varphi - \cos^2 \theta)^2 dx = \int_{\Omega} \xi_1 (\mathbf{n} \cdot \nabla \varphi - \cos^2 \theta) \frac{d(\mathbf{n} \cdot \nabla \varphi)}{dt} dx \\
 &= - \int_{\Omega} a \{ \nabla \cdot [\xi_1 (\mathbf{n} \cdot \nabla \varphi - \cos^2 \theta) \mathbf{n}] \}^2 dx \\
 &\quad - \int_{\Omega} a \nabla \cdot [\xi_1 (\mathbf{n} \cdot \nabla \varphi - \cos^2 \theta) \mathbf{n}] \cdot \nabla \cdot [\xi_2 (|\nabla \varphi|^2 - \cos^2 \theta) \nabla \varphi] dx \\
 &\quad - \int_{\Omega} 2K (\nabla \mathbf{n} : \nabla \mathbf{n}) \nabla \cdot [\xi_1 (\mathbf{n} \cdot \nabla \varphi - \cos^2 \theta) \mathbf{n}] dx + \int_{\Omega} \nabla \cdot [\xi_1 (\mathbf{n} \cdot \nabla \varphi - \cos^2 \theta) \mathbf{n} \otimes \nabla \varphi] \cdot \mathbf{u} dx.
 \end{aligned} \tag{4.5}$$

Similarly, using (3.9), we can calculate the energy of the layers as follows:

$$\begin{aligned}
 & \frac{1}{4} \frac{d}{dt} \int_{\Omega} \xi_2 (|\nabla \varphi|^2 - \cos^2 \theta)^2 dx = \int_{\Omega} \xi_2 (|\nabla \varphi|^2 - \cos^2 \theta) |\nabla \varphi| \frac{d(|\nabla \varphi|)}{dt} dx \\
 &= - \int_{\Omega} a \{ \nabla \cdot [\xi_2 (|\nabla \varphi|^2 - \cos^2 \theta) \nabla \varphi] \}^2 dx \\
 &\quad - \int_{\Omega} a \nabla \cdot [\xi_1 (\mathbf{n} \cdot \nabla \varphi - \cos^2 \theta) \mathbf{n}] \cdot \nabla \cdot [\xi_2 (|\nabla \varphi|^2 - \cos^2 \theta) \nabla \varphi] dx \\
 &\quad - \int_{\Omega} 2K (\nabla \mathbf{n} : \nabla \mathbf{n}) \nabla \cdot [\xi_2 (|\nabla \varphi|^2 - \cos^2 \theta) \nabla \varphi] dx \\
 &\quad + \int_{\Omega} \nabla \cdot [\xi_2 (|\nabla \varphi|^2 - \cos^2 \theta) \nabla \varphi] \mathbf{u} \cdot \nabla \varphi dx.
 \end{aligned} \tag{4.6}$$

Then, using (3.10), we can deduce the energy of the molecule field:

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \int_{\Omega} K (\nabla \mathbf{n} : \nabla \mathbf{n})^2 dx = \int_{\Omega} K (\nabla \mathbf{n} : \nabla \mathbf{n}) \frac{d}{dt} (\nabla \mathbf{n} : \nabla \mathbf{n}) dx \\
 &= \int_{\Omega} 2K (\nabla \mathbf{n} : \nabla \mathbf{n}) \nabla \cdot [\xi_1 (\mathbf{n} \cdot \nabla \varphi - \cos^2 \theta) \cdot \mathbf{n} + \xi_2 (|\nabla \varphi|^2 - \cos^2 \theta) \nabla \varphi] dx \\
 &\quad - \int_{\Omega} 4dK^2 (\nabla \mathbf{n} : \nabla \mathbf{n})^2 dx - \int_{\Omega} 2K (\nabla \mathbf{n} : \nabla \mathbf{n}) [(\nabla (\mathbf{u} \cdot \nabla \mathbf{n}) : \nabla \mathbf{n})] dx.
 \end{aligned}$$

Next we calculate the last term on the right-hand side. Since $[(\nabla (\mathbf{u} \cdot \nabla \mathbf{n}) : \nabla \mathbf{n})] = (\nabla \mathbf{u} \nabla \mathbf{n} : \nabla \mathbf{n}) + \frac{1}{2} \mathbf{u} \cdot \nabla (\nabla \mathbf{n} : \nabla \mathbf{n})$, we obtain

$$\int_{\Omega} K (\nabla \mathbf{n} : \nabla \mathbf{n}) \mathbf{u} \cdot \nabla (\nabla \mathbf{n} : \nabla \mathbf{n}) dx = \int_{\Omega} \frac{1}{2} K \mathbf{u} \cdot \nabla ((\nabla \mathbf{n} : \nabla \mathbf{n})^2) dx = 0.$$

Therefore,

$$\begin{aligned}
 - \int_{\Omega} 2K(\nabla \mathbf{n} : \nabla \mathbf{n})(\nabla(\mathbf{u} \cdot \nabla \mathbf{n}) : \nabla \mathbf{n}) dx &= - \int_{\Omega} 2K(\nabla \mathbf{n} : \nabla \mathbf{n})(\nabla \mathbf{u} \nabla \mathbf{n} : \nabla \mathbf{n}) dx \\
 &= \int_{\Omega} 2K \nabla \cdot [(\nabla \mathbf{n} : \nabla \mathbf{n}) \nabla \mathbf{n} (\nabla \mathbf{n})^T] \cdot \mathbf{u} dx.
 \end{aligned}$$

So the energy of the molecule field satisfies

$$\begin{aligned}
 &\frac{1}{2} \frac{d}{dt} \int_{\Omega} K(\nabla \mathbf{n} : \nabla \mathbf{n})^2 dx \\
 &= \int_{\Omega} 2K(\nabla \mathbf{n} : \nabla \mathbf{n}) \nabla \cdot [\xi_1(\mathbf{n} \cdot \nabla \varphi - \cos^2 \theta) \cdot \mathbf{n} + \xi_2(|\nabla \varphi|^2 - \cos^2 \theta) \nabla \varphi] dx \\
 &\quad - \int_{\Omega} 4dK^2(\nabla \mathbf{n} : \nabla \mathbf{n})^2 dx + \int_{\Omega} 2K \nabla \cdot [(\nabla \mathbf{n} : \nabla \mathbf{n}) \nabla \mathbf{n} (\nabla \mathbf{n})^T] \cdot \mathbf{u} dx. \tag{4.7}
 \end{aligned}$$

Adding (4.4), (4.5), (4.6), and (4.7), we get (4.3) immediately.

From the energy law (4.3), it is not hard to find the molecular kinetic energy (1st term), the tilt energy of the orientation (2nd term), the energy of the layer (3rd term), and the deformation energy of molecules (4th term) on the left-hand side. All on the right-hand side are the negative terms. Therefore, the isothermal incompressible model of the smectic-C phase (3.7)–(3.11) is an energy dissipation system. That is to say, we have proven the rationality of the model from the energy law. \square

5. Incompressible Limit

First, we must clarify that the compressibility of the fluid, and the compressibility of the layers are separate issues. Bulk compression of the fluid gives rise to the conventional hydrodynamic pressure which acts as a Lagrange multiplier for incompressible fluids; i.e.,

$$\nabla \cdot \mathbf{u} = 0.$$

The compression of the layers gives rise to the term associated with the compression modulus ξ_2 in the elastic stress. In contrast to the hydrodynamic pressure, which is isotropic, this force acts only in the direction normal to the layers. The constraint associated with the incompressibility of the layers is

$$|\nabla \varphi| = \cos \theta.$$

Next, we will derive the characteristic of the dynamic equations of the smectic-C phase for incompressible fluid, and liquid crystal layers from separate situations.

5.1. Derivation of the incompressible-limit equations using asymptotic analysis. The isothermal dual incompressible model we obtain in Remark 3.1 can also be derived from the equations for compressible layers by using systematic asymptotic analysis in the limit as $\xi_2 \rightarrow \infty$. We present this derivation for the isothermal case where the algebra is slightly easier.

Let $\varepsilon = \frac{1}{\xi_2}$. We look for solutions of (3.7) of the following form:

$$\begin{aligned}
 \varphi &= \varphi_0 + \varepsilon \varphi_1 + O(\varepsilon^2), \\
 \mathbf{n} &= \mathbf{n}_0 + \varepsilon \mathbf{n}_1 + O(\varepsilon^2), \\
 \mathbf{u} &= \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + O(\varepsilon^2),
 \end{aligned}$$

$$p = p_0 + \varepsilon p_1 + O(\varepsilon^2).$$

Substituting these into (3.7), we find

$$\begin{aligned} \tau &= \frac{1}{\varepsilon} \tau_{-1} + \tau_0 + O(\varepsilon), \\ \sigma^e &= \frac{1}{\varepsilon} \sigma_{-1}^e + \sigma_0^e + O(\varepsilon), \end{aligned}$$

where

$$\begin{aligned} \tau_{-1} &= -(|\nabla\varphi_0|^2 - \cos^2\theta)\nabla\varphi_0, \\ \tau_0 &= -\xi_1(\mathbf{n}_0\nabla\varphi_0 - \cos^2\theta)\mathbf{n}_0 - 2(\nabla\varphi_0 \cdot \nabla\varphi_1)\nabla\varphi_0 - (|\nabla\varphi_0|^2 - \cos^2\theta)\nabla\varphi_1, \\ \sigma_{-1}^e &= -(|\nabla\varphi_0|^2 - \cos^2\theta)\nabla\varphi_0 \otimes \nabla\varphi_0, \\ \sigma_0^e &= \tau_0 \otimes \nabla\varphi_0 + \tau_{-1} \otimes \nabla\varphi_1 - 2K(\nabla\mathbf{n}_0 : \nabla\mathbf{n}_0)\nabla\mathbf{n}_0(\nabla\mathbf{n}_0)^T. \end{aligned}$$

Manipulating the $O(\frac{1}{\varepsilon})$ terms in the momentum equations, we get

$$\nabla \cdot \sigma_{-1}^e = 0.$$

Integrating it along thin tubes of the integral curves of $\nabla\varphi_0$, we conclude that $(|\nabla\varphi_0|^2 - \cos^2\theta)\nabla\varphi_0$ is constant along the integral curves of $\nabla\varphi_0$. There are only two possibilities. Either $|\nabla\varphi_0| \equiv \cos\theta$ along the integral curve, or $\nabla\varphi_0$ is a constant vector along the integral curve. The latter implies that the integral curve is a straight line, and $|\nabla\varphi_0|$ is also constant along the integral curve.

Let Σ be an iso-surface of φ_0 . In a neighborhood Ω of Σ , we define $g(x) = \text{dist}(\mathbf{x}, \Sigma)$. Assume that $\nabla\varphi_0$ is not identically equal to $\cos\theta$ in Ω . Then, because the integral curves of ∇g are also straight lines perpendicular to the tangent planes of Σ , there exists a function h such that $g = h(\varphi_0)$. Differentiating $h(\varphi_0)$ with respect to φ_0 , we get

$$|h'(\varphi_0)| = \frac{1}{|\nabla\varphi_0|}.$$

The left-hand side is a function that is constant along the iso-surfaces of φ_0 . The function on the right-hand side is constant along the integral curves of $\nabla\varphi_0$. Therefore we conclude that

$$|\nabla\varphi_0| \equiv C_0$$

in Ω , where C_0 is a constant. The terms $O(\frac{1}{\varepsilon})$ in the φ -equation give

$$\nabla \cdot \tau_{-1} = 0, \quad \text{i.e.,} \quad (|\nabla\varphi_0|^2 - \cos^2\theta)\nabla \cdot \nabla\varphi_0 = 0.$$

If we allow the layers to be curved, the term $\nabla \cdot \nabla\varphi_0$ is not equal to 0. Hence

$$|\nabla\varphi_0| = \cos\theta.$$

Then the expressions for the terms with $O(1)$ can be simplified as

$$\begin{aligned} \tau_0 &= -\xi_1(\mathbf{n}_0\nabla\varphi_0 - \cos^2\theta)\mathbf{n}_0, \\ \sigma_0^e &= \tau_0 \otimes \nabla\varphi_0 - 2K(\nabla\mathbf{n}_0 : \nabla\mathbf{n}_0)\nabla\mathbf{n}_0(\nabla\mathbf{n}_0)^T. \end{aligned}$$

Balancing the terms $O(1)$ in the equations, we obtain the isothermal dual incompressible model which is the system (3.7)–(3.11) with $|\nabla\varphi| = \cos\theta$.

Here the results obtained by the asymptotic expression of ε are consistent with the incompressible liquid crystal layer model. It is verified that the choice of the Lagrange multiplier is reasonable. We can also use a similar method to validate the rationality of the other Lagrange multipliers in the model.

6. Conclusion

We present a nonlinear continuum theory of smectic-C liquid crystals through the construction of the hydrostatic, and hydrodynamic theory, which involves the molecular orientation, layer order parameter, flow velocity, and hydrostatic pressure fields. First, we establish the free energy depending on the molecular orientation feature of smectic-C liquid crystals. The free energy includes the Oseen–Frank energy of nematic liquid crystals and the smectic-C energy. Then, we develop hydrodynamic equations for smectic-C liquid crystals from their permeation force, elastic stress, and the liquid conservation laws. We also obtain the simplified model through the parity of our variables under time reversal. In order to prove the rationality of the model, we verify the energy law. The kinetic energy, layer energy, molecular tilting energy, and distortion energy are found by our model, and we derive the energy dissipative relation. In addition, we verify that the system can also be obtained by using asymptotic analysis when both the fluid and layers are incompressible. The method to establish the model of smectic-C liquid crystals can be extended to other smectic liquid crystals.

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REFERENCES

- [1] G.I. Blake and F.M. Leslie, *A backflow effect in smectic-C liquid crystals*, *Liq. Cryst.*, 25, 319–327, 1998.
- [2] M.C. Calderer and S. Joo, *A continuum theory of chiral smectic-C liquid crystals*, *SIAM J. Appl. Math.*, 69, 787–809, 2008.
- [3] M.C. Calderer and C. Liu, *Liquid crystal flow: dynamic and static configurations*, *SIAM J. Appl. Math.*, 60, 1925–1949, 2000.
- [4] M.C. Calderer, C. Liu, and K. Voss, *Smectic-A liquid crystal configurations with interface defects*, *Math. Meth. Appl. Sci.*, 24, 473–489, 2001.
- [5] S. Chandrasekhar, *Liquid Crystals*, Second Edition, Cambridge University Press, 1992.
- [6] T. Carlson, N.A. Clark, and Z. Zou, *Theoretical studies of the influence of backflow on the dynamical behavior of Fredricks transition of a ferroelectric smectic-C* liquid crystal in the bookshelf geometry*, *Liq. Cryst.*, 15, 461–477 1993.
- [7] J.H. Chen and T.C. Lubensky, *Landau–Ginzburg mean-field theory for the nematic to smectic-C and nematic to smectic-A phase transitions*, *Phys. Rev. A*, 14, 1202–1207, 1976.
- [8] P.A. Cruz, M.F. Tome, I.W. Stewart, and S. McKee, *Numerical solution of the Ericksen–Leslie dynamic equations for two-dimensional nematic liquid crystal flows*, *J. Comput. Phys.*, 247, 109–136, 2013.
- [9] W.N. E, *Nonlinear continuum theory of smectic-A liquid crystals*, *Arch. Rat. Mech. Anal.*, 137, 159–175, 1997.
- [10] W.N. E and P. Palffy-Muhoray, *Dynamics of filaments in the isotropic-smectic-A phase transition*, *J. Nonlin. Sci.*, 9, 417–437, 1999.
- [11] J.L. Ericksen, *Hydrostatic theory of liquid crystals*, *Arch. Rat. Mech. Anal.*, 9, 371–378, 1962.
- [12] J.L. Ericksen, *Continuum theory of liquid crystals of nematic type*, *Molec. Cryst. Liq. Cryst.*, 7, 153–164, 1969.
- [13] J.L. Ericksen, *Continuum theory of nematic liquid crystals*, *Continuum Mech. Thermodyn.*, 4, 167–175, 1992.
- [14] P.G. de Gennes and J. Prost, *The Physics of Liquid Crystals*, Second Edition, Science Publications, New York, 1993.
- [15] E.I. Kats and V.V. Lebedev, *Fluctuational Effects in the Dynamics of Liquid Crystals*, Springer, 1993.
- [16] F.H. Lin and C. Liu, *Nonparabolic dissipative systems modeling the flow of liquid crystals*, *Commun. Pure Appl. Math.*, 48, 501–537, 1995.
- [17] F.H. Lin and C. Liu, *Existence of solutions for the Ericksen–Leslie System*, *Arch. Rat. Mech. Anal.*, 154, 135–156, 2000.
- [18] F.J. Leslie, *Theory of flow phenomena in liquid crystals*, *Adv. Liq. Cryst.*, 4, 1–81, 1979.

- [19] C. Liu, *Dynamic theory for incompressible smectic-A liquid crystals: existence and regularity*, Disc. Contin. Dyn. Sys., 6, 591–608, 2000.
- [20] G. McKay, *Modelling smectics in confined geometries*, J. Non-Newtonian Fluid Mech., 119, 115–122, 2004.
- [21] G. McKay and F.M. Leslie, *A continuum theory for smectic liquid crystals allowing layer dilation and compression*, Euro. J. Appl. Math., 8, 273–280, 1997.
- [22] M. Nakagawa, *A hydrodynamic theory of compressible SmC* liquid crystals*, J. Non-Newtonian Fluid Mech., 119, 123–129, 2004.
- [23] I.W. Stewart and E.J. Wigham, *Dynamics of cylindrical domain walls in smectic-C liquid crystals*, J. Phys. A: Math. Theor., 42(23), 235501, 2009.
- [24] G. Warnecke and H. Zhang, *Steady states of the 1D Doi-Onsager model in the strong shear flow*, Commun. Math. Sci., 8, 721–734, 2010.
- [25] H. Zhang and P. Zhang, *A theoretical and numerical study for the rod-like model of a polymeric fluid*, J. Comput. Math., 22, 319–330, 2004.
- [26] H. Zhang and P.W. Zhang, *Stable dynamic states at the nematic liquid crystals in weak shear flow*, Physica D, 232, 156–165, 2007.
- [27] H. Zhang and P.W. Zhang, *Review on Doi-Onsager model in polymeric fluids*, Recent progress in scientific computing, Wenbin Liu, Michael Ng and Zhong-Ci Shi (eds.), 155–167, 2007.
- [28] H. Zhang and P.W. Zhang, *On the new multiscale rodlike model of polymeric fluids*, SIAM J. Math. Anal., 40, 1246–1271, 2008.
- [29] H. Zhang and Q.C. Bai, *Numerical investigation of tumbling phenomena based on a macroscopic model for hydrodynamic nematic liquid crystals*, Commun. Comput. Phys., 7, 317–332, 2010.
- [30] S.P. Zhang, C. Liu, and H. Zhang, *Numerical simulations of hydrodynamics of nematic liquid crystals: effects of kinematic transports*, Commun. Comput. Phys., 9, 974–993, 2011.