COMMUN. MATH. SCI.
Vol. 13, No. 7, pp. 1711–1740

EXISTENCE OF REGULAR SOLUTIONS TO AN ERICKSEN–LESLIE MODEL OF THE LIQUID CRYSTAL SYSTEM*

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Abstract. We study a general Ericksen–Leslie system with non-constant density, which describes the flow of nematic liquid crystal. In particular the model investigated here is associated with Parodi's relation. We prove that in two dimension, the solutions are globally regular with general data, and in three dimension, the solutions are globally regular with small initial data or for a short time with large data. Moreover, a weak-strong type of uniqueness result is obtained.

Key words. Liquid crystals, Parodi's relation, regularity.

AMS subject classifications. 76D03, 3Q35.

1. Introduction

The flows of nematic liquid crystals can be treated as slow moving particles where the fluid velocity and the alignment of the particles influence each other. The hydrodynamic theory of liquid crystals was established by Ericksen [15, 16] and Leslie [33, 34] in the 1960's. As Leslie points out in his 1968 paper, "liquid crystals are states of matter which are capable of flow, and in which the molecular arrangements give rise to a preferred direction". The full Ericksen–Leslie system consists of the following equations (cf. [15, 33, 34, 38]):

$$\begin{aligned} &\rho_t+\nabla\cdot(\rho u)=0,\\ &\rho\dot{u}=\rho F+\nabla\cdot\hat{\sigma},\\ &\rho_1\dot{\omega}=\rho_1 G+\hat{g}+\nabla\cdot\pi, \end{aligned} \tag{1.1}$$

in $\Omega\times(0,T)$ where Ω is a domain in R^n with $n=2,3$. The three equations in system (1.1) describe the conservation of mass, linear momentum, and angular momentum, respectively. The anisotropic feature of liquid crystal materials is exhibited in the third equation, and the nonlinear coupling is represented in second equation. In the above equations, $\rho:\Omega\times[0,T]\to\mathbb{R}$ is the fluid density, $u:\Omega\times[0,T]\to\mathbb{R}^n$ is the fluid velocity, $d:\Omega\times[0,T]\to\mathbb{R}^n$ is the director field representing the alignment of the molecules, ρ_1 is the inertial constant, $\hat{g}$ is the intrinsic force associated with d, π is the director stress, and F and G are external body force and external director body force, respectively. In this paper, we consider the incompressible flow with $\nabla\cdot u=0$. The superposed dot denotes the material derivative $\partial_t+u\cdot\nabla$. The notations

$$\begin{aligned} &A=\frac{1}{2}(\nabla u+\nabla^T u), \qquad \Omega=\frac{1}{2}(\nabla u-\nabla^T u),\\ &\omega=\dot{d}=d_t+(u\cdot\nabla)d, \qquad N=\omega-\Omega d, \end{aligned}$$

represent the rate of the strain tensor, the skew-symmetric part of the strain rate, the material derivative of d, and the rigid rotation part of director changing rate by fluid vorticity, respectively.

*Received: April 9, 2014; accepted (in revised form): August 29, 2014. Communicated by Chun Liu.

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We have the following constitutive relations for $\hat{\sigma}$, π and $\hat{g}$ in (1.1):

$$\begin{aligned}\hat{\sigma}_{ij} &= -P\delta_{ij} - \frac{\partial(\rho W)}{\partial d_{k,i}} d_{k,j} + \sigma_{ij},\\ \pi_{ij} &= \beta_i d_j + \frac{\partial(\rho W)}{\partial d_{j,i}},\\ \hat{g}_i &= \gamma d_i - \beta_j d_{i,j} - \frac{\partial(\rho W)}{\partial d_i} + g_i,\end{aligned} \tag{1.2}$$

with

$$\begin{aligned} g_i &= \lambda_1 N_i + \lambda_2 d_j A_{ji},\\ \sigma_{ij} &= \mu_1 d_k A_{kp} d_p d_i d_j + \mu_2 N_i d_j + \mu_3 d_i N_j + \mu_4 A_{ij} + \mu_5 A_{ik} d_k d_j + \mu_6 d_i A_{jk} d_k,\\ N_i &= \omega_i + \Omega_{ki} d_k.\end{aligned} \tag{1.3}$$

Here the scalar function P represents the pressure. The vector $\beta = (\beta_1, \beta_2, \beta_3)^T$ and the scalar function γ are Lagrangian multipliers due to the constraint $|d| = 1$. The term ρW denotes the Oseen–Frank energy functional for the equilibrium configuration of a unit director field. To relax the constraint $|d| = 1$, we consider the elastic energy with a penalty term:

$$\rho W = \frac{1}{2}|\nabla d|^2 + \frac{1}{4\eta^2}(|d|^2 - 1)^2 \tag{1.4}$$

with constant $\eta > 0$. This procedure is usually called Ginzburg–Landau approximation. In (1.3), the constants λ_1, λ_2 represent the molecular shape and $\mu_1, \ldots, \mu_6$ are Leslie coefficients with regard to certain local correlations in the fluid (cf. [14]). They satisfy (cf. [33])

$$\lambda_1 = \mu_2 - \mu_3, \quad \lambda_2 = \mu_5 - \mu_6, \tag{1.5}$$

$$\mu_5 - \mu_6 = -(\mu_2 + \mu_3). \tag{1.6}$$

The relations in (1.5) arise from the second law of thermodynamics. The relation (1.6) is called Parodi's condition which is derived from the Onsager's reciprocal relation.

To further simplify the model, we take $\rho_1 = 0$, $\beta = 0$, $\gamma = 0$, and $F = G = 0$. Thus, the incompressible Ericksen–Leslie system (1.1) is reformulated as

$$\rho_t + (u \cdot \nabla)\rho = 0, \tag{1.7}$$

$$\rho u_t + \rho(u \cdot \nabla)u + \nabla P = -\nabla \cdot (\nabla d \otimes \nabla d) + \nabla \cdot \sigma, \tag{1.8}$$

$$d_t + (u \cdot \nabla)d - \Omega d + \frac{\lambda_2}{\lambda_1} A d = -\frac{1}{\lambda_1}(\Delta d - f(d)), \tag{1.9}$$

$$\nabla \cdot u = 0, \tag{1.10}$$

with

$$\rho \mathcal{F}(d) = \frac{1}{4\eta^2}(|d|^2 - 1)^2, \quad f(d) = \nabla_d(\rho \mathcal{F}(d)) = \frac{1}{\eta^2}(|d|^2 - 1)d.$$

The force term $\nabla d \otimes \nabla d$ in the equation of the conservation of momentum denotes the 3×3 matrix whose ij-th entry is given by "$\nabla_i d \cdot \nabla_j d$" for $1 \le i, j \le 3$.

There is a vast literature on the hydrodynamics of the liquid crystal system. For background, we list a few names with no intention of being complete: [3, 4, 5, 6, 7, 8, 9, 12, 13, 32, 36, 18, 20, 21, 24, 25, 26, 35, 37, 38, 42, 45, 49, 50]. Among these works, it is worth mentioning some recent results on a simplified liquid crystal model. Without the Ginzberg–Landau approximation, the elastic energy is usually taken as $\rho W = \frac{1}{2}|\nabla d|^2$ instead of (1.4). In this case, a relatively simplified liquid crystal model (with some corresponding physical parameters equal zero in (1.1-1.3)) reads as

$$\begin{aligned}
&u_t + u\cdot\nabla u - \Delta u + \nabla P = -\cdot(\nabla d \otimes \nabla d),\\
&d_t + u\cdot\nabla d = \Delta d + |\nabla d|^2 d,\\
&\nabla\cdot u = 0, \qquad |d| = 1.
\end{aligned}$$

The existence of global weak solutions to the above system in the three dimensional case is a long standing open problem due to the strong nonlinearity $|\nabla d|^2 d$. However, this problem in the two dimensional case was solved recently by Lin, Lin, and Wang [36] which was important progress in the study of the liquid crystal system. Namely, under smallness conditions, the authors first established both interior and boundary regularity; applying the regularity theory, they obtained the existence of global weak solutions on a bounded smooth domain. Moreover, they proved the weak solutions are smooth everywhere with the possible exceptions of finitely many singular times. Later, under a natural geometric angle condition, Lei, Li, and Zhang [32] obtained the global well-posedness of smooth solutions for a class of large initial data by a different method. In their proof, a rigidity theorem is the main technical tool which produces the coercivity of the harmonic energy; the combination of a frequency localization argument and the concentration-compactness approach also plays an important role. On the other hand, Dong and Lei [13] are able to construct a family of exact strong solutions to the above simplified liquid crystal model with large initial data. The strong solutions are global in time, and the director d on $\mathbb{S}^2$ may shrink to a single point as time goes to infinity. The success of such a construction relies on the choice of rotationally symmetric initial velocity fields and symmetric initial orientations.

Recently, Wu, Xu, and Liu [50] studied a general Ericksen–Leslie system which is similar to the model (1.7-1.10) but with constant density. In their work, the global regularity and long-time behavior of solutions are obtained under the assumption that the viscosity coefficient is sufficiently large (3D). With Parodi's relation, the authors established the global well-posedness and Lyapunov stability near local energy minimizers. The authors also discussed the connection between Parodi's relation and linear stability. On the other hand, since the density is not constant, a relatively full model for the dynamic of Smectic-A liquid crystals is studied in [41]. In this model, several terms in σ are assumed to be zero, and the term $-\Omega d + \frac{\lambda_2}{\lambda_1} A d$ does not appear in Equation (1.9). The author proved the existence of global classical solutions in both two and three dimensional cases. In 2D, no additional assumption is needed while in 3D the flow viscosity coefficient is assumed to be sufficiently large. Moreover, a "weak-strong" type of uniqueness result, a long-time behavior, and stability of solutions are obtained. Also, with a non-constant density, a simplified Ericksen–Leslie model is studied in [12]. The authors indicated that a regular solution exists globally in 2D with general data, while a general solution exists globally in 3D with small initial data or for a short time with general data.

In the present paper, the consideration is given to the model (1.7)–(1.10) with non-constant density and under Parodi's relation (1.6). We establish that, in 2D, there exists a global regular solution to the system with general data (no extra condition), and, in 3D, there exists a global regular solution with small initial data or a local (short time) regular solution with general data. We also show that a "weak-strong" type of uniqueness result holds with certain assumptions on the weak solution. Namely, if there exists a regular solution satisfying a higher order energy estimate and a weak solution satisfying the basic energy estimate and two auxiliary estimates, they must be identical.

For the simplified liquid crystal model with constant density (cf. [35]) or non-constant density (cf. [12]) and for the general Ericksen–Leslie system (1.7)–(1.10) with the artificial assumption $\lambda_2=0$ (cf. [38]) the transport equation of d satisfies a certain type of maximum principle. In the present paper, for the general model (1.7)–(1.10) with $\lambda_2\neq 0$, the stretching effect causes the loss of the maximum principle for d (cf. [49]). However, in the analysis of the sequel, the estimate

$$d\in L^\infty(0,T;L^\infty)$$

turns out to be essential in the derivation of higher-order energy estimates and, hence, to assure that the stress term $\nabla\cdot\sigma$ can be handled successfully. Therefore, as in [49], we consider the periodic boundary conditions which help to avoid the difficulties from boundary terms when deriving the higher order energy estimates. As such, we restrict ourselves to the following boundary conditions:

$$u(x+e_i,t)=u(x,t),\quad d(x+e_i,t)=d(x,t),\qquad \text{for } (x,t)\in\partial Q\times\mathbb{R}^+ \tag{1.11}$$

with Q a unit square in $\mathbb{R}^2$ or cube in $\mathbb{R}^3$. The model also satisfies the initial conditions:

$$\rho(x,0)=\rho_0(x),\quad 0<M_1\leq\rho_0(x)\leq M_2, \tag{1.12}$$

$$u(x,0)=u_0(x),\qquad \nabla\cdot u_0=0,\qquad \text{and}\qquad d(x,0)=d_0(x). \tag{1.13}$$

Note that in 2D, we usually assume either

$$(i)\ \mu_1\geq 0,\ \lambda_2=0\ \text{ or }\ (ii)\ \mu_1=0,\ \lambda_2\neq 0.$$

For the literature of the Ericksen–Leslie system with assumption (i), we refer the readers to [38]; for the work with assumption (ii), we refer to [8, 23, 45, 49].

To guarantee the dissipation of the director field, it is assumed that (cf. [17, 34])

$$\begin{cases} \lambda_1<0\\ \mu_1\geq 0,\quad \mu_4>0,\\ \mu_5+\mu_6\geq 0, \end{cases} \tag{1.14}$$

and in addition

$$\frac{\lambda_2^2}{-\lambda_1}\leq\mu_5+\mu_6. \tag{1.15}$$

In the rest of the introduction we describe our main results.

THEOREM 1.1. *(2D) Suppose that $Q\subset\mathbb{R}^2$ is a unit square. Let ρ_0, u_0, and d_0 satisfy (1.12) and (1.13). Suppose that $\rho_0\in C^1(Q)$, $u_0\in H_p^2$, and $d_0\in H_p^3$. Then, system (1.7)–(1.10) has a global classical solution (ρ,u,d). That is, for all $T>0$ and some $\alpha\in(0,1)$*

$$\begin{aligned} u&\in C^{1+\alpha/2,2+\alpha}((0,T)\times Q),\\ \nabla p&\in C^{\alpha/2,\alpha}((0,T)\times Q),\\ d&\in C^{1+\alpha/2,2+\alpha}((0,T)\times Q),\\ \rho&\in C^1((0,T)\times Q). \end{aligned} \tag{1.16}$$

The notations H_p^2 and H_p^3 will be introduced in Section 2.

Provided we have sufficiently small data or we work with sufficiently short time, we also obtain the regularity in the three dimensional case.

THEOREM 1.2. *(3D) Suppose that $Q\subset\mathbb{R}^3$ is a unit cube. Let ρ_0, u_0, and d_0 satisfy (1.12) and (1.13). Assume that $\rho_0\in C^1(Q)$, $u_0\in H_p^2$, and $d_0\in H_p^3$. Then*

1. *There is a positive small number ϵ_0 such that if*

$$\rho_0\|u_0\|_{H_p^1}^2+\|d_0\|_{H_p^1}^2+\|\Delta d_0-f(d_0)\|_{L_p^2}^2\le\epsilon_0, \tag{1.17}$$

then system (1.7)–(1.10) has a classical solution (ρ,u,d) in the time period $(0,T)$ for all $T>0$. That is, (1.16) holds for some $\alpha\in(0,1)$.
2. *For general data, there exists a positive number $\delta_0=\delta_0(\rho_0,u_0,d_0)$ such that (1.16) holds in the interval $(0,T)$ for some small $T\le\delta_0$.*

REMARK 1.3. There are two main differences between the current work and the work of Wu, Xu, and Liu [50]. Firstly, our model contains non-constant density which satisfies a transport equation. Due to the presence of the density, it generates more terms in the estimate of higher order energy in Subsection 3.2, and it requires the study of the regularity of the density, as in Subsection 3.4, which was also discussed in [12] with more details. Secondly and most importantly, our result in 3D is obtained with the small data assumption (1.17) while in [50] the authors obtained the global regularity in 3D with a large viscosity assumption. It is pointed out in [42, 50] that the small initial data assumption is not equivalent to the large viscosity assumption for the Ericksen–Leslie system (1.7)–(1.10) due to its much more complicated structure. Thus it is particularly interesting to investigate the regularity of solutions to the non-constant density Ericksen–Leslie system under the small initial data assumption.

REMARK 1.4. In contrast to the initial condition in [12] (Theorem 1.3) where $(u_0,\nabla d_0)\in H^1\times H^1$, we require a higher order condition on the initial data here, that is $(u_0,\nabla d_0)\in H_p^2\times H_p^2$. The reason is that, in the full system, the term $\nabla\cdot\sigma$ in (1.8) contains the high order term $\nabla\cdot(Nd)$ which is not in the simplified system in [12]. To deal with the high order term $\nabla\cdot(Nd)$, we need a higher order energy estimate compared to the case of the simplified system. But to obtain the higher order energy estimate, the smallness assumption on $(u_0,\nabla d_0)$ in $H_p^1\times H_p^1$ is not sufficient.

The basic idea of the regularity proof is to get some high order energy estimates (Ladyzhenskaya method) as described in [35, 12]. To fully utilize the smallness assumption (1.17), we follow the principle in [12] by keeping the potentially small terms $\|u\|_{L^2}$ and $\|\nabla d\|_{L^2}$ instead of throwing them away. However, we point out that the situation for the full system in the present work is much more complicated than that of the simplified system in [12], because the velocity field Equation (1.8) and director field Equation (1.9) are coupled not only through $\nabla\cdot(\nabla d\otimes\nabla d)$, but also through $\nabla\cdot\sigma$ and $-\Omega d+\frac{\lambda_2}{\lambda_1}Ad$ while σ consists of six non-zero terms.

In the following, we state the uniqueness of solutions,

THEOREM 1.5. *Let (ρ,u,d) be a solution to systems (1.7)–(1.10) and (1.11)–(1.13) obtained in Theorem 1.1 for the two dimensional case or, in Theorem 1.2, for the three* *dimensional case. Let* $(\bar{\rho},\bar{u},\bar{d})$ *be a weak solution to system (1.7)–(1.10) with (1.11)–(1.13) satisfying the following energy inequalities:*

$$\int_Q |\bar{\rho}|^2 dx \leq \int_Q |\rho_0|^2 dx, \tag{1.18}$$

$$\int_Q \frac{1}{2}\bar{\rho}|\bar{u}|^2 + \frac{1}{2}|\nabla \bar{d}|^2 + \bar{\rho}F(\bar{d})dx + \int_0^T \int_Q \mu_4 |\nabla \bar{u}|^2 - \frac{1}{\lambda_1}|\Delta \bar{d} - f(\bar{d})|^2 dxdt$$
$$\leq \int_Q \frac{1}{2}\rho_0|u_0|^2 + \frac{1}{2}|\nabla d_0|^2 + \rho_0 F(d_0)dx. \tag{1.19}$$

In addition, $|\bar{d}|$ *is bounded. Then* $(\rho,u,d) \equiv (\bar{\rho},\bar{u},\bar{d})$.

The uniqueness result is achieved by a standard approach where one establishes a Gronwall inequality for the difference of the two solutions. A uniqueness proof for the simplified system was presented in [12]. For system (1.7)–(1.10) the process will be similar. Since the computation work is huge and tedious, we give the proof of Theorem 1.5 in the appendix (Section A).

The rest of the paper is organized as follows. In Section 2, we introduce some notations that shall be used throughout the paper and the basic energy law governing the full system (1.7)–(1.10). In Section 3, we prove theorems 1.1 and 1.2 by several steps. We devote the appendix to proving Theorem 1.5.

2. Preliminary

2.1. Notations. We adopt the standard functional settings and notations for periodic problems (cf. [48]) in the following:

$$\begin{aligned}
H_p^m(Q) &= \{u \in H^m(\mathbb{R}^n,\mathbb{R}^n) | u(x+e_i) = u(x)\},\\
\dot{H}_p^m &= H_p^m(Q) \cap \left\{\int_Q u(x)dx = 0\right\},\\
H &= \left\{u \in L_p^2(Q) | \nabla \cdot u = 0\right\} \qquad \text{with } L_p^2(Q) = H_p^0(Q),\\
V &= \left\{u \in \dot{H}_p^1(Q) | \nabla \cdot u = 0\right\},\\
V' &= \text{ the dual space of } V.
\end{aligned}$$

At certain places in the paper, we also denote the space of scalar functions by $H_p^m(Q)$.

The inner product on $L_p^2(Q)$ and H is denoted by $(\cdot,\cdot)$ and the associated norm by $\|\cdot\|$. For simplicity, the space $H_p^m(Q)$ is denoted by H_p^m. The inner product on H_p^m is defined as $<u,v>_m = \Sigma_{|k|=0}^m (D^k u, D^k v)$ with $k = (k_1,\ldots,k_n)$ being the multi-index of length $|k| = \Sigma_{i=1}^n k_i$ and $D^k = \partial_{x_1}^{k_1} \ldots \partial_{x_1}^{k_1}$.

2.2. Definition of weak solution. The weak formulation of the problem is given as follows.

Definition 2.1. *The triplet* (ρ,u,d) *is called a weak solution to the system (1.7)–(1.10) in* $Q_T = Q \times (0,T)$ *subject to the boundary and initial conditions (1.11)–(1.13) if it satisfies*

$$\begin{aligned}
&0 < M_1 \leq \rho \leq M_2,\\
&u \in L^\infty(0,T;H) \cap L^2(0,T;V),\\
&d \in L^\infty(0,T;H_p^1 \cap L_p^\infty) \cap L^2(0,T;H_p^2),
\end{aligned}$$

and moreover, for any smooth function $\psi(t)$ with $\psi(T)=0$ and $\phi(x)\in H_p^1$, the following integral equations hold:

$$\int_0^T (\rho,\psi_t\phi)dt-\int_0^T (\rho u,\psi\nabla\phi)dt=(\rho_0,\phi)\psi(0),$$

$$\begin{aligned}&\int_0^T (u,\psi_t\phi)dt-\int_0^T (u\cdot\nabla u,\psi\phi)dt\\ &=(u_0,\phi)\psi(0)-\int_0^T (\nabla d\otimes\nabla d,\psi\nabla\phi)dt+\int_0^T (\sigma,\psi\nabla\phi)dt,\end{aligned}$$

$$\begin{aligned}&\int_0^T (d,\psi_t\phi)dt-\int_0^T (u\cdot\nabla d,\psi\phi)dt+\int_0^T (\Omega d,\psi\phi)dt-\frac{\lambda_2}{\lambda_1}\int_0^T (Ad,\psi\phi)dt\\ &=(d_0,\phi)\psi(0)+\frac{1}{\lambda_1}\int_0^T (\Delta d-f(d),\psi\phi)dt.\end{aligned}$$

2.3. Basic energy law. The total energy of system (1.7)–(1.10) is given by

$$\mathcal{E}(t)=\int_Q \frac{1}{2}\rho|u|^2+\frac{1}{2}|\nabla d|^2+\rho\mathcal{F}(d)dx \tag{2.1}$$

which consists of kinetic and potential energies. Formally, a smooth solution (ρ,u,d) satisfies (cf. [38, 41])

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t)=-\int_Q &\mu_1|d^TAd|^2+\mu_4|\nabla u|^2+(\mu_5+\mu_6)|Ad|^2dx\\ &+\lambda_1\|N\|^2+(\lambda_2-\mu_2-\mu_3)(N,Ad).\end{aligned}\tag{2.2}$$

With the Parodi's relation (1.6) we have the following basic energy law.

LEMMA 2.2. *Suppose that (1.5), (1.6), (1.14), and (1.15) are satisfied. Then the total energy $\mathcal{E}(t)$ for smooth solution satisfies*

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t)=-\int_Q &\mu_1|d^TAd|^2+\mu_4|\nabla u|^2dx+\frac{1}{\lambda_1}\|\Delta d-f(d)\|^2\\ &-\left(\mu_5+\mu_6+\frac{\lambda_2^2}{\lambda_1}\right)\|Ad\|^2\le 0.\end{aligned}\tag{2.3}$$

The proof of Lemma 2.2 is identical to the proof of Lemma 2.1 in [49], and thus it is omitted here.

3. Regular solutions

3.1. Galerkin approximate solutions. We construct a sequence of Galerkin approximating solutions that satisfy both the basic energy estimate and a higher order energy estimate. The higher order energy estimate, obtained through the Ladyzhenskaya method, yields a subsequence that will converge to the classical solution.

We first introduce the functional settings of the Galerkin approximation method. Let

$$\mathcal{H}(Q)=\text{ closure of }\{f\in C_0^\infty(Q,\mathbb{R}^n):\nabla\cdot f=0\}\text{ in }H.$$

Let $\{\phi_i\}_{i=1}^{\infty}$ be the unit eigenvectors of the Stokes problem in the periodic case with zero mean:

$$-\triangle\phi_i+\nabla\pi_i=k_i\phi_i \text{ in } Q,\quad \int_Q \phi_i(x)dx=0$$

with $\pi_i\in L^2$ and $0<k_1\le k_2\le\dots$ eigenvalues. It is known that each ϕ_i is smooth and $\{\phi_i\}_{i=1}^{\infty}$ forms an orthogonal basis of $\mathcal{H}$ (see [48]). Let

$$P_m:\mathcal{H}\to\mathcal{H}_m=\operatorname{span}\{\phi_1,\dots,\phi_m\}$$

be the orthonormal projection. We seek approximate solutions (ρ^m,u^m,d^m) with $u^m\in\mathcal{H}_m$ that satisfy the following equations:

$$\rho_t^m+u^m\cdot\nabla\rho^m=0, \tag{3.1}$$

$$P_m(\rho^m\frac{\partial}{\partial t}u^m)=P_m(\triangle u^m-\rho^m u^m\cdot\nabla u^m-\nabla\cdot(\nabla d^m\otimes\nabla d^m)-\nabla\cdot\sigma^m), \tag{3.2}$$

$$d_t^m+u^m\cdot\nabla d^m-\Omega^m d^m+\frac{\lambda_2}{\lambda_1}A^m d^m=-\frac{1}{\lambda_1}(\Delta d^m-f(d^m)), \tag{3.3}$$

with

$$\begin{aligned}
\Omega^m=&\frac{1}{2}(\nabla u^m-\nabla^T u^m),\quad A^m=\frac{1}{2}(\nabla u^m+\nabla^T u^m),\\
N^m=&\partial_t d^m+(u^m\cdot\nabla)d^m+\Omega^m d^m,\\
\sigma^m=&\mu_1((d^m)^T A^m d^m)d^m\otimes d^m+\mu_2 N^m\otimes d^m+\mu_3 d^m\otimes N^m+\mu_4 A^m\\
&+\mu_5 A^m d^m\otimes d^m+\mu_6 d^m\otimes A^m d^m,
\end{aligned}$$

and with the initial and boundary conditions

$$\rho^m(x,0)=\rho_0(x),\quad u^m(x,0)=P_m u_0(x),\quad d^m(x,0)=d_0(x),$$

$$u^m(x+e_i,t)=u^m(x,t),\quad d^m(x+e_i,t)=d^m(x,t).$$

Let

$$u^m(x,t)=\sum_{l=1}^m g_i^m(t)\phi_i(x),$$

with $g_i^m(t)\in C^1[0,T]$. Hence (3.2) is equivalent to the following system of ordinary differential equations:

$$\sum_{i=1}^m\mathcal{A}_i^{mj}(t)\frac{d}{dt}g_i^m(t)=-\sum_{i,k}^m\mathcal{B}_{ik}^{mj}(t)g_i^m(t)g_k^m(t)-\sum_{i=1}^m\mathcal{C}_i^j g_i^m(t)+\mathcal{D}^{mj}(t), \tag{3.4}$$

for $j=1,2,\dots,m$, where

$$\begin{cases}
\mathcal{A}_i^{mj}(t)=\int_\Omega\rho^m(t)\phi_i(x)\phi_j(x)dx,\\
\mathcal{B}_{ik}^{mj}(t)=\int_\Omega\rho^m(t)(\phi_i(x)\cdot\nabla\phi_k(x))\phi_j(x)dx,\\
\mathcal{C}_i^j=\int_\Omega\nabla\phi_i(x)\cdot\nabla\phi_j(x)dx,\\
\mathcal{D}^{mj}(t)=\int_\Omega\sum_{k,l}(\frac{\partial}{\partial x_k}d^m\cdot\frac{\partial}{\partial x_l}d^m+\sigma_{kl}^m)\frac{\partial}{\partial x_l}\phi_j^k(x)dx.
\end{cases}$$

Here $\phi_j^k(x)$ is the k-th component of the vector $\phi_j(x)$. And

$$u^m(\cdot,0)=\sum_{i=1}^m g_i^m(0)\phi_i(x), \quad \text{where} \quad g_i^m(0)=\int_\Omega u_0(x)\phi_i(x)dx.$$

Lemma 3.1. *There exists a solution (ρ^m,u^m,d^m) to the problem (3.1)–(3.3) in $Q_T=Q\times[0,T]$ for any $T\in(0,\infty)$ satisfying*

$$M_1\le\rho^m\le M_2,$$
$$u^m\in L^\infty(0,T;H)\cap L^2(0,T;V),$$
$$d^m\in L^\infty(0,T;H_p^1\cap L_p^\infty)\cap L^2(0,T;H_p^2).$$

Moreover, (ρ^m,u^m,d^m) is smooth in the interior of Q_T and satisfies the basic energy equality,

$$\begin{aligned}\frac{d}{dt}\mathcal{E}^m(t)=&-\int_Q \mu_1|(d^m)^TA^md^m|^2+\mu_4|\nabla u^m|^2dx+\frac{1}{\lambda_1}\|\Delta d^m-f(d^m)\|^2\\&-\left(\mu_5+\mu_6+\frac{\lambda_2^2}{\lambda_1}\right)\|A^md^m\|^2\le 0,\end{aligned} \tag{3.5}$$

with

$$\mathcal{E}^m(t)=\int_Q\frac{1}{2}\rho^m|u^m|^2+\frac{1}{2}|\nabla d^m|^2+\rho^m\mathcal{F}(d^m)dx.$$

Proof. The proof of the existence of weak solutions is based on an application of the Leray–Schauder fixed point theorem. Let $v^m=\sum_{i=1}^m h_i^m\phi_i\in C^1(0,T;\mathcal{H}_m)$. For each m, let ρ^m be a solution to

$$\rho+v^m\cdot\nabla\rho=0 \tag{3.6}$$

with initial condition $\rho(\cdot,0)=\rho_0$. Let d^m be a solution to

$$\begin{aligned}d_t+v^m\cdot\nabla d-\tilde{\Omega}^md+\frac{\lambda_2}{\lambda_1}\tilde{A}^md&=-\frac{1}{\lambda_1}(\Delta d-f(d)),\\ \tilde{\Omega}^m=\frac{1}{2}(\nabla v^m-\nabla^Tv^m),\quad \tilde{A}^m&=\frac{1}{2}(\nabla v^m+\nabla^Tv^m)\end{aligned}$$

with initial condition $d(\cdot,0)=d_0$ and boundary condition $d(x+e_i,t)=d(x,t)$. The reason the transport equation (3.6) is solvable for $v^m\in C^1(0,T;\mathcal{H}_m)$ is due to the regularity of the eigenfunctions of the Stokes operators (cf. [47] [30]). Let $u^m=\sum_{i=1}^m g_i^m\phi_i\in C^1(0,T;\mathcal{H}_m)$ be the solution of the system of linear equations

$$\sum_{i=1}^m\mathcal{A}_i^{mj}(t)\frac{d}{dt}g_i^m(t)=-\sum_{i,k}^m\mathcal{B}_{ik}^{mj}(t)h_i^m(t)g_k^m(t)-\sum_{i=1}^m\mathcal{C}_i^jg_i^m(t)+\mathcal{D}^{mj}(t).$$

This system of linear equations is solvable because the eigenvalues of the matrix of the coefficients $\mathcal{A}_i^{mj}(t)$ are bounded from below since

$$\mathcal{A}_i^{mj}\gamma_i\gamma_j=\int_\Omega\rho|\psi|^2dx\ge M_1\int_\Omega|\psi|^2dx \quad \text{where } \psi=\sum_{i=1}^m\gamma_i\phi_i. \tag{3.7}$$

Thus we constructed a mapping $\mathcal{M}$ with $\mathcal{M}(v^m)=u^m$. The energy estimate (3.5) will be obtained as in Lemma 2.2, and it is the key that allows one to apply the Leray–Schauder fixed point theorem to $\mathcal{M}$. ∎

3.2. Higher order energy estimates. As a first step, we use the Ladyzhenskaya energy method [30] to show that $u^m \in L^\infty(0,T;V)\bigcap L^2(0,T;H_p^2)$ and $d^m \in L^\infty(0,T;H_p^2) \bigcap L^2(0,T;H_p^3)$ provided $u_0 \in H_p^1$ and $d_0 \in H_p^2$. In 2D, these estimates will be obtained with general initial data; in 3D, they will be obtained under the assumption of small initial data. We then pass to the limit for the Galerkin approximating solutions (ρ^m,u^m,d^m) to yield weak solutions for the system (1.7)–(1.10). The key inequalities used often in this paper are the following Gagliardo–Nirenberg inequalities (cf. [19]).

LEMMA 3.2 (Gagliardo–Nirenberg). *If Ω is a smooth bounded domain in $\mathbb{R}^n$, then*

$$\|u\|_{L^4(\Omega)}^4 \le C\|u\|_{L^2(\Omega)}^2(\|\nabla u\|_{L^2(\Omega)}^2+\|u\|_{L^2(\Omega)}^2), \qquad n=2,$$

$$\|u\|_{L^4(\Omega)}^4 \le C\|u\|_{L^2(\Omega)}(\|\nabla u\|_{L^2(\Omega)}^2+\|u\|_{L^2(\Omega)}^2)^{\frac{3}{2}}, \qquad n=3.$$

By a similar strategy as in [35, 12], we will establish uniform estimates for the following energy quantity:

$$\Phi_m^2(t)=\|\sqrt{\rho^m}\nabla u^m\|_{L^2}^2+\|\triangle d^m-f(d^m)\|_{L^2}^2.$$

LEMMA 3.3 (3D small data assumption). *Let $Q\subset\mathbb{R}^3$ and suppose that the initial data (ρ_0,u_0,d_0) satisfies (1.12) and (1.13). Suppose that $\Phi^2(0)=\|\sqrt{\rho_0}\nabla u_0\|_2^2+\|\Delta d_0-f(d_0)\|_2^2<\infty$. There is $\epsilon_0>0$ such that if*

$$(\frac{M_2}{\mu_4}+1)\rho_0\|u_0\|_{H^1}^2+\|d_0\|_{H^1}^2+\|\Delta d_0-f(d_0)\|_2^2\le\epsilon_0, \tag{3.8}$$

then the approximating solutions (ρ^m,u^m,d^m) obtained in Lemma 3.1 satisfy

$$\|\sqrt{\rho^m}\nabla u^m\|_{L^2}^2+\|\Delta d^m-f(d^m)\|_{L^2}^2\le C(\Phi^2(0)+C) \tag{3.9}$$

for all $t\in[0,T]$ and

$$\int_0^T \|\Delta u^m\|_2^2+\|\nabla(\Delta d^m-f(d^m))\|_2^2+\mu_1\|d^{mT}\nabla A^m d^m\|_2^2 dt\le C \tag{3.10}$$

where the constants C depend only on initial data, Q, M_1, M_2, and the physical coefficients in system (1.7)–(1.10).

Proof. Note that the approximating solutions (ρ^m,u^m,d^m) obtained in Lemma 3.1 also satisfy equations (1.7)–(1.9) point-wise. To simplify the notation, throughout the proof, we drop the approximating index m and denote (ρ,u,d) as the Galerkin approximating solutions (ρ^m,u^m,d^m). We also denote $\Phi_m(t)$ by $\Phi(t)$. By Equation (1.7) and (1.10), applying the periodic boundary conditions (1.11) and integrating by parts, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Phi^2(t)&=\int_Q\frac{1}{2}\rho_t|\nabla u|^2+\rho\nabla u\nabla u_t dx+\int_Q(\Delta d-f(d))\cdot(\Delta d_t-f'(d)d_t)dx\\&=\int_Q\frac{1}{2}\rho(u\cdot\nabla)|\nabla u|^2-\rho u_t\Delta u dx+\int_Q(\Delta d-f(d))\Delta d_t dx\\&\quad-\int_Q(\Delta d-f(d))f'(d)d_t dx.\end{aligned} \tag{3.11}$$

It follows from Equation (1.8) that

$$
\begin{aligned}
&\int_Q \frac{1}{2}\rho(u\cdot\nabla)|\nabla u|^2-\rho u_t\Delta u dx\\
=&\int_Q \frac{1}{2}\rho(u\cdot\nabla)|\nabla u|^2+\rho(u\cdot\nabla)u\Delta u+\Delta u\nabla d\Delta d-\Delta u\nabla\cdot\sigma dx\\
=&-\mu_4\int_Q|\Delta u|^2dx+\int_Q 2\rho(u\cdot\nabla)u\Delta u+\Delta u\nabla d\Delta d-\Delta u\nabla\cdot\bar{\sigma}dx\\
=&-\mu_4\int_Q|\Delta u|^2dx+I_1+I_2+I_3
\end{aligned}
\tag{3.12}
$$

with $\bar{\sigma}=\sigma-\mu_4 A$. Taking the Laplacian of Equation (1.9) gives

$$
\Delta d_t+\Delta(u\cdot\nabla d)-\Delta(\Omega d)+\frac{\lambda_2}{\lambda_1}\Delta(Ad)=-\frac{1}{\lambda_1}\Delta(\Delta d-f(d)).
$$

Thus,

$$
\begin{aligned}
\int_\Omega(\Delta d-f(d))\Delta d_t dx=&\frac{1}{\lambda_1}\int_\Omega|\nabla(\Delta d-f(d))|^2dx-\int_\Omega(\Delta d-f(d))\Delta(u\cdot\nabla d)dx\\
&+\int_\Omega(\Delta d-f(d))\Delta(\Omega d)dx-\frac{\lambda_2}{\lambda_1}\int_\Omega(\Delta d-f(d))\Delta(Ad)x\\
=&\frac{1}{\lambda_1}\int_\Omega|\nabla(\Delta d-f(d))|^2dx+I_4+I_5+I_6.
\end{aligned}
\tag{3.13}
$$

We also have the following from Equation (1.9):

$$
\begin{aligned}
&-\int_\Omega(\Delta d-f(d))f'(d)d_t dx\\
=&\frac{1}{\lambda_1}\int_\Omega f'(d)|\Delta d-f(d)|^2dx+(\Delta d-f(d),f'(d)(u\cdot\nabla)d)-(\Delta d-f(d),f'(d)(\Omega d-\frac{\lambda_2}{\lambda_1}Ad))\\
=&I_7+I_8+I_9.
\end{aligned}
\tag{3.14}
$$

Combining (3.11), (3.12), (3.13), and (3.14) yields

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\Phi^2(t)=&-\mu_4\int_Q|\Delta u|^2dx+\frac{1}{\lambda_1}\int_Q|\nabla(\Delta d-f(d))|^2dx\\
&+I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8+I_9,
\end{aligned}
\tag{3.15}
$$

with I_k defined as in the above equations for $k=1,\dots,9$. In the following, we shall estimate these I_k term by term.

By Hölder's inequality and the Gagliardo–Nirenberg inequality in Lemma 3.2, it follows that

$$
\begin{aligned}
|I_1|\le& M_2\int_\Omega|u||\nabla u||\Delta u|dx\\
\le&\epsilon\|\Delta u\|_2^2+C\|u\|_4^2\|\nabla u\|_4^2\\
\le&\epsilon\|\Delta u\|_2^2+C\|u\|_2^{1/2}\|\nabla u\|_2^2\|\Delta u\|_2^{3/2}\\
\le&\epsilon\|\Delta u\|_2^2+C\|u\|_2^2\|\nabla u\|_2^8
\end{aligned}
$$

$$\leq \epsilon\|\Delta u\|_2^2 + CM_1^{-4}\|u\|_2^2\|\sqrt{\rho}\nabla u\|_2^8. \tag{3.16}$$

By Agmon's inequality in 3D, we have

$$\|d\|_\infty \leq C\|\nabla d\|_2^{1/2}\|\Delta d\|_2^{1/2}. \tag{3.17}$$

Thus we infer that

$$\|\Delta d\|_2 \leq \|\Delta d - f(d)\|_2 + \|f(d)\|_2 \leq \|\Delta d - f(d)\|_2 + C, \tag{3.18}$$

$$\begin{aligned}
\|\nabla\Delta d\|_2 &\leq \|\nabla(\Delta d - f(d))\|_2 + \|\nabla f(d)\|_2 \\
&\leq \|\nabla(\Delta d - f(d))\|_2 + \|f'(d)\|_\infty\|\nabla d\|_2 \\
&\leq \|\nabla(\Delta d - f(d))\|_2 + (1 + \|d\|_\infty^2)\|\nabla d\|_2 \\
&\leq \|\nabla(\Delta d - f(d))\|_2 + (1 + C\|\nabla d\|_2\|\Delta d\|_2)\|\nabla d\|_2 \\
&\leq \|\nabla(\Delta d - f(d))\|_2 + C(1 + \|\nabla d\|_2\|\Delta d - f(d)\|_2)\|\nabla d\|_2.
\end{aligned}$$

Therefore, applying Hölder's inequality, the Gagliardo–Nirenberg inequality, and (3.19) yields

$$\begin{aligned}
|I_2| \leq& \epsilon\|\Delta u\|_2^2 + C\|\nabla d\|_4^2\|\Delta d\|_4^2 \\
\leq& \epsilon\|\Delta u\|_2^2 + C\|\nabla d\|_2^{1/2}\|\Delta d\|_2^2\|\nabla\Delta d\|_2^{3/2} \\
\leq& \epsilon\|\Delta u\|_2^2 + \epsilon\|\nabla\Delta d\|_2^2 + C\|\nabla d\|_2^2\|\Delta d\|_2^8 \\
\leq& \epsilon\|\Delta u\|_2^2 + \epsilon\|\nabla(\Delta d - f(d))\|_2^2 + C\|\nabla d\|_2^2(\|\Delta d - f(d)\|_2^8 + 1) \\
&+ C\|\nabla d\|_2^2(1 + \|\nabla d\|_2^2\|\Delta d - f(d)\|_2^2).
\end{aligned} \tag{3.19}$$

From the equation of σ in (1.3), we have

$$\begin{aligned}
I_3 =& -\mu_1\int_Q \nabla_j(d_kd_pA_{kp}d_id_j)\nabla_l\nabla_l u_i dx - \mu_2\int_Q \nabla_j(d_jN_i)\nabla_l\nabla_l u_i dx \\
&- \mu_3\int_Q \nabla_j(d_iN_j)\nabla_l\nabla_l u_i dx - \mu_5\int_Q \nabla_j(d_jd_kA_{ki})\nabla_l\nabla_l u_i dx \\
&- \mu_6\int_Q \nabla_j(d_id_kA_{kj})\nabla_l\nabla_l u_i dx.
\end{aligned}$$

Integrating by parts yields

$$\begin{aligned}
-\mu_1\int_Q \nabla_j(d_kd_pA_{kp}d_id_j)\nabla_l\nabla_l u_i dx =& \mu_1\int_Q (d_kd_pA_{kp}d_id_j)\nabla_l\nabla_l(A_{ij} + \Omega_{ij})dx \\
=& \mu_1\int_Q (d_kd_pA_{kp}d_id_j)\nabla_l\nabla_l A_{ij}dx,
\end{aligned}$$

where we use the fact that Ω is antisymmetric, which implies

$$\begin{aligned}
\mu_1\int_Q (d_kd_pA_{kp}d_id_j)\nabla_l\nabla_l\Omega_{ij}dx =& -\mu_1\int_Q \nabla_l(d_kd_pA_{kp}d_id_j)\nabla_l\Omega_{ij}dx \\
=& \mu_1\int_Q \nabla_l(d_kd_pA_{kp}d_id_j)\nabla_l\Omega_{ji}dx
\end{aligned}$$

$$=\mu_1\int_Q \nabla_l(d_kd_pA_{kp}d_jd_i)\nabla_l\Omega_{ij}dx$$
$$=-\mu_1\int_Q (d_kd_pA_{kp}d_jd_i)\nabla_l\nabla_l\Omega_{ij}dx$$

hence $\int_Q(d_kd_pA_{kp}d_id_j)\nabla_l\nabla_l\Omega_{ij}dx=0$. Therefore,

$$\begin{aligned}-\mu_1\int_Q \nabla_j(d_kd_pA_{kp}d_id_j)\nabla_l\nabla_l u_i dx &= -\mu_1\int_Q |d^T\nabla Ad|^2dx-\mu_1\int_Q \nabla(d\otimes d)d\otimes dA\nabla Adx\\ &= -\mu_1\int_Q |d^T\nabla Ad|^2dx+I_{31}.\end{aligned}\tag{3.20}$$

Integration by parts also yields

$$\begin{aligned}&-\mu_2\int_Q \nabla_j(d_jN_i)\nabla_l\nabla_l u_idx-\mu_3\int_Q\nabla_j(d_iN_j)\nabla_l\nabla_l u_idx\\ =&\mu_2\int_Q d_jN_i\Delta(A_{ij}+\Omega_{ij})dx+\mu_3\int_Q d_iN_j\Delta(A_{ij}+\Omega_{ij})dx\\ =&(\mu_2+\mu_3)\int_Q d_jN_i\Delta A_{ij}dx+(\mu_2-\mu_3)\int_Q d_jN_i\Delta\Omega_{ij}dx\\ =&I_{32}+I_{33},\ \text{and}\end{aligned}\tag{3.21}$$

$$\begin{aligned}&-\mu_5\int_Q \nabla_j(d_jd_kA_{ki})\nabla_l\nabla_l u_idx-\mu_6\int_Q\nabla_j(d_id_kA_{kj})\nabla_l\nabla_l u_idx\\ =&(\mu_5+\mu_6)\int_Q d_jd_kA_{ki}\Delta A_{ij}dx+(\mu_5-\mu_6)\int_Q d_jd_kA_{ki}\Delta\Omega_{ij}dx\\ =&-2(\mu_5+\mu_6)\int_Q Ad\nabla d\nabla Adx+(\mu_5-\mu_6)\int_Q Ad\Delta\Omega ddx\\ &-(\mu_5+\mu_6)\int_Q|d\nabla A|^2dx=I_{34}+I_{35}+I_{36}.\end{aligned}\tag{3.22}$$

Using integration by parts and Equation (1.9), we infer

$$\begin{aligned}I_5=&\int_Q(\Delta d-f(d))\Delta\Omega ddx+2\int_Q(\Delta d-f(d))\nabla\Omega\nabla ddx+\int_Q(\Delta d-f(d))\Omega\Delta ddx\\ =&-\lambda_1(N,\Delta\Omega d)-\lambda_2(Ad,\Delta\Omega d)+2\int_Q(\Delta d-f(d))\nabla\Omega\nabla ddx\\ &-\int_Q(\Delta d-f(d))\nabla\Omega\nabla ddx-\int_Q\nabla(\Delta d-f(d))\Omega\nabla ddx\\ =&-\lambda_1(N,\Delta\Omega d)-\lambda_2(Ad,\Delta\Omega d)+\int_Q(\Delta d-f(d))\nabla\Omega\nabla ddx-\int_Q\nabla(\Delta d-f(d))\Omega\nabla ddx\\ =&I_{51}+I_{52}+I_{53}+I_{54},\ \text{and}\end{aligned}\tag{3.23}$$

$$
\begin{aligned}
I_6 =&\lambda_2 \int_Q N\Delta(Ad)dx + \frac{\lambda_2^2}{\lambda_1}\int_Q Ad\Delta(Ad)dx\\
=&\lambda_2 \int_Q N\Delta A d dx + 2\lambda_2 \int_Q N\nabla A\nabla d dx\\
&+\lambda_2 \int_Q NA\Delta d dx - \frac{\lambda_2^2}{\lambda_1}\int_Q |d\nabla A|^2 dx - \frac{\lambda_2^2}{\lambda_1}\int_Q |A\nabla d|^2 dx\\
=&I_{61}+I_{62}+I_{63}+I_{64}+I_{65}.
\end{aligned}
\tag{3.24}
$$

Note that there is cancelation among I_3, I_5, and I_6, due to the parameter relations (1.5) and (1.6) and assumption (1.15). Indeed,

$$
I_{32}+I_{61}=0,\ I_{33}+I_{51}=0,\ I_{35}+I_{52}=0,\ I_{36}+I_{64}\le 0.
$$

In the following, we estimate the rest of the terms in I_3, I_5, and I_6. Applying Hölder's inequality, the Gagliardo–Nirenberg's inequality, and Agmon's inequality, (3.17) yields

$$
\begin{aligned}
|I_{31}| \le&\epsilon\|\Delta u\|_2^2 + C\|d\|_\infty^6\|\nabla u\|_4^2\|\nabla d\|_4^2\\
\le&\epsilon\|\Delta u\|_2^2 + C\|\nabla d\|_2^3\|\Delta d\|_2^3\|\nabla u\|_2^{1/2}\|\Delta u\|_2^{3/2}\|\nabla d\|_2^{1/2}\|\Delta d\|_2^{3/2}\\
\le&\epsilon\|\Delta u\|_2^2 + C\|\nabla d\|_2^{14}\|\Delta d\|_2^{18}\|\nabla u\|_2^2\\
\le&\epsilon\|\Delta u\|_2^2 + C\|\nabla d\|_2^{14}(\|\Delta d - f(d)\|_2^{18}+1)\|\nabla u\|_2^2,
\end{aligned}
\tag{3.25}
$$

$$
|I_{34}| \le \epsilon\|\Delta u\|_2^2 + C\|\nabla d\|_2^6(\|\Delta d - f(d)\|_2^{10}+1)\|\nabla u\|_2^2,
\tag{3.26}
$$

$$
|I_{53}| \le \epsilon(\|\Delta u\|_2^2 + \|\nabla(\Delta d - f)\|_2^2) + C\|\nabla d\|_2^2(\|\Delta d - f(d)\|_2^6+1)\|\Delta d - f(d)\|_2^2,
\tag{3.27}
$$

$$
|I_{54}| \le \epsilon(\|\Delta u\|_2^2 + \|\nabla(\Delta d - f)\|_2^2) + C\|\nabla d\|_2^2(\|\Delta d - f\|_2^6+1)\|\nabla u\|_2^2,
\tag{3.28}
$$

and

$$
\begin{aligned}
|I_{62}| \le& \int_Q |Ad\Delta u\nabla d| dx + \int_Q |(\Delta d - f(d))\Delta u\nabla d| dx\\
\le&\epsilon(\|\Delta u\|_2^2 + \|\nabla(\Delta d - f)\|_2^2) + C\|\nabla d\|_2^2\\
&\cdot(\|\Delta d - f\|_2^{10} + \|\Delta d - f\|_2^6 + 1)\left(\|u\|_2^2 + \|\Delta d - f\|_2^2\right)
\end{aligned}
\tag{3.29}
$$

since the two integrals are similar to I_{34} and I_{53} respectively. Moreover, from integration by parts we have

$$
\begin{aligned}
I_{63}+I_{65} =& -\lambda_2 \int_Q N\nabla A\nabla d dx - \lambda_2\int_Q \nabla NA\nabla d dx - \frac{\lambda_2^2}{\lambda_1}\int_Q |A\nabla d|^2 dx\\
=& -\lambda_2 \int_Q N\nabla A\nabla d dx + \frac{\lambda_2^2}{\lambda_1}\int_Q \nabla(Ad)A\nabla d dx\\
&+\frac{\lambda_2}{\lambda_1}\int_Q \nabla(\Delta d - f)A\nabla d dx - \frac{\lambda_2^2}{\lambda_1}\int_Q |A\nabla d|^2 dx\\
=& -\lambda_2 \int_Q N\nabla A\nabla d dx + \frac{\lambda_2^2}{\lambda_1}\int_Q \nabla A dA\nabla d dx + \frac{\lambda_2}{\lambda_1}\int_Q \nabla(\Delta d - f)A\nabla d dx
\end{aligned}
$$

where the three integrals are similar to I_{62}, I_{34} and I_{54} respectively. Thus

$$\begin{aligned}|I_{63}+I_{65}| \le & \epsilon(\|\Delta u\|_2^2+\|\nabla(\Delta d-f)\|_2^2)+C\|\nabla d\|_2^2 \\ & \cdot(\|\Delta d-f\|_2^{10}+\|\Delta d-f\|_2^6+1)\left(\|u\|_2^2+\|\Delta d-f\|_2^2\right).\end{aligned} \tag{3.30}$$

The estimate for I_4+I_8 is as follows. The facts $\nabla d\cdot f(d)=\nabla F(d)$ and $\nabla\cdot u=0$ imply that $(f,\Delta u\cdot\nabla d)=0$. Also, since $\nabla\cdot u=0$, we have $(\Delta d-f,u\cdot\nabla(\Delta d-f))=0$. Thus,

$$\begin{aligned}I_4 &= -(\Delta d-f,\Delta u\cdot\nabla d)-2\int_\Omega(\Delta d_i-f_i)\nabla_l v_j\nabla_l\nabla_j d_i dx-(\Delta d-f,u\cdot\nabla\Delta d) \\ &= -(\Delta d\nabla d,\Delta u)+2\int_\Omega\nabla_j(\Delta d_i-f_i)\nabla_l v_j\nabla_l d_i dx-(\Delta d-f,u\cdot\nabla f(d)) \\ &= -(\Delta d\nabla d,\Delta u)+2\int_\Omega\nabla_j(\Delta d_i-f_i)\nabla_l v_j\nabla_l d_i dx-I_8.\end{aligned}$$

Therefore,

$$I_4+I_8=-(\Delta d\nabla d,\Delta u)+2\int_\Omega\nabla_j(\Delta d_i-f_i)\nabla_l v_j\nabla_l d_i dx,$$

where the two terms are similar to I_2 and I_{54} respectively. Hence, from (3.19) and (3.28), we have

$$|I_4+I_8|\le\epsilon\left(\|\Delta u\|_2^2+\|\nabla(\Delta d-f)\|_2^2\right)+C\|\nabla d\|_2^2\left(\Phi^8+\Phi^6+1\right). \tag{3.31}$$

By Agmon's inequality, the terms I_7 and I_9 are estimated as

$$\begin{aligned}|I_7| &\le C\|f'(d)\|_\infty\|\Delta d-f(d)\|_2^2\le C\|d\|_\infty^2\|\Delta d-f(d)\|_2^2 \\ &\le C\|\nabla d\|_2(\|\Delta d-f(d)\|_2+1)\|\Delta d-f(d)\|_2^2 \text{ and}\end{aligned} \tag{3.32}$$

$$\begin{aligned}|I_9| &\le C\|f'(d)\|_\infty\|d\|_\infty\left(\|\Delta d-f(d)\|_2^2+\|\nabla u\|_2^2\right) \\ &\le C\|\nabla d\|_2^{3/2}(\|\Delta d-f(d)\|_2^{3/2}+1)\Phi^2.\end{aligned} \tag{3.33}$$

Denote $\mathcal{D}=\|\Delta d-f(d)\|_2$. Combining (3.15), (3.16), and (3.19)–(3.33) yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Phi^2(t)+(\mu_4-\epsilon)\int_Q|\Delta u|^2dx+\left(-\frac{1}{\lambda_1}-\epsilon\right)\int_\Omega|\nabla(\Delta d-f(d))|^2dx+\mu_1\int_Q|d^T\nabla Ad|^2dx \\ &\le CM_1^{-4}\|u\|_2^2\|\sqrt{\rho}\nabla u\|_2^8+C\|\nabla d\|_2^2(\mathcal{D}^8+\mathcal{D}^2+1) \\ &\quad+C\|\nabla d\|_2^2(\mathcal{D}^{18}+\mathcal{D}^{10}+\mathcal{D}^6+1)(\|\nabla u\|_2^2+\mathcal{D}^2) \\ &\quad+C\|\nabla d\|_2(\mathcal{D}+1)\mathcal{D}^2+C\|\nabla d\|_2^{3/2}(\mathcal{D}^{3/2}+1)(\|\nabla u\|_2^2+\mathcal{D}^2) \\ &\le C\|\nabla d\|_2^2+(\|u\|_2+\|\nabla d\|_2)(\Phi^{18}+\Phi^{10}+\Phi^6+\Phi^2+\Phi+1)\Phi^2,\end{aligned} \tag{3.34}$$

where we have used the facts from the basic energy estimate that $\|\nabla u\|_2^2\le M_1^{-1}\|\sqrt{\rho}\nabla u\|_2^2$ and $\|u\|_2+\|\nabla d\|_2\le C$.

Set

$$\tilde{\Phi}^2=\Phi^2+\|u\|_2+\|\nabla d\|_2$$

and observe from (3.34) that

$$\frac{d}{dt}\tilde{\Phi}^2 \le C(1+(\|u\|_2+\|\nabla d\|_2)\tilde{\Phi}^{18})\tilde{\Phi}^2. \tag{3.35}$$

Recall that $\|u\|_2+\|\nabla d\|_2$ is small by the basic energy estimate (3.5) and the smallness assumption (3.8). Thus, following a similar argument as in [12], we infer that for a small enough $\epsilon_0>0$ in assumption (3.8), the conclusions of the lemma will be proven. Suppose that

$$(\frac{M_2}{\mu_4}+1)\rho_0\|u_0\|_{H^1}^2+\|d_0\|_{H^1}^2+\|\Delta d_0-f(d_0)\|_2^2=\epsilon_0. \tag{3.36}$$

By the basic energy estimate, we have

$$\|u\|_2+\|\nabla d\|_2 \le \|u_0\|_2+\|\nabla d_0\|_2 \le \sqrt{2\epsilon_0}.$$

We claim that,if ϵ_0 is so small that

$$\sqrt{2\epsilon_0}(4e^{2C}\sqrt{\epsilon_0})^9<1, \tag{3.37}$$

then

$$\tilde{\Phi}^2 < 4e^{2C}\sqrt{\epsilon_0}, \text{ for all } t>0.$$

First, we prove the claim for $t\in[0,1]$. Assume otherwise; then there must be $t_0\in(0,1)$ such that

$$\begin{cases}\tilde{\Phi}^2(t_0)=4e^{2C}\sqrt{\epsilon_0}\\ \tilde{\Phi}^2(t)\le 4e^{2C}\sqrt{\epsilon_0}, \text{ for all } t\in(0,t_0].\end{cases} \tag{3.38}$$

Therefore, from (3.35), by the choice of ϵ_0 in (3.37), we have

$$\frac{d}{dt}\tilde{\Phi}^2 \le 2C\tilde{\Phi}^2$$

for all $t\in(0,t_0)$ and $\tilde{\Phi}^2(0)\le\epsilon_0+\sqrt{2\epsilon_0}<3\sqrt{\epsilon_0}$ which implies that

$$\tilde{\Phi}^2(t_0)\le e^{2C}\tilde{\Phi}^2(0)\le 3e^{2C}\sqrt{\epsilon_0}$$

and thus contradicts (3.38). For $t>1$, we simply observe that from the basic energy inequality (3.5) and assumption (3.36),

$$\int_{t-1}^t \tilde{\Phi}^2(t)dt \le 3\sqrt{\epsilon_0}$$

which implies that there is $t_0\in(t-1,t)$ such that

$$\tilde{\Phi}^2(t_0)\le 3\sqrt{\epsilon_0}.$$

One may repeat the above argument to conclude that

$$\tilde{\Phi}^2(t)\le 4e^{2C}\sqrt{\epsilon_0}, \text{ for all } t>0.$$

The inequality (3.10) follows directly from the basic energy estimate (3.5) and (3.35). This completes the proof of the lemma. ◻

Without the assumption of small initial data, the higher order energy estimate holds for a short time.

LEMMA 3.4 (3D short time). *Let $Q\subset\mathbb{R}^3$, and supopse that the initial data (ρ_0,u_0,d_0) satisfies (1.12) and (1.13). Suppose that $\Phi(0)^2=\|\sqrt{\rho_0}\nabla u_0\|_2^2+\|\Delta d_0-f(d_0)\|_2^2<\infty$. Then the approximating solutions (ρ^m,u^m,d^m) obtained in Lemma 3.1 satisfy (3.9) and (3.10) on the time interval $(0,T]$ for some $T<\delta$.*

Proof . Set

$$\tilde{\Phi}_m^2=\Phi_m^2+\|u^m\|_{L^2}+\|\nabla d^m\|_{L^2}+1.$$

By (3.34) and the uniform estimate $\|u^m\|_2+\|\nabla d^m\|_2\le C$, we have

$$\frac{d}{dt}\tilde{\Phi}_m^2\le C\tilde{\Phi}_m^{20}.$$

Hence,

$$\int\frac{d\tilde{\Phi}_m^2}{\tilde{\Phi}_m^{18}}\le C\int_0^T\tilde{\Phi}_m^2dt$$

which produces for, any $t\in(0,T]$,

$$\tilde{\Phi}_m^{16}(t)\le\frac{\tilde{\Phi}_m^{16}(0)}{1-C\tilde{\Phi}_m^{16}(0)\int_0^T\tilde{\Phi}_m^2(0)dt}.$$

The basic energy estimate (3.5) indicates that there exists a small time $T<\delta$ such that

$$C\tilde{\Phi}_m^{16}(0)\int_0^T\tilde{\Phi}_m^2(0)dt\le\frac{1}{2}$$

which consequently implies

$$\tilde{\Phi}_m^2(t)\le C\tilde{\Phi}_m^2(0),\ \text{ for all } t\in(0,T].$$

The conclusion of the lemma thus holds true. ◻

Next we shall establish the higher order energy estimate with general data in the two dimensional case.

LEMMA 3.5 (2D). *Let $Q\subset\mathbb{R}^2$, and suppose that the initial data (ρ_0,u_0,d_0) satisfies (1.12) and (1.13). Suppose that $\Phi(0)^2=\|\sqrt{\rho_0}\nabla u_0\|_2^2+\|\Delta d_0-f(d_0)\|_2^2<\infty$. Assume either (i) or (ii) holds. Then the approximating solutions (ρ^m,u^m,d^m) obtained in Lemma 3.1 satisfy*

$$\|\sqrt{\rho^m}\nabla u^m\|_{L^2}^2+\|\Delta d^m-f(d^m)\|_{L^2}^2\le(\Phi^2(0)+1)e^{C(\rho_0\|u_0\|_2^2+\|\nabla d_0\|_2^2+1)}\tag{3.39}$$

with $t\in[0,T]$ and

$$\int_0^T\|\Delta u^m\|_2^2+\|\nabla(\Delta d^m-f(d^m))\|_2^2+\mu_1\|d^{mT}\nabla A^m d^m\|_2^2dt\le C\tag{3.40}$$

for all $T>0$. The constants C depend only on initial data, Q, M_1, M_2, and the physical coefficients in system (1.7)–(1.10).

Proof .
Case (i). Since $\lambda_2=0$, a maximum principle for d holds (cf. [38]). Agmon's inequality is not needed here. The highest nonlinear term I_{31} is estimated as

$$\begin{aligned}|I_{31}| &\le C\|d^m\|_\infty^3\int_Q |\nabla d^m \nabla u^m \Delta u^m|dx\\ &\le \epsilon\|\Delta u^m\|_2^2+C\|\nabla u^m\|_4^2\|\nabla d^m\|_4^2\\ &\le \epsilon\|\Delta u^m\|_2^2+C\|\nabla u^m\|_2^2(\|\Delta d^m-f(d^m)\|_2^2+1)\end{aligned}$$

where we used the 2D Gagliardo–Nirenberg inequality in Lemma 3.2. Similarly, we apply Hölder's equality, the Gagliardo–Nirenberg inequality, the basic energy estimate, and $\|d^m\|_\infty\le C$ to all the other terms in (3.11) and obtain that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Phi_m^2(t)+(\mu_4-\epsilon)\int_Q|\Delta u^m|^2dx\\ &\quad+\left(-\frac{1}{\lambda_1}-\epsilon\right)\int_\Omega|\nabla(\Delta d^m-f(d^m))|^2dx+\mu_1\int_Q|d^{mT}\nabla A^m d^m|^2dx\\ &\le C(\Phi_m^4+\Phi_m^2+1).\end{aligned}\tag{3.41}$$

Let $\tilde\Phi_m^2=\Phi_m^2+1$. We have

$$\begin{aligned}\frac{d}{dt}\tilde\Phi_m^2(t)+\mu_4\int_Q|\Delta u^m|^2dx-\frac{1}{\lambda_1}\int_\Omega|\nabla(\Delta d^m-f(d^m))|^2dx&\\ +2\mu_1\int_Q|d^{mT}\nabla A^m d^m|^2dx\le C\tilde\Phi_m^4.&\end{aligned}$$

Thus,

$$\frac{1}{\tilde\Phi_m^2}\frac{d}{dt}\tilde\Phi_m^2\le C\tilde\Phi_m^2.$$

The conclusion of the lemma follows immediately from the above inequality and the basic energy estimate (3.5).

Case (ii). Without loss of the generality, we assume $|d^m|$ is large since otherwise we have $\|d^m\|_\infty\le C$, and hence it can be handled similarly to case (i). Thus there exists a constant C_1 such that

$$\int_Q|d^m|^2dx\le|Q|^{1/2}\left(\int_Q|d^m|^4dx\right)^{1/2}\le C_1|Q|^{1/2}\left(\int_Q(|d^m|^2-1)^2dx\right)^{1/2}\le C$$

where we used the that fact $\int_Q F(d^m)dx$ is bounded by the initial data from the basic energy estimate. It then follows from Agmon's inequality in 2D that

$$\|d^m\|_\infty^2\le C\|d^m\|_2\|\Delta d^m\|_2\le C\|\Delta d^m\|_2.$$

Hence, slightly different from (3.19), we have

$$\|\nabla\Delta d^m\|_2\le\|\nabla(\Delta d^m-f(d^m))\|_2+C(1+\|\Delta d^m-f(d^m)\|_2)\|\nabla d^m\|_2.$$

Keep in mind that $\mu_1 = 0$, and thus the highest nonlinear term disappears here, and we have the Gagliardo–Nirenberg inequality in 2D. A computation analagous to the one in the proof of Lemma 3.3 gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Phi_m^2(t) + (\mu_4 - \epsilon)\int_Q |\Delta u^m|^2 dx + \left(-\frac{1}{\lambda_1} - \epsilon\right)\int_\Omega |\nabla(\Delta d^m - f(d^m))|^2 dx\\ \leq &\epsilon \|\sqrt{\rho^m}\nabla u^m\|_2^2 \|\Delta u^m\|_2^2 + C(\Phi_m^4 + \Phi_m^2 + 1).\end{aligned}$$

Set $\tilde{\Phi}_m^2 = \Phi_m^2 + 1$. It follows

$$\frac{d}{dt}\tilde{\Phi}_m^2(t) \leq -\left(\mu_4 - \epsilon\tilde{\Phi}_m^2\right)\int_Q |\Delta u^m|^2 dx + C\tilde{\Phi}_m^4. \tag{3.42}$$

For ϵ small enough such that

$$\epsilon 2\tilde{\Phi}^2(0) e^{C(\rho_0\|u_0\|_2^2 + \|\nabla d_0\|_2^2 + 1)} \leq \frac{\mu_4}{2}, \tag{3.43}$$

we have

$$\tilde{\Phi}_m^2(t) < 2\tilde{\Phi}^2(0) e^{C(\rho_0\|u_0\|_2^2 + \|\nabla d_0\|_2^2 + 1)}, \text{ for all } t > 0.$$

First we prove the claim for $t \in [0,1]$. Assume otherwise; then there must be $t_0 \in (0,1)$ such that

$$\begin{cases}\tilde{\Phi}_m^2(t_0) = 2\tilde{\Phi}^2(0) e^{C(\rho_0\|u_0\|_2^2 + \|\nabla d_0\|_2^2 + 1)}\\ \tilde{\Phi}_m^2(t) \leq 2\tilde{\Phi}^2(0) e^{C(\rho_0\|u_0\|_2^2 + \|\nabla d_0\|_2^2 + 1)}, \text{ for all } t \in (0, t_0].\end{cases} \tag{3.44}$$

Therefore, from (3.42), by the choice of ϵ in (3.43), we have

$$\frac{1}{\tilde{\Phi}_m^2}\frac{d}{dt}\tilde{\Phi}_m^2 \leq C\tilde{\Phi}_m^2$$

for all $t \in (0, t_0)$ which implies that

$$\tilde{\Phi}_m^2(t_0) \leq \tilde{\Phi}^2(0) e^{C\int_0^1 \tilde{\Phi}_m^2(t)dt} \leq \tilde{\Phi}^2(0) e^{C(\rho_0\|u_0\|_2^2 + \|\nabla d_0\|_2^2 + 1)}$$

and thus contradicts (3.44). For $t > 1$, we apply an argument similar to the proof of Lemma 3.3 to show that

$$\tilde{\Phi}_m^2(t) \leq 2\tilde{\Phi}^2(0) e^{C(\rho_0\|u_0\|_2^2 + \|\nabla d_0\|_2^2 + 1)}.$$

This completes the proof of the lemma. ∎

3.3. Passage of the limit to the weak solutions. In Subsection 3.2, the higher order energy estimates in lemmas 3.3, 3.4, and 3.5 are all independent of the approximation index m and time t. This implies that the following uniform estimates hold:

$$\begin{aligned}u^m &\in L^\infty(0,T;V) \cap L^2(0,T;H_p^2),\\ d^m &\in L^\infty(0,T;H_p^2) \cap L^2(0,T;H_p^3).\end{aligned}$$

By the mass conservation equation (1.7), we also have

$$0 < M_1 \leq \rho^m \leq M_2.$$

Moreover, the positive lower bound of density ρ^m combined with the uniform estimates for u^m and d^m indicate that

$$u_t^m \in L^2(0,T;L_p^2),\ d_t^m \in L^2(0,T;H_p^1).$$

Therefore, with these higher order energy estimates, a standard procedure will show that the approximating solutions (ρ^m,u^m,d^m) (a subsequence) converge to a limit (ρ,u,d) such that the limit is a weak solution of system (1.7)–(1.10) and satisfies

$$0<M_1\le\rho\le M_2, \tag{3.45}$$

$$u\in L^\infty(0,T;V)\cap L^2(0,T;H_p^2), \tag{3.46}$$

$$d\in L^\infty(0,T;H_p^2)\cap L^2(0,T;H_p^3). \tag{3.47}$$

Thus we have obtained the existence of weak solutions.

Theorem 3.6 (3D). *Let $Q\subset\mathbb{R}^3$ and suppose that the initial data (ρ_0,u_0,d_0) satisfies conditions (1.12) and (1.13). Suppose that $\Phi^2(0)=\|\sqrt{\rho_0}\nabla u_0\|_{L^2}^2+\|\Delta d_0-f(d_0)\|_{L^2}^2<\infty$.*
(I) There is $\epsilon_0>0$ such that if

$$(\frac{M_2}{\mu_4}+1)\rho_0\|u_0\|_{H^1}^2+\|d_0\|_{H^1}^2+\|\Delta d_0-f(d_0)\|_2^2\le\epsilon_0,$$

then system (1.7)–(1.10) with the periodic boundary condition (1.11) has a weak solution (ρ,u,d) satisfying the basic energy estimate inequality

$$\begin{aligned}\mathcal{E}(t)+\int_{t_0}^t\int_Q \mu_1|(d)^TAd|^2+\mu_4|\nabla u|^2-\frac{1}{\lambda_1}|\Delta d-f(d)|^2dx \\ +\left(\mu_5+\mu_6+\frac{\lambda_2^2}{\lambda_1}\right)\int_{t_0}^t\int_Q|Ad|^2dx\le\mathcal{E}(t_0),\end{aligned} \tag{3.48}$$

for all $t>t_0$ and almost every t_0. Moreover, the weak solution (ρ,u,d) satisfies the higher order energy estimates (3.45)–(3.47) for all $T>0$.
(II) On the other hand, without the smallness assumption on the initial data, there exists a weak solution to the system satisfying (3.48) for all $t>t_0$ and a.e. t_0, but satisfying (3.45)–(3.47) only for a short time $T<\delta$, for some $\delta>0$.

Theorem 3.7 (2D). *Let $Q\subset\mathbb{R}^2$ and suppose that the initial data (ρ_0,u_0,d_0) satisfies conditions (1.12) and (1.13). Suppose that $\Phi^2(0)=\|\sqrt{\rho_0}\nabla u_0\|_{L^2}^2+\|\Delta d_0-f(d_0)\|_{L^2}^2<\infty$. Assume in addition that either*

$$(i)\,\mu_1\ge 0,\ \lambda_2=0,\ \ or\ \ (ii)\,\mu_1=0,\ \lambda_2\ne 0.$$

The system (1.7)–(1.10) with the periodic boundary condition (1.11) has a weak solution (ρ,u,d) satisfying the basic energy estimate inequality (3.48) with $t\in(0,T]$ and the higher order energy estimates (3.45)–(3.47) for all the time $T>0$.

Applying a similar method to obtain the higher order energy estimates as in Subsection 3.2, the weak solution (ρ,u,d) satisfies even higher order energy estimates provided more regular initial data.

COROLLARY 3.8. *Let $Q\subset\mathbb{R}^n$ with $n=2,3$. Assume the initial data (ρ_0,u_0,d_0) satisfies the conditions in Theorem 3.6 and Theorem 3.7, respectively, for $n=3,2$. Assume, in addition, that $u_0\in H_p^2$ and $d_0\in H_p^3$. Then there exists a weak solution (ρ,u,d) to system (1.7)–(1.10) satisfying the basic energy estimate (3.48) and (3.45). Meanwhile,*

$$u\in L^\infty(0,T;H_p^2)\cap L^2(0,T;H_p^3), \tag{3.49}$$

$$d\in L^\infty(0,T;H_p^3)\cap L^2(0,T;H_p^4). \tag{3.50}$$

Indeed, compared to the simplified Ericksen–Leslie model studied in [12], the full model contains higher order derivative terms, for example, the term $\nabla\cdot(Nd)$ in Equation (1.8). When applying the L^p theory of parabolic equations on (1.8) to improve the regularity of velocity, the higher order energy estimates (3.49) and (3.50) are needed. Please see a detailed discussion in Subsection 3.5.

3.4. Auxiliary estimates on density. As a consequence of the higher order energy estimates obtained in Subsection 3.3 for the weak solution (ρ,u,d), the regularity of the density can be improved in both the two and three dimensional cases.

LEMMA 3.9 ([2]). *Assume that $\rho(0)\in C^1(Q)$ and that $Q\subset\mathbb{R}^2$ is smooth and bounded. Suppose $u\in L^\infty(0,T;H^1)\bigcap L^2(0,T;H^2)$ and*

$$\rho_t+u\cdot\nabla\rho=0$$

in $Q\times(0,T)$. Then $\rho\in C^\alpha(\Omega\times[0,T])$ for some $\alpha\in(0,1)$ which depends only on the initial data, T and Q.

LEMMA 3.10 ([12]). *Assume that $\rho(0)\in C^1(Q)$ and that $Q\subset\mathbb{R}^3$ is smooth and bounded. Suppose $u\in L^\infty(0,T;H^1)\bigcap L^2(0,T;H^2)$ and*

$$\rho_t+u\cdot\nabla\rho=0$$

in $Q\times(0,T)$. Let $t_1\in(0,T)$ and $p\in Q$. Define

$$\mathcal{A}_{(p,t_1)}=(B_{r_0}(p)\cap Q)\times([t_1-r_0,t_1+r_0]\cap[0,T]).$$

Then, for any $\epsilon>0$, there exists $r_0>0$ such that for $p\in Q$ and all $T>0$,

$$\sup_{(q,t_2)\in\mathcal{A}_{(p,t_1)}}|\rho(q,t_2)-\rho(p,t_1)|\le\epsilon.$$

We refer the readers to [12] for a detailed proof of this lemma.

REMARK 3.11. In 2D, the Hölder continuity for the fluid density guarantees that a frozen coefficient method (cf. [31]) can be applied to the Navier–Stokes equation (1.8), and hence the regularity of the velocity u will be improved through a standard L^p theory for the parabolic equation. In 3D, the density has the property of small oscillations over small balls in $Q\times[0,T]$ provided that either the initial data is small or for a short time. This turns out to be enough to carry out the frozen coefficient method to improve the regularity of the fluid velocity too. We refer the reader to [31] for a general idea of the frozen coefficient method. Also a more specific discussion relevant to the current model can be found in [12] (Appendix 6).

3.5. Classical solution. In this subsection, using the the estimates (3.45)–(3.47), we apply the frozen coefficient method to improve regularities for ρ, p, u, d by a bootstrapping argument among the three equations (1.7)–(1.9). Hence Theorem 1.1 and 1.2 will be proved. The process of obtaining regular solutions in the two dimensional case is much easier than the one in the three dimensional case. And based on the previous work for the simplified Ericksen–Leslie model in [12], we just briefly show the steps needed to obtain regular solutions in 3D in the following.

We notice from (3.46) and (3.47)

$$u \in L^\infty(0,T;L^6_p), \quad \nabla d \in L^\infty(0,T;L^6_p),$$

which implies

$$u \cdot \nabla d \in L^\infty(0,T;L^3_p).$$

In the mean time, (3.46) and (3.47) indicate that

$$\Omega d, Ad \in L^\infty(0,T;L^q_p), \text{ for all } q > 1.$$

By the standard parabolic estimates on Equation (1.9) (cf. [29] and [1]), we have

$$d \in W^{1,r}(W^{2,3}_p), \text{ for all } r > 1$$

which implies that $\nabla d \in L^\infty(0,T;L^q_p)$ for any $q \in (1,\infty)$. Thus, we have

$$u \cdot \nabla d \in L^\infty(0,T;L^q_p), \quad \forall q \in (1,6).$$

Applying the same standard parabolic estimates on (1.9) again yields

$$d \in W^{1,r}(W^{2,q}_p), \forall r \in (1,\infty) \text{ and } q \in (1,6), \tag{3.51}$$

which implies that $d \in C^{\alpha/2,1+\alpha}([0,T] \times \bar{Q})$ for some $\alpha \in (0,1)$ and

$$\nabla d \Delta d \in L^\infty(0,T;L^q_p), \quad \forall q \in (1,6).$$

In the Navier–Stokes equation (1.8), the estimates for the conservation of momentum with constant density can be extended to the non-constant density case when Lemma 3.10 is available. This is done via the frozen coefficient method.

The estimates (3.46) and (3.47) imply

$$u \cdot \nabla u \in L^\infty(0,T;L^{\frac{3}{2}}_p),$$

while (3.49) and (3.50) implies

$$\nabla \cdot \sigma \in L^\infty(0,T;L^{\frac{3}{2}}_p).$$

Now we apply the frozen coefficient method using the oscillation estimates for the density as in Lemma 3.10 to yield

$$u \in W^{1,q}(W^{2,3/2}_p), \quad \forall q \in (1,\infty).$$

Thus $u \in L^\infty(0,T;W^{1,3})$ and $u \cdot \nabla u \in L^\infty(0,T;L^2_p)$. Repeating the above argument yields

$$u \in W^{1,q}(W^{2,2}_p), \quad \forall q \in (1,\infty),$$

from which it follows that $u\in C^{\alpha}([0,T]\times Q)$.

Back to Equation (1.9), we conclude that

$$d\in C^{1+\frac{\alpha}{2},2+\alpha}((0,T)\times Q),\ \text{ for some } \alpha\in(0,1).$$

Finally $\rho\in C^1((0,T)\times Q)$ follows from the regularity of u, and the regularity of the pressure p follows easily from the regularity of (ρ,u,d) similarly to [2]. This completes the proof of Theorem 1.2.

Appendix A. Proof of uniqueness. In this appendix we sketch the proof of Theorem 1.5. For the full Ericksen–Leslie system with constant density, Wu, Xu, and Liu [50] proved that the regular solution (u,d) is unique in the sense that two regular solutions starting from the same initial data are identical. In the present paper, for the full Ericksen–Leslie system with non-constant density, we show a weak-strong type of uniqueness result with certain conditions on the weak solution. The idea is to calculate the energy law satisfied by the difference of the regular and weak solutions and establish a Gronwall inequality. In our case, to calculate the energy law of the difference of the regular and the weak solutions we need some extra terms involving the density. We shall perform a calculation analogous to one we did in [12] to achieve our goal. Similarly, the estimates are more involved requiring additional bounds on the strong solution (ρ,u,d) to yield a Gronwall inequality. In 2D, we need

$$\nabla\rho,\nabla u\in L^{\infty}((0,T)\times\Omega),\quad u_t,u\cdot\nabla u\in L^{\infty}(0,T;L^q(\Omega)),\, q>2. \tag{A.1}$$

In 3D, we need

$$\nabla\rho,\nabla u\in L^{\infty}((0,T)\times\Omega),\quad u_t,u\cdot\nabla u\in L^{\infty}(0,T;L^3(\Omega)). \tag{A.2}$$

With the assumption on data, $\rho_0\in C^1(\bar{\Omega})$, $u_0\in C^{2+\alpha}(\bar{\Omega})$, and $d_0\in C^{2+\alpha}(\bar{\Omega})$, the solution (ρ,u,d) from Theorem 1.1 or Theorem 1.2 satisfies (A.1) or (A.2), respectively .

Proof of Theorem 1.5. First recall that the regular solution (ρ,u,d) from Theorem 1.1 or Theorem 1.2 satisfies the energy equality:

$$\begin{aligned}&\int_{\Omega}\frac{1}{2}\rho|u|^2+\frac{1}{2}|\nabla d|^2+\rho F(d)dx\\&+\int_0^T\int_Q\mu_4|\nabla u|^2-\frac{1}{\lambda_1}|\Delta d-f(d)|^2+\mu_1|d^TAd|^2+(\mu_5+\mu_6+\frac{\lambda_2^2}{\lambda_1})|Ad|^2dxdt\\=&\int_Q\frac{1}{2}\rho_0|u_0|^2+\frac{1}{2}|\nabla d_0|^2+\rho_0F(d_0)dx.\end{aligned} \tag{A.3}$$

The density ρ is the strong solution of the transport equation, hence it satisfies

$$\int_Q\rho^2dx=\int_Q\rho_0^2dx.$$

On the other hand, $\bar{\rho}$ is a weak solution of the transport equation and $M_1\le\bar{\rho}\le M_2$. We have by hypothesis that

$$\int_{\Omega}\bar{\rho}^2dx\le\int_{\Omega}\rho_0^2dx. \tag{A.4}$$

Thus,

$$\frac{1}{2}\int_\Omega |\rho-\bar\rho|^2 dx = \frac{1}{2}\int_\Omega \rho^2 dx + \frac{1}{2}\int_\Omega \bar\rho^2 dx - \int_\Omega \rho\bar\rho dx$$
$$\le \int_\Omega \rho_0^2 dx - \int_\Omega \rho\bar\rho dx. \tag{A.5}$$

Since $\rho \in C^1([0,T]\times\bar\Omega)$, we can take ρ as a test function. Thus, multiplying

$$\bar\rho_t + \bar u\cdot\nabla\bar\rho = 0$$

by ρ and integrating by parts yields

$$\int_\Omega \rho_0^2 dx - \int_\Omega \rho\bar\rho dx = -\int_0^t\int_\Omega \bar\rho\rho_t dsdx - \int_0^t\int_\Omega (\bar u\cdot\nabla\rho)\bar\rho dsdx$$
$$= \int_0^t\int_\Omega (u\cdot\nabla\rho)\bar\rho dsdx - \int_0^t\int_\Omega (\bar u\cdot\nabla\rho)\bar\rho dsdx. \tag{A.6}$$

Here we used again that ρ is a classical solution of the transport equation. Substituting (A.6) into (A.5) gives

$$\frac{1}{2}\int_\Omega |\rho-\bar\rho|^2 dx \le \int_0^t\int_\Omega \bar\rho(u-\bar u)\nabla\rho dsdx$$
$$= \int_0^t\int_\Omega (\rho-\bar\rho)(u-\bar u)\nabla\rho dsdx. \tag{A.7}$$

Next, calculate the following term:

$$\frac{1}{2}\int_\Omega \bar\rho|u-\bar u|^2 dx + \frac{1}{2}\int_\Omega |\nabla d-\nabla\bar d|^2 dx$$
$$= \frac{1}{2}\int_\Omega (\bar\rho-\rho)|u|^2 dx + \frac{1}{2}\int_\Omega \rho|u|^2 dx + \frac{1}{2}\int_\Omega \bar\rho|\bar u|^2 dx - \int_\Omega \bar\rho u\cdot\bar u dx$$
$$+ \frac{1}{2}\int_\Omega |\nabla d|^2 dx + \frac{1}{2}\int_\Omega |\nabla\bar d|^2 dx - \int_\Omega \nabla d\otimes\nabla\bar d dx, \tag{A.8}$$

where $\nabla d\otimes\nabla\bar d$ denotes the 3×3 matrix whose ij-th entry is given by $\nabla_i d\cdot\nabla_j\bar d$ for $1\le i,j\le 3$.

Using energy equality (A.3) for the regular solution (ρ,u,d) and inequality (1.18) for the weak solution $(\bar\rho,\bar u,\bar d)$ combined with the last equation yields

$$\frac{1}{2}\int_Q \bar\rho|u-\bar u|^2 dx + \frac{1}{2}\int_Q |\nabla d-\nabla\bar d|^2 dx$$
$$\le -\int_0^t\int_Q \mu_4|\nabla u-\nabla\bar u|^2 - \frac{1}{\lambda_1}|\Delta d-\Delta\bar d|^2 + \mu_1|d^TAd-\bar d^T\bar A\bar d|^2 dxdt$$
$$-\int_0^T\int_Q\left(\mu_5+\mu_6+\frac{\lambda_2^2}{\lambda_1}\right)|Ad-\bar A\bar d|^2 dxdt + \frac{1}{2}\int_Q (\bar\rho-\rho)|u|^2 dx$$
$$-\int_Q \bar\rho u\otimes\bar u - \rho_0|u_0|^2 dx - \int_Q \nabla d\otimes\nabla\bar d - |\nabla d_0|^2 dx$$
$$-\int_Q \rho F(d) + \bar\rho F(\bar d) - 2\rho_0 F(d_0) dx - 2\mu_1\int_0^t\int_Q d^TAd\bar d^T\bar A\bar d dxdt$$

$$-2\mu_4\int_0^t\int_Q \nabla u\nabla\bar{u}dxdt+\frac{2}{\lambda_1}\int_0^t\int_Q \Delta d\Delta\bar{d}dxdt-2\left(\mu_5+\mu_6+\frac{\lambda_2^2}{\lambda_1}\right)\int_0^t\int_Q Ad\bar{A}\bar{d}dxdt$$
$$+\frac{1}{\lambda_1}\int_0^T\int_Q |f(d)|^2+|f(\bar{d})|^2dxdt-\frac{2}{\lambda_1}\int_0^t\int_Q \Delta df(d)+\Delta\bar{d}f(\bar{d})dxdt. \tag{A.9}$$

Since $u,d\in C^{1+\alpha/2,2+\alpha}([0,T]\times\bar{Q})$, we can take u,d as test functions for the weak solution $\bar{u},\bar{d}$. Thus it follows that

$$\int_Q \bar{\rho}u\otimes\bar{u}-\rho_0|u_0|^2dx=\int_0^t\int_Q \bar{\rho}\bar{u}u_tdxdt+\int_0^t\int_Q \bar{\rho}\bar{u}(\bar{u}\cdot\nabla u)dxdt$$
$$+\int_0^t\int_Q \nabla\bar{d}\nabla\bar{d}\nabla udxdt+\int_0^t\int_Q \nabla\cdot\bar{\sigma}udxdt, \tag{A.10}$$

$$\int_Q \nabla d\otimes\nabla\bar{d}-|\nabla d_0|^2dx$$
$$=-\int_0^T\int_Q \Delta\bar{d}d_t\,dxdt+\int_0^T\int_Q \Delta d(\bar{u}\cdot\nabla\bar{d})dxdt-\int_0^T\int_Q \Delta d\bar{\Omega}\bar{d}dxdt$$
$$+\frac{\lambda_2}{\lambda_1}\int_0^T\int_Q \Delta d\bar{A}\bar{d}dxdt+\frac{1}{\lambda_1}\int_0^T\int_Q \Delta d(\Delta\bar{d}-f(\bar{d}))\,dxdt. \tag{A.11}$$

Substituting (A.10) and (A.11) into (A.9) and adding (A.7) yields

$$\frac{1}{2}\int_Q|\rho-\bar{\rho}|^2dx+\frac{1}{2}\int_Q\bar{\rho}|u-\bar{u}|^2dx+\frac{1}{2}\int_Q|\nabla d-\nabla\bar{d}|^2dx$$
$$\leq-\int_0^t\int_Q \mu_4|\nabla u-\nabla\bar{u}|^2-\frac{1}{\lambda_1}|\Delta d-\Delta\bar{d}|^2+\mu_1|d^TAd-\bar{d}^T\bar{A}\bar{d}|^2dxdt$$
$$-\int_0^T\int_Q\left(\mu_5+\mu_6+\frac{\lambda_2^2}{\lambda_1}\right)|Ad-\bar{A}\bar{d}|^2dxdt+\frac{1}{2}\int_Q(\bar{\rho}-\rho)|u|^2dx$$
$$-\int_Q\rho F(d)+\bar{\rho}F(\bar{d})-2\rho_0F(d_0)dx+\int_0^t\int_Q(\bar{\rho}-\rho)(u-\bar{u})\nabla\rho dxdt$$
$$-\int_0^t\int_Q\bar{\rho}\bar{u}u_t\,dxdt-\int_0^t\int_Q\bar{\rho}\bar{u}\bar{u}\nabla u\,dxdt-\int_0^t\int_Q\nabla\bar{d}\nabla\bar{d}\nabla u\,dxdt$$
$$-\int_0^t\int_Q u\nabla\cdot\bar{\sigma}'\,dxdt+\mu_4\int_0^t\int_Q\nabla\bar{u}\nabla u\,dxdt+\int_0^t\int_Q\Delta\bar{d}d_t\,dxdt$$
$$-\int_0^t\int_Q\bar{u}\nabla\bar{d}\Delta d\,dxdt+\int_0^t\int_Q\bar{\Omega}\bar{d}\Delta d\,dxdt-\frac{\lambda_2}{\lambda_1}\int_0^t\int_Q\bar{A}\bar{d}\Delta d\,dxdt$$
$$-2\mu_1\int_0^t\int_Q d^TAd\bar{d}^T\bar{A}\bar{d}dxdt-2\mu_4\int_0^t\int_Q\nabla u\nabla\bar{u}dxdt$$
$$+\frac{2}{\lambda_1}\int_0^t\int_Q\Delta d\Delta\bar{d}dxdt-2\left(\mu_5+\mu_6+\frac{\lambda_2^2}{\lambda_1}\right)\int_0^t\int_Q Ad\bar{A}\bar{d}dxdt$$
$$+\frac{1}{\lambda_1}\int_0^T\int_Q|f(d)|^2+|f(\bar{d})|^2dxdt-\frac{2}{\lambda_1}\int_0^t\int_Q\Delta df(d)+\Delta\bar{d}f(\bar{d})dxdt. \tag{A.12}$$

Note that

$$
\begin{aligned}
-\int_0^t\int_Q \bar\rho\bar u u_t\,dxdt = &-\int_0^t\int_Q (\bar\rho-\rho)(\bar u-u)u_t\,dxdt\\
&-\int_0^t\int_Q (\bar\rho-\rho)uu_t\,dxdt-\int_0^t\int_Q \rho\bar u u_t\,dxdt,
\end{aligned}
$$

while

$$
\begin{aligned}
-\int_0^t\int_Q(\bar\rho-\rho)uu_t\,dxdt &= -\frac{d}{dt}\int_0^t\int_Q(\bar\rho-\rho)\frac{|u|^2}{2}\,dxdt+\int_0^t\int_Q(\bar\rho-\rho)_t\frac{|u|^2}{2}\,dxdt\\
&= -\int_0^t\int_Q(\bar\rho-\rho)\frac{|u|^2}{2}\,dxdt-\int_0^t\int_Q\operatorname{div}(\bar\rho\bar u-\rho u)\frac{|u|^2}{2}\,dxdt\\
&= -\int_0^t\int_Q(\bar\rho-\rho)\frac{|u|^2}{2}\,dxdt+\int_0^t\int_Q\bar\rho\bar u u\nabla u\,dxdt\\
&\quad -\int_0^t\int_Q(\rho-\bar\rho)uu\nabla u\,dxdt-\int_0^t\int_Q\bar\rho uu\nabla u\,dxdt
\end{aligned}
$$

and

$$
\begin{aligned}
-\int_0^t\int_Q\rho\bar u u_t\,dxdt = &\,\mu_4\int_0^t\int_Q\nabla u\nabla\bar u\,dxdt+\int_0^t\int_Q\rho u\bar u\nabla u\,dxdt\\
&+\int_0^t\int_Q\nabla d\Delta d\bar u\,dxdt-\int_0^t\int_Q\bar u\nabla\cdot\sigma'\,dxdt.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\int_0^t\int_Q\Delta\bar d d_t\,dxdt = &-\int_0^t\int_Q\Delta\bar d u\nabla d\,dxdt+\int_0^t\int_Q\Delta\bar d\Omega d\,dxdt-\frac{\lambda_2}{\lambda_1}\int_0^t\int_Q\Delta\bar d A d\,dxdt\\
&-\frac{1}{\lambda_1}\int_0^t\int_Q\Delta\bar d\Delta d\,dxdt+\frac{1}{\lambda_1}\int_0^t\int_Q\Delta\bar d f(d)\,dxdt.
\end{aligned}
$$

Thus, with some cancelation, (A.12) becomes

$$
\begin{aligned}
&\frac12\int_Q|\rho-\bar\rho|^2dx+\frac12\int_Q\bar\rho|u-\bar u|^2dx+\frac12\int_Q|\nabla d-\nabla\bar d|^2dx\\
&\le -\int_0^t\int_Q\mu_4|\nabla u-\nabla\bar u|^2-\frac{1}{\lambda_1}|\Delta d-\Delta\bar d|^2+\mu_1|d^TAd-\bar d^T\bar A\bar d|^2dxdt\\
&\quad-\int_0^T\int_Q\left(\mu_5+\mu_6+\frac{\lambda_2^2}{\lambda_1}\right)|Ad-\bar A\bar d|^2dxdt\\
&\quad+\int_0^t\int_Q(\bar\rho-\rho)(u-\bar u)\nabla\rho\,dxdt-\int_0^t\int_Q(\bar\rho-\rho)(\bar u-u)u_t\,dxdt\\
&\quad-\int_0^t\int_Q\bar\rho\bar u\bar u\nabla u-\bar\rho\bar u u\nabla u+\bar\rho uu\nabla u-\bar\rho\bar u u\nabla u\,dxdt\\
&\quad-\int_0^t\int_Q(\rho-\bar\rho)uu\nabla u-(\rho-\bar\rho)\bar u u\nabla u\,dxdt
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t\int_Q \nabla\bar{d}\Delta\bar{d}u+\nabla d\Delta d\bar{u}-\nabla\bar{d}\Delta d\bar{u}-\nabla d\Delta\bar{d}u\,dxdt-\int_0^t\int_Q u\nabla\cdot\bar{\sigma}'+\bar{u}\nabla\cdot\sigma'\,dxdt\\
&-\int_0^t\int_Q 2\mu_1 d^TAd\bar{d}^T\bar{A}\bar{d}+2\left(\mu_5+\mu_6+\frac{\lambda_2^2}{\lambda_1}\right)Ad\bar{A}\bar{d}\,dxdt\\
&+\int_0^t\int_Q \Omega d\Delta\bar{d}+\bar{\Omega}\bar{d}\Delta d-\frac{\lambda_2}{\lambda_1}(\bar{A}\bar{d}\Delta d+Ad\Delta\bar{d})\,dxdt\\
&+\frac{1}{\lambda_1}\int_0^t\int_Q \Delta\bar{d}f(d)+\Delta df(\bar{d})-2\Delta df(d)-2\Delta\bar{d}f(\bar{d})dxdt\\
&-\int_Q \rho F(d)+\bar{\rho}F(\bar{d})-2\rho_0F(d_0)dx+\frac{1}{\lambda_1}\int_0^T\int_Q |f(d)|^2+|f(\bar{d})|^2dxdt\\
:=&-G_1-G_2+I_1+I_2+\ldots+I_{11}.
\end{aligned}
\tag{A.13}
$$

Note that

$$
\begin{aligned}
I_3&=-\int_0^t\int_Q \bar{\rho}\nabla u|u-\bar{u}|^2dxdt,\\
I_4&=\int_0^t\int_Q (\rho-\bar{\rho})u\nabla u(\bar{u}-u)dxdt,\\
I_5&=\int_0^t\int_Q u(\nabla d-\nabla\bar{d})(\Delta d-\Delta\bar{d})dxdt-\int_0^t\int_Q(\bar{u}-u)(\nabla\bar{d}-\nabla d)\Delta d\,dxdt.
\end{aligned}
$$

Using the relation $\lambda_2=\mu_2+\mu_3$, I_6+I_7 can be reorganized as

$$
\begin{aligned}
I_6+I_7=&\mu_1\int_0^t\int_Q d^TAd(d\otimes d-\bar{d}\otimes\bar{d})(\nabla\bar{u}-\nabla u)dxdt\\
&+\mu_1\int_0^t\int_Q (d^TAd-\bar{d}^T\bar{A}\bar{d})(d\otimes d-\bar{d}\otimes\bar{d})\nabla u dxdt\\
&+\left(\mu_5+\mu_6+\frac{\lambda_2^2}{\lambda_1}\right)\int_0^t\int_Q d^TA(d-\bar{d})(\nabla\bar{u}-\nabla u)dxdt\\
&+\left(\mu_5+\mu_6+\frac{\lambda_2^2}{\lambda_1}\right)\int_0^t\int_Q (d^TA-\bar{d}^T\bar{A})(d-\bar{d})\nabla u dxdt.
\end{aligned}
$$

Applying (1.9) and the fact that $\int_Q u\cdot\nabla df(d)dx=0$, we derive

$$
\begin{aligned}
I_8+I_9+I_{10}+I_{11}=&-\int_0^t\int_Q\left(\Omega-\frac{\lambda_2}{\lambda_1}A\right)(d-\bar{d})(\Delta d-\Delta\bar{d})dxdt\\
&+\int_0^t\int_Q\left(\left(\Omega-\frac{\lambda_2}{\lambda_1}A\right)-\left(\bar{\Omega}-\frac{\lambda_2}{\lambda_1}\bar{A}\right)\right)(d-\bar{d})\Delta d\,dxdt\\
&-\frac{1}{\lambda_1}\int_0^t\int_Q(\Delta d-\Delta\bar{d})(f(d)-f(\bar{d}))dxdt.
\end{aligned}
\tag{A.14}
$$

Recall that the regular solution (ρ,u,d) satisfies (A.1) in 2D and (A.2) in 3D. By the Hölder and Gagliardo–Nirenberg inequalities on the terms of $I_1,\ldots,I_{11}$, it follows that

$$\begin{aligned}&\frac{1}{2}\int_Q |\rho(t)-\bar{\rho}(t)|^2+\bar{\rho}(t)|u(t)-\bar{u}(t)|^2+|\nabla d(t)-\nabla\bar{d}(t)|^2dx\\&\leq C\int_0^t\int_Q |\rho-\bar{\rho}|^2+\bar{\rho}|u-\bar{u}|^2+|\nabla d-\nabla\bar{d}|^2dxdt.\end{aligned}\tag{A.15}$$

To handle the last integral in (A.14), we used the fact that $|d|\leq 1$ and the hypothesis that $|\bar{d}|$ is bounded, which imply $|f(d)-f(\bar{d})|\leq C|d-\bar{d}|$ by the definition of $f(d)$. Thus,

$$\int_0^t\int_Q |f(d)-f(\bar{d})|^2dxdt\leq C\int_0^t\int_Q |d-\bar{d}|^2dxdt\leq C(Q)\int_0^t\int_Q |\nabla d-\nabla\bar{d}|^2dxdt$$

where the constant C depends on the space domain Q and not on the time T.

Thanks to the fact $\bar{\rho}\geq M_1>0$, applying Gronwall's inequality to (A.15), we obtain

$$\begin{aligned}&\frac{1}{2}\int_\Omega |\rho(t)-\bar{\rho}(t)|^2+\bar{\rho}|u(t)-\bar{u}(t)|^2+|\nabla d(t)-\nabla\bar{d}(t)|^2dx\\&\leq\int_\Omega |\rho(0)-\bar{\rho}(0)|^2+\bar{\rho}(0)|u(0)-\bar{u}(0)|^2+|\nabla d(0)-\nabla\bar{d}(0)|^2dxe^{Ct}\\&=0\end{aligned}$$

for all $t>0$ which implies that

$$\bar{\rho}-\rho=\bar{u}-u=\bar{d}-d\equiv 0.$$

This completes the proof of Theorem 1.5. ∎

Acknowledgement. The author would like to thank Hao Wu for bringing this project to her attention, and thank the anonymous referees for their careful review and valuable suggestions which helped to improve the paper a lot.

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