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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO THE COMPRESSIBLE BIPOLAR EULER–MAXWELL SYSTEM IN $\mathbb{R}^{3}$*

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Abstract. We study the large time behavior of solutions near a constant equilibrium state to the compressible bipolar Euler–Maxwell system in $\mathbb{R}^3$. We first refine the global existence of solutions by assuming that the initial data is small in the H^3 norm, but its higher order derivatives could be large. If, further, the initial data belongs to $\dot{H}^{-s}$ $(0\leqslant s<3/2)$ or $\dot{B}^{-s}_{2,\infty}$ $(0<s\leqslant 3/2)$, then we obtain the various time decay rates of the solution and its higher order derivatives. As an immediate byproduct, the L^p–L^2 $(1\leqslant p\leqslant 2)$ type of the decay rates follows without requiring the smallness for L^p norm of initial data. So far, our decay results are most comprehensive ones for the bipolar Euler–Maxwell system in $\mathbb{R}^3$.

Key words. Compressible bipolar Euler–Maxwell system, time decay rates, ensergy method, interpolation.

AMS subject classifications. 82D10, 35A01, 35B40, 35Q35, 35Q61.

1. Introduction

We consider the compressible isentropic bipolar Euler–Maxwell system in three space dimensions [1, 18, 23]:

$$\begin{cases}\partial_t\tilde{n}_\pm+\operatorname{div}(\tilde{n}_\pm\tilde{u}_\pm)=0,\\ \partial_t(\tilde{n}_\pm\tilde{u}_\pm)+\operatorname{div}(\tilde{n}_\pm\tilde{u}_\pm\otimes\tilde{u}_\pm)+\nabla p_\pm(\tilde{n}_\pm)=\pm\tilde{n}_\pm(\tilde{E}+\varepsilon\tilde{u}_\pm\times\tilde{B})-\frac{1}{\tau_\pm}\tilde{n}_\pm\tilde{u}_\pm,\\ \varepsilon\lambda^2\partial_t\tilde{E}-\nabla\times\tilde{B}=\varepsilon(\tilde{n}_-\tilde{u}_--\tilde{n}_+\tilde{u}_+),\\ \varepsilon\partial_t\tilde{B}+\nabla\times\tilde{E}=0,\\ \lambda^2\operatorname{div}\tilde{E}=\tilde{n}_+-\tilde{n}_-,\quad \operatorname{div}\tilde{B}=0,\\ (\tilde{n}_\pm,\tilde{u}_\pm,\tilde{E},\tilde{B})|_{t=0}=(\tilde{n}_{\pm0},\tilde{u}_{\pm0},\tilde{E}_0,\tilde{B}_0).\end{cases}\tag{1.1}$$

Here the unknown functions are the charged density $\tilde{n}_\pm$, the velocity $\tilde{u}_\pm$, the electric field $\tilde{E}$, and the magnetic field $\tilde{B}$ with the subscripts $+$ and $-$ representing ions and electrons respectively. We assume the pressure $p_\pm(\tilde{n}_\pm)=A_\pm\tilde{n}_\pm^\gamma$ with constants $A_\pm>0$ and $\gamma\geqslant 1$ the adiabatic exponent. The $\tau_\pm>0$ are the velocity relaxation time of ions and electrons respectively, $\lambda>0$ is the Debye length, and $\varepsilon=1/c$ with c the speed of light.

Although the compressible Euler–Maxwell system plays an important role in plasma physics and semiconductor physics, there are few mathematical results about it due to its mathematical complexity. For the unipolar case, Chen, Jerome, and Wang [2] showed the global existence of entropy weak solutions to the initial-boundary value problem for arbitrarily large initial data in $L^\infty(\mathbb{R})$, Guo and Tahvildar-Zadeh [11] showed a blow-up criterion for a spherically symmetric Euler–Maxwell system. Recently, there have been some results on the global existence and the asymptotic behavior of smooth solutions with small amplitudes, see [3, 26, 30, 31]. For the asymptotic limits that derive simplified models starting from the Euler–Maxwell system, we refer to [13, 22, 35] for the relaxation limit, and to [35] for the non-relativistic limit, [20, 21] for the quasi-neutral limit, [28, 29] for WKB asymptotics and the references therein. For the bipolar

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case, Duan et al. [4] showed the global existence and time-decay rates of solutions near a constant steady state with the vanishing electromagnetic field. Peng [19] considered the Cauchy problem and the periodic problem and proved their global existence and long-time behavior of smooth solutions near a given constant equilibrium state. Xu et al. [36] studied the well-posedness in critical Besov spaces. Since the unipolar or bipolar Euler–Maxwell system is a symmetrizable hyperbolic system, the Cauchy problem in $\mathbb{R}^3$ has a local unique smooth solution when the initial data is smooth, see Kato [17] and Jerome [15] for instance. Besides, we can refer to [27, 32, 33] for the non-isentropic case.

In this paper, our main purpose is to derive some various time decay rates of the solution as well as its spatial derivatives of any order. Meantime, we also establish a refined global existence of smooth solutions near the constant equilibrium state $(n_\infty, n_\infty, 0, 0, 0, B_\infty)$ to the compressible bipolar Euler–Maxwell system, compared with [4]. We should notice that the relaxation term of the velocity plays an important role in the whole paper. The non-relaxation case is much more difficult; we refer to [5, 7, 9, 10, 14] for such a case. For the compressible unipolar Euler–Maxwell system [26], we do not need that the initial electron density belongs to a negative Sobolev space $\dot{H}^{-s}$ or a negative Besov spaces $\dot{B}^{-s}_{2,\infty}$ when deriving the optimal decay rates of solutions. However, in Theorem 1.2 the initial total densities n_{10} must belong to $\dot{H}^{-s}$ or $\dot{B}^{-s}_{2,\infty}$ due to the cancellation between two carriers. In fact, in Theorem 1.2 the assumption for the initial difference of densities n_{20} could be deleted given [26]. Compared with the unipolar case [26], there are two new major difficulties in addition to the computational complexity. First of all, the bipolar system (1.1) could be reformulated equivalently as the damped Euler equation coupled with the one-fluid Euler–Maxwell equation (1.5). Then, the total density n_1 in the damped Euler equation is degenerately dissipative because of the cancellation between two carriers. It is difficult to close the energy estimates due to the degenerate dissipation of n_1. We manage to obtain the effective energy estimates by dealing carefully with those terms involved with n_1 in the proofs of lemmas 2.8–2.9. The other difficulty is caused by the nonlinear function $f(\frac{n_1 \pm n_2}{2})$. Since n_1 and n_2 have different dissipative structures, we must be careful with the function $f(\frac{n_1 \pm n_2}{2})$. Here we overcome such an obstacle by some detailed calculi. Without loss of generality, we take all the physical constants $\tau_\pm, \varepsilon, \lambda, A_\pm, n_\infty$ in (1.1) to be 1.

We define

$$\begin{cases} n_\pm(x,t) = \frac{2}{\gamma-1}\left\{\left[\tilde{n}_\pm\left(x, \frac{t}{\sqrt{\gamma}}\right)\right]^{\frac{\gamma-1}{2}} - 1\right\}, \quad u_\pm(x,t) = \frac{1}{\sqrt{\gamma}}\tilde{u}_\pm\left(x, \frac{t}{\sqrt{\gamma}}\right), \\ E(x,t) = \frac{1}{\sqrt{\gamma}}\tilde{E}\left(x, \frac{t}{\sqrt{\gamma}}\right), \qquad B(x,t) = \frac{1}{\sqrt{\gamma}}\tilde{B}\left(x, \frac{t}{\sqrt{\gamma}}\right) - B_\infty. \end{cases} \tag{1.2}$$

Then the Euler–Maxwell system (1.1) is reformulated equivalently as

$$\begin{cases} \partial_t n_\pm + \operatorname{div} u_\pm = -u_\pm \cdot \nabla n_\pm - \mu n_\pm \operatorname{div} u_\pm, \\ \partial_t u_\pm + \nu u_\pm \mp u_\pm \times B_\infty + \nabla n_\pm \mp \nu E = -u_\pm \cdot \nabla u_\pm - \mu n_\pm \nabla n_\pm \pm u_\pm \times B, \\ \partial_t E - \nu \nabla \times B + \nu(u_+ - u_-) = \nu(f(n_-)u_- - f(n_+)u_+), \\ \partial_t B + \nu \nabla \times E = 0, \\ \operatorname{div} E = \nu(f(n_+) - f(n_-)), \quad \operatorname{div} B = 0, \\ (n_\pm, u_\pm, E, B)|_{t=0} = (n_{\pm 0}, u_{\pm 0}, E_0, B_0). \end{cases}$$

Here $\mu := \frac{\gamma-1}{2}$, $\nu := \frac{1}{\sqrt{\gamma}}$, and the nonlinear function $f(n_\pm)$ is defined by

$$f(n_\pm) := \left(1 + \frac{\gamma-1}{2} n_\pm\right)^{\frac{2}{\gamma-1}} - 1. \tag{1.3}$$

In fact, we have assumed $\gamma>1$ in (1.2). If $\gamma=1$, then we instead define $n_\pm:=\ln\tilde n_\pm$.

Let

$$n_1=n_++n_-,\ n_2=n_+-n_-,\ u_1=u_++u_-,\ u_2=u_+-u_-,$$

that is

$$n_+=\frac{n_1+n_2}{2},\ n_-=\frac{n_1-n_2}{2},\ u_+=\frac{u_1+u_2}{2},\ u_-=\frac{u_1-u_2}{2}. \tag{1.4}$$

Then $U:=(n_1,n_2,u_1,u_2,E,B)$ satisfies

$$\begin{cases}\partial_t n_1+\operatorname{div}u_1=g_1,\\ \partial_t u_1+\nu u_1-u_2\times B_\infty+\nabla n_1=g_2+u_2\times B,\\ \partial_t n_2+\operatorname{div}u_2=g_3,\\ \partial_t u_2+\nu u_2-u_1\times B_\infty+\nabla n_2-2\nu E=g_4+u_1\times B,\\ \partial_t E-\nu\nabla\times B+\nu u_2=g_5,\\ \partial_t B+\nu\nabla\times E=0,\\ \operatorname{div}E=\nu\left(f(\frac{n_1+n_2}{2})-f(\frac{n_1-n_2}{2})\right),\quad \operatorname{div}B=0,\\ U|_{t=0}=U_0:=(n_{10},n_{20},u_{10},u_{20},E_0,B_0).\end{cases} \tag{1.5}$$

Here

$$\begin{aligned}g_1&=-\frac12(u_1\cdot\nabla n_1+u_2\cdot\nabla n_2)-\frac{\mu}{2}(n_1\operatorname{div}u_1+n_2\operatorname{div}u_2),\\ g_2&=-\frac12(u_1\cdot\nabla u_1+u_2\cdot\nabla u_2)-\frac{\mu}{2}(n_1\nabla n_1+n_2\nabla n_2),\\ g_3&=-\frac12(u_1\cdot\nabla n_2+u_2\cdot\nabla n_1)-\frac{\mu}{2}(n_1\operatorname{div}u_2+n_2\operatorname{div}u_1),\\ g_4&=-\frac12(u_1\cdot\nabla u_2+u_2\cdot\nabla u_1)-\frac{\mu}{2}(n_1\nabla n_2+n_2\nabla n_1),\\ g_5&=\nu\left(f\left(\frac{n_1-n_2}{2}\right)\frac{u_1-u_2}{2}-f\left(\frac{n_1+n_2}{2}\right)\frac{u_1+u_2}{2}\right).\end{aligned} \tag{1.6}$$

Notation: In this paper, we use $H^s(\mathbb{R}^3)$, $s\in\mathbb{R}$ to denote the usual Sobolev spaces with norm $\|\cdot\|_{H^s}$ and $L^p(\mathbb{R}^3)$, $1\leqslant p\leqslant\infty$ to denote the usual L^p spaces with norm $\|\cdot\|_{L^p}$. The symbol ∇^ℓ with an integer $\ell\geqslant 0$ stands for the usual spatial derivatives of order ℓ. When $\ell<0$ or ℓ is not a positive integer, ∇^ℓ stands for Λ^ℓ defined by $\Lambda^\ell f:=\mathscr{F}^{-1}(|\xi|^\ell\mathscr{F}f)$ where $\mathscr{F}$ is the usual Fourier transform operator and $\mathscr{F}^{-1}$ is its inverse. We use $\dot H^s(\mathbb{R}^3)$, $s\in\mathbb{R}$ to denote the homogeneous Sobolev spaces with norm $\|\cdot\|_{\dot H^s}$ defined by $\|f\|_{\dot H^s}:=\|\Lambda^s f\|_{L^2}$. We then recall the homogeneous Besov spaces. Let $\phi\in C_c^\infty(R^3_\xi)$ be such that $\phi(\xi)=1$ when $|\xi|\leqslant 1$ and $\phi(\xi)=0$ when $|\xi|\geqslant 2$. Let $\varphi(\xi)=\phi(\xi)-\phi(2\xi)$ and $\varphi_j(\xi)=\varphi(2^{-j}\xi)$ for $j\in\mathbb{Z}$. Then by the construction, $\sum_{j\in\mathbb{Z}}\varphi_j(\xi)=1$ if $\xi\neq 0$. We define $\dot\Delta_j f:=\mathscr{F}^{-1}(\varphi_j)*f$; then for $s\in\mathbb{R}$ and $1\leqslant p,r\leqslant\infty$, we define the homogeneous Besov spaces $\dot B^s_{p,r}(\mathbb{R}^3)$ with norm $\|\cdot\|_{\dot B^s_{p,r}}$ by

$$\|f\|_{\dot B^s_{p,r}}:=\Big(\sum_{j\in\mathbb{Z}}2^{rsj}\|\dot\Delta_j f\|_{L^p}^r\Big)^{\frac1r}.$$

Particularly, if $r=\infty$, then

$$\|f\|_{\dot B^s_{p,\infty}}:=\sup_{j\in\mathbb{Z}}2^{sj}\left\|\dot\Delta_j f\right\|_{L^p}.$$

Throughout this paper, we let C denote some positive universal constants. We will use $a \lesssim b$ if $a \leqslant Cb$, and $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$. We use C_0 to denote the constants depending on the initial data. For simplicity, we write $\|(A,B)\|_X := \|A\|_X + \|B\|_X$ and $\int f := \int_{\mathbb{R}^3} f\,dx$. We use $(*) \times \varepsilon + (**)$ to denote the process of multiplying $(*)$ by a sufficiently small but fixed factor ε and then adding it to $(**)$.

For $N \geqslant 3$, we define the energy functional by

$$\mathcal{E}_N(t) := \sum_{l=0}^{N} \left\|\nabla^l U\right\|_{L^2}^2$$

and the corresponding dissipation rate by

$$\mathcal{D}_N(t) := \sum_{l=1}^{N} \left\|\nabla^l n_1\right\|_{L^2}^2 + \sum_{l=0}^{N} \left\|\nabla^l (n_2, u_1, u_2)\right\|_{L^2}^2 + \sum_{l=0}^{N-1} \left\|\nabla^l E\right\|_{L^2}^2 + \sum_{l=1}^{N-1} \left\|\nabla^l B\right\|_{L^2}^2.$$

In the process of deriving the time decay rates of the solution to the system (1.5), we may first refine a global existence theorem as stated in the following.

THEOREM 1.1. *Assume the initial data satisfies the compatible conditions*

$$\operatorname{div} E_0 = \nu \left(f\left(\frac{n_{10}+n_{20}}{2}\right) - f\left(\frac{n_{10}-n_{20}}{2}\right)\right), \quad \operatorname{div} B_0 = 0.$$

There exists a sufficiently small $\delta_0 > 0$ such that if $\mathcal{E}_3(0) \leqslant \delta_0$, then there exists a unique global solution $U(t)$ to the Euler–Maxwell system (1.5) *satisfying*

$$\sup_{0 \leqslant t \leqslant \infty} \mathcal{E}_3(t) + \int_0^\infty \mathcal{D}_3(\tau)\,d\tau \lesssim \mathcal{E}_3(0). \tag{1.7}$$

Furthermore, if $\mathcal{E}_N(0) < +\infty$ for any $N \geqslant 3$, then there exists an increasing continuous function $P_N(\cdot)$ with $P_N(0) = 0$ such that the unique solution satisfies

$$\sup_{0 \leqslant t \leqslant \infty} \mathcal{E}_N(t) + \int_0^\infty \mathcal{D}_N(\tau)\,d\tau \lesssim P_N\left(\mathcal{E}_N(0)\right). \tag{1.8}$$

In the proof of Theorem 1.1, the new major difficulties are caused by the degenerate dissipation for the total density n_1 and the regularity-loss of the electromagnetic field (E,B). We will do the refined energy estimates stated in lemmas 2.8–2.9 which allow us to deduce

$$\frac{d}{dt}\mathcal{E}_3 + \mathcal{D}_3 \lesssim \sqrt{\mathcal{E}_3}\mathcal{D}_3$$

and, for $N \geqslant 4$,

$$\frac{d}{dt}\mathcal{E}_N + \mathcal{D}_N \lesssim \mathcal{D}_{N-1}\mathcal{E}_N.$$

Then Theorem 1.1 follows in the fashion of [8, 34].

The main purpose of this paper is to derive some various time decay rates of the solution to the system (1.5) as follows.

THEOREM 1.2. *Assume that $U(t)$ is the solution to the Euler–Maxwell system* (1.5) *constructed in Theorem 1.1 with $N \geqslant 4$. There exists a sufficiently small $\delta_0 = \delta_0(N)$ such that $\mathcal{E}_N(0) \leqslant \delta_0$, and assuming that $U_0 \in \dot{H}^{-s}$ for some $s \in [0,3/2)$ or $U_0 \in \dot{B}^{-s}_{2,\infty}$ for some $s \in (0,3/2]$, then we have*

$$\|U(t)\|_{\dot{H}^{-s}} \leqslant C_0 \quad \text{or} \quad \|U(t)\|_{\dot{B}^{-s}_{2,\infty}} \leqslant C_0. \tag{1.9}$$

Moreover, for any fixed integer $k \geqslant 0$, if $N \geqslant 2k+2+s$, then

$$\left\|\nabla^k U(t)\right\|_{L^2} \leqslant C_0(1+t)^{-\frac{k+s}{2}}. \tag{1.10}$$

Furthermore, for any fixed integer $k \geqslant 0$, if $N \geqslant 2k+4+s$, then

$$\left\|\nabla^k (n_2,u_1,u_2,E)(t)\right\|_{L^2} \leqslant C_0(1+t)^{-\frac{k+1+s}{2}}, \tag{1.11}$$

if $N \geqslant 2k+6+s$, then

$$\left\|\nabla^k n_2(t)\right\|_{L^2} \leqslant C_0(1+t)^{-\frac{k+2+s}{2}}, \tag{1.12}$$

and if $N \geqslant 2k+12+s$ and $B_\infty = 0$, then

$$\left\|\nabla^k (n_2,\operatorname{div} u_2)(t)\right\|_{L^2} \leqslant C_0(1+t)^{-(\frac{k}{2}+\frac{7}{4}+s)}. \tag{1.13}$$

In the proof of Theorem 1.2, we mainly use the regularity interpolation method developed by Strain and Guo in [25] and by Guo and Wang in [12]. To prove the optimal time decay rate of the dissipative equations in the whole space, Guo and Wang [12] developed a general energy method of using a family of scaled energy estimates with minimum derivative counts and interpolations among them. However, this method cannot be applied directly to the compressible bipolar Euler–Maxwell system which is of regularity-loss. To overcome this obstacle caused by the regularity-loss of the electromagnetic field, we deduce from lemmas 2.8–2.9 that

$$\frac{d}{dt}\mathcal{E}_k^{k+2} + \mathcal{D}_k^{k+2} \lesssim \|(n_1,n_2,u_1,u_2)\|_{L^\infty} \left\|\nabla^{k+2}(n_1,n_2,u_1,u_2)\right\|_{L^2} \left\|\nabla^{k+2}(E,B)\right\|_{L^2},$$

where $\mathcal{E}_k^{k+2}$ and $\mathcal{D}_k^{k+2}$ with minimum derivative counts are defined by (3.5) and (3.6) respectively. Then combining the methods of [12] and a trick of [25] to treat the electromagnetic field, we manage to conclude the time decay rate (1.10). The decay rate of the magnetic field B in (1.10) is optimal since it is consistent with the linear one proved in Duan et al. [4]. Indeed, the decay rate of B is the slowest one among all the components of the solution. In this sense, if in view of the whole solution, then the decay rate (1.10) can be regarded as being optimal. The higher decay rates (1.11)–(1.13) follow by revisiting the equations carefully. In particular, we will use a bootstrap argument to derive (1.13).

By Theorem 1.2 and lemmas 2.4–2.5, we have the following corollary of the usual L^p–L^2 type of the decay results:

COROLLARY 1.1. *Under the assumptions of Theorem 1.2, except that we replace the $\dot{H}^{-s}$ or $\dot{B}^{-s}_{2,\infty}$ assumption by $U_0 \in L^p$ for some $p \in [1,2]$, for any fixed integer $k \geqslant 0$, if $N \geqslant 2k+2+s_p$, then*

$$\left\|\nabla^k U(t)\right\|_{L^2} \leqslant C_0(1+t)^{-\frac{k+s_p}{2}}. \tag{1.14}$$

Here the number $s_p := 3\left(\frac{1}{p}-\frac{1}{2}\right)$.

Furthermore, for any fixed integer $k \geqslant 0$*, if* $N \geqslant 2k+4+s_p$*, then*

$$\left\|\nabla^k(n_2,u_1,u_2,E)(t)\right\|_{L^2} \leqslant C_0(1+t)^{-\frac{k+1+s_p}{2}}, \tag{1.15}$$

if $N \geqslant 2k+6+s_p$*, then*

$$\left\|\nabla^k n_2(t)\right\|_{L^2} \leqslant C_0(1+t)^{-\frac{k+2+s_p}{2}}, \tag{1.16}$$

and if $N \geqslant 2k+12+s_p$ *and* $B_\infty = 0$*, then*

$$\left\|\nabla^k(n_2,\operatorname{div} u_2)(t)\right\|_{L^2} \leqslant C_0(1+t)^{-(\frac{k}{2}+\frac{7}{4}+s_p)}. \tag{1.17}$$

The following are several remarks on theorems 1.1 and 1.2 and Corollary 1.1.

Remark 1.2. In Theorem 1.1, we only assume that the initial data is small in the H^3 norm, but the higher order derivatives could be large. Notice that in Theorem 1.2, the $\dot{H}^{-s}$ and $\dot{B}^{-s}_{2,\infty}$ norms of the solution are preserved along the time evolution; however, in Corollary 1.1 it is difficult to show that the L^p norm of the solution can be preserved. Note that the L^2 decay rate of the higher order spatial derivatives of the solution are obtained. Then the general optimal L^q $(2 \leqslant q \leqslant \infty)$ decay rates of the solution follow by the Sobolev interpolation.

Remark 1.3. In Theorem 1.2, the space $\dot{H}^{-s}$ or $\dot{B}^{-s}_{2,\infty}$ is introduced to enhance the decay rates. By the usual embedding theorem, we know that for $p \in (1,2]$, $L^p \subset \dot{H}^{-s}$ with $s = 3(\frac{1}{p}-\frac{1}{2}) \in [0,3/2)$. Meantime, we note that the endpoint embedding $L^1 \subset \dot{B}^{-\frac{3}{2}}_{2,\infty}$ holds. Hence the L^p–$L^2 (1 \leqslant p \leqslant 2)$ type of the optimal decay rates follows as a corollary.

Remark 1.4. We remark that Corollary 1.1 not only provides an alternative way to derive the L^p–L^2 type of the optimal decay rates, but also improves the previous results of the L^p–L^2 approach in Duan et al. [4]. Assuming that $B_\infty = 0$ and $\|U_0\|_{L^1}$ is sufficiently small, by combining the energy method and the linear decay analysis, Duan et al. [4] proved that

$$\|n_2(t)\|_{L^2} \leqslant C_0(1+t)^{-\frac{5}{2}}, \quad \|(u_1,u_2,E)(t)\|_{L^2} \leqslant C_0(1+t)^{-\frac{5}{4}},$$

$$\text{and} \quad \|(n_1,B)(t)\|_{L^2} \leqslant C_0(1+t)^{-\frac{3}{4}}.$$

We could find that their computation for the decay estimate is essentially based on the requirement $B_\infty = 0$ and do not apply to the case $B_\infty \neq 0$. However, our decay results (1.14)–(1.16) follow for a general B_∞. In particular, we don't require that $\|U_0\|_{L^p}$ $(1 \leqslant p \leqslant 2)$ is sufficiently small. Even if $B_\infty = 0$, then our decay rate of $n_2(t)$ will reach $(1+t)^{-13/4}$ for $p = 1$ in (1.17).

The rest of our paper is structured as follows. In Section 2, we establish the refined energy estimates for the solution and derive the negative Sobolev and Besov estimates. Theorem 1.1 and Theorem 1.2 are proved in Section 3.

2. Nonlinear energy estimates

In this section, we will derive the a priori energy estimates by assuming that $\|n_{\pm}(t)\|_{H^3}\leqslant\delta\ll 1$. Recall the expression (1.3) of $f(n_{\pm})$ and (1.4). Then by Taylor's formula and Sobolev's inequality, we have

$$f\left(\frac{n_1\pm n_2}{2}\right)\sim\frac{n_1\pm n_2}{2} \text{ and } \left|f^{(k)}\left(\frac{n_1\pm n_2}{2}\right)\right|\leqslant C \text{ for any } k\geqslant 1. \tag{2.1}$$

2.1. Preliminary. In this subsection, we collect some analytic tools used later in this paper.

Lemma 2.1. *Let* $2\leqslant p\leqslant+\infty$ *and* $\alpha,m,\ell\geqslant 0$. *Then we have*

$$\|\nabla^{\alpha}f\|_{L^p}\lesssim\|\nabla^m f\|_{L^2}^{1-\theta}\left\|\nabla^{\ell}f\right\|_{L^2}^{\theta}.$$

Here $0\leqslant\theta\leqslant 1$ *(if* $p=+\infty$, *then we require that* $0<\theta<1$*) and* α *satisfies*

$$\alpha+3\left(\frac{1}{2}-\frac{1}{p}\right)=m(1-\theta)+\ell\theta.$$

Proof. For the case $2\leqslant p<+\infty$, we refer to Lemma A.1 in [12]; for the case $p=+\infty$, we refer to Exercise 6.1.2 in [6] (p. 421). □

Lemma 2.2. *For any integer* $k\geqslant 0$, *we have*

$$\left\|\nabla^k f(n)\right\|_{L^\infty}\lesssim\left\|\nabla^{k+1}n\right\|_{L^2}^{1/2}\left\|\nabla^{k+2}n\right\|_{L^2}^{1/2} \tag{2.2}$$

and

$$\left\|\nabla^k f(n)\right\|_{L^2}\lesssim\left\|\nabla^k n\right\|_{L^2}.$$

Proof. See Lemma 2.2 in [26]. □

We recall the following commutator and product estimates:

Lemma 2.3. *Let* $k\geqslant 1$ *be an integer and define the commutator*

$$\left[\nabla^k,g\right]h=\nabla^k(gh)-g\nabla^k h. \tag{2.3}$$

Then we have

$$\left\|\left[\nabla^k,g\right]h\right\|_{L^{p_0}}\lesssim\|\nabla g\|_{L^{p_1}}\left\|\nabla^{k-1}h\right\|_{L^{p_2}}+\left\|\nabla^k g\right\|_{L^{p_3}}\|h\|_{L^{p_4}}. \tag{2.4}$$

In addition, we have that for $k\geqslant 0$,

$$\left\|\nabla^k(gh)\right\|_{L^{p_0}}\lesssim\|g\|_{L^{p_1}}\left\|\nabla^k h\right\|_{L^{p_2}}+\left\|\nabla^k g\right\|_{L^{p_3}}\|h\|_{L^{p_4}}. \tag{2.5}$$

In the above, $p_0,p_2,p_3\in(1,+\infty)$ *such that*

$$\frac{1}{p_0}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}+\frac{1}{p_4}.$$

Proof. We refer to Lemma 3.1 in [16]. □

Notice that when using the commutator estimates in this paper, we usually will not consider the case when $k=0$ since it is trivial. We have the following L^p embeddings:

LEMMA 2.4. *Let $0\leqslant s<3/2$, $1<p\leqslant 2$ with $1/2+s/3=1/p$. Then*

$$\|f\|_{\dot{H}^{-s}}\lesssim\|f\|_{L^p}.$$

Proof. It follows from the Hardy–Littlewood–Sobolev Theorem, see [6]. □

LEMMA 2.5. *Let $0<s\leqslant 3/2$, $1\leqslant p<2$ with $1/2+s/3=1/p$. Then*

$$\|f\|_{\dot{B}^{-s}_{2,\infty}}\lesssim\|f\|_{L^p}.$$

Proof. See Lemma 4.1 in [24]. □

It is important to use the following special interpolation estimates:

LEMMA 2.6. *Let $s\geqslant 0$ and $\ell\geqslant 0$. Then we have*

$$\left\|\nabla^\ell f\right\|_{L^2}\leqslant\left\|\nabla^{\ell+1}f\right\|_{L^2}^{1-\theta}\|f\|_{\dot{H}^{-s}}^{\theta},\ \textit{where}\ \theta=\frac{1}{\ell+1+s}.$$

Proof. It follows directly by the Parseval Theorem and Hölder's inequality. □

LEMMA 2.7. *Let $s>0$ and $\ell\geqslant 0$, then we have*

$$\left\|\nabla^\ell f\right\|_{L^2}\leqslant\left\|\nabla^{\ell+1}f\right\|_{L^2}^{1-\theta}\|f\|_{\dot{B}^{-s}_{2,\infty}}^{\theta},\ \textit{where}\ \theta=\frac{1}{\ell+1+s}.$$

Proof. We can refer to Lemma 4.2 in [24] by noting that $\dot{B}^{-s}_{2,p}\subset\dot{B}^{-s}_{2,q}$ for $p\leqslant q$. □

2.2. Energy estimates. In this subsection, we will derive the basic energy estimates for the solution to the Euler–Maxwell system (1.5). We begin with the standard energy estimates.

LEMMA 2.8. *For any integer $k\geqslant 0$ and some fixed positive constant λ, we have*

$$\begin{aligned}&\frac{d}{dt}\sum_{l=k}^{k+2}\left\|\nabla^l U\right\|_{L^2}^2+\lambda\sum_{l=k}^{k+2}\left\|\nabla^l(u_1,u_2)\right\|_{L^2}^2\\ &\lesssim F\left(\sum_{l=k+1}^{k+2}\left\|\nabla^l n_1\right\|_{L^2}^2+\sum_{l=k}^{k+2}\left\|\nabla^l(n_2,u_1,u_2)\right\|_{L^2}^2+\sum_{l=k}^{k+1}\left\|\nabla^l E\right\|_{L^2}^2+\left\|\nabla^{k+1}B\right\|_{L^2}^2\right)\\ &\qquad+\|(n_1,n_2,u_1,u_2)\|_{L^\infty}\left\|\nabla^{k+2}(n_1,n_2,u_1,u_2)\right\|_{L^2}\left\|\nabla^{k+2}(E,B)\right\|_{L^2},\end{aligned}\tag{2.6}$$

where F is defined by

$$F=F(n_1,n_2,u_1,u_2,B):=\|\nabla n_1\|_{H^2}+\|n_2\|_{H^3}+\|(u_1,u_2)\|_{H^{k+1}\cap H^3}+\|\nabla B\|_{H^1}.$$

Proof. The standard ∇^l $(l=k,k+1,k+2)$ energy estimates on the system (1.5) yield

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\left|\nabla^l(n_1,n_2,u_1,u_2)\right|^2+\frac{d}{dt}\int\left|\nabla^l(E,B)\right|^2+\nu\left\|\nabla^l(u_1,u_2)\right\|_{L^2}^2\\ &\quad=\int\nabla^l g_1\nabla^l n_1+\nabla^l g_2\cdot\nabla^l u_1+\nabla^l g_3\nabla^l n_2+\nabla^l g_4\cdot\nabla^l u_2\end{aligned}$$

$$
\begin{aligned}
&+\int \nabla^l (u_2 \times B)\cdot \nabla^l u_1 + \nabla^l (u_1 \times B)\cdot \nabla^l u_2 + 2\int \nabla^l g_5 \cdot \nabla^l E \\
&:= I_1 + I_2 + 2I_3.
\end{aligned} \tag{2.7}
$$

We now estimate $I_1 - I_3$. First, by (1.6), we split I_1 as:

$$
\begin{aligned}
I_1 = &-\frac{1}{2}\int \nabla^l (u_1\cdot\nabla n_1)\nabla^l n_1 + \nabla^l (u_1\cdot\nabla u_1)\cdot\nabla^l u_1 \\
&-\frac{1}{2}\int \nabla^l (u_1\cdot\nabla n_2)\nabla^l n_2 + \nabla^l (u_1\cdot\nabla u_2)\cdot\nabla^l u_2 \\
&-\frac{1}{2}\int \nabla^l (u_2\cdot\nabla n_2)\nabla^l n_1 + \nabla^l (u_2\cdot\nabla n_1)\nabla^l n_2 \\
&-\frac{1}{2}\int \nabla^l (u_2\cdot\nabla u_2)\cdot\nabla^l u_1 + \nabla^l (u_2\cdot\nabla u_1)\cdot\nabla^l u_2 \\
&-\frac{\mu}{2}\int \nabla^l (n_1 \operatorname{div} u_1)\nabla^l n_1 + \nabla^l (n_1\nabla n_1)\cdot\nabla^l u_1 \\
&-\frac{\mu}{2}\int \nabla^l (n_1 \operatorname{div} u_2)\nabla^l n_2 + \nabla^l (n_1\nabla n_2)\cdot\nabla^l u_2 \\
&-\frac{\mu}{2}\int \nabla^l (n_2 \operatorname{div} u_2)\nabla^l n_1 + \nabla^l (n_2\nabla n_1)\cdot\nabla^l u_2 \\
&-\frac{\mu}{2}\int \nabla^l (n_2 \operatorname{div} u_1)\nabla^l n_2 + \nabla^l (n_2\nabla n_2)\cdot\nabla^l u_1 \\
:= &\frac{1}{2}(I_{11}+I_{12}+I_{13}+I_{14}) + \frac{\mu}{2}(I_{15}+I_{16}+I_{17}+I_{18}).
\end{aligned} \tag{2.8}
$$

We shall estimate the terms $I_{11} - I_{18}$. We must be careful with the terms involving n_1 since n_1 is degenerately dissipative. First we estimate I_{11}. We have to distinguish the arguments by the value of l. For $l=k$ or $k+1$, using the product estimates (2.5), we have

$$
\begin{aligned}
&-\int \nabla^l (u_1\cdot\nabla n_1)\nabla^l n_1 \lesssim \left\|\nabla^l (u_1\cdot\nabla n_1)\right\|_{L^{6/5}} \left\|\nabla^l n_1\right\|_{L^6} \\
&\lesssim \left(\|u_1\|_{L^3}\left\|\nabla^{l+1} n_1\right\|_{L^2} + \|\nabla n_1\|_{L^3}\left\|\nabla^l u_1\right\|_{L^2}\right)\left\|\nabla^{l+1} n_1\right\|_{L^2} \\
&\lesssim \left(\|\nabla n_1\|_{H^2} + \|u_1\|_{H^3}\right)\left(\left\|\nabla^{l+1} n_1\right\|_{L^2}^2 + \left\|\nabla^l u_1\right\|_{L^2}^2\right).
\end{aligned} \tag{2.9}
$$

Now for $l=k+2$, by integrating by parts and from the commutator estimates (2.4), we have

$$
\begin{aligned}
&-\int \nabla^{k+2}(u_1\cdot\nabla n_1)\nabla^{k+2} n_1 = -\int \left[\nabla^{k+2}, u_1\right]\cdot\nabla n_1 \nabla^{k+2} n_1 - \int u_1\cdot\nabla\nabla^{k+2} n_1 \nabla^{k+2} n_1 \\
&\lesssim \left(\|\nabla u_1\|_{L^\infty}\left\|\nabla^{k+2} n_1\right\|_{L^2} + \left\|\nabla^{k+2} u_1\right\|_{L^2}\|\nabla n_1\|_{L^\infty}\right)\left\|\nabla^{k+2} n_1\right\|_{L^2} \\
&\quad -\frac{1}{2}\int u_1\cdot\nabla\left(\nabla^{k+2} n_1 \nabla^{k+2} n_1\right) \\
&\lesssim \|\nabla(n_1,u_1)\|_{L^\infty}\left\|\nabla^{k+2}(n_1,u_1)\right\|_{L^2}^2 + \frac{1}{2}\int \operatorname{div} u_1 \left|\nabla^{k+2} n_1\right|^2 \\
&\lesssim \left(\|\nabla n_1\|_{H^2} + \|u_1\|_{H^3}\right)\left\|\nabla^{k+2}(n_1,u_1)\right\|_{L^2}^2.
\end{aligned} \tag{2.10}
$$

On the other hand, like (2.10), we have for $l=k,\ k+1,\ k+2$,

$$
-\int \nabla^l (u_1\cdot\nabla u_1)\cdot\nabla^l u_1 = -\int \left(u_1\cdot\nabla\nabla^l u_1 + \left[\nabla^l, u_1\right]\cdot\nabla u_1\right)\cdot\nabla^l u_1
$$

$$=-\int \frac{1}{2}u_1\cdot\nabla\left(\nabla^l u_1\cdot\nabla^l u_1\right)+\left[\nabla^l,u_1\right]\cdot\nabla u_1\cdot\nabla^l u_1\lesssim\|\nabla u_1\|_{L^\infty}\left\|\nabla^l u_1\right\|_{L^2}^2. \tag{2.11}$$

Hence, by (2.9)–(2.11), we have for $l=k,\ k+1$,

$$I_{11}\lesssim(\|\nabla n_1\|_{H^2}+\|u_1\|_{H^3})\left(\left\|\nabla^{l+1}n_1\right\|_{L^2}^2+\left\|\nabla^l u_1\right\|_{L^2}^2\right),$$

and for $l=k+2$,

$$I_{11}\lesssim(\|\nabla n_1\|_{H^2}+\|u_1\|_{H^3})\left\|\nabla^{k+2}(n_1,u_1)\right\|_{L^2}^2.$$

Like (2.10), we have for $l=k,\ k+1,\ k+2$,

$$I_{12}\lesssim\|(n_2,u_1,u_2)\|_{H^3}\left\|\nabla^l(n_2,u_1,u_2)\right\|_{L^2}^2,\ I_{14}\lesssim\|(u_1,u_2)\|_{H^3}\left\|\nabla^l(u_1,u_2)\right\|_{L^2}^2.$$

As in (2.9)–(2.10), we have for $l=k,\ k+1$,

$$I_{13}\lesssim(\|\nabla n_1\|_{H^2}+\|(n_2,u_2)\|_{H^3})\left(\left\|\nabla^{l+1}(n_1,n_2)\right\|_{L^2}^2+\left\|\nabla^l u_1\right\|_{L^2}^2\right),$$

and for $l=k+2$,

$$I_{13}\lesssim(\|\nabla n_1\|_{H^2}+\|(n_2,u_2)\|_{H^3})\left\|\nabla^{k+2}(n_1,n_2,u_1)\right\|_{L^2}^2.$$

We next estimate the term I_{15}. For $l=k$ or $k+1$, using the commutator notation (2.3), we can split I_{15} as:

$$\begin{aligned}
I_{15}=&-\int\nabla^l(n_1\mathrm{div}u_1)\nabla^l n_1+\nabla^l(n_1\nabla n_1)\cdot\nabla^l u_1\\
=&-\sum_{\ell=0}^{l}\int C_l^\ell\nabla^{l-\ell}n_1\nabla^\ell\,\mathrm{div}\,u_1\nabla^l n_1-\int\left(\left[\nabla^l,n_1\right]\nabla n_1+n_1\nabla^{l+1}n_1\right)\cdot\nabla^l u_1\\
=&-\sum_{\ell=0}^{l-1}\int C_l^\ell\nabla^{l-\ell}n_1\nabla^\ell\,\mathrm{div}\,u_1\nabla^l n_1-\int\left[\nabla^l,n_1\right]\nabla n_1\cdot\nabla^l u_1\\
&-\int n_1\,\mathrm{div}\,\nabla^l u_1\nabla^l n_1+n_1\nabla^{l+1}n_1\cdot\nabla^l u_1\\
:=&\,I_{151}+I_{152}+I_{153}.
\end{aligned}\tag{2.12}$$

First we estimate the term I_{153}. Using integration by parts, we obtain

$$\begin{aligned}
I_{153}=&-\int n_1\,\mathrm{div}\,\nabla^l u_1\nabla^l n_1+n_1\nabla^{l+1}n_1\cdot\nabla^l u_1\\
=&-\int n_1\,\mathrm{div}\left(\nabla^l u_1\nabla^l n_1\right)=\int\nabla n_1\nabla^l u_1\nabla^l n_1\\
\lesssim&\|\nabla n_1\|_{L^3}\left\|\nabla^l u_1\right\|_{L^2}\left\|\nabla^l n_1\right\|_{L^6}\lesssim\|\nabla n_1\|_{H^2}\left(\left\|\nabla^{l+1}n_1\right\|_{L^2}^2+\left\|\nabla^l u_1\right\|_{L^2}^2\right).
\end{aligned}\tag{2.13}$$

Next we estimate the term I_{151}. By Hölder's and Sobolev's inequalities, we obtain

$$I_{151}=-\sum_{\ell=0}^{l-1}\int C_l^\ell\nabla^{l-\ell}n_1\nabla^\ell\,\mathrm{div}\,u_1\nabla^l n_1$$

$$
\begin{aligned}
&\lesssim \sum_{\ell=0}^{l-1} \left\|\nabla^{l-\ell} n_1 \nabla^\ell \operatorname{div} u_1\right\|_{L^{6/5}} \left\|\nabla^l n_1\right\|_{L^6} \\
&\lesssim \sum_{\ell=0}^{l-1} \left\|\nabla^{l-\ell} n_1 \nabla^\ell \operatorname{div} u_1\right\|_{L^{6/5}} \left\|\nabla^{l+1} n_1\right\|_{L^2}.
\end{aligned} \tag{2.14}
$$

If $0 \leqslant \ell \leqslant \left[\frac{l}{2}\right]$, by Hölder's inequality and Lemma 2.1, then we have

$$
\begin{aligned}
\left\|\nabla^{l-\ell} n_1 \nabla^\ell \operatorname{div} u_1\right\|_{L^{6/5}} &\lesssim \left\|\nabla^{l-\ell} n_1\right\|_{L^3} \left\|\nabla^{\ell+1} u_1\right\|_{L^2} \\
&\lesssim \|\nabla n_1\|_{L^2}^{\frac{2\ell+1}{2l}} \left\|\nabla^{l+1} n_1\right\|_{L^2}^{\frac{2l-2\ell-1}{2l}} \|\nabla^\alpha u_1\|_{L^2}^{\frac{2l-2\ell-1}{2l}} \left\|\nabla^l u_1\right\|_{L^2}^{\frac{2\ell+1}{2l}} \\
&\lesssim (\|\nabla n_1\|_{L^2} + \|u_1\|_{H^3}) \left(\left\|\nabla^{l+1} n_1\right\|_{L^2} + \left\|\nabla^l u_1\right\|_{L^2}\right),
\end{aligned} \tag{2.15}
$$

where α is defined by

$$
\ell+1 = \alpha \times \frac{2l-2\ell-1}{2l} + \frac{2\ell+1}{2} \Longrightarrow \alpha = \frac{l}{2l-2\ell-1} \in \left(\frac{1}{2}, 3\right) \text{ since } \ell \leqslant \frac{l}{2};
$$

if $\left[\frac{l}{2}\right]+1 \leqslant \ell \leqslant l-1$, by Hölder's inequality and Lemma 2.1 again, then we have

$$
\begin{aligned}
\left\|\nabla^{l-\ell} n_1 \nabla^\ell \operatorname{div} u_1\right\|_{L^{6/5}} &\lesssim \left\|\nabla^{l-\ell} n_1\right\|_{L^3} \left\|\nabla^{\ell+1} u_1\right\|_{L^2} \\
&\lesssim \|\nabla^\alpha n_1\|_{L^2}^{\frac{\ell+1}{l}} \left\|\nabla^{l+1} n_1\right\|_{L^2}^{\frac{l-\ell-1}{l}} \|u_1\|_{L^2}^{\frac{l-\ell-1}{l}} \left\|\nabla^l u_1\right\|_{L^2}^{\frac{\ell+1}{l}} \\
&\lesssim (\|\nabla n_1\|_{H^2} + \|u_1\|_{H^3}) \left(\left\|\nabla^{l+1} n_1\right\|_{L^2} + \left\|\nabla^l u_1\right\|_{L^2}\right),
\end{aligned} \tag{2.16}
$$

where α is defined by

$$
l-\ell+\frac{1}{2} = \alpha \times \frac{\ell+1}{l} + (l+1) \times \frac{l-\ell-1}{l} \Longrightarrow \alpha = 1 + \frac{l}{2\ell+2} \in \left[\frac{3}{2}, 3\right) \text{ since } \ell \geqslant \frac{l+1}{2}.
$$

In light of (2.15)–(2.16), we deduce from (2.14) that

$$
I_{151} \lesssim (\|\nabla n_1\|_{H^2} + \|u_1\|_{H^3}) \left(\left\|\nabla^{l+1} n_1\right\|_{L^2}^2 + \left\|\nabla^l u_1\right\|_{L^2}^2\right). \tag{2.17}
$$

Finally, we estimate the term I_{152}. Using the commutator estimates (2.4), we obtain

$$
\begin{aligned}
I_{152} &= -\int \left[\nabla^l, n_1\right] \nabla n_1 \cdot \nabla^l u_1 \lesssim \left\|\left[\nabla^l, n_1\right] \nabla n_1\right\|_{L^2} \left\|\nabla^l u_1\right\|_{L^2} \\
&\lesssim \|\nabla n_1\|_{L^3} \left\|\nabla^l n_1\right\|_{L^6} \left\|\nabla^l u_1\right\|_{L^2} \lesssim \|\nabla n_1\|_{H^2} \left(\left\|\nabla^{l+1} n_1\right\|_{L^2}^2 + \left\|\nabla^l u_1\right\|_{L^2}^2\right).
\end{aligned} \tag{2.18}
$$

Hence, by (2.13), (2.17)–(2.18), we deduce from (2.12) that for $l = k, k+1$,

$$
I_{15} \lesssim (\|\nabla n_1\|_{H^2} + \|u_1\|_{H^3}) \left(\left\|\nabla^{l+1} n_1\right\|_{L^2}^2 + \left\|\nabla^l u_1\right\|_{L^2}^2\right).
$$

For $l = k+2$, like (2.10), we have

$$
\begin{aligned}
I_{15} &= -\int \nabla^{k+2}(n_1 \operatorname{div} u_1) \nabla^{k+2} n_1 + \nabla^{k+2}(n_1 \nabla n_1) \cdot \nabla^{k+2} u_1 \\
&= -\int \left[\nabla^{k+2}, n_1\right] \operatorname{div} u_1 \nabla^{k+2} n_1 + \left[\nabla^{k+2}, n_1\right] \nabla n_1 \cdot \nabla^{k+2} u_1
\end{aligned}
$$

$$
\begin{aligned}
&-\int n_1 \operatorname{div} \nabla^{k+2} u_1 \nabla^{k+2} n_1 + n_1 \nabla\nabla^{k+2} n_1 \cdot \nabla^{k+2} u_1 \\
&\lesssim \left(\|\nabla n_1\|_{L^\infty} \left\|\nabla^{k+2} u_1\right\|_{L^2} + \left\|\nabla^{k+2} n_1\right\|_{L^2} \|\nabla u_1\|_{L^\infty}\right) \left\|\nabla^{k+2} n_1\right\|_{L^2} \\
&\quad -\int n_1 \operatorname{div}\left(\nabla^{k+2} u_1 \nabla^{k+2} n_1\right) \\
&\lesssim \|\nabla(n_1,u_1)\|_{L^\infty} \left\|\nabla^{k+2}(n_1,u_1)\right\|_{L^2}^2 + \int \nabla n_1 \nabla^{k+2} u_1 \nabla^{k+2} n_1 \\
&\lesssim \left(\|\nabla n_1\|_{H^2} + \|u_1\|_{H^3}\right) \left\|\nabla^{k+2}(n_1,u_1)\right\|_{L^2}^2 .
\end{aligned}
$$

Applying similar arguments to the terms $I_{16}-I_{18}$, we deduce that for $l=k$ or $k+1$,

$$
I_{16}+I_{17}+I_{18} \lesssim \left(\|\nabla n_1\|_{H^2} + \|(n_2,u_1,u_2)\|_{H^3}\right)\left(\left\|\nabla^{l+1}(n_1,n_2)\right\|_{L^2}^2 + \left\|\nabla^{l}(u_1,u_2)\right\|_{L^2}^2\right),
$$

and for $l=k+2$,

$$
I_{16}+I_{17}+I_{18} \lesssim \left(\|\nabla n_1\|_{H^2} + \|(n_2,u_1,u_2)\|_{H^3}\right)\left\|\nabla^{k+2}(n_1,n_2,u_1,u_2)\right\|_{L^2}^2 .
$$

Hence, by the estimates for $I_{11}-I_{18}$, we deduce for $l=k,k+1$,

$$
I_1 \lesssim \left(\|\nabla n_1\|_{H^2} + \|(n_2,u_1,u_2)\|_{H^3}\right)\left(\left\|\nabla^{l+1}(n_1,n_2)\right\|_{L^2}^2 + \left\|\nabla^{l}(u_1,u_2)\right\|_{L^2}^2\right), \tag{2.19}
$$

and for $l=k+2$

$$
I_1 \lesssim \left(\|\nabla n_1\|_{H^2} + \|(n_2,u_1,u_2)\|_{H^3}\right)\left\|\nabla^{k+2}(n_1,n_2,u_1,u_2)\right\|_{L^2}^2 .
$$

Now we estimate the term I_2, and we must be much more careful with this term since the magnetic field B has the weakest dissipative estimates. First of all, by Hölder's inequality, we have

$$
\begin{aligned}
I_2 &= \int \nabla^l(u_2\times B)\cdot\nabla^l u_1 + \nabla^l(u_1\times B)\cdot\nabla^l u_2 \\
&\lesssim \left\|\nabla^l(u_2\times B)\right\|_{L^2}\left\|\nabla^l u_1\right\|_{L^2} + \left\|\nabla^l(u_1\times B)\right\|_{L^2}\left\|\nabla^l u_2\right\|_{L^2} .
\end{aligned} \tag{2.20}
$$

We again have to distinguish the arguments by the value of l. For $i=1,2$, we make good use of the product estimates (2.5) to bound

$$
\begin{aligned}
\left\|\nabla^l(u_i\times B)\right\|_{L^2} &\lesssim \|B\|_{L^\infty}\left\|\nabla^l u_i\right\|_{L^2} + \|u_i\|_{L^3}\left\|\nabla^l B\right\|_{L^6} \\
&\lesssim \|\nabla B\|_{H^1}\left\|\nabla^l u_i\right\|_{L^2} + \|u_i\|_{H^1}\left\|\nabla^{l+1} B\right\|_{L^2} \quad \text{for} \quad l=k, \\
\left\|\nabla^l(u_i\times B)\right\|_{L^2} &\lesssim \|B\|_{L^\infty}\left\|\nabla^l u_i\right\|_{L^2} + \|u_i\|_{L^\infty}\left\|\nabla^l B\right\|_{L^2} \\
&\lesssim \|\nabla B\|_{H^1}\left\|\nabla^l u_i\right\|_{L^2} + \|u_i\|_{H^2}\left\|\nabla^l B\right\|_{L^2} \quad \text{for} \quad l=k+1, \\
\left\|\nabla^l(u_i\times B)\right\|_{L^2} &\lesssim \|B\|_{L^\infty}\left\|\nabla^l u_i\right\|_{L^2} + \|u_i\|_{L^\infty}\left\|\nabla^l B\right\|_{L^2} \\
&\lesssim \|\nabla B\|_{H^1}\left\|\nabla^l u_i\right\|_{L^2} + \|u_i\|_{L^\infty}\left\|\nabla^l B\right\|_{L^2} \quad \text{for} \quad l=k+2.
\end{aligned}
$$

Hence by Young's inequality, we deduce from (2.20) that for $l=k$,

$$
I_2 \lesssim \left(\|(u_1,u_2)\|_{H^1} + \|\nabla B\|_{H^1}\right)\left(\left\|\nabla^k(u_1,u_2)\right\|_{L^2}^2 + \left\|\nabla^{k+1} B\right\|_{L^2}^2\right), \tag{2.21}
$$

for $l=k+1$,

$$
I_2 \lesssim \left(\|(u_1,u_2)\|_{H^2} + \|\nabla B\|_{H^1}\right)\left(\left\|\nabla^{k+1}(u_1,u_2)\right\|_{L^2}^2 + \left\|\nabla^{k+1} B\right\|_{L^2}^2\right),
$$

and for $l=k+2$,

$$I_2\lesssim \|\nabla B\|_{H^1}\left\|\nabla^{k+2}(u_1,u_2)\right\|_{L^2}^2+\|(u_1,u_2)\|_{L^\infty}\left\|\nabla^{k+2}B\right\|_{L^2}\left\|\nabla^{k+2}(u_1,u_2)\right\|_{L^2}.$$

We now estimate the last term I_3. First, we split I_3 as:

$$\begin{aligned}
I_3=&\nu\sum_{\ell=0}^{l}C_l^\ell\int\left[\nabla^\ell f\left(\frac{n_1-n_2}{2}\right)\nabla^{l-\ell}\left(\frac{u_1-u_2}{2}\right)\right.\\
&\qquad\qquad\left.-\nabla^\ell f\left(\frac{n_1+n_2}{2}\right)\nabla^{l-\ell}\left(\frac{u_1+u_2}{2}\right)\right]\cdot\nabla^l E\\
=&\frac{\nu}{2}\sum_{\ell=0}^{l}C_l^\ell\int\nabla^\ell f\left(\frac{n_1-n_2}{2}\right)\nabla^{l-\ell}u_1\cdot\nabla^l E-\frac{\nu}{2}\sum_{\ell=0}^{l}C_l^\ell\int\nabla^\ell f\left(\frac{n_1-n_2}{2}\right)\nabla^{l-\ell}u_2\cdot\nabla^l E\\
&-\frac{\nu}{2}\sum_{\ell=0}^{l}C_l^\ell\int\nabla^\ell f\left(\frac{n_1+n_2}{2}\right)\nabla^{l-\ell}u_1\cdot\nabla^l E-\frac{\nu}{2}\sum_{\ell=0}^{l}C_l^\ell\int\nabla^\ell f\left(\frac{n_1+n_2}{2}\right)\nabla^{l-\ell}u_2\cdot\nabla^l E\\
:=&\frac{\nu}{2}I_{31}+\frac{\nu}{2}I_{32}+\frac{\nu}{2}I_{33}+\frac{\nu}{2}I_{34}.
\end{aligned}\tag{2.22}$$

We still have to distinguish the arguments by the value of l. For $l=k$ or $k+1$, we only estimate the first term I_{31} on the right-hand side of (2.22), then the terms I_{32}–I_{34} can be estimated similarly. If $0\leqslant\ell\leqslant l-1$, by Lemma 2.1 and the estimate (2.2) of Lemma 2.2, we obtain

$$\begin{aligned}
&\left\|\nabla^\ell f\left(\frac{n_1-n_2}{2}\right)\nabla^{l-\ell}u_1\right\|_{L^2}\leqslant\left\|\nabla^\ell f\left(\frac{n_1-n_2}{2}\right)\right\|_{L^\infty}\left\|\nabla^{l-\ell}u_1\right\|_{L^2}\\
&\lesssim\left\|\nabla^{\ell+1}n_1\right\|_{L^2}^{\frac12}\left\|\nabla^{\ell+2}n_1\right\|_{L^2}^{\frac12}\left\|\nabla^{l-\ell}u_1\right\|_{L^2}+\left\|\nabla^{\ell+1}n_2\right\|_{L^2}^{\frac12}\left\|\nabla^{\ell+2}n_2\right\|_{L^2}^{\frac12}\left\|\nabla^{l-\ell}u_1\right\|_{L^2}\\
&\lesssim\left(\|\nabla n_1\|_{L^2}^{\frac{l-\ell}{l}}\left\|\nabla^{l+1}n_1\right\|_{L^2}^{\frac{\ell}{l}}\right)^{\frac12}\left(\|\nabla n_1\|_{L^2}^{\frac{l-\ell-1}{l}}\left\|\nabla^{l+1}n_1\right\|_{L^2}^{\frac{\ell+1}{l}}\right)^{\frac12}\left\|\nabla^{l-\ell}u_1\right\|_{L^2}\\
&\qquad+\left(\|\nabla n_2\|_{L^2}^{\frac{l-\ell}{l}}\left\|\nabla^{l+1}n_2\right\|_{L^2}^{\frac{\ell}{l}}\right)^{\frac12}\left(\|\nabla n_2\|_{L^2}^{\frac{l-\ell-1}{l}}\left\|\nabla^{l+1}n_2\right\|_{L^2}^{\frac{\ell+1}{l}}\right)^{\frac12}\left\|\nabla^{l-\ell}u_1\right\|_{L^2}\\
&\lesssim\|\nabla n_1\|_{L^2}^{1-\frac{2\ell+1}{2l}}\left\|\nabla^{l+1}n_1\right\|_{L^2}^{\frac{2\ell+1}{2l}}\|\nabla^\alpha u_1\|_{L^2}^{\frac{2\ell+1}{2l}}\left\|\nabla^l u_1\right\|_{L^2}^{1-\frac{2\ell+1}{2l}}\\
&\qquad+\|\nabla n_2\|_{L^2}^{1-\frac{2\ell+1}{2l}}\left\|\nabla^{l+1}n_2\right\|_{L^2}^{\frac{2\ell+1}{2l}}\|\nabla^\alpha u_1\|_{L^2}^{\frac{2\ell+1}{2l}}\left\|\nabla^l u_1\right\|_{L^2}^{1-\frac{2\ell+1}{2l}},
\end{aligned}$$

where α is defined by

$$l-\ell=\alpha\times\frac{2\ell+1}{2l}+l\times\left(1-\frac{2\ell+1}{2l}\right)\Longrightarrow\alpha=\frac{l}{2\ell+1}\leqslant l;$$

if $\ell=l$, then we deduce from the estimate (2.2) that

$$\left\|\nabla^l f\left(\frac{n_1-n_2}{2}\right)u_1\right\|_{L^2}\lesssim\left\|\nabla^l f\left(\frac{n_1-n_2}{2}\right)\right\|_{L^6}\|u_1\|_{L^3}\lesssim\left\|\nabla^{l+1}(n_1,n_2)\right\|_{L^2}\|u_1\|_{H^1}.$$

We thus have that for $l=k$ or $k+1$,

$$I_{31}\lesssim\left(\|\nabla(n_1,n_2)\|_{L^2}+\|u_1\|_{H^l\cap H^1}\right)\left(\left\|\nabla^{l+1}(n_1,n_2)\right\|_{L^2}^2+\left\|\nabla^l u_1\right\|_{L^2}^2+\left\|\nabla^l E\right\|_{L^2}^2\right).$$

Hence, we have that for $l=k$ or $k+1$,

$$\begin{aligned}I_3 \lesssim & (\|\nabla(n_1,n_2)\|_{L^2}+\|(u_1,u_2)\|_{H^l\cap H^1})\\ &\times\left(\left\|\nabla^{l+1}(n_1,n_2)\right\|_{L^2}^2+\left\|\nabla^l(u_1,u_2)\right\|_{L^2}^2+\left\|\nabla^l E\right\|_{L^2}^2\right).\end{aligned} \tag{2.23}$$

Now for $l=k+2$, we rewrite $I_{31}+I_{33}$ as

$$\begin{aligned}I_{31}+I_{33}=&\sum_{\ell=0}^{k+2}C_{k+2}^{\ell}\int\nabla^\ell g\nabla^{k+2-\ell}u_1\cdot\nabla^{k+2}E\\ =&\int\left(g\nabla^{k+2}u_1+\nabla^{k+2}gu_1\right)\cdot\nabla^{k+2}E-\sum_{\ell=1}^{k+1}C_{k+2}^{\ell}\int\nabla\left(\nabla^{k+2-\ell}g\nabla^\ell u_1\right)\cdot\nabla^{k+1}E\\ =&\int\left(g\nabla^{k+2}u_1+\nabla^{k+2}gu_1\right)\cdot\nabla^{k+2}E-(k+2)\int\left(\nabla^{k+2}g\nabla u_1+\nabla g\nabla^{k+2}u_1\right)\cdot\nabla^{k+1}E\\ &-\sum_{\ell=2}^{k+1}C_{k+2}^{\ell}\int\nabla^{k+3-\ell}g\nabla^\ell u_1\cdot\nabla^{k+1}E\quad-\sum_{\ell=1}^{k}C_{k+2}^{\ell}\int\nabla^{k+2-\ell}g\nabla^{\ell+1}u_1\cdot\nabla^{k+1}E\\ :=&I_{311}+I_{312}+I_{313}+I_{314},\end{aligned}$$

where the function g is defined by

$$g:=f\left(\frac{n_1-n_2}{2}\right)-f\left(\frac{n_1+n_2}{2}\right). \tag{2.24}$$

By Lemma 2.2 and (2.1), we have

$$\begin{aligned}I_{311}&\lesssim\left(\|g\|_{L^\infty}\left\|\nabla^{k+2}u_1\right\|_{L^2}+\left\|\nabla^{k+2}g\right\|_{L^2}\|u_1\|_{L^\infty}\right)\left\|\nabla^{k+2}E\right\|_{L^2}\\ &\lesssim\|(n_2,u_1)\|_{L^\infty}\left\|\nabla^{k+2}(n_2,u_1)\right\|_{L^2}\left\|\nabla^{k+2}E\right\|_{L^2}\end{aligned}$$

and

$$\begin{aligned}I_{312}&\lesssim\left(\left\|\nabla^{k+2}g\right\|_{L^2}\|\nabla u_1\|_{L^\infty}+\|\nabla g\|_{L^\infty}\left\|\nabla^{k+2}u_1\right\|_{L^2}\right)\left\|\nabla^{k+1}E\right\|_{L^2}\\ &\lesssim\|\nabla(n_2,u_1)\|_{L^\infty}\left\|\nabla^{k+2}(n_2,u_1)\right\|_{L^2}\left\|\nabla^{k+1}E\right\|_{L^2}.\end{aligned}$$

As for the cases $l=k,k+1$ for I_3, we can bound I_{313} and I_{314} by

$$\begin{aligned}I_{313}+I_{314}\lesssim&(\|\nabla(n_1,n_2)\|_{L^2}+\|u_1\|_{H^{k+1}})\\ &\times\left(\left\|\nabla^{k+2}(n_1,n_2)\right\|_{L^2}^2+\left\|\nabla^{k+1}u_1\right\|_{L^2}^2+\left\|\nabla^{k+1}E\right\|_{L^2}^2\right).\end{aligned}$$

Hence, we have that for $l=k+2$,

$$\begin{aligned}I_{31}+I_{33}\lesssim&(\|\nabla(n_1,n_2)\|_{H^2}+\|u_1\|_{H^{k+1}\cap H^3})\\ &\times\left(\left\|\nabla^{k+1}u_1\right\|_{L^2}^2+\left\|\nabla^{k+2}(n_1,n_2,u_1)\right\|_{L^2}^2+\left\|\nabla^{k+1}E\right\|_{L^2}^2\right)\\ &+\|(n_2,u_1)\|_{L^\infty}\left\|\nabla^{k+2}(n_2,u_1)\right\|_{L^2}\left\|\nabla^{k+2}E\right\|_{L^2}.\end{aligned}$$

Similarly, we can estimate $I_{32}+I_{34}$ for $l=k+2$. So, we have for $l=k+2$,

$$
\begin{aligned}
I_3 \lesssim & (\|\nabla(n_1,n_2)\|_{H^2}+\|(u_1,u_2)\|_{H^{k+1}\cap H^3}) \\
& \times\left(\sum_{l=k+1}^{k+2}\left\|\nabla^l(n_1,n_2,u_1,u_2)\right\|_{L^2}^2+\left\|\nabla^{k+1}E\right\|_{L^2}^2\right) \\
& +\|(n_1,n_2,u_1,u_2)\|_{L^\infty}\left\|\nabla^{k+2}(n_1,n_2,u_1,u_2)\right\|_{L^2}\left\|\nabla^{k+2}E\right\|_{L^2}.
\end{aligned}
$$

Consequently, plugging the estimates for I_1-I_3 into (2.7) with $l=k,k+1,k+2$, and then summing them up, we deduce (2.6). □

Note that in Lemma 2.8 we only derive the dissipative estimates of u_1 and u_2. We now recover the dissipative estimates of n_1, n_2, E and B by constructing some interactive energy functionals in the following lemma.

LEMMA 2.9. *For any integer $k\geqslant 0$ and some fixed positive constant λ, we have that for any small fixed $\eta>0$,*

$$
\begin{aligned}
&\frac{d}{dt}\left(\sum_{l=k}^{k+1}\int\nabla^l u_1\cdot\nabla\nabla^l n_1+\nabla^l u_2\cdot\nabla\nabla^l n_2-\sum_{l=k}^{k+1}\int\nabla^l u_2\cdot\nabla^l E-\eta\int\nabla^k E\cdot\nabla^k\nabla\times B\right) \\
&\quad+\lambda\left(\sum_{l=k+1}^{k+2}\left\|\nabla^l n_1\right\|_{L^2}^2+\sum_{l=k}^{k+2}\left\|\nabla^l n_2\right\|_{L^2}^2+\sum_{l=k}^{k+1}\left\|\nabla^l E\right\|_{L^2}^2+\left\|\nabla^{k+1}B\right\|_{L^2}^2\right) \\
&\lesssim\sum_{l=k}^{k+2}\left\|\nabla^l(u_1,u_2)\right\|_{L^2}^2+G\left(\sum_{l=k+1}^{k+2}\left\|\nabla^l n_1\right\|_{L^2}^2+\sum_{l=k}^{k+2}\left\|\nabla^l(n_2,u_1,u_2)\right\|_{L^2}^2+\left\|\nabla^{k+1}B\right\|_{L^2}^2\right),
\end{aligned}
\tag{2.25}
$$

where G is defined by

$$
G=G(n_1,n_2,u_1,u_2,B):=\|\nabla n_1\|_{H^2}^2+\|n_2\|_{H^3}^2+\|(u_1,u_2)\|_{H^{k+1}\cap H^3}^2+\|\nabla B\|_{H^1}^2.
$$

Proof. We divide the proof into four steps.

Step 1: Dissipative estimates of n_1,n_2.

Applying ∇^l $(l=k,k+1)$ to $(1.5)_2,(1.5)_4$ and then taking the L^2 inner product with $\nabla\nabla^l n_1,\nabla\nabla^l n_2$ respectively, we obtain

$$
\begin{aligned}
&\int\partial_t\nabla^l u_1\cdot\nabla\nabla^l n_1+\partial_t\nabla^l u_2\cdot\nabla\nabla^l n_2+\left\|\nabla\nabla^l(n_1,n_2)\right\|_{L^2}^2 \\
&\leqslant 2\nu\int\nabla^l E\cdot\nabla\nabla^l n_2+C\left\|\nabla^l(u_1,u_2)\right\|_{L^2}\left\|\nabla^{l+1}n_1\right\|_{L^2}+C\left\|\nabla^l(u_1,u_2)\right\|_{L^2}\left\|\nabla^{l+1}n_2\right\|_{L^2} \\
&\quad+\left\|\nabla^l(u_1\cdot\nabla u_2+u_2\cdot\nabla u_1+n_1\nabla n_2+n_2\nabla n_1+u_1\times B)\right\|_{L^2}\left\|\nabla^{l+1}n_2\right\|_{L^2} \\
&\quad+\left\|\nabla^l(u_1\cdot\nabla u_1+u_2\cdot\nabla u_2+n_1\nabla n_1+n_2\nabla n_2+u_2\times B)\right\|_{L^2}\left\|\nabla^{l+1}n_1\right\|_{L^2}.
\end{aligned}
\tag{2.26}
$$

The delicate first term on the left-hand side of (2.26) involves $\partial_t\nabla^l(u_1,u_2)$, and the key idea is to integrate by parts in the t-variable and use Equation $(1.5)_1$ and Equation $(1.5)_3$. Thus integrating by parts for both the t and x-variables, we obtain

$$\begin{aligned}
&\int \nabla^l \partial_t u_1 \cdot \nabla\nabla^l n_1 + \nabla^l \partial_t u_2 \cdot \nabla\nabla^l n_2 \\
&= \frac{d}{dt}\int \nabla^l u_1 \cdot \nabla\nabla^l n_1 + \nabla^l u_2 \cdot \nabla\nabla^l n_2 - \int \nabla^l u_1 \cdot \nabla\nabla^l \partial_t n_1 + \nabla^l u_2 \cdot \nabla\nabla^l \partial_t n_2 \\
&= \frac{d}{dt}\int \nabla^l u_1 \cdot \nabla\nabla^l n_1 + \nabla^l u_2 \cdot \nabla\nabla^l n_2 + \int \nabla^l \operatorname{div} u_1 \nabla^l \partial_t n_1 + \nabla^l \operatorname{div} u_2 \nabla^l \partial_t n_2 \\
&= \frac{d}{dt}\int \nabla^l u_1 \cdot \nabla\nabla^l n_1 + \nabla^l u_2 \cdot \nabla\nabla^l n_2 - \left\|\nabla^l \operatorname{div}(u_1,u_2)\right\|_{L^2}^2 \\
&\quad + \int \nabla^l \operatorname{div} u_1 \nabla^l g_1 + \nabla^l \operatorname{div} u_2 \nabla^l g_3 \\
&\geqslant \frac{d}{dt}\int \nabla^l u_1 \cdot \nabla\nabla^l n_1 + \nabla^l u_2 \cdot \nabla\nabla^l n_2 - C\left\|\nabla^{l+1}(u_1,u_2)\right\|_{L^2}^2 \\
&\quad - C\left\|\nabla^l(u_1\cdot\nabla n_1, u_2\cdot\nabla n_1, n_1 \operatorname{div} u_1, n_1 \operatorname{div} u_2)\right\|_{L^2}^2 \\
&\quad - C\left\|\nabla^l(u_1\cdot\nabla n_2, u_2\cdot\nabla n_2, n_2 \operatorname{div} u_2, n_2 \operatorname{div} u_1)\right\|_{L^2}^2 .
\end{aligned}$$

Using the product estimates (2.5), we obtain

$$\begin{aligned}
\left\|\nabla^l(u_1\cdot\nabla n_1)\right\|_{L^2} &\lesssim \|u_1\|_{L^\infty}\left\|\nabla^{l+1} n_1\right\|_{L^2} + \left\|\nabla^l u_1\right\|_{L^6}\|\nabla n_1\|_{L^3} \\
&\lesssim (\|\nabla n_1\|_{H^2} + \|u_1\|_{H^3})\left(\left\|\nabla^{l+1} n_1\right\|_{L^2} + \left\|\nabla^{l+1} u_1\right\|_{L^2}\right)
\end{aligned} \tag{2.27}$$

and

$$\begin{aligned}
\left\|\nabla^l(n_1 \operatorname{div} u_1)\right\|_{L^2} &\lesssim \|n_1\|_{L^\infty}\left\|\nabla^l \operatorname{div} u_1\right\|_{L^2} + \left\|\nabla^l n_1\right\|_{L^6}\|\operatorname{div} u_1\|_{L^3} \\
&\lesssim (\|\nabla n_1\|_{H^2} + \|u_1\|_{H^3})\left(\left\|\nabla^{l+1} n_1\right\|_{L^2} + \left\|\nabla^{l+1} u_1\right\|_{L^2}\right).
\end{aligned} \tag{2.28}$$

Similarly, we also obtain

$$\begin{aligned}
&\left\|\nabla^l(u_2\cdot\nabla n_1, n_1 \operatorname{div} u_2, u_1\cdot\nabla n_2, u_2\cdot\nabla n_2, n_2 \operatorname{div} u_2, n_2 \operatorname{div} u_1)\right\|_{L^2} \\
&\lesssim (\|\nabla n_1\|_{H^2} + \|(n_2,u_1,u_2)\|_{H^3})\left\|\nabla^{l+1}(n_1,n_2,u_1,u_2)\right\|_{L^2}.
\end{aligned} \tag{2.29}$$

Hence, we obtain

$$\begin{aligned}
&\int \nabla^l \partial_t u_1 \cdot \nabla\nabla^l n_1 + \nabla^l \partial_t u_2 \cdot \nabla\nabla^l n_2 \\
&\geqslant \frac{d}{dt}\int \nabla^l u_1 \cdot \nabla^l \nabla n_1 + \nabla^l u_2 \cdot \nabla^l \nabla n_2 - C\left\|\nabla^{l+1}(u_1,u_2)\right\|_{L^2}^2 \\
&\quad - C\left(\|\nabla n_1\|_{H^2}^2 + \|(n_2,u_1,u_2)\|_{H^3}^2\right)\left\|\nabla^{l+1}(n_1,n_2,u_1,u_2)\right\|_{L^2}^2 .
\end{aligned} \tag{2.30}$$

Next, integrating by parts and using Equation $(1.5)_7$, we have

$$\begin{aligned}
&2\nu\int \nabla^l E \cdot \nabla\nabla^l n_2 = -2\nu\int \nabla^l \operatorname{div} E \nabla^l n_2 \\
&= -2\nu^2 \int \nabla^l \left(f\left(\frac{n_1+n_2}{2}\right) - f\left(\frac{n_1-n_2}{2}\right)\right)\nabla^l n_2 \\
&= -2\nu^2 \int \nabla^l \left[n_2 + f\left(\frac{n_1+n_2}{2}\right) - f\left(\frac{n_1-n_2}{2}\right) - n_2 \right]\nabla^l n_2
\end{aligned}$$

$$\lesssim -\left\|\nabla^l n_2\right\|_{L^2}^2+(\|\nabla n_1\|_{H^2}+\|n_2\|_{H^3})\left(\left\|\nabla^{l+1}n_1\right\|_{L^2}^2+\left\|\nabla^l n_2\right\|_{L^2}^2\right). \tag{2.31}$$

Here we have used the estimate

$$\begin{aligned}&\left\|\nabla^l\left[f\left(\frac{n_1+n_2}{2}\right)-f\left(\frac{n_1-n_2}{2}\right)-n_2\right]\right\|_{L^2}\\ &\lesssim(\|\nabla n_1\|_{H^2}+\|n_2\|_{H^3})\left(\left\|\nabla^{l+1}n_1\right\|_{L^2}+\left\|\nabla^l n_2\right\|_{L^2}\right).\end{aligned}$$

In fact, by noticing that $f(\frac{n_1+n_2}{2})-f(\frac{n_1-n_2}{2})-n_2\sim n_1n_2$ and lemmas 2.2 and 2.3, we have

$$\begin{aligned}&\left\|\nabla^l\left[f\left(\frac{n_1+n_2}{2}\right)-f\left(\frac{n_1-n_2}{2}\right)-n_2\right]\right\|_{L^2}\lesssim\left\|\nabla^l(n_1n_2)\right\|_{L^2}\\ &\lesssim\|n_1\|_{L^\infty}\left\|\nabla^l n_2\right\|_{L^2}+\left\|\nabla^l n_1\right\|_{L^6}\|n_2\|_{L^3}\\ &\lesssim(\|\nabla n_1\|_{H^2}+\|n_2\|_{H^3})\left(\left\|\nabla^{l+1}n_1\right\|_{L^2}+\left\|\nabla^l n_2\right\|_{L^2}\right).\end{aligned}$$

As in (2.27)–(2.29), we have

$$\begin{aligned}&\left\|\nabla^l(u_1\cdot\nabla u_2+u_2\cdot\nabla u_1+n_1\nabla n_2+n_2\nabla n_1)\right\|_{L^2}\\ &+\left\|\nabla^l(u_1\cdot\nabla u_1+u_2\cdot\nabla u_2+n_1\nabla n_1+n_2\nabla n_2)\right\|_{L^2}\\ &\lesssim(\|\nabla n_1\|_{H^2}+\|(n_2,u_1,u_2)\|_{H^3})\left\|\nabla^{l+1}(n_1,n_2,u_1,u_2)\right\|_{L^2}.\end{aligned}\tag{2.32}$$

From the estimate of I_2 in Lemma 2.8, we have that for $l=k$ or $k+1$,

$$\left\|\nabla^l(u_1\times B,u_2\times B)\right\|_{L^2}\lesssim(\|(u_1,u_2)\|_{H^2}+\|\nabla B\|_{H^1})\left(\left\|\nabla^l(u_1,u_2)\right\|_{L^2}+\left\|\nabla^{k+1}B\right\|_{L^2}\right). \tag{2.33}$$

Plugging the estimates (2.30)–(2.33) into (2.26), by Cauchy's inequality, we obtain

$$\begin{aligned}&\frac{d}{dt}\sum_{l=k}^{k+1}\int\nabla^l u_1\cdot\nabla\nabla^l n_1+\nabla^l u_2\cdot\nabla\nabla^l n_2+\lambda\left(\sum_{l=k+1}^{k+2}\left\|\nabla^l n_1\right\|_{L^2}^2+\sum_{l=k}^{k+2}\left\|\nabla^l n_2\right\|_{L^2}^2\right)\\ &\lesssim\sum_{l=k}^{k+2}\left\|\nabla^l(u_1,u_2)\right\|_{L^2}^2+G\left(\sum_{l=k+1}^{k+2}\left\|\nabla^l n_1\right\|_{L^2}^2+\sum_{l=k}^{k+2}\left\|\nabla^l(n_2,u_1,u_2)\right\|_{L^2}^2+\left\|\nabla^{k+1}B\right\|_{L^2}^2\right).\end{aligned}\tag{2.34}$$

Here G is well-defined. This completes the dissipative estimates for n_1,n_2.

Step 2: Dissipative estimate of E.

Applying ∇^l $(l=k,k+1)$ to $(1.5)_4$ and then taking the L^2 inner product with $-\nabla^l E$, we obtain

$$\begin{aligned}-\int\nabla^l\partial_t u_2\cdot\nabla^l E+2\nu\left\|\nabla^l E\right\|_{L^2}^2&\leqslant\int\nabla\nabla^l n_2\cdot\nabla^l E+C\left\|\nabla^l(u_1,u_2)\right\|_{L^2}\left\|\nabla^l E\right\|_{L^2}\\ &+\left\|\nabla^l(u_1\cdot\nabla u_2+u_2\cdot\nabla u_1+n_1\nabla n_2+n_2\nabla n_1+u_1\times B)\right\|_{L^2}\left\|\nabla^l E\right\|_{L^2}.\end{aligned}\tag{2.35}$$

Again, the delicate first term on the left-hand side of (2.35) involves $\partial_t\nabla^l u_2$, and the key idea is to integrate by parts in the t-variable and use Equation $(1.5)_5$ in the Maxwell system. Thus we obtain

$$-\int\nabla^l\partial_t u_2\cdot\nabla^l E=-\frac{d}{dt}\int\nabla^l u_2\cdot\nabla^l E+\int\nabla^l u_2\cdot\nabla^l\partial_t E$$

$$=-\frac{d}{dt}\int\nabla^l u_2\cdot\nabla^l E-\nu\left\|\nabla^l u_2\right\|_{L^2}^2+\int\nabla^l u_2\cdot\nabla^l(g_5+\nu\nabla\times B).\tag{2.36}$$

From the estimates of I_3 in Lemma 2.8, we have that

$$\left\|\nabla^l g_5\right\|_{L^2}\lesssim(\|\nabla(n_1,n_2)\|_{L^2}+\|(u_1,u_2)\|_{H^l\cap H^1})\left(\left\|\nabla^{l+1}(n_1,n_2)\right\|_{L^2}+\left\|\nabla^l(u_1,u_2)\right\|_{L^2}\right).\tag{2.37}$$

We must be much more careful with the remaining term in (2.36) since there is no small factor in front of it. The key is to use Cauchy's inequality and distinguish the cases of $l=k$ and $l=k+1$ due to the weakest dissipative estimate of B. For $l=k$, we have

$$\nu\int\nabla^k u_2\cdot\nabla\times\nabla^k B\leqslant\varepsilon\left\|\nabla^{k+1}B\right\|_{L^2}^2+C_\varepsilon\left\|\nabla^k u_2\right\|_{L^2}^2;\tag{2.38}$$

for $l=k+1$, integrating by parts, we obtain

$$\begin{aligned}\nu\int\nabla^{k+1}u_2\cdot\nabla\times\nabla^{k+1}B&=\nu\int\nabla\times\nabla^{k+1}u_2\cdot\nabla^{k+1}B\\&\leqslant\varepsilon\left\|\nabla^{k+1}B\right\|_{L^2}^2+C_\varepsilon\left\|\nabla^{k+2}u_2\right\|_{L^2}^2.\end{aligned}\tag{2.39}$$

Plugging the estimates (2.36)–(2.39) and (2.31)–(2.33) from Step 1 into (2.35), by Cauchy's inequality, we then obtain

$$\begin{aligned}-\frac{d}{dt}\sum_{l=k}^{k+1}\int\nabla^l u_2\cdot\nabla^l E+\lambda\sum_{l=k}^{k+1}\left\|\nabla^l E\right\|_{L^2}^2&\leqslant\varepsilon\left\|\nabla^{k+1}B\right\|_{L^2}^2+C_\varepsilon\sum_{l=k}^{k+2}\left\|\nabla^l(u_1,u_2)\right\|_{L^2}^2\\&+CG\left(\sum_{l=k+1}^{k+2}\left\|\nabla^l n_1\right\|_{L^2}^2+\sum_{l=k}^{k+2}\left\|\nabla^l(n_2,u_1,u_2)\right\|_{L^2}^2+\left\|\nabla^{k+1}B\right\|_{L^2}^2\right).\end{aligned}\tag{2.40}$$

This completes the dissipative estimate for E.

Step 3: Dissipative estimate of B.

Applying ∇^k to $(1.5)_5$ and then taking the L^2 inner product with $-\nabla\times\nabla^k B$, we obtain

$$\begin{aligned}&-\int\nabla^k\partial_t E\cdot\nabla\times\nabla^k B+\nu\left\|\nabla\times\nabla^k B\right\|_{L^2}^2\\&\leqslant\nu\left\|\nabla^k u_2\right\|_{L^2}\left\|\nabla\times\nabla^k B\right\|_{L^2}+\left\|\nabla^k g_5\right\|_{L^2}\left\|\nabla\times\nabla^k B\right\|_{L^2}.\end{aligned}\tag{2.41}$$

Integrating by parts for both the t and x-variables and using Equation $(1.5)_6$, we have

$$\begin{aligned}-\int\nabla^k\partial_t E\cdot\nabla\times\nabla^k B&=-\frac{d}{dt}\int\nabla^k E\cdot\nabla\times\nabla^k B+\int\nabla\times\nabla^k E\cdot\nabla^k\partial_t B\\&=-\frac{d}{dt}\int\nabla^k E\cdot\nabla\times\nabla^k B-\nu\left\|\nabla\times\nabla^k E\right\|_{L^2}^2.\end{aligned}$$

From the estimate of I_3 in Lemma 2.8, we have that

$$\left\|\nabla^k g_5\right\|_{L^2}\lesssim(\|\nabla(n_1,n_2)\|_{L^2}+\|(u_1,u_2)\|_{H^k\cap H^1})\left(\left\|\nabla^{k+1}(n_1,n_2)\right\|_{L^2}+\left\|\nabla^k(u_1,u_2)\right\|_{L^2}\right).$$

Plugging the estimates above into (2.41) and by Cauchy's inequality, since $\operatorname{div}B=0$, we then obtain

$$\begin{aligned}&-\frac{d}{dt}\int\nabla^k E\cdot\nabla^k\nabla\times B+\lambda\left\|\nabla^{k+1}B\right\|_{L^2}^2\lesssim\left\|\nabla^k u_2\right\|_{L^2}^2+\left\|\nabla^{k+1}E\right\|_{L^2}^2\\&+\left(\|\nabla(n_1,n_2)\|_{L^2}^2+\|(u_1,u_2)\|_{H^k\cap H^1}^2\right)\left(\left\|\nabla^{k+1}(n_1,n_2)\right\|_{L^2}^2+\left\|\nabla^k(u_1,u_2)\right\|_{L^2}^2\right).\end{aligned}\tag{2.42}$$

This completes the dissipative estimate for B.

Step 4: Conclusion.

Multiplying (2.42) by a small enough but fixed constant η and then adding it to (2.40) so that the second term on the right-hand side of (2.42) can be absorbed, then choosing ε small enough so that the first term on the right-hand side of (2.40) can be absorbed, we obtain

$$\frac{d}{dt}\left(\sum_{l=k}^{k+1}\int\nabla^l u_2\cdot\nabla^l E-\eta\int\nabla^k E\cdot\nabla^k\nabla\times B\right)+\lambda\left(\sum_{l=k}^{k+1}\left\|\nabla^l E\right\|_{L^2}^2+\left\|\nabla^{k+1}B\right\|_{L^2}^2\right)$$
$$\lesssim\sum_{l=k}^{k+2}\left\|\nabla^l u_2\right\|_{L^2}^2+G\left(\sum_{l=k+1}^{k+2}\left\|\nabla^l n_1\right\|_{L^2}^2+\sum_{l=k}^{k+2}\left\|\nabla^l(n_2,u_1,u_2)\right\|_{L^2}^2+\left\|\nabla^{k+1}B\right\|_{L^2}^2\right).$$

Adding the inequality above to (2.34), we get (2.25). □

2.3. Negative Sobolev estimates. In this subsection, we will derive the evolution of the negative Sobolev norms of $U:=(n_1,n_2,u_1,u_2,E,B)$. In order to estimate the nonlinear terms, we need to restrict $s\in(0,3/2)$. We will establish the following lemma.

LEMMA 2.10. *For $s\in(0,1/2]$, we have*

$$\frac{d}{dt}\|U\|_{\dot H^{-s}}^2+C\|(u_1,u_2)\|_{\dot H^{-s}}^2\lesssim\left(\|(n_2,u_1,u_2)\|_{H^2}^2+\|\nabla(n_1,B)\|_{H^1}^2\right)\|U\|_{\dot H^{-s}},\tag{2.43}$$

and for $s\in(1/2,3/2)$, we have

$$\frac{d}{dt}\|U\|_{\dot H^{-s}}^2+C\|(u_1,u_2)\|_{\dot H^{-s}}^2\lesssim\|(\nabla n_1,n_2,u_1,u_2)\|_{H^1}^2\|U\|_{\dot H^{-s}}$$
$$+\|(n_1,B)\|_{L^2}^{s-1/2}\|\nabla(n_1,B)\|_{L^2}^{3/2-s}\|(\nabla n_1,\nabla n_2,u_1,u_2,\nabla u_1,\nabla u_2)\|_{L^2}\|U\|_{\dot H^{-s}}.\tag{2.44}$$

Proof. The Λ^{-s} $(s>0)$ energy estimates of $(1.5)_1$–$(1.5)_6$ yield

$$\frac{d}{dt}\left(\frac12\|(n_1,n_2,u_1,u_2)\|_{\dot H^{-s}}^2+\|(E,B)\|_{\dot H^{-s}}^2\right)+\nu\|(u_1,u_2)\|_{\dot H^{-s}}^2$$
$$=\int\Lambda^{-s}g_1\cdot\Lambda^{-s}n_1+\int\Lambda^{-s}(g_2+u_2\times B)\cdot\Lambda^{-s}u_1$$
$$+\int\Lambda^{-s}g_3\cdot\Lambda^{-s}n_2+\int\Lambda^{-s}(g_4+u_1\times B)\cdot\Lambda^{-s}u_2+2\int\Lambda^{-s}g_5\cdot\Lambda^{-s}E$$
$$\lesssim\|g_1\|_{\dot H^{-s}}\|n_1\|_{\dot H^{-s}}+\|g_2+u_2\times B\|_{\dot H^{-s}}\|u_1\|_{\dot H^{-s}}$$
$$+\|g_3\|_{\dot H^{-s}}\|n_2\|_{\dot H^{-s}}+\|g_4+u_1\times B\|_{\dot H^{-s}}\|u_2\|_{\dot H^{-s}}+\|g_5\|_{\dot H^{-s}}\|E\|_{\dot H^{-s}}.\tag{2.45}$$

We now restrict the value of s in order to estimate the nonlinear terms on the right-hand side of (2.45). If $s\in(0,1/2]$, then $1/2+s/3<1$ and $3/s\geqslant6$. Then applying Lemma 2.4, together with Hölder's, Sobolev's, and Young's inequalities, we obtain

$$\|u_1\cdot\nabla u_2\|_{\dot H^{-s}}\lesssim\|u_1\cdot\nabla u_2\|_{L^{\frac{1}{1/2+s/3}}}\lesssim\|u_1\|_{L^{3/s}}\|\nabla u_2\|_{L^2}$$
$$\lesssim\|\nabla u_1\|_{L^2}^{1/2+s}\left\|\nabla^2 u_1\right\|_{L^2}^{1/2-s}\|\nabla u_2\|_{L^2}\lesssim\|\nabla u_1\|_{H^1}^2+\|\nabla u_2\|_{L^2}^2.$$

We can similarly bound the other terms in the g_1-g_5 and $(u_1+u_2)\times B$. So we have

$$\sum_{i=1}^{5}\|g_i\|_{\dot H^{-s}}+\|(u_1+u_2)\times B\|_{\dot H^{-s}}\lesssim\|(n_2,u_1,u_2)\|_{H^2}^2+\|\nabla(n_1,B)\|_{H^1}^2.\tag{2.46}$$

Now if $s\in(1/2,3/2)$, then we will estimate the right-hand side of (2.45) in a different way. Since $s\in(1/2,3/2)$, we have that $1/2+s/3<1$ and $2<3/s<6$. Then applying Lemma 2.4 and using (a different) Sobolev's inequality, we have

$$\begin{aligned}&\|u_1\cdot\nabla u_2\|_{\dot H^{-s}}\lesssim\|u_1\|_{L^{3/s}}\|\nabla u_2\|_{L^2}\\&\lesssim\|u_1\|_{L^2}^{s-1/2}\|\nabla u_1\|_{L^2}^{3/2-s}\|\nabla u_2\|_{L^2}\lesssim\|u_1\|_{H^1}^2+\|\nabla u_2\|_{L^2}^2.\end{aligned}$$

In particular, we must be careful with the terms involving n_1 and B since they are both degenerately dissipative. For example,

$$\begin{aligned}&\|n_1\nabla n_2\|_{\dot H^{-s}}\lesssim\|n_1\|_{L^2}^{s-1/2}\|\nabla n_1\|_{L^2}^{3/2-s}\|\nabla n_2\|_{L^2},\\&\|u_2\times B\|_{\dot H^{-s}}\lesssim\|B\|_{L^2}^{s-1/2}\|\nabla B\|_{L^2}^{3/2-s}\|u_2\|_{L^2}.\end{aligned}$$

Then, we have

$$\begin{aligned}\sum_{i=1}^5\|g_i\|_{\dot H^{-s}}+\|(u_1+u_2)\times B\|_{\dot H^{-s}}&\lesssim\|(\nabla n_1,n_2,u_1,u_2)\|_{H^1}^2\\&+\|(n_1,B)\|_{L^2}^{s-1/2}\|\nabla(n_1,B)\|_{L^2}^{3/2-s}\|(\nabla n_1,\nabla n_2,u_1,u_2,\nabla u_1,\nabla u_2)\|_{L^2}.\end{aligned}\tag{2.47}$$

Hence, we deduce (2.43) from (2.46) and (2.44) from (2.47). ∎

2.4. Negative Besov estimates. In this subsection, we will derive the evolution of the negative Besov norms of $U:=(n_1,n_2,u_1,u_2,E,B)$. The argument is similar to the previous subsection.

LEMMA 2.11. *For $s\in(0,1/2]$, we have*

$$\frac{d}{dt}\|U\|_{\dot B^{-s}_{2,\infty}}^2+C\|(u_1,u_2)\|_{\dot B^{-s}_{2,\infty}}^2\lesssim\left(\|(n_2,u_1,u_2)\|_{H^2}^2+\|\nabla(n_1,B)\|_{H^1}^2\right)\|U\|_{\dot B^{-s}_{2,\infty}},$$

and for $s\in(1/2,3/2]$, we have

$$\begin{aligned}&\frac{d}{dt}\|U\|_{\dot B^{-s}_{2,\infty}}^2+C\|(u_1,u_2)\|_{\dot B^{-s}_{2,\infty}}^2\lesssim\|(\nabla n_1,n_2,u_1,u_2)\|_{H^1}^2\|U\|_{\dot B^{-s}_{2,\infty}}\\&\quad+\|(n_1,B)\|_{L^2}^{s-1/2}\|\nabla(n_1,B)\|_{L^2}^{3/2-s}\|(\nabla n_1,\nabla n_2,u_1,u_2,\nabla u_1,\nabla u_2)\|_{L^2}\|U\|_{\dot B^{-s}_{2,\infty}}.\end{aligned}$$

Proof. The $\dot\Delta_j$ energy estimates of $(1.5)_1$–$(1.5)_6$ yield, with multiplication of 2^{-2sj} and then taking the supremum over $j\in\mathbb{Z}$,

$$\begin{aligned}&\frac{d}{dt}\left(\frac12\|(n_1,n_2,u_1,u_2)\|_{\dot B^{-s}_{2,\infty}}^2+\|(E,B)\|_{\dot B^{-s}_{2,\infty}}^2\right)+\nu\|(u_1,u_2)\|_{\dot B^{-s}_{2,\infty}}^2\\&\lesssim\sup_{j\in\mathbb{Z}}2^{-2sj}\int\dot\Delta_jg_1\cdot\dot\Delta_jn_1+\dot\Delta_j(g_2+u_2\times B)\cdot\dot\Delta_ju_1\\&\quad+\sup_{j\in\mathbb{Z}}2^{-2sj}\int\dot\Delta_jg_3\cdot\dot\Delta_jn_2+\dot\Delta_j(g_4+u_1\times B)\cdot\dot\Delta_ju_2+2\dot\Delta_jg_5\cdot\dot\Delta_jE\\&\lesssim\|g_1\|_{\dot B^{-s}_{2,\infty}}\|n_1\|_{\dot B^{-s}_{2,\infty}}+\|g_2+u_2\times B\|_{\dot B^{-s}_{2,\infty}}\|u_1\|_{\dot B^{-s}_{2,\infty}}\\&\quad+\|g_3\|_{\dot B^{-s}_{2,\infty}}\|n_2\|_{\dot B^{-s}_{2,\infty}}+\|g_4+u_1\times B\|_{\dot B^{-s}_{2,\infty}}\|u_2\|_{\dot B^{-s}_{2,\infty}}+\|g_5\|_{\dot B^{-s}_{2,\infty}}\|E\|_{\dot B^{-s}_{2,\infty}}.\end{aligned}$$

Then the proof is exactly the same as the proof of Lemma 2.10 except that we should apply Lemma 2.5 instead to estimate the $\dot B^{-s}_{2,\infty}$ norm. Note that we allow $s=3/2$. ∎

3. Proof of theorems

3.1. Proof of Theorem 1.1. In this subsection, we will prove the unique global solution to the system (1.5), and the key point is that we only assume that the H^3 norms of initial data are small.

Step 1. Global small $\mathcal{E}_3$ solution.

We first close the energy estimates at the H^3 level by assuming a priori that $\sqrt{\mathcal{E}_3(t)} \leqslant \delta$ is sufficiently small. Taking $k=0,1$ in (2.6) of Lemma 2.8 and then summing them up, we obtain

$$\frac{d}{dt}\sum_{l=0}^{3}\left\|\nabla^l U\right\|_{L^2}^2+\lambda\sum_{l=0}^{3}\left\|\nabla^l(u_1,u_2)\right\|_{L^2}^2 \lesssim \sqrt{\mathcal{E}_3}\mathcal{D}_3+\sqrt{\mathcal{D}_3}\sqrt{\mathcal{D}_3}\sqrt{\mathcal{E}_3} \lesssim \delta\mathcal{D}_3. \tag{3.1}$$

Taking $k=0,1$ in (2.25) of Lemma 2.9 and then summing them up, we obtain

$$\begin{aligned}&\frac{d}{dt}\left(\sum_{l=0}^{2}\int\nabla^l u_1\cdot\nabla\nabla^l n_1+\nabla^l u_2\cdot\nabla\nabla^l n_2+\sum_{l=0}^{2}\int\nabla^l u_2\cdot\nabla^l E-\eta\sum_{l=0}^{1}\int\nabla^l E\cdot\nabla^l\nabla\times B\right)\\&\quad+\lambda\left(\sum_{l=1}^{3}\left\|\nabla^l n_1\right\|_{L^2}^2+\sum_{l=0}^{3}\left\|\nabla^l n_2\right\|_{L^2}^2+\sum_{l=0}^{2}\left\|\nabla^l E\right\|_{L^2}^2+\sum_{l=1}^{2}\left\|\nabla^l B\right\|_{L^2}^2\right)\\&\lesssim\sum_{l=0}^{3}\left\|\nabla^l(u_1,u_2)\right\|_{L^2}^2+\delta^2\mathcal{D}_3.\end{aligned} \tag{3.2}$$

Since δ is small, we deduce from $(3.2)\times\varepsilon+(3.1)$ that there exists an instant energy functional $\widetilde{\mathcal{E}}_3$ equivalent to $\mathcal{E}_3$ such that

$$\frac{d}{dt}\widetilde{\mathcal{E}}_3+\mathcal{D}_3\leqslant 0.$$

Integrating the inequality above directly in time, we obtain (1.7). By a standard continuity argument, we then close the a priori estimates if we assume at initial time that $\mathcal{E}_3(0)\leqslant\delta_0$ is sufficiently small. This concludes the unique global small $\mathcal{E}_3$ solution.

Step 2. Global $\mathcal{E}_N$ solution.

We will prove this by an induction on $N\geqslant 3$. By (1.7), (1.8) is valid for $N=3$. Assume (1.8) holds for $N-1$ (then $N\geqslant 4$). Taking $k=0,\dots,N-2$ in (2.6) of Lemma 2.8 and then summing them up, we obtain

$$\frac{d}{dt}\sum_{l=0}^{N}\left\|\nabla^l U\right\|_{L^2}^2+\lambda\sum_{l=0}^{N}\left\|\nabla^l(u_1,u_2)\right\|_{L^2}^2\lesssim\sqrt{\mathcal{D}_{N-1}}\sqrt{\mathcal{D}_N}\sqrt{\mathcal{E}_N}. \tag{3.3}$$

Here we have used the fact that $3\leqslant N-2+1=N-1$ since $N\geqslant 4$. Note that it is important that we have put the two first factors in (2.6) into the dissipation.

Taking $k=0,\dots,N-2$ in (2.25) of Lemma 2.9 and then summing them up, we obtain

$$\begin{aligned}\frac{d}{dt}&\left(\sum_{l=0}^{N-1}\int\nabla^l u_1\cdot\nabla\nabla^l n_1+\nabla^l u_2\cdot\nabla\nabla^l n_2+\sum_{l=0}^{N-1}\int\nabla^l u_2\cdot\nabla^l E\right.\\&\left.\quad-\eta\sum_{l=0}^{N-2}\int\nabla^l E\cdot\nabla\times\nabla^l B\right)\end{aligned}$$

$$+\lambda\left(\sum_{l=1}^{N}\left\|\nabla^l n_1\right\|_{L^2}^2+\sum_{l=0}^{N}\left\|\nabla^l n_2\right\|_{L^2}^2+\sum_{l=0}^{N-1}\left\|\nabla^l E\right\|_{L^2}^2+\sum_{l=1}^{N-1}\left\|\nabla^l B\right\|_{L^2}^2\right)$$
$$\lesssim\sum_{l=0}^{N}\left\|\nabla^l(u_1,u_2)\right\|_{L^2}^2+\sqrt{\mathcal{D}_{N-1}}\sqrt{\mathcal{D}_N}\sqrt{\mathcal{E}_N}. \tag{3.4}$$

We deduce from (3.4) $\times\varepsilon+$(3.3) that there exists an instant energy functional $\widetilde{\mathcal{E}}_N$ equivalent to $\mathcal{E}_N$ such that, by Cauchy's inequality,

$$\frac{d}{dt}\widetilde{\mathcal{E}}_N+\mathcal{D}_N\lesssim\sqrt{\mathcal{D}_N}\sqrt{\mathcal{D}_{N-1}}\sqrt{\mathcal{E}_N}\lesssim\tilde{\varepsilon}\mathcal{D}_N+C_{\tilde{\varepsilon}}\mathcal{D}_{N-1}\mathcal{E}_N.$$

This implies

$$\frac{d}{dt}\widetilde{\mathcal{E}}_N+\frac{1}{2}\mathcal{D}_N\lesssim\mathcal{D}_{N-1}\mathcal{E}_N.$$

We then use the standard Gronwall lemma and the induction hypothesis to deduce that

$$\begin{aligned}&\mathcal{E}_N(t)+\int_0^t\mathcal{D}_N(\tau)\,d\tau\lesssim\mathcal{E}_N(0)e^{\int_0^t\mathcal{D}_{N-1}(\tau)\,d\tau}\\&\lesssim\mathcal{E}_N(0)e^{P_{N-1}(\mathcal{E}_{N-1}(0))}\lesssim\mathcal{E}_N(0)e^{P_{N-1}(\mathcal{E}_N(0))}\equiv P_N(\mathcal{E}_N(0)).\end{aligned}$$

This concludes the global $\mathcal{E}_N$ solution. The proof of Theorem 1.1 is completed. □

3.2. Proof of Theorem 1.2. In this subsection, we will prove the various time decay rates of the unique global solution to the system (1.5) obtained in Theorem 1.1. Fix $N\geqslant 4$. We need to assume that $\mathcal{E}_N(0)\leqslant\delta_0=\delta_0(N)$ is small. Then Theorem 1.1 implies that there exists a unique global $\mathcal{E}_N$ solution, and $\mathcal{E}_N(t)\leqslant P_N(\mathcal{E}_N(0))\leqslant\delta_0$ is small for all time t.

Step 1. Basic decay.

For convenience of presentation, we define a family of energy functionals and the corresponding dissipation rates with *minimum derivative counts* as

$$\mathcal{E}_k^{k+2}=\sum_{l=k}^{k+2}\left\|\nabla^l U\right\|_{L^2}^2 \tag{3.5}$$

and

$$\mathcal{D}_k^{k+2}=\sum_{l=k+1}^{k+2}\left\|\nabla^l n_1\right\|_{L^2}^2+\sum_{l=k}^{k+2}\left\|\nabla^l(n_2,u_1,u_2)\right\|_{L^2}^2+\sum_{l=k}^{k+1}\left\|\nabla^l E\right\|_{L^2}^2+\left\|\nabla^{k+1}B\right\|_{L^2}^2. \tag{3.6}$$

By Lemma 2.8, we have that for $k=0,\dots,N-2$,

$$\frac{d}{dt}\sum_{l=k}^{k+2}\left\|\nabla^l U\right\|_{L^2}^2+\lambda\sum_{l=k}^{k+2}\left\|\nabla^l(u_1,u_2)\right\|_{L^2}^2$$
$$\lesssim\sqrt{\delta_0}\mathcal{D}_k^{k+2}+\|(n_1,n_2,u_1,u_2)\|_{L^\infty}\left\|\nabla^{k+2}(n_1,n_2,u_1,u_2)\right\|_{L^2}\left\|\nabla^{k+2}(E,B)\right\|_{L^2}. \tag{3.7}$$

By Lemma 2.9, we have that for $k=0,\dots,N-2$,

$$\frac{d}{dt}\left(\sum_{l=k}^{k+1}\int\nabla^l u_1\cdot\nabla\nabla^l n_1+\nabla^l u_2\cdot\nabla\nabla^l n_2+\sum_{l=k}^{k+1}\int\nabla^l u_2\cdot\nabla^l E-\eta\int\nabla^k E\cdot\nabla^k\nabla\times B\right)$$

$$+\lambda\left(\sum_{l=k+1}^{k+2}\left\|\nabla^l n_1\right\|_{L^2}^2+\sum_{l=k}^{k+2}\left\|\nabla^l n_2\right\|_{L^2}^2+\sum_{l=k}^{k+1}\left\|\nabla^l E\right\|_{L^2}^2+\left\|\nabla^{k+1}B\right\|_{L^2}^2\right)$$
$$\lesssim\sum_{l=k}^{k+2}\left\|\nabla^l(u_1,u_2)\right\|_{L^2}^2. \tag{3.8}$$

Since δ_0 is small, we deduce from $(3.8)\times\varepsilon+(3.7)$ that there exists an instant energy functional $\widetilde{\mathcal{E}}_k^{k+2}$ equivalent to $\mathcal{E}_k^{k+2}$ such that

$$\frac{d}{dt}\widetilde{\mathcal{E}}_k^{k+2}+\mathcal{D}_k^{k+2}\lesssim\|(n_1,n_2,u_1,u_2)\|_{L^\infty}\left\|\nabla^{k+2}(n_1,n_2,u_1,u_2)\right\|_{L^2}\left\|\nabla^{k+2}(E,B)\right\|_{L^2}. \tag{3.9}$$

Note that we cannot absorb the right-hand side of (3.9) by the dissipation $\mathcal{D}_k^{k+2}$ since it does not contain $\left\|\nabla^{k+2}(E,B)\right\|_{L^2}^2$. We will distinguish the arguments by the value of k. If $k=0$, then we bound $\left\|\nabla^{k+2}(E,B)\right\|_{L^2}$ by the energy. Thus we have that for $k=0$,

$$\frac{d}{dt}\widetilde{\mathcal{E}}_k^{k+2}+\mathcal{D}_k^{k+2}\lesssim\sqrt{\mathcal{D}_k^{k+2}}\sqrt{\mathcal{D}_k^{k+2}}\sqrt{\mathcal{E}_3}\lesssim\sqrt{\delta_0}\mathcal{D}_k^{k+2},$$

which implies

$$\frac{d}{dt}\widetilde{\mathcal{E}}_k^{k+2}+\mathcal{D}_k^{k+2}\leqslant 0.$$

If $k\geqslant 1$, then we have to bound $\left\|\nabla^{k+2}(E,B)\right\|_{L^2}$ in terms of $\left\|\nabla^{k+1}(E,B)\right\|_{L^2}$ since $\sqrt{\mathcal{D}_k^{k+2}}$ cannot control $\|(n_1,n_2,u_1,u_2)\|_{L^\infty}$. The key point is to use the regularity interpolation method developed in [25]. By Lemma 2.1, we have

$$\begin{aligned}&\|(n_1,n_2,u_1,u_2)\|_{L^\infty}\left\|\nabla^{k+2}(n_1,n_2,u_1,u_2)\right\|_{L^2}\left\|\nabla^{k+2}(E,B)\right\|_{L^2}\\&\lesssim\left\|\nabla^{k+2}(n_1,n_2,u_1,u_2)\right\|_{L^2}\|(n_1,n_2,u_1,u_2)\|_{L^2}^{\frac{2k-1}{2k+2}}\left\|\nabla^{k+1}(n_1,n_2,u_1,u_2)\right\|_{L^2}^{\frac{3}{2k+2}}\\&\quad\times\left\|\nabla^{k+1}(E,B)\right\|_{L^2}^{\frac{2k-1}{2k+2}}\|\nabla^\alpha(E,B)\|_{L^2}^{\frac{3}{2k+2}},\end{aligned} \tag{3.10}$$

where α is defined by

$$k+2=(k+1)\times\frac{2k-1}{2k+2}+\alpha\times\frac{3}{2k+2}\Longrightarrow\alpha=\frac{5}{3}k+\frac{5}{3}.$$

Hence, for $k\geqslant 1$, if $N\geqslant\frac{5}{3}k+\frac{5}{3}\Longleftrightarrow 1\leqslant k\leqslant\frac{3}{5}N-1$, then by (3.10), we deduce from (3.9) that

$$\frac{d}{dt}\widetilde{\mathcal{E}}_k^{k+2}+\mathcal{D}_k^{k+2}\lesssim\sqrt{\mathcal{E}_N}\mathcal{D}_k^{k+2}\lesssim\sqrt{\delta_0}\mathcal{D}_k^{k+2},$$

which allows us to deduce that for any integer k with $0\leqslant k\leqslant\frac{3}{5}N-1$ (note that $N-2\geqslant\frac{3}{5}N-1\geqslant 1$ since $N\geqslant 4$), we have

$$\frac{d}{dt}\widetilde{\mathcal{E}}_k^{k+2}+\mathcal{D}_k^{k+2}\leqslant 0. \tag{3.11}$$

We now begin to derive the decay rate from (3.11). In fact, we have proved (1.9) in a similar fashion to [26] by utilizing lemmas 2.10–2.11. Using Lemma 2.6, we have that for $s\geqslant 0$ and $k+s>0$,

$$\left\|\nabla^k(n_1,B)\right\|_{L^2}\leqslant\|(n_1,B)\|_{\dot{H}^{-s}}^{\frac{1}{k+1+s}}\left\|\nabla^{k+1}(n_1,B)\right\|_{L^2}^{\frac{k+s}{k+1+s}}\leqslant C_0\left\|\nabla^{k+1}(n_1,B)\right\|_{L^2}^{\frac{k+s}{k+1+s}}.$$

Similarly, using Lemma 2.7, we have that for $s>0$ and $k+s>0$,

$$\left\|\nabla^k(n_1,B)\right\|_{L^2}\leqslant\|(n_1,B)\|_{\dot B^{-s}_{2,\infty}}^{\frac{1}{k+1+s}}\left\|\nabla^{k+1}(n_1,B)\right\|_{L^2}^{\frac{k+s}{k+1+s}}\leqslant C_0\left\|\nabla^{k+1}(n_1,B)\right\|_{L^2}^{\frac{k+s}{k+1+s}}.$$

On the other hand, for $k+2<N$, we have

$$\left\|\nabla^{k+2}(E,B)\right\|_{L^2}\leqslant\left\|\nabla^{k+1}(E,B)\right\|_{L^2}^{\frac{N-k-2}{N-k-1}}\left\|\nabla^{N}(E,B)\right\|_{L^2}^{\frac{1}{N-k-1}}\leqslant C_0\left\|\nabla^{k+1}(E,B)\right\|_{L^2}^{\frac{N-k-2}{N-k-1}}.$$

Then we deduce from (3.11) that

$$\frac{d}{dt}\widetilde{\mathcal{E}}_k^{k+2}+\left\{\mathcal{E}_k^{k+2}\right\}^{1+\vartheta}\leqslant 0,$$

where $\vartheta=\max\left\{\frac{1}{k+s},\frac{1}{N-k-2}\right\}$. Solving this inequality directly, we obtain in particular that

$$\begin{aligned}\mathcal{E}_k^{k+2}(t)&\leqslant\left\{\left[\mathcal{E}_k^{k+2}(0)\right]^{-\vartheta}+\vartheta t\right\}^{-1/\vartheta}\\&\leqslant C_0(1+t)^{-1/\vartheta}=C_0(1+t)^{-\min\{k+s,N-k-2\}}.\end{aligned}\tag{3.12}$$

Notice that (3.12) holds also for $k+s=0$ or $k+2=N$. So, if we want to obtain the optimal decay rate of the whole solution for the spatial derivatives of order k, then we only need to assume N large enough (for fixed k and s) so that $k+s\leqslant N-k-2$. Thus we should require that

$$N\geqslant\max\left\{k+2,\frac{5}{3}k+\frac{5}{3},2k+2+s\right\}=2k+2+s.$$

This proves the optimal decay (1.10).

Step 2. Higher decay.

We first prove (1.11) and (1.12). First, noticing that $-\nu g=\operatorname{div}E$, by (1.10) and Lemma 2.2, if $N\geqslant 2k+4+s$, then

$$\left\|\nabla^k n_2(t)\right\|_{L^2}\lesssim\left\|\nabla^k g(t)\right\|_{L^2}\lesssim\left\|\nabla^{k+1}E(t)\right\|_{L^2}\leqslant C_0(1+t)^{-\frac{k+1+s}{2}}.\tag{3.13}$$

Next, applying ∇^k to $(1.5)_2,(1.5)_4,(1.5)_5$ and then taking the L^2 inner product with $\nabla^k u_1$, $\nabla^k u_2$, $\nabla^k E$ respectively, we obtain

$$\begin{aligned}&\frac{d}{dt}\int\left(\frac12\left|\nabla^k(u_1,u_2)\right|^2+\left|\nabla^k E\right|^2\right)+\nu\left\|\nabla^k(u_1,u_2)\right\|_{L^2}^2\\&=\int\nabla^k(-\nabla n_1+g_2+u_2\times B)\cdot\nabla^k u_1+\int\nabla^k(-\nabla n_2+g_4+u_1\times B)\cdot\nabla^k u_2\\&\qquad+2\nu\int\nabla^k(\nabla\times B+g_5)\cdot\nabla^k E\\&\lesssim\left\|\nabla^{k+1}n_1\right\|_{L^2}\left\|\nabla^k u_1\right\|_{L^2}+\left\|\nabla^k(g_2+u_2\times B)\right\|_{L^2}\left\|\nabla^k u_1\right\|_{L^2}\\&\qquad+\left\|\nabla^{k+1}n_2\right\|_{L^2}\left\|\nabla^k u_2\right\|_{L^2}+\left\|\nabla^k(g_4+u_1\times B)\right\|_{L^2}\left\|\nabla^k u_2\right\|_{L^2}\\&\qquad+\left\|\nabla^k(\nabla\times B+g_5)\right\|_{L^2}\left\|\nabla^k E\right\|_{L^2}.\end{aligned}\tag{3.14}$$

On the other hand, taking $l=k$ in (2.35), we have

$$-\int \nabla^k \partial_t u_2 \cdot \nabla^k E + 2\nu \left\|\nabla^k E\right\|_{L^2}^2 \leqslant \int \nabla\nabla^k n_2 \cdot \nabla^k E + C\left\|\nabla^k (u_1,u_2)\right\|_{L^2} \left\|\nabla^k E\right\|_{L^2} + \left\|\nabla^k (g_4 + u_1\times B)\right\|_{L^2} \left\|\nabla^k E\right\|_{L^2}. \tag{3.15}$$

Substituting (2.36) with $l=k$ into (3.15), we then have

$$\begin{aligned} & -\frac{d}{dt}\int \nabla^k u_2 \cdot \nabla^k E + 2\nu \left\|\nabla^k E\right\|_{L^2}^2 \\ & \lesssim \left\|\nabla^k u_2\right\|_{L^2}^2 + \left(\left\|\nabla^{k+1} n_2\right\|_{L^2} + \left\|\nabla^k (u_1,u_2)\right\|_{L^2}\right)\left\|\nabla^k E\right\|_{L^2} \\ & \quad + \left\|\nabla^k (\nabla\times B + g_5)\right\|_{L^2} \left\|\nabla^k u_2\right\|_{L^2} + \left\|\nabla^k (g_4 + u_1\times B)\right\|_{L^2} \left\|\nabla^k E\right\|_{L^2}. \end{aligned} \tag{3.16}$$

Since ε is small, we deduce from $(3.16)\times\varepsilon + (3.14)$ that there exists $\mathcal{F}_k(t) \sim \left\|\nabla^k (u_1,u_2,E)(t)\right\|_{L^2}^2$ such that, by Cauchy's inequality, lemmas 2.2–2.3, (1.10), and (3.13),

$$\begin{aligned} & \frac{d}{dt}\mathcal{F}_k(t) + \mathcal{F}_k(t) \lesssim \left\|\nabla^{k+1}(n_1,n_2)\right\|_{L^2}^2 + \left\|\nabla^{k+1} B\right\|_{L^2}^2 + \left\|\nabla^k (u_1\times B + u_2\times B)\right\|_{L^2}^2 \\ & \quad + \left\|\nabla^k (g_2+g_4)\right\|_{L^2}^2 + \left\|\nabla^k g_5\right\|_{L^2}^2 \\ & \lesssim \left\|\nabla^{k+1}(n_1,n_2,B)\right\|_{L^2}^2 + \left(\|(u_1,u_2)\|_{H^1} + \|\nabla B\|_{H^1}\right)^2 \left\|\nabla^{k+1} B\right\|_{L^2}^2 \\ & \quad + \|(n_1,n_2,u_1,u_2)\|_{L^\infty}^2 \left\|\nabla^{k+1}(n_1,n_2,u_1,u_2)\right\|_{L^2}^2 + \|\nabla(n_1,n_2)\|_{L^\infty}^2 \left\|\nabla^k (n_1,n_2)\right\|_{L^2}^2 \\ & \quad + \|(u_1,u_2)\|_{L^\infty}^2 \left\|\nabla^k (n_1,n_2)\right\|_{L^2}^2 \\ & \leqslant C_0 (1+t)^{-(k+1+s)}, \end{aligned} \tag{3.17}$$

where we require $N \geqslant 2k+4+s$. Applying the standard Gronwall lemma to (3.17), we obtain

$$\mathcal{F}_k(t) \leqslant \mathcal{F}_k(0) e^{-t} + C_0 \int_0^t e^{-(t-\tau)} (1+\tau)^{-(k+1+s)}\, d\tau \leqslant C_0 (1+t)^{-(k+1+s)}.$$

This implies

$$\left\|\nabla^k (u_1,u_2,E)(t)\right\|_{L^2} \lesssim \sqrt{\mathcal{F}_k(t)} \leqslant C_0 (1+t)^{-\frac{k+1+s}{2}}.$$

We thus complete the proof of (1.11). Notice that (1.12) now follows by (3.13) with the improved decay rate of E in (1.11), just requiring $N \geqslant 2k+6+s$.

Now we prove (1.13). Assume for the moment that $B_\infty = 0$. Then we can extract the following system from $(1.5)_3$–$(1.5)_4$, denoting $\psi = \operatorname{div} u_2$,

$$\begin{cases} \partial_t n_2 + \psi = g_3, \\ \partial_t \psi + \nu\psi - 2\nu^2 n_2 = -\Delta n_2 - \operatorname{div}(g_4 + u_1\times B) + 2\nu^2(-g - n_2). \end{cases} \tag{3.18}$$

Here g is defined in (2.24). Applying ∇^k to (3.18) and then taking the L^2 inner product with $2\nu^2\nabla^k n_2$, $\nabla^k \psi$ respectively, we obtain

$$\begin{aligned} & \frac{d}{dt}\int \nu^2 \left|\nabla^k n_2\right|^2 + \frac{1}{2}\left|\nabla^k \psi\right|^2 + \nu \left\|\nabla^k \psi\right\|_{L^2}^2 \\ & = 2\nu^2 \int \nabla^k g_3 \nabla^k n_2 - \int \nabla^k \Delta n_2 \nabla^k \psi \end{aligned}$$

$$-\int \nabla^k \left[\operatorname{div}(g_4+u_1\times B)-2\nu^2(-g-n_2)\right]\nabla^k\psi. \tag{3.19}$$

Applying ∇^k to $(3.18)_2$, multiplying by $-\nabla^k n_2$, integrating by parts over t and x-variables, and using Equation $(3.18)_1$, we obtain

$$\begin{aligned}-\frac{d}{dt}\int \nabla^k\psi\nabla^k n_2+2\nu^2\left\|\nabla^k n_2\right\|_{L^2}^2&=\left\|\nabla^k\psi\right\|_{L^2}^2+\nu\int\nabla^k n_2\nabla^k\psi-\int\nabla^k g_3\nabla^k\psi\\&+\int\nabla^k\left[\Delta n_2+\operatorname{div}(g_4+u_1\times B)-2\nu^2(-g-n_2)\right]\nabla^k n_2.\end{aligned} \tag{3.20}$$

Since ε is small, we deduce from $(3.20)\times\varepsilon+(3.19)$ that there exists $\mathcal{G}_k(t)\sim\left\|\nabla^k(n_2,\psi)\right\|_{L^2}^2$ such that, by Cauchy's inequality,

$$\frac{d}{dt}\mathcal{G}_k(t)+\mathcal{G}_k(t)\lesssim\left\|\nabla^{k+2}n_2\right\|_{L^2}^2+\left\|\nabla^k g_3\right\|_{L^2}^2+\left\|\nabla^{k+1}(g_4,u_1\times B)\right\|_{L^2}^2+\left\|\nabla^k(n_1n_2)\right\|_{L^2}^2, \tag{3.21}$$

where we have used $-g-n_2\sim n_1n_2$. By the product estimates (2.5), we obtain

$$\begin{aligned}\left\|\nabla^k(n_1n_2)\right\|_{L^2}^2&\lesssim\|n_1\|_{L^\infty}^2\left\|\nabla^k n_2\right\|_{L^2}^2+\left\|\nabla^k n_1\right\|_{L^2}^2\|n_2\|_{L^\infty}^2\\&\lesssim\delta_0\left\|\nabla^k n_2\right\|_{L^2}^2+\|n_2\|_{L^\infty}^2\left\|\nabla^k n_1\right\|_{L^2}^2\end{aligned}$$

and

$$\left\|\nabla^{k+1}(u_1\times B)\right\|_{L^2}^2\lesssim\|u_1\|_{L^\infty}^2\left\|\nabla^{k+1}B\right\|_{L^2}^2+\left\|\nabla^{k+1}u_1\right\|_{L^2}^2\|B\|_{L^\infty}^2.$$

The remaining terms on the right-hand side of (3.21) can be estimated similarly. Hence, we deduce from (3.21) that, by (1.10)–(1.12),

$$\begin{aligned}\frac{d}{dt}\mathcal{G}_k(t)+\mathcal{G}_k(t)&\lesssim\left\|\nabla^{k+2}n_2\right\|_{L^2}^2+\|u_1\|_{L^\infty}^2\left\|\nabla^{k+1}B\right\|_{L^2}^2\\&+\|B\|_{L^\infty}^2\left\|\nabla^{k+1}u_1\right\|_{L^2}^2+\|n_2\|_{L^\infty}^2\left\|\nabla^k n_1\right\|_{L^2}^2\\&+\|(n_1,n_2,u_1,u_2)\|_{L^\infty}^2\left\|\nabla^{k+2}(n_1,n_2,u_1,u_2)\right\|_{L^2}^2\\&+\|\nabla(n_1,n_2,u_1,u_2)\|_{L^\infty}^2\left\|\nabla^{k+1}(n_1,n_2,u_1,u_2)\right\|_{L^2}^2\\&\leqslant C_0\left((1+t)^{-(k+3+s)}+(1+t)^{-(k+7/2+2s)}+(1+t)^{-(k+11/2+2s)}\right)\\&\leqslant C_0(1+t)^{-(k+3+s)},\end{aligned} \tag{3.22}$$

where we required $N\geqslant 2k+8+s$. Applying the Gronwall lemma to (3.22) again, we obtain

$$\mathcal{G}_k(t)\leqslant\mathcal{G}_k(0)e^{-t}+C_0\int_0^t e^{-(t-\tau)}(1+\tau)^{-(k+3+s)}\,d\tau\leqslant C_0(1+t)^{-(k+3+s)}.$$

This implies

$$\left\|\nabla^k(n_2,\psi)(t)\right\|_{L^2}\lesssim\sqrt{\mathcal{G}_k(t)}\leqslant C_0(1+t)^{-\frac{k+3+s}{2}}. \tag{3.23}$$

If we require $N\geqslant 2k+12+s$, then by (3.23), we have

$$\left\|\nabla^{k+2}n_2(t)\right\|_{L^2}\leqslant C_0(1+t)^{-\frac{k+5+s}{2}}.$$

Having obtained such a faster decay rate, we can then improve (3.22) to

$$\frac{d}{dt}\mathcal{G}_k(t)+\mathcal{G}_k(t)\leqslant C_0\left((1+t)^{-(k+5+s)}+(1+t)^{-(k+7/2+2s)}\right)\leqslant C_0(1+t)^{-(k+7/2+2s)}.$$

Applying the Gronwall lemma again, we obtain

$$\left\|\nabla^k(n_2,\psi)(t)\right\|_{L^2}\lesssim\sqrt{\mathcal{G}_k(t)}\leqslant C_0(1+t)^{-(k/2+7/4+s)}.$$

We thus complete the proof of (1.13). The proof of Theorem 1.2 then follows.

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REFERENCES

[1] F. Chen, *Introduction to Plasma Physics and Controlled Fusion*, Plenum Press, New York, 1, 1984.

[2] G.Q. Chen, J.W. Jerome, and D.H. Wang, *Compressible Euler–Maxwell equations*, Transp. Theory, Statist. Phys., 29, 311–331, 2000.

[3] R.J. Duan, *Global smooth flows for the compressible Euler–Maxwell system: The relaxation case*, J. Hyp. Diff. Eqs., 8, 375–413, 2011.

[4] R.J. Duan, Q.Q. Liu, and C.J. Zhu, *The Cauchy problem on the compressible two-fluids Euler–Maxwell equations*, SIAM J. Math. Anal., 44, 102–133, 2012.

[5] P. Germain and N. Masmoudi, *Global existence for the Euler–Maxwell system*, Ann. Sci. Èc. Norm. Supèr, 47(4), 469–503, 2014.

[6] L. Grafakos, *Classical and Modern Fourier Analysis*, Pearson Education, Inc., Prentice Hall, 2004.

[7] Y. Guo, *Smooth irrotational flows in the large to the Euler–Poisson system in* $\mathbb{R}^{3+1}$, Commun. Math. Phys., 195, 249–265, 1998.

[8] Y. Guo, *The Vlasov–Poisson–Landau system in a periodic box*, J. Amer. Math. Soc., 25, 759–812, 2012.

[9] Y. Guo, A.D. Ionescu, and B. Pausader, *Global solutions of the Euler–Maxwell two-fluid system in* $3D$, preprint, arXiv: 1303.1060.

[10] Y. Guo and B. Pausader, *Global smooth ion dynamics in the Euler-Poisson system*, Commun. Math. Phys., 303, 89–125, 2011.

[11] Y. Guo and A.S. Tahvildar-Zadeh, *Formation of singularities in relativistic fluid dynamics and in spherically symmetric plasma dynamics*, in Nonlinear partial differential equations (Evanston, IL, 1998), Contemp. Math., Providence, RI: Amer. Math. Soc., 238, 151–161, 1999.

[12] Y. Guo and Y.J. Wang, *Decay of dissipative equations and negative Sobolev spaces*, Commun. Part. Diff. Eqs., 37, 2165–2208, 2012.

[13] M.L. Hajjej and Y.J. Peng, *Initial layers and zero-relaxation limits of Euler–Maxwell equations*, J. Diff. Eqs., 252, 1441–1465, 2012.

[14] A.D. Ionescu and B. Pausader, *Global solutions of quasilinear systems of Klein-Gordon equations in* $3D$, J. Eur. Math. Soc., 16, 2355–2431, 2014.

[15] J.W. Jerome, *The Cauchy problem for compressible hydrodynamic-Maxwell systems: A local theory for smooth solutions*, Diff. Int. Eqs., 16, 1345–1368, 2003.

[16] N. Ju, *Existence and uniqueness of the solution to the dissipative* $2D$ *quasi-geostrophic equations in the Sobolev space*, Commun. Math. Phys., 251, 365–376, 2004.

[17] T. Kato, *The Cauchy problem for quasi-linear symmetric hyperbolic systems*, Arch. Rat. Mech. Anal., 58, 181–205, 1975.

[18] P.A. Markowich, C. Ringhofer, and C. Schmeiser, *Semiconductor Equations*, Springer-Verlag, Vienna, 1990.

[19] Y.J. Peng, *Global existence and long-time behavior of smooth solutions of two-fluid Euler–Maxwell equations*, Ann. Inst. Henri Poincare Anal. Non Linear, 29, 737–759, 2012.

[20] Y.J. Peng and S. Wang, *Convergence of compressible Euler–Maxwell equations to compressible Euler equations*, Commun. Part. Diff. Eqs., 33, 349–376, 2008.

[21] Y.J. Peng and S. Wang, *Rigorous derivation of incompressible e-MHD equations from compressible Euler–Maxwell equations*, SIAM J. Math. Anal., 40, 540–565, 2008.

[22] Y.J. Peng, S. Wang, and Q.L. Gu, *Relaxation limit and global existence of smooth solutions of compressible Euler–Maxwell equations*, SIAM J. Math. Anal., 43, 944–970, 2011.

[23] H. Rishbeth and O.K. Garriott, *Introduction to Ionospheric Physics*, Academic Press, 1969.
[24] V. Sohinger and R.M. Strain, *The Boltzmann equation, Besov spaces, and optimal time decay rates in* $\mathbb{R}^n_x$, Adv. Math., 261, 274–332, 2014.
[25] R.M. Strain and Y. Guo, *Almost exponential decay near Maxwellian*, Commun. Part. Diff. Eqs., 31, 417–429, 2006.
[26] Z. Tan, Y.J. Wang, and Y. Wang, *Decay estimates of solutions to the compressible Euler–Maxwell system in* $\mathbb{R}^3$, J. Diff. Eqs., 257, 2846–2873, 2014.
[27] Z. Tan and Y. Wang, *Large-time behavior of solutions to the compressible non-isentropic Euler–Maxwell system in* $\mathbb{R}^3$, Nonlinear Anal. Real World Appl., 15, 187–204, 2014.
[28] B. Texier, *WKB asymptotics for the Euler–Maxwell equations*, Asymptot. Anal., 42, 211–250, 2005.
[29] B. Texier, *Derivation of the Zakharov equations*, Arch. Rat. Mech. Anal., 184, 121–183, 2007.
[30] Y. Ueda and S. Kawashima, *Decay property of regularity-loss type for the Euler–Maxwell system*, Meth. Appl. Anal., 18, 245–268, 2011.
[31] Y. Ueda, S. Wang, and S. Kawashima, *Dissipative structure of the regularity-loss type and time asymptotic decay of solutions for the Euler–Maxwell system*, SIAM J. Math. Anal., 44, 2002–2017, 2012.
[32] S. Wang, Y.H. Feng, and X. Li, *The asymptotic behavior of globally smooth solutions of bipolar non-isentropic compressible Euler–Maxwell system for plasma*, SIAM J. Math. Anal., 44, 3429–3457, 2012.
[33] S. Wang, Y.H. Feng, and X. Li, *The asymptotic behavior of globally smooth solutions of non-isentropic Euler–Maxwell equations for plasmas*, Appl. Math. Comput., 231, 299–306, 2014.
[34] Y.J. Wang, *Global solution and time decay of the Vlasov–Poisson–Landau system in* $\mathbb{R}^3$, SIAM J. Math. Anal., 44, 3281–3323, 2012.
[35] J. Xu, *Global classical solutions to the compressible Euler–Maxwell equations*, SIAM J. Math. Anal., 43, 2688–2718, 2011.
[36] J. Xu, J. Xiong, and S. Kawashima, *Global well-posedness in critical Besov spaces for two-fluid Euler–Maxwell equations*, SIAM J. Math. Anal., 45, 1422–1447, 2013.