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ON THE COMPLETE PHASE SYNCHRONIZATION FOR THE KURAMOTO MODEL IN THE MEAN-FIELD LIMIT*

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Abstract. We study the Kuramoto model for coupled oscillators. For the case of identical natural frequencies, we give a new proof of the complete frequency synchronization for all initial data; extending this result to the continuous version of the model, we manage to prove the complete phase synchronization for any non-atomic measure-valued initial datum. We also discuss the relation between the boundedness of the entropy and the convergence to an incoherent state for the case of non identical natural frequencies.

Key words. Kuramoto model, complete synchronization, coupled oscillators.

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1. Introduction

The Kuramoto model is a mean-field model of coupled oscillators which exhibits spontaneous synchronization in a certain range of parameters (see [5, 8]). The equations for the phases of the oscillators are

$$\dot{\vartheta}_i(t) = \omega_i - \frac{K}{N}\sum_{j=1}^{N}\sin(\vartheta_i(t)-\vartheta_j(t)), \quad i=1,\dots,N, \tag{1.1}$$

where the phases ϑ_i can be considered in the one-dimensional torus $\mathcal{T}$, i.e. defined mod 2π. The parameters ω_i are the 'natural frequencies' of the oscillators, and $K>0$ is the coupling intensity. It can be useful to represent the system (1.1) in the unitary circle in the complex plane by considering N particles with position $\mathrm{e}^{\mathrm{i}\vartheta_i(t)}$. The center of mass is in the point

$$R(t)e^{\mathrm{i}\varphi(t)} = \frac{1}{N}\sum_{j=1}^{N} e^{\mathrm{i}\vartheta_j(t)}, \tag{1.2}$$

where $0 \le R(t) \le 1$ and $\varphi(t)$ are well defined only if $R(t)>0$. Using this definition, the system (1.1) can be rewritten as

$$\dot{\vartheta}_i(t) = \omega_i - KR(t)\sin(\vartheta_i(t)-\varphi(t)), \quad i=1,\dots,N, \tag{1.3}$$

as follows from easy calculations. The interaction term here becomes an attraction term towards the center of mass, and the intensity of the attraction is given by $R(t)$ which grows when the particles get closer.

As shown in [1, 4, 8], for large values of K, this system exhibits *complete frequency synchronization*; i.e. for all i and j

$$\vartheta_i - \vartheta_j \to \text{const. and } \ R \to \text{const. in } (0,1],$$

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as $t \to +\infty$ and all the phases asymptotically rotate with the mean frequency

$$\omega = \frac{1}{N}\sum_{i=1}^{N}\omega_i.$$

In the case of *identical oscillators*, i.e. if $\omega_i = \omega$ for all i, *complete phase synchronization* is possible, i.e. that $\vartheta_i - \vartheta_j \to 0$ and $R \to 1$.

For $K=0$, Equation (1.3) describes a free motion on the N-dimensional torus (*incoherent state*). For intermediate values, the asymptotic behaviour is more complex: some oscillators can synchronize while others move following their natural frequencies. The asymptotic behaviour of R is strictly related to the synchronization, so it is the "order parameter" for this phenomenon.

The complete synchronization has been studied with various methods (see [4] and [3] and references therein). In [4], the authors also consider the case of identical oscillators: they prove the exponential convergence to a complete phase synchronized state for initial data supported in an arc of $\mathcal{T}$ with length less then π. This bound is optimal: $\vartheta_1(t) \equiv 0$ and $\vartheta_2(t) \equiv \pi$ is a stationary solution of (1.1) if $\omega_1 = \omega_2 = 0$. In [3], the authors prove the complete frequency synchronization of identical oscillators for any initial datum.

In this work, in Section 2, we preliminarily prove the complete frequency synchronization of identical oscillators with a different method than in [3], and we also analyze the case in which we obtain complete phase synchronization showing that it is, in effect, the "typical" behaviour of the system of identical oscillators (see Theorem 2.4).

Our method also works for the model obtained in the limit of infinitely many identical oscillators in which the unknown is a measure $\rho(t,\vartheta)$ on $\mathcal{T}$. In Section 3, we prove the complete frequency synchronization for any initial datum ρ_0 and the complete phase synchronization if ρ_0 is non-atomic, i.e. if it gives zero measure to the points (see Theorem 3.4). In this sense, we extend a results of [2] in which the authors prove the complete phase synchronization if ρ_0 has support in a half circle.

In Section 4, we analyze the case of non-identical oscillators with the partial results of Proposition 4.1. Finally, we discuss the relation between the boundedness of the entropy and the convergence to an incoherent state.

2. N identical oscillators

Without loss of generality, we can choose $\omega_i = \omega = 0$ for all i because we can subtract ωt from the phases. Moreover, scaling the time, we can set $K=1$. The system now reads as

$$\dot{\vartheta}_i(t) = -\frac{1}{N}\sum_{j=1}^{N}\sin(\vartheta_i(t) - \vartheta_j(t)) = -R(t)\sin(\vartheta_i(t) - \varphi(t)), \tag{2.1}$$

where R and φ are defined in (1.2). Equation (1.1) is a gradient system. Namely,

$$\dot{\vartheta}_i = \frac{\partial U}{\partial \vartheta_i}, \quad \text{where} \quad U(\vartheta_1, \ldots \vartheta_N) = \frac{1}{2N}\sum_{h,j=1}^{N}\cos(\vartheta_h - \vartheta_j). \tag{2.2}$$

Note that U is a function of R:

$$U = \frac{N}{2}R^2,$$

which follows from the following identities obtained by (1.2):

$$R = \frac{1}{N}\sum_{i=1}^{N}\cos(\vartheta_i - \phi) \quad \text{and} \quad \frac{1}{N}\sum_{i=1}^{N}\sin(\vartheta_i - \phi) = 0. \tag{2.3}$$

The system is invariant under translation, and the mean phase is conserved, which follows by direct computation:

$$\frac{1}{N}\sum_{j=1}^{N}\vartheta_j(t) = \frac{1}{N}\sum_{j=1}^{N}\vartheta_j(0). \tag{2.4}$$

Without loss of generality, we assume that the r.h.s. is zero.

It is simple to find the stationary solutions of the system by remembering that we are in the framework of zero mean frequency.

PROPOSITION 2.1. *We have that* $\{\vartheta_i^*\}_{i=1}^N$ *is a stationary solution of* (2.1) *iff one of the following properties holds:*

(1) $R \equiv 0$

(2) $\{\vartheta_i^*\}_{i=1}^N$ *is of type* $(N-k,k)$*, that is there exists* φ^* *such that:*
$\vartheta_i^* = \varphi^*$ *mod* 2π*, for* $i \in I$*,*
$\vartheta_i^* = \varphi^* + \pi$ *mod* 2π*, for* $i \in I^c$
where $I \subseteq \{1,\ldots,N\}$ *is a subset of indices with* $|I| = k > N/2$.

The first case corresponds to an incoherent state: the center of mass is in the origin and φ is undefined. These solutions form translational invariant submanifolds of the torus $\mathcal{T}^N$ of dimension $N-1$. In the second case, $R = 1 - 2k/N$ and $\varphi = \varphi^*$. If $k = 0$, the solution is a complete phase synchronized state, but if $k \geq 1$ the solution is only a complete frequency synchronized state.

It is easy to prove that the absolute maximum of the function U is achieved by complete synchronization states, i.e. stationary solutions of the type $(N,0)$, which are the only stable solutions of the system. Removing the translational invariance by fixing the mean phase, all the critical point of U are isolated except for the minima which corresponds to $R = 0$. Moreover, it can be proven that the values of ϑ_i are bounded in time (see [3]). The gradient structure (2.2) allows the authors in [3] to prove the complete frequency synchronization of the system for any initial data. As a consequence, it is easy to prove that the solutions must converge to a complete phase synchronized state, up to a measure zero set of initial data which corresponds to the unstable stationary solutions and their stable manifolds.

This kind of argument cannot be used in the limit $N \to +\infty$, so we use a different method which is based on the analysis of the asymptotic behaviour of R and φ.

PROPOSITION 2.2. *If* $\{\vartheta_i(t)\}_{i=1}^N$ *is not a stationary solution, then*

(1) $\dot{R}(t) > 0, \quad \forall t > 0,$

(2) $R(t) \xrightarrow[t\to\infty]{} R^* \in (0,1],$

(3) $\varphi(t)$ *is well defined for all* $t > 0$.

Proof. Deriving in t Equation (1.2), after some manipulations, we obtain

$$\dot{R}(t) = \left[\frac{1}{N}\sum_{j=1}^{N}\sin^2(\vartheta_j(t) - \varphi(t))\right] R(t). \tag{2.5}$$

Because $\{\vartheta_i(t)\}_{i=1}^N$ is not a stationary solution, by Proposition 2.1 and the uniqueness theorem for ODEs, both the factors in the r.h.s. of (2.5) are strictly positive for all time, so (1) is true. Additionally (2) is a direct consequence of (1), and (3) follows from the positivity of R for all time. ∎

In the sequel, we use the following calculus lemma.

LEMMA 2.3. *Let f be a C^1 function $f:[0,+\infty)\to\mathbb{R}$ with $|f'(t)|\le C$. If the integral $\int_0^\infty f(s)\mathrm{d}s$ exists and is finite, then $f(t)\xrightarrow[t\to\infty]{}0$.*

Now we have all the ingredients to prove the main result of this section.

THEOREM 2.4. *If $\vartheta_i(t)$, $i=1,\dots,N$, is not a stationary solution, then it converges to a completely frequency synchronized state of type $(N-k,k)$.*

Moreover, if $\vartheta_i(0)\neq\vartheta_j(0) \mod 2\pi$ when $i\neq j$, the solution converges to a stationary solution of type $(N,0)$ or $(N-1,1)$.

Proof. Since $R(t)\to R^*$, as stated in Proposition 2.2, $\dot R$ verifies the hypothesis of Lemma 2.3. Then, using Equation (2.5),

$$\frac{1}{N}\sum_{j=1}^N \sin^2(\vartheta_j(t)-\varphi(t))\to 0. \tag{2.6}$$

Therefore,

$$\sin(\vartheta_j(t)-\varphi(t))\to 0, \quad j=1,\dots N. \tag{2.7}$$

Since $\sin x$ has isolated zeros,

$$\vartheta_j(t)-\varphi(t)\to k_j\pi, \tag{2.8}$$

for some $k_j\in\mathbb{Z}$. Using that, from Assumption (2.4), the mean phase is zero:

$$\varphi(t)=-\frac{1}{N}\sum_{j=1}^N[\vartheta_j(t)-\varphi(t)]\to -\frac{1}{N}\sum_{j=1}^N k_j\pi =: \varphi^*. \tag{2.9}$$

Finally, $\vartheta_j(t)$ converges for $j=1,\dots,N$:

$$\vartheta_j(t)=\vartheta_j(t)-\varphi(t)+\varphi(t)\to k_j\pi+\varphi^*. \tag{2.10}$$

In order to prove the second part of the theorem, we assume $k\ge 2$. Then there exist i and j such that

$$\vartheta_i(t)\to\varphi^*+(2k_i+1)\pi, \quad \vartheta_j(t)\to\varphi^*+(2k_j+1)\pi, \quad k_i,k_j\in\mathbb{Z}. \tag{2.11}$$

We can write

$$\vartheta_h(t)-\varphi(t)=\xi_h(t)+(2k_h+1)\pi, \quad h=i,j, \tag{2.12}$$

where $\xi_h\to 0$ and $\dot\xi_h(t)=R(t)\sin(\xi_h(t))-\dot\varphi(t)$, which follows from (2.1). Then, using $x\sin(x/2)\ge x^2/\pi$ when $x\in[-\pi,\pi]$ and that ξ_i and ξ_j go to zero,

$$\frac{d}{dt}[\xi_i(t)-\xi_j(t)]^2=2(\xi_i(t)-\xi_j(t))[R(t)\sin(\xi_i(t))-R(t)\sin(\xi_j(t))]$$

$$= 4(\xi_i(t)-\xi_j(t))R(t)\sin\left(\frac{\xi_i(t)-\xi_j(t)}{2}\right)\cos\left(\frac{\xi_i(t)+\xi_j(t)}{2}\right)$$
$$\geq C(\xi_i(t)-\xi_j(t))^2, \tag{2.13}$$

which contradicts the fact that ξ_i and ξ_j go to zero. ◻

It is not possible to exclude that the limit point is a stationary solution of type $(N-1,1)$. In fact its stable manifold is clearly non-empty, which can be easily verified. For instance, consider the case of three oscillators with $\vartheta_1(t)=-\vartheta_2(t)=\delta(t)$, $\vartheta_3(t)\equiv\pi$, where $\delta(t)$ satisfies the equation

$$\dot{\delta} = \frac{2}{3}\sin\delta\left(\frac{1}{2}-\cos\delta\right).$$

The asymptotic behaviour depends on the initial data point $\delta(0)=\delta_0$:

$$\lim_{t\to+\infty}\delta(t) = \begin{cases} 0 \text{ if } \delta_0\in[0,\pi/3) \\ \pi/3 \text{ if } \delta_0=\pi/3 \\ \pi \text{ if } \delta_0\in(\pi/3,\pi] \end{cases}.$$

In the first case, the solution tends to a stationary solution of type (2,1) which is a complete frequency synchronized state. In the second case, the system is in an incoherent state. In the last case, we have complete phase synchronization.

3. The kinetic model for identical oscillators

We now consider the dynamics induced by (2.1). In the limit $N\to+\infty$ for a density of phases $\rho(t,\vartheta)$ defined on $\mathcal{T}$ (see [2]). The equation for ρ is a conservation law of current v depending non-locally on ρ:

$$\begin{cases} \partial_t\rho(t,\vartheta)+\partial_\vartheta(v(t,\vartheta)\rho(t,\vartheta))=0 \\ v(t,\vartheta)=-\int_{\mathcal{T}}\sin(\vartheta-\vartheta')\rho(t,\vartheta')\,\mathrm{d}\vartheta' \end{cases}. \tag{3.1}$$

This equation has a weak form for which existence and uniqueness results for the measure valued solution have been proven in [6] (see also [2]):

$$\begin{cases} \dot{\Theta}(t,\vartheta)=-R(t)\sin(\Theta(t,\vartheta)-\varphi(t)), \quad \text{with } \Theta(0,\vartheta)=\vartheta \\ R(t)e^{\mathrm{i}\varphi(t)}=\displaystyle\int_{\mathcal{T}}e^{\mathrm{i}\vartheta}\rho(t,\vartheta)d\vartheta \\ \displaystyle\int_{\mathcal{T}}h(\vartheta)\rho(t,\vartheta)\,\mathrm{d}\vartheta=\int_{\mathcal{T}}h(\Theta(t,\vartheta))\rho_0(\vartheta)\,\mathrm{d}\vartheta, \end{cases} \tag{3.2}$$

where the measure $\rho_0(\vartheta)$ is the initial datum and h is any regular 2π-periodic observable.

The order parameters verify the identities

$$R(t)=\int_{\mathcal{T}}\cos(\eta-\varphi(t))\rho(t,\eta)\,\mathrm{d}\eta \quad \text{and} \quad \int_{\mathcal{T}}\sin(\eta-\varphi(t))\rho(t,\eta)\,\mathrm{d}\eta=0. \tag{3.3}$$

Their time derivatives are

$$\begin{aligned} \dot{R}(t)&=R(t)\int_{\mathcal{T}}\sin^2(\eta-\varphi(t))\rho(t,\eta)d\eta, \\ \dot{\varphi}(t)&=-\int_{\mathcal{T}}\sin(\eta-\varphi(t))\cos(\eta-\varphi(t))\rho(t,\eta)d\eta. \end{aligned} \tag{3.4}$$

Also, in this case, the mean phase is constant in time,

$$\int_{[-\pi,\pi)} \Theta(t,\vartheta)\rho_0(\vartheta)d\vartheta = \int_{[-\pi,\pi)} \vartheta\rho_0(\vartheta)d\vartheta, \tag{3.5}$$

because its time derivative is zero which follows from (3.3). Note that ϑ is not an observable on $\mathcal{T}$, so $\int_{[-\pi,\pi)} \vartheta\rho(t,\vartheta)d\vartheta$ is different from $\int_{[-\pi,\pi)} \vartheta\rho_0(\vartheta)d\vartheta$.

The stationary solutions of (3.1) are a generalization of the ones relative to the discrete system.

PROPOSITION 3.1. *We have that $\rho(t,\vartheta)\equiv\rho^*(\vartheta)$ is a stationary solution of* (3.1) *if and only if it verifies one of the following identities:*

(1) $R=0$

(2) $\rho^*(\vartheta)$ *is of type* (c_1,c_2)*, that is* $\rho^*(\vartheta)=c_1\delta(\vartheta-\varphi^*)+c_2\delta(\vartheta-\varphi^*-\pi)$*, where* $c_1>c_2\geq 0$ *and* $c_1+c_2=1$.

Proof. Let $\rho^*(\vartheta)$ be a stationary solution and R^*, and let φ^* be the corresponding order parameter. The current is

$$v(t,\theta)=v(\vartheta)=-R^*\sin(\vartheta-\varphi^*).$$

The product $v\rho^*$ must be constant. Then $R^*=0$ or ρ^* is supported where $\sin(\vartheta-\varphi^*)$ is zero. ∎

The discrete model is a particular case of the kinetic model (3.1), but for the second one, the proof of the convergence is a little more difficult, so we have to adapt our argument. We start proving the existence of the asymptotic value of $\varphi(t)$.

PROPOSITION 3.2. *The value of $\varphi(t)$ converges when t goes to infinity.*

Proof. As in the discrete case, for a non-stationary solution $R(t)\to R^*\in(0,1]$ which follows from the first equation of (3.4). Applying Lemma 2.3, we obtain

$$\int_{\mathcal{T}} \sin^2(\eta-\varphi(t))\rho(t,\eta)d\eta \xrightarrow[t\longrightarrow+\infty]{} 0, \tag{3.6}$$

and from the second equation of (3.4), we have

$$\begin{aligned} |\dot\varphi(t)| &\leq \int_{\mathcal{T}} |\sin(\eta-\varphi(t))\cos(\eta-\varphi(t))|\rho(t,\eta)d\eta \\ &\leq \left[\int_{\mathcal{T}} \sin^2(\eta-\varphi(t))\rho(t,\eta)d\eta\right]^{\frac{1}{2}} \xrightarrow[t\longrightarrow+\infty]{} 0, \end{aligned} \tag{3.7}$$

which implies that $\int_0^\infty \ddot\varphi(s)ds$ is finite.

Again, using Lemma 2.3 and doing some calculations, we have

$$\begin{aligned} \ddot\varphi(t) &= \frac{d}{dt}\left[-\int_{\mathcal{T}} \sin(\eta-\varphi(t))\cos(\eta-\varphi(t))\rho(t,\eta)d\eta\right] \\ &= \int_{\mathcal{T}} \cos(2(\eta-\varphi(t)))[\dot\varphi(t)+R(t)\sin(\eta-\varphi(t))]\rho(t,\eta)d\eta \\ &= \int_{\mathcal{T}} (1-2\sin^2(\eta-\varphi(t)))[\dot\varphi(t)+R(t)\sin(\eta-\varphi(t))]\rho(t,\eta) \end{aligned}$$

$$= \dot{\varphi}(t) - 2\int_{\mathcal{T}} \sin^2(\eta - \varphi(t))[\dot{\varphi}(t) + R(t)\sin(\eta - \varphi(t))]\rho(t,\eta), \tag{3.8}$$

where, in the last identity, we used the second equation of (3.3). The second term is bounded by

$$4\int_{\mathcal{T}} \sin^2(\eta - \varphi(t))\rho(t,\eta)d\eta,$$

which is summable in $t \in [0,+\infty)$. Then

$$\phi^* := \lim_{t\to\infty} \varphi(t) = \phi(0) + \int_0^\infty \dot{\phi}(s)ds \tag{3.9}$$

exists and is finite. ∎

Using this result, we can prove the convergence of the characteristics $\Theta(t,\vartheta)$.

PROPOSITION 3.3. *There exists* $\alpha \in \mathcal{T}$ *such that*

$$\lim_{t\to+\infty} \Theta(t,\vartheta) = \begin{cases} \phi^* & \text{for } \vartheta \in \mathcal{T}\setminus\{\alpha\} \\ \phi^* + \pi & \text{for } \vartheta = \alpha \end{cases}.$$

(These identities are taken mod 2π*).*

Proof. Consider $\mathcal{T}$ as $\phi^* + [-\pi,\pi]$, and for $n \geq 1$, define the partition of $\mathcal{T}$

$$A_n^0 = \phi^* + [-1/n, 1/n], \quad A_n^\pi = \phi^* + [\pi - 1/n, \pi] \cup [-\pi, -\pi + 1/n],$$
$$B_n^+ = \phi^* + (1/n, \pi - 1/n), \quad B_n^- = \phi^* + (-\pi + 1/n, -1/n).$$

Since $R(t) \to R^*$ and $\varphi(t) \to \varphi^*$, there exists an increasing, diverging sequence t_n such that, for $t \geq t_n$,

$$R(t)|\sin(\vartheta - \varphi(t))| \geq \frac{R^*}{2}|\sin(\vartheta - \varphi^*)| \quad \text{for } \vartheta \in B_n^+ \cup B_n^-. \tag{3.10}$$

The subsets $G_n = A_n^0 \cup B_n^+ \cup B_n^-$ are invariant in the sense that if for $\bar{t} \geq t_n$, $\Theta(\bar{t},\vartheta) \in G_n$, then $\Theta(t,\vartheta) \in G_n$ for all $t \geq \bar{t}$. Note that A_n^0 is invariant. As a consequence, if $n > m$, then

$$\Theta(-t_n, A_n^\pi) \subseteq \Theta(-t_m, A_m^\pi).$$

Since $\Theta(-t_n, A_n^\pi)$ are arcs of $\mathcal{T}$, we can uniquely define the arc

$$[\alpha_1, \alpha_2] = \bigcap_{n\geq 1} \Theta(-t_n, A_n^\pi).$$

By definition, if $\vartheta \notin [\alpha_1,\alpha_2]$, then $\Theta(t,\vartheta) \in G_n$ for all $t \geq t_n$ definitely in n. For a finite τ independent of n, $\Theta(t_n + \tau, \vartheta) \in A_n^0$. Using the invariance of A_n^0, we obtain that $\Theta(t,\vartheta) \to \varphi^*$.

If $\vartheta \in [\alpha_1,\alpha_2]$, then $\Theta(t_n,\vartheta) \in A_n^\pi$ for all n. Suppose now that there exists $\bar{t} > t_n$ such that $\Theta(\bar{t},\vartheta) \notin A_n^\pi$. By the invariance of G_n, for all m such that $t_m \geq \bar{t}$, $\Theta(t_m,\vartheta) \in G_n \subset G_m$, we have that $\vartheta \in \Theta(-t_m, G_m)$ in contrast to the hypothesis on $\vartheta \in [\alpha_1,\alpha_2]$. We conclude that $\Theta(t,\vartheta) \to \varphi^* + \pi$.

Finally, we can repeat the same argument of the proof of the second part of Theorem 2.4, showing that asymptotically,

$$\frac{d}{dt}[\Theta(t,\alpha_1)-\Theta(t,\alpha_2)]^2 \geq C[\Theta(t,\alpha_1)-\Theta(t,\alpha_2)]^2.$$

Hence $\alpha_1=\alpha_2=\alpha$. ∎

THEOREM 3.4. *If $\rho(\vartheta,t)$ is not a stationary solution, then*

$$\rho(t,\vartheta) \xrightarrow[t\to\infty]{weak-*} \rho^*(\vartheta), \tag{3.11}$$

where $\rho^*(\vartheta)$ is a stationary solution of type (c_1,c_2).

Moreover, if $\rho_0(\vartheta)$ is non-atomic, then $\rho^*=\delta(\vartheta-\varphi^*)$; i.e. it is a complete phase synchronized state.

Proof. Let be h a regular periodic observable. Using Proposition 3.3,

$$\int_{\mathcal{T}} h(\vartheta)\rho(t,\vartheta)\,\mathrm{d}\vartheta = \int_{\mathcal{T}} h(\Theta(t,\vartheta))\rho_0(\vartheta)\,\mathrm{d}\vartheta \to c_1 h(\phi^*)+c_2 h(\phi^*+\pi),$$

where

$$c_1=\int_{\mathcal{T}\setminus\{\alpha\}} \rho_0(\vartheta)\,\mathrm{d}\vartheta, \quad c_2=1-c_1,$$

and c_2 is the measure that ρ_0 gives to the point α which is zero if $\rho_0(\vartheta)$ gives zero measure to the points. ∎

4. Some considerations on the kinetic model for non identical oscillators

The following equations describe the dynamic of infinitely many non-identical oscillators in the kinetic limit:

$$\begin{cases}\partial_t f(t,\vartheta,\omega)+\partial_\vartheta(v(t,\vartheta,\omega)f(t,\vartheta,\omega))=0,\\ v(t,\vartheta,\omega)=\omega-K\int_{\mathcal{T}\times\mathbb{R}}\sin(\vartheta-\vartheta')f(t,\vartheta',\omega')d\vartheta'd\omega',\end{cases} \tag{4.1}$$

where $f(t,\vartheta,\omega)$ is a positive 2π-periodic function in ϑ which represents the probability density of oscillators with phase ϑ and frequency ω. The marginal $\rho(t,\vartheta)=\int_{\mathbb{R}} f(t,\vartheta,\omega)d\omega$ is the probability density of the phases. The distribution of the natural frequencies is $g(\omega)=\int_{\mathcal{T}} f(t,\vartheta,\omega)d\vartheta$, which is a conserved quantity.

A reference for existence and uniqueness results for this equation is [6] where the kinetic model (4.1) is rigorously derived by taking the limit as $N\to\infty$ of (1.1). A weak formulation of (4.1) can be given in terms of the characteristics $\Theta(t,\vartheta,\omega)$:

$$\begin{cases}\dot{\Theta}(t,\vartheta,\omega)=\omega-KR(t)\sin(\Theta(t,\vartheta,\omega)-\varphi(t)),\ \text{with}\ \Theta(0,\vartheta,\omega)=\vartheta,\\ R(t)\mathrm{e}^{\mathrm{i}\varphi(t)}=\displaystyle\int_{\mathcal{T}\times\mathbb{R}}\mathrm{e}^{\mathbf{i}\vartheta}f(t,\vartheta,\omega)d\vartheta d\omega,\\ \displaystyle\int_{\mathcal{T}\times\mathbb{R}} f(t,\vartheta,\omega)h(\vartheta,\omega)d\vartheta d\omega=\int_{\mathcal{T}\times\mathbb{R}} f_0(\eta,\omega)h(\Theta(t,\eta,\omega),\omega)d\eta d\omega,\end{cases} \tag{4.2}$$

where h is any regular function of $(\vartheta,\varphi)\in\mathcal{T}\times\mathbb{R}$. Without loss of generality, we can assume

$$\langle\omega\rangle=\int_{\mathbb{R}}\omega g(\omega)d\omega=0 \quad \text{and} \quad \langle\vartheta\rangle=\int_{[-\pi,\pi)}\vartheta\rho_0(\vartheta)d\vartheta=0. \tag{4.3}$$

By the previous assumptions, it follows that

$$\int_{[-\pi,\pi)\times\mathbb{R}} \Theta(t,\vartheta,\omega) f_0(\vartheta,\omega) d\vartheta d\omega = 0. \tag{4.4}$$

As shown in [1], when g has compact support and K is sufficiently large, there exist stationary solutions f^* which are, in some sense, the analogues of the two delta solutions for the case of identical oscillators described in Proposition 3.1. Imposing the current $v = \omega - KR\sin(\theta - \varphi)$ to be zero, we obtain

$$\begin{aligned} &f^*(\vartheta,\omega) = g^+(\omega)\delta(\vartheta - \vartheta^+(\omega)) + g^-(\omega)\delta(\vartheta - \vartheta^-(\omega)), \\ &\vartheta^+(\omega) = \varphi^* + \arcsin\left(\frac{\omega}{KR}\right), \ \vartheta^-(\omega) = \pi + \varphi^* - \arcsin\left(\frac{\omega}{KR}\right), \\ &g^+, g^- \geq 0, \ g^+ + g^- = g, \end{aligned} \tag{4.5}$$

with R satisfying the following equation of self-consistency (which has solutions for K large enough):

$$KR^2 = \int_{\mathbb{R}} \sqrt{(KR)^2 - \omega^2}\Big[g^+(\omega) - g^-(\omega)\Big] d\omega. \tag{4.6}$$

Taking the marginal of f^*, the particle density ρ^* is

$$\begin{aligned} \rho^*(\vartheta) = &KR|\cos(\vartheta - \varphi^*)| g^+(KR\sin(\vartheta - \varphi^*)) \mathcal{X}_{|\vartheta - \varphi^*| < \frac{\pi}{2}} \\ &+ KR|\cos(\vartheta - \varphi^*)| g^-(KR\sin(\vartheta - \varphi^*)) \mathcal{X}_{|\vartheta - (\pi + \varphi^*)| < \frac{\pi}{2}}. \end{aligned} \tag{4.7}$$

Particularly relevant are the stable solutions (see [2]) which are the solutions with $g^+(\omega) = g(\omega)$ and $g^-(\omega) = 0$:

$$f^*(\vartheta,\omega) = g(\omega)\delta\left(\vartheta - \vartheta^+(\omega)\right), \tag{4.8}$$

where $R \in (0,1]$ is the largest solution of

$$KR^2 = \int_{\mathbb{R}} \sqrt{(KR)^2 - \omega^2} g(\omega) d\omega. \tag{4.9}$$

To the authors' knowledge, the best result of convergence to an equilibrium of this kind is in [2], where complete frequency synchronization is proven for initial phases lying in a compact subset of $(-\frac{\pi}{2}, \frac{\pi}{2})$, although the expression of equilibrium density $\rho^*(\vartheta) = KR\cos(\vartheta - \varphi^*) g(KR\sin(\vartheta - \varphi^*))$ is not explicitly written in the paper.

In the case of non identical oscillators, the order parameter $R(t)$ is no more increasing in general:

$$\dot{R}(t) = KR(t)\int_{\mathcal{T}} \sin^2(\eta - \varphi(t))\rho(t,\eta) d\eta - \int_{\mathcal{T}\times\mathbb{R}} \omega \sin(\eta - \varphi(t)) f(t,\eta,\omega) d\eta d\omega, \tag{4.10}$$

and

$$\begin{aligned} R(t)\dot{\varphi}(t) = &-KR(t)\int_{\mathcal{T}} \sin(\eta - \varphi(t))\cos(\eta - \varphi(t))\rho(t,\eta) d\eta \\ &+ \int_{\mathcal{T}\times\mathbb{R}} \omega \cos(\eta - \varphi(t)) f(t,\eta,\omega) d\eta d\omega. \end{aligned} \tag{4.11}$$

Then we cannot extend the convergence result of the previous section to this case. Nevertheless, we can characterize the possible limits of the solution, excluding in this case the "two delta solutions" as generic asymptotic behaviour.

PROPOSITION 4.1. *Suppose that $f_0(\vartheta,\omega)\mathrm{d}\vartheta$ is non-atomic for any ω. If as $t\to+\infty$, $R(t)\to R^*>0$ with* supp $g\subset[-KR^*,KR^*]$ *and $\varphi(t)\to\varphi^*$, then f converges weakly to f^* given by* (4.8) *and R^* solves* (4.9).

The proof follows as in Proposition 3.3 and Theorem 3.4. We first prove the convergence of $\Theta(t,\vartheta,\omega)$ to $\vartheta^{\pm}(\omega)$ mod 2π for $|\omega|<KR^*$. Then we show that there exists only one value of $\vartheta\in\mathcal{T}$ such that $\Theta(t,\vartheta,\omega)\to\vartheta^{-}(\omega)$ mod 2π. Finally, we prove the weak convergence of f using the convergence of the characteristics.

Note that Equation (4.9) can have two solutions (see [9, 7]). Then Proposition 4.1 does not assure the convergence to the stable stationary solution.

The asymptotic behaviour in the case of non identical oscillators can be complex even if the system is still of gradient type (in a different space). The functional is

$$\mathcal{H}_f(t)=\int_{[-\pi,\pi)\times\mathbb{R}}\Theta(t,\vartheta,\omega)\omega f_0(\vartheta,\omega)d\vartheta d\omega+K\frac{R^2(t)}{2}, \tag{4.12}$$

which is non decreasing along the solutions. In contrast to the case of identical oscillators, this functional is not well defined on the function on $\mathcal{T}\times R$, and it is unbounded.

There is another functional with monotone behaviour, related to the entropy.

PROPOSITION 4.2. *If $\Theta(t,\vartheta,\omega)$ is a solution of* (4.2), *then*

$$\frac{d}{dt}\int_{\mathcal{T}\times\mathbb{R}}\ln\left(\frac{\partial}{\partial\vartheta}\Theta(t,\vartheta,\omega)\right)f_0(\vartheta,\omega)d\vartheta d\omega=-KR^2(t). \tag{4.13}$$

Proof. The r.h.s. of (4.13) is

$$\begin{aligned}&\int_{\mathcal{T}\times\mathbb{R}}\left[\frac{\partial\Theta}{\partial\vartheta}\right]^{-1}\frac{d}{dt}\frac{\partial\Theta}{\partial\vartheta}f_0(\vartheta,\omega)d\vartheta d\omega\\&=-\int_{\mathcal{T}\times\mathbb{R}}KR(t)\cos(\Theta(t,\vartheta,\omega)-\varphi(t))f_0(\vartheta,\omega)d\vartheta d\omega=-KR^2(t).\end{aligned} \tag{4.14}$$

□

Proposition 4.2 makes sense for any initial data points of (4.1) and shows the tendency of the system to shrink the solution in the ϑ variabile. If f_0 is absolute continuous with respect to the Lebesgue measure, this functional can be rewritten, modulo a constant that explodes if f becomes singular, in a more elegant way.

PROPOSITION 4.3. *The entropy is non-decreasing along the solutions of* (4.1):

$$\frac{d}{dt}\int_{\mathcal{T}\times\mathbb{R}}f(t,\vartheta,\omega)\ln(f(t,\vartheta,\omega))d\vartheta d\omega=KR^2(t). \tag{4.15}$$

In the hypothesis of the results presented in [2] and in that of Proposition 4.1, as f approaches f^* the entropy grows to infinity.

If the entropy does not diverge, $R\to 0$, so the system behaves as an incoherent state. More precisely, we can prove the following two propositions.

PROPOSITION 4.4. *If the entropy does not diverge, then the functional $\mathcal{H}_f$ asymptotically grows as in the case of incoherent states: the limit*

$$\lim_{t\to\infty}\left[\int_{[-\pi,\pi)\times\mathbb{R}}\Theta(t,\vartheta,\omega)\omega f_0(\vartheta,\omega)d\vartheta d\omega-\left(\int_{[-\pi,\pi)\times\mathbb{R}}\omega^2 f_0(\vartheta,\omega)d\vartheta d\omega\right)t\right] \tag{4.16}$$

is finite. In other words, the functional $\mathcal{H}_f$ grows as in the case of free flows ($K=0$).

Proof. First, we write the derivative

$$\begin{aligned}&\frac{d}{dt}\left[\int_{[-\pi,\pi)\times\mathbb{R}}(\Theta(t,\vartheta,\omega)-\omega t)\omega f_0(\vartheta,\omega)d\vartheta d\omega\right]\\&=-KR(t)\int_{\mathcal{T}\times\mathbb{R}}\omega\sin(\vartheta-\varphi(t))f(t,\vartheta,\omega)d\vartheta d\omega.\end{aligned} \tag{4.17}$$

Now we write the derivative of $\frac{R^2(t)}{2}$:

$$\begin{aligned}\frac{d}{dt}\frac{R^2(t)}{2}=&\left[\int_{\mathcal{T}}\sin^2(\eta-\varphi(t))\rho(t,\eta)d\eta\right]KR^2(t)\\&-\left[\int_{\mathcal{T}\times\mathbb{R}}\omega\sin(\eta-\varphi(t))f(t,\eta,\omega)d\eta d\omega\right]R(t).\end{aligned} \tag{4.18}$$

Integrating the last identity between 0 and t, we get

$$\begin{aligned}\frac{R^2(t)}{2}-\frac{R^2(0)}{2}=&\int_0^t\left[\int_{\mathcal{T}}\sin^2(\eta-\varphi(s))\rho(s,\eta)d\eta\right]KR^2(s)ds\\&-\int_0^t\left[\int_{\mathcal{T}\times\mathbb{R}}\omega\sin(\eta-\varphi(s))f(s,\eta,\omega)d\eta d\omega\right]R(s)ds.\end{aligned} \tag{4.19}$$

Using both (4.18) and (4.19), we get

$$\begin{aligned}\int_{[-\pi,\pi)\times\mathbb{R}}(\Theta(t,\vartheta,\omega)-\omega t)\omega f_0(\vartheta,\omega)d\vartheta d\omega=&\int_{[-\pi,\pi)\times\mathbb{R}}\vartheta\omega f_0(\vartheta,\omega)d\vartheta d\omega+\frac{R^2(t)}{2}-\frac{R^2(0)}{2}\\&-\int_0^t\left[\int_{\mathcal{T}}\sin^2(\eta-\varphi(s))\rho(s,\eta)d\eta\right]KR^2(s)ds.\end{aligned} \tag{4.20}$$

We are done because $R^2(t)$ is integrable and $R(t)\xrightarrow[t\to\infty]{}0$ (for Lemma 2.3). ☐

PROPOSITION 4.5. *If the entropy does not diverge, the only possible limit points of $f(t,\vartheta,\omega)$ are incoherent states*

Proof. First step: if the entropy does not diverge, then

$$\int_{\mathcal{T}\times\mathbb{R}}e^{\mathbf{i}\vartheta}\omega^k f(t,\vartheta,\omega)d\vartheta d\omega\to 0,\forall k\in\mathbb{N}.$$

The proof is done by induction. The first step is the fact that if the entropy does not diverge, then $R(t)$ vanishes by Lemma 2.3. Now we do the inductive step:

$$\int_{\mathcal{T}\times\mathbb{R}}e^{\mathbf{i}\vartheta}\omega^k f(t,\vartheta,\omega)d\vartheta d\omega\to 0\Rightarrow\int_{\mathcal{T}\times\mathbb{R}}e^{\mathbf{i}\vartheta}\omega^{k+1}f(t,\vartheta,\omega)d\vartheta d\omega\to 0. \tag{4.21}$$

We write the derivative of the l.h.s.

$$\frac{d}{dt}\left[\int_{\mathcal{T}\times\mathbb{R}} e^{\mathbf{i}\vartheta}\omega^k f(t,\vartheta,\omega)d\vartheta d\omega\right]$$
$$=\int_{\mathcal{T}\times\mathbb{R}} \mathbf{i}e^{\mathbf{i}\vartheta}\omega^k\left[\omega - KR(t)\sin(\vartheta-\varphi(t))\right]f(t,\vartheta,\omega)d\vartheta d\omega. \tag{4.22}$$

This quantity satisfies the hypothesis of Lemma 2.3, so it goes to zero which implies that

$$\int_{\mathcal{T}\times\mathbb{R}} e^{\mathbf{i}\vartheta}\omega^{k+1} f(t,\vartheta,\omega)d\vartheta d\omega \to 0. \tag{4.23}$$

Second step: the limit points are incoherent states.

Let's call $\bar{f}(\vartheta,\omega)$ a limit point of $f(t,\vartheta,\omega)$. Then we have

$$\int e^{\mathbf{i}\vartheta}\omega^k \bar{f}(\vartheta,\omega)d\vartheta d\omega = 0, \quad \forall k\in\mathbb{N}. \tag{4.24}$$

The solution of Equation (4.1) with $\bar{f}$ as initial datum is $\bar{f}(\vartheta-\omega t,\omega)$. In fact, $R(t)$ generated by this density is zero:

$$R(t) = \int_{\mathcal{T}\times\mathbb{R}} e^{\mathbf{i}\vartheta}\bar{f}(\vartheta-\omega t,\omega)d\vartheta d\omega = \int_{\mathcal{T}\times\mathbb{R}} e^{\mathbf{i}(\eta+\omega t)}\bar{f}(\eta,\omega)d\eta d\omega \tag{4.25}$$
$$= \int_{\mathcal{T}\times\mathbb{R}} e^{\mathbf{i}\eta}e^{\mathbf{i}\omega t}\bar{f}(\eta,\omega)d\eta d\omega = \int_{\mathcal{T}\times\mathbb{R}} e^{\mathbf{i}\eta}\sum_{k=0}^{\infty}\frac{(\mathbf{i}\omega)^k}{k!}\bar{f}(\eta,\omega)d\eta d\omega \tag{4.26}$$
$$= \sum_{k=0}^{\infty}\frac{\mathbf{i}^k}{k!}\int_{\mathcal{T}\times\mathbb{R}} e^{\mathbf{i}\eta}\omega^k\bar{f}(\eta,\omega)d\eta d\omega = 0.$$

□

Note that a density $\bar{f}(\vartheta,\omega)$ is an incoherent state if and only its first Fourier coefficient in ϑ is zero for any ω.

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